

Question 5

Consider the exact equation

$$-2x \, dx + 2y \, dy = 0, \quad y(0) = 0.$$

In this problem a solution means a real, continuously differentiable function $y(x)$ on an open interval containing 0, satisfying $-2x + 2yy' = 0$ at every point, including 0.

Tasks

1. Find a potential and the level selected by the initial condition.
2. Classify all solution graphs on such an interval. Justify why no additional sign-switching choices are differentiable at 0.
3. Check $y = x$, $y = -x$, $y = |x|$ and $y = -|x|$ against the stated definition.
4. Explain why neither division by y nor the usual nonzero- F_y implicit-function criterion settles the initial point.

Question 5 – Solution

Strategy. The selected level has crossing branches. Analyze differentiability at their intersection directly in the original equation.

Step 1: Find the level. Since $M_y = N_x = 0$, a potential is $F = y^2 - x^2$. Along any solution, $dF/dx = 0$, and the initial condition gives

$$\boxed{y^2 - x^2 = 0}.$$

Thus for each $x \neq 0$, y/x is either 1 or -1 .

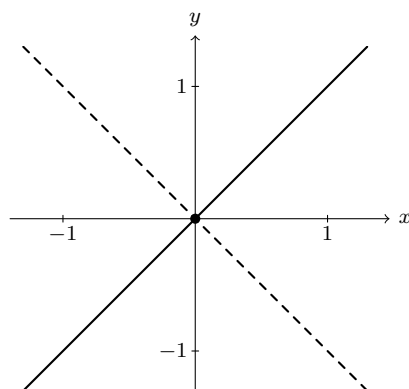
Step 2: Classify differentiable choices. Continuity makes y/x constant on each of the two connected pieces $x < 0$ and $x > 0$: a continuous function taking only the values ± 1 cannot switch between them. Write those constants as s_- and s_+ . At 0, the left and right difference quotients are s_- and s_+ . Differentiability requires them to agree. Therefore the only possibilities are

$$\boxed{y = x \quad \text{or} \quad y = -x}.$$

Both extend to all of $\mathbb{R}$.

Step 3: Verify and identify the failures. For either line, $yy' = x$ everywhere, including 0. The choices $|x|$ and $-|x|$ satisfy the level identity and the equation away from 0, but their one-sided derivatives disagree at 0. They fail the required definition.

Step 4: Explain the degenerate point. Dividing gives $y' = x/y$, which is undefined at the initial point and discards information about solutions of the original form there. Also $F_y = 2y = 0$ at $(0, 0)$, so the nonzero- F_y criterion does not apply. Its failure does not prove nonexistence: here two valid graph branches pass through the point.



Two differentiable solutions cross at $(0, 0)$: solid $y = x$, dashed $y = -x$.