

Question 4

The coefficient $N(x, y)$ is unknown in

$$(2x + y) dx + N(x, y) dy = 0.$$

Assume N is continuously differentiable on $\mathbb{R}^2$, the form is exact there, and the calibration data are $N(0, y) = 2y$ for every real y .

Tasks

1. Recover N uniquely and construct a potential.
2. Select the solution through $(0, 1)$ and find an explicit formula for its graph near that point.
3. Determine the maximal open interval of this graph and its slope at $x = 0$.
4. Find the vertical-tangent points of the full selected level curve. Explain why the full curve is not a single differentiable function $y(x)$.

Question 4 – Solution

Strategy. Use exactness as an equation for the missing coefficient, then distinguish a complete level curve from its graph branches.

Step 1: Recover the form. Exactness gives $N_x = M_y = 1$, so $N = x + h(y)$. The data force $h(y) = 2y$, uniquely. A potential is

$$N = x + 2y, \quad F = x^2 + xy + y^2.$$

Its derivatives are $2x + y$ and $x + 2y$, as required.

Step 2: Select a branch. The initial point gives $F = 1$. Solving the quadratic in y and selecting $y(0) = 1$ yields

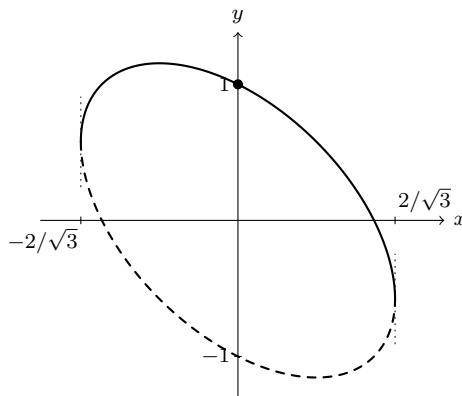
$$y = \frac{-x + \sqrt{4 - 3x^2}}{2}, \quad I = (-2/\sqrt{3}, 2/\sqrt{3}).$$

At $(0, 1)$, $y' = -(2x + y)/(x + 2y) = -1/2$.

Step 3: Interpret the endpoints and full curve. On this branch, $N = \sqrt{4 - 3x^2} > 0$. At the two endpoints, $N = 0$, while $M = 3x/2 \neq 0$. Thus a finite y' cannot satisfy $M + Ny' = 0$ there. The full ellipse has vertical tangents at

$$(2/\sqrt{3}, -1/\sqrt{3}), \quad (-2/\sqrt{3}, 1/\sqrt{3}).$$

It remains a smooth curve at these points, but cannot be continued through either as a differentiable graph over x . Most vertical lines through the ellipse meet both quadratic branches, so the complete ellipse is not one function $y(x)$.



Solid: IVP branch, excluding its vertical-tangent endpoints. Dashed: the other half.