

Question 3

Let $a, b \in \mathbb{R}$. Consider

$$(3x^2y^2 + 2bxy + e^y) dx + (2ax^3y + 5x^2 + xe^y + 4y^3) dy = 0.$$

Tasks

1. Determine every pair (a, b) for which the form is exact on $\mathbb{R}^2$. Explain why testing equality at a single point is insufficient.
2. For those parameters, construct a potential and verify both of its first partial derivatives.
3. Find the implicit solution through $(0, 1)$ and its tangent line there.
4. A student checks $M_y = N_x$ only at $(0, 0)$ and accepts every pair (a, b) . Show why that check cannot distinguish any of the parameters.

Question 3 – Solution

Strategy. Exactness requires an identity on a region, so compare the variable-dependent terms before constructing a potential.

Step 1: Determine the parameters. The cross partials are

$$M_y = 6x^2y + 2bx + e^y, \quad N_x = 6ax^2y + 10x + e^y.$$

Their difference is $6(1-a)x^2y + (2b-10)x$. For this to vanish for all (x, y) , first put $y = 0$ and $x = 1$, obtaining $b = 5$. Then put $x = y = 1$, obtaining $a = 1$. Conversely, those choices make the difference identically zero. Hence

$$\boxed{a = 1, \quad b = 5}.$$

Step 2: Construct and verify the potential. With these values, integrating M in x yields

$$F = x^3y^2 + 5x^2y + xe^y + h(y).$$

Matching F_y with N gives $h'(y) = 4y^3$. Thus choose

$$F = x^3y^2 + 5x^2y + xe^y + y^4.$$

Direct differentiation gives $F_x = 3x^2y^2 + 10xy + e^y = M$ and $F_y = 2x^3y + 5x^2 + xe^y + 4y^3 = N$.

Step 3: Apply the data and find the tangent. Since $F(0, 1) = 1$, the selected level is $\boxed{F(x, y) = 1}$. At that point, $N = 4 \neq 0$, so it determines a local solution graph. Its slope is $-M/N = -e/4$, giving

$$\boxed{y - 1 = -\frac{e}{4}x}.$$

Step 4: Explain the failed one-point test. At $(0, 0)$, both cross partials equal 1 for every a, b : all parameter-dependent terms vanish there. This agreement is necessary at that point but says nothing about agreement on a neighborhood. Exactness cannot be certified from that single sample.