

Question 1

Consider the differential equation

$$(2xy + 1) dx + (x^2 + y^2) dy = 0, \quad y(1) = 1.$$

A potential for $M dx + N dy$ is a function F satisfying $F_x = M$ and $F_y = N$; its level curves $F = C$ give implicit solutions.

Tasks

1. Check exactness and construct a potential, retaining the unknown function that arises when integrating with respect to x .
2. Apply the initial condition and verify the resulting implicit solution by differentiation.
3. Find the slope and tangent line at $(1, 1)$.
4. Prove that the selected level determines a unique differentiable function $y(x)$ on all of $\mathbb{R}$, despite the vanishing of $x^2 + y^2$ at the origin.

Question 1 – Solution

Strategy. Recover a potential from both coefficients, then examine its derivative with respect to y on the selected level.

Step 1: Test and integrate. Here $M_y = 2x = N_x$, so the form is exact on $\mathbb{R}^2$. Integrating $F_x = M$ gives

$$F = x^2y + x + h(y), \quad F_y = x^2 + h'(y).$$

Matching N requires $h'(y) = y^2$, so we may choose $F = x^2y + x + y^3/3$. At $(1, 1)$, $F = 7/3$. Thus

$$\boxed{x^2y + x + \frac{y^3}{3} = \frac{7}{3}}.$$

Step 2: Verify and find the tangent. Differentiating this identity gives

$$2xy + 1 + (x^2 + y^2)y' = 0,$$

exactly the original equation for a graph. At $(1, 1)$ this yields $y' = -3/2$, hence $\boxed{y - 1 = -\frac{3}{2}(x - 1)}$.

Step 3: Establish a global graph. For fixed x , the function $u \mapsto x^2u + u^3/3$ is strictly increasing and ranges over all real numbers. This remains true at $x = 0$, where the cubic alone is strictly increasing. Therefore the equation

$$x^2u + \frac{u^3}{3} = \frac{7}{3} - x$$

has exactly one real root for every real x .

The derivative $F_y = x^2 + y^2$ can vanish only at $(0, 0)$, but that point has $F = 0$, not $7/3$. It is consequently nonzero everywhere on the selected level. The implicit function theorem gives a differentiable local graph at each point; the uniqueness of each root makes those graphs agree. They form one differentiable solution on all of $\mathbb{R}$. The isolated degeneracy on a different level does not obstruct this IVP.