

Question 10

Design an affine coefficient $g(x) = \alpha + \beta x$ in the separable equation

$$y' = g(x)(1 - y^2)$$

so that one solution satisfies

$$y(0) = 0, \quad y(1) = \frac{1}{2}, \quad y'(1) = 0.$$

The constants α and β are real.

Tasks

1. Find α and β and justify why the three requirements determine them uniquely.
2. Find an explicit formula for the selected solution. You may use exponentials instead of hyperbolic functions.
3. Determine its maximal interval, every zero, its global maximum, and its limits as $x \rightarrow \pm\infty$.
4. Verify the three requirements directly and list the constant solutions omitted when separating by $1 - y^2$.

Question 10 – Solution

Strategy. Use the zero terminal slope to constrain the coefficient, then use the separated integral to enforce the target height.

Step 1: Determine the coefficient. Since $1 - y(1)^2 = 3/4$, $y'(1) = 0$ forces $\alpha + \beta = 0$. The smooth right-hand side gives local uniqueness, so the solution starting at 0 cannot meet either constant solution $y = \pm 1$. On its branch $-1 < y < 1$,

$$\frac{1}{2} \ln \frac{1+y}{1-y} = \alpha x + \frac{\beta x^2}{2}.$$

At $x = 1$, this becomes $\frac{1}{2} \ln 3 = \alpha + \beta/2$. Hence

$$\boxed{\alpha = \ln 3, \quad \beta = -\ln 3}.$$

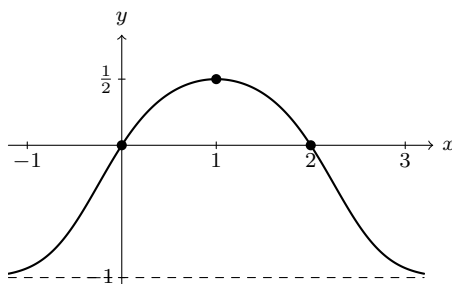
These two independent linear conditions have exactly one pair of coefficients.

Step 2: Solve and interpret the curve. Set $E(x) = 3^{2x-x^2} > 0$. Exponentiating and isolating y gives

$$\boxed{y(x) = \frac{E(x) - 1}{E(x) + 1}, \quad I = \mathbb{R}}.$$

Since $E + 1 > 0$, the formula is globally smooth and lies strictly between -1 and 1 . Its zeros are $x = 0, 2$. The exponent $2x - x^2 = 1 - (x - 1)^2$ has its unique maximum at 1 , and $(E - 1)/(E + 1)$ increases with E . Thus the unique global maximum is $y(1) = 1/2$, while both limits at infinity are -1 .

Step 3: Verify and restore constants. Here $E' = 2(\ln 3)(1 - x)E$ and $1 - y^2 = 4E/(E + 1)^2$, so $y' = (\ln 3)(1 - x)(1 - y^2)$. Also $E(0) = 1$ and $E(1) = 3$ verify both values and the zero slope. The omitted constants are $\boxed{y \equiv 1, y \equiv -1}$; neither satisfies $y(0) = 0$.



The unique maximum is $(1, 1/2)$; both tails approach the dashed level $y = -1$.