

Question 9

Let A and B be prescribed real numbers. Seek a real function y that is continuously differentiable on $[-1, 1]$ and satisfies

$$y' = 2xy^2, \quad y(-1) = A, \quad y(1) = B.$$

At the endpoints, derivatives are understood as one-sided derivatives.

Tasks

1. For $A \neq 0$, separate the equation and express the candidate solution in terms of A . Also treat $A = 0$.
2. Determine all pairs (A, B) for which a solution exists on the entire interval, and prove your conditions are sufficient as well as necessary.
3. Show why matching the two endpoint values in a formula does not by itself prove existence. Use $A = B = -2$ as a test case.
4. For every admissible pair, find $y(0)$ and decide whether the solution is unique.

Question 9 – Solution

Strategy. Separate from the left endpoint, then check the denominator on the whole prescribed interval before accepting the right endpoint.

Step 1: Obtain the candidate. For a nonzero solution,

$$-\frac{1}{y} = x^2 + C, \quad \frac{1}{y} = \frac{1}{A} + 1 - x^2, \quad \boxed{y = \frac{A}{1 + A(1 - x^2)}}.$$

For $A = 0$, $y \equiv 0$ is a solution. No solution can leave or cross zero: the right-hand side $2xy^2$ has a continuous y -derivative, $4xy$, so the usual local uniqueness theorem applies at every finite point. Consequently the separated formula applies to any solution with $A \neq 0$ throughout its interval.

Step 2: Test the entire interval. On $[-1, 1]$, the quantity $1 - x^2$ ranges over $[0, 1]$. If $A \geq 0$, the denominator is at least 1. If $-1 < A < 0$, it is at least $1 + A > 0$. For $A = -1$ it vanishes at $x = 0$; for $A < -1$ it vanishes at

$$x = \pm\sqrt{1 + 1/A} \in (-1, 1).$$

The nonzero numerator cannot cancel these zeros. Any admissible formula gives $y(1) = A$, so the necessary conditions are

$$\boxed{B = A \quad \text{and} \quad A > -1}.$$

They are also sufficient: the displayed formula, including $A = 0$, is smooth on a neighborhood of $[-1, 1]$. Its derivative is $2xA^2/[1 + A(1 - x^2)]^2 = 2xy^2$, and it gives both specified endpoint values.

Step 3: Expose the false endpoint test. For $A = B = -2$, the candidate is $y = -2/(2x^2 - 1)$. It has the desired values at $x = \pm 1$ but poles at $x = \pm 1/\sqrt{2}$. Hence it is not a solution on the entire interval.

Step 4: Give the center value and uniqueness. For every admissible pair, $\boxed{y(0) = A/(1 + A)}$. The initial value at -1 already fixes the candidate uniquely; for $A = 0$, local uniqueness also forbids any departure from zero. The second endpoint is a compatibility condition, not an extra free integration constant.