

Question 8

Consider the parameterized initial-value problem

$$y' = (\sin x)y^2, \quad y(0) = a, \quad a > 0.$$

Tasks

1. Solve by separation and determine the maximal open interval containing 0 for every positive a .
2. Find the precise threshold separating global bounded solutions from finite-endpoint blow-up. Include the threshold case.
3. For the global solutions, find the minimum, maximum, and a period.
4. A student argues that $\int_0^{2\pi} \sin x \, dx = 0$ guarantees $y(2\pi) = a$ for every $a > 0$. Explain exactly when this reasoning is valid and why it can fail.

Question 8 – Solution

Strategy. Track the denominator of the separated formula throughout an interval, not merely at its endpoints.

Step 1: Integrate. Since $a > 0$, separation is valid near 0:

$$-\frac{1}{y} = -\cos x + C, \quad \boxed{y = \frac{a}{1 - a + a \cos x}}.$$

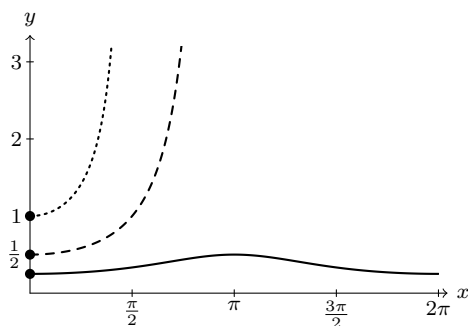
Its derivative is $a^2 \sin x / (1 - a + a \cos x)^2 = (\sin x)y^2$, and $y(0) = a$.

Step 2: Classify all positive parameters. The denominator $D = 1 - a + a \cos x$ ranges from $1 - 2a$ to 1.

- If $0 < a < 1/2$, then $D > 0$ everywhere: $I = \mathbb{R}$ and y is bounded.
- If $a = 1/2$, then $y = 1/(1 + \cos x)$ and $I = (-\pi, \pi)$.
- If $a > 1/2$, let $\alpha = \arccos(1 - 1/a) \in (0, \pi)$. Then $I = (-\alpha, \alpha)$.

In the last two cases $D \downarrow 0$ at either endpoint from inside I , so $y \rightarrow +\infty$. Thus the threshold itself already blows up.

Step 3: Determine extrema and audit the return claim. For $0 < a < 1/2$, $\min y = a$ at $x = 2n\pi$ and $\max y = a/(1 - 2a)$ at $x = (2n + 1)\pi$, $n \in \mathbb{Z}$; 2π is a period. Only this parameter range gives a solution through the full cycle. For $a \geq 1/2$, a pole occurs before reaching 2π . Substituting 2π into the algebraic formula gives a value on a disconnected branch, not a continuation of the initial solution.



Solid: $a = 1/4$. Dashed: $a = 1/2$. Dotted: $a = 1$. Only the solid curve completes a cycle.