

Question 7

A positive concentration obeys a power-law decay model

$$\frac{dy}{dt} = -ky^p, \quad k > 0, \quad p \in \mathbb{R}.$$

Use fixed units for time and concentration. Exact observations are

$$y(0) = 16, \quad y(1) = 8, \quad y(2) = 4.$$

Assume one pair of constants (k, p) describes the entire observation interval.

Tasks

1. Express each of the two observed one-unit decay times as a separated definite integral, without assuming $p = 1$.
2. Prove that these observations determine p uniquely. Treat the case $p = 1$ without dividing by $1 - p$.
3. Find k and the resulting solution, and predict when the concentration first reaches 1.
4. Explain why the first two observations alone would not determine p .

Question 7 – Solution

Strategy. Compare passage-time integrals on two concentration intervals related by scaling; the unknown constant k then cancels.

Step 1: Write the two passage times. Since $y > 0$ and $k > 0$, the concentration decreases strictly. Separation gives

$$1 = \frac{1}{k} \int_8^{16} s^{-p} ds, \quad 1 = \frac{1}{k} \int_4^8 s^{-p} ds.$$

These integrals are positive and finite for every real p .

Step 2: Recover the exponent. In the first integral put $s = 2u$:

$$\int_8^{16} s^{-p} ds = 2^{1-p} \int_4^8 u^{-p} du.$$

The observed equal passage times force $2^{1-p} = 1$, hence $\boxed{p = 1}$. Conversely, $p = 1$ makes the two integrals equal. No division by $1 - p$ was used, so the logarithmic case was retained throughout.

Step 3: Determine the rate and prediction. For $p = 1$,

$$k = \int_8^{16} \frac{ds}{s} = \ln 2, \quad \ln \frac{y}{16} = -t \ln 2.$$

Therefore $\boxed{y(t) = 16 2^{-t}}$. It gives exactly 16, 8, 4 at the measured times, and differentiating gives $y' = -(\ln 2)y$. It is positive for every real t . The value 1 is first reached at $\boxed{t = 4}$ because the solution is strictly decreasing.

Step 4: Identify the information supplied by the third observation. With only $y(0) = 16$ and $y(1) = 8$, each real p could be paired with

$$k(p) = \int_8^{16} s^{-p} ds > 0.$$

The corresponding separated solution decreases from 16 to 8 in one unit of time and stays positive during that interval. Thus infinitely many exponents fit the first two data; the second measured halving time identifies the exponent within this model family.