

Question 5

Consider these two equations for real x and y :

$$(A) \quad y' = x + y, \quad (B) \quad y' = xy + x + y + 1.$$

Here “directly separable” means that the right-hand side can be written $g(x)h(y)$ on the region in question, without a change of variables.

Tasks

1. Prove that any product $f(x, y) = g(x)h(y)$ satisfies

$$f(a, c)f(b, d) = f(a, d)f(b, c).$$

2. Use $a = c = 1$ and $b = d = 2$ to show that (A) is not directly separable on the rectangle $[1, 2] \times [1, 2]$.
3. Factor the right-hand side of (B), then find all its real solutions on intervals and list any constant solution separately.
4. Solve (B) with $y(0) = -2$, verify it, and explain why division by $x + 1$ is unnecessary even at $x = -1$.

Question 5 – Solution

Strategy. Test whether variables actually factor before performing separation; a sum of terms can still conceal a product.

Step 1: Establish the necessary product test. If $f(x, y) = g(x)h(y)$, then both products in the stated identity equal $g(a)g(b)h(c)h(d)$. This argument does not divide by any factor and remains valid if some factors vanish.

Step 2: Reject direct separation for (A). For $f(x, y) = x + y$, the proposed rectangle gives

$$f(1, 1)f(2, 2) = 2 \cdot 4 = 8, \quad f(1, 2)f(2, 1) = 3 \cdot 3 = 9.$$

The necessary identity fails, so there is no product representation on this rectangle. This conclusion concerns direct separation on that region; it makes no claim about other methods of solving (A).

Step 3: Solve the factored equation (B). Here $xy + x + y + 1 = (x + 1)(y + 1)$. First, $y \equiv -1$ is a constant solution. For $y \neq -1$,

$$\frac{dy}{y + 1} = (x + 1) dx, \quad \ln |y + 1| = \frac{x^2}{2} + x + C.$$

Absorbing the fixed sign into a nonzero constant gives

$$\boxed{y = -1 + Ae^{x^2/2+x}}.$$

Allowing $A = 0$ includes the constant solution as well. These are all solutions: for any differentiable solution, the product rule applied to $(y + 1)e^{-x^2/2-x}$ gives derivative zero, even at $y = -1$.

Step 4: Apply and check the data. The initial condition requires $A = -1$, so $\boxed{y = -1 - e^{x^2/2+x}}$ on all of $\mathbb{R}$. Its derivative is $-(x + 1)e^{x^2/2+x} = (x + 1)(y + 1)$, and $y(0) = -2$. Separation integrates the factor $x + 1$; it never requires its reciprocal. The point $x = -1$ is an ordinary point with $y' = 0$.