

Question 4

An explicit elementary formula is not required in this problem:

$$y' = \frac{x}{1 + e^y}, \quad y(0) = 0.$$

Tasks

1. Use separation to find an implicit equation for the solution.
2. Prove that it determines exactly one real value of y for every real x , and that the resulting function is differentiable.
3. Determine its symmetry, monotonicity, and minimum. Prove the bound $0 \leq y(x) \leq x^2/4$ and identify every equality case.
4. Verify the initial condition and differential equation, and explain why failure to isolate y in elementary functions is not a failure to solve the problem.

Question 4 – Solution

Strategy. Treat the left side of the integrated equation as an invertible function rather than forcing an elementary inverse.

Step 1: Integrate. Multiplication and integration give

$$(1 + e^y) dy = x dx, \quad \boxed{y + e^y = 1 + \frac{x^2}{2}}.$$

The constant is 1 because $y(0) = 0$.

Step 2: Prove global existence and uniqueness of the implicit value. Let $F(u) = u + e^u$. It is continuous and strictly increasing because $F'(u) = 1 + e^u > 0$. Also $F(u) \rightarrow -\infty$ as $u \rightarrow -\infty$ and $F(u) \rightarrow \infty$ as $u \rightarrow \infty$. It therefore has an inverse on all of $\mathbb{R}$. Since its derivative never vanishes, the inverse is differentiable. Thus

$$y(x) = F^{-1}(1 + x^2/2)$$

defines a unique differentiable solution for every real x . Any solution through $(0, 0)$ must satisfy the same integrated identity, establishing uniqueness for the initial-value problem.

Step 3: Extract shape and an exact bound. Uniqueness of the implicit value and the dependence on x^2 show $y(-x) = y(x)$. Since $F(0) = 1$, $y \geq 0$, with equality only at $x = 0$. The differential equation makes y decreasing for $x < 0$ and increasing for $x > 0$, so its unique minimum is 0.

Using $e^y \geq 1 + y$, with equality only at $y = 0$, gives

$$1 + \frac{x^2}{2} = y + e^y \geq 1 + 2y, \quad \boxed{0 \leq y \leq \frac{x^2}{4}}.$$

Equality in either bound occurs only at $x = 0$.

Step 4: Verify and interpret. Implicit differentiation yields $(1 + e^y)y' = x$, exactly the given equation. At $x = 0$, strict monotonicity of F forces $y = 0$. The implicit formula uniquely specifies every solution value, its full domain, and its derivative; an elementary inverse is unnecessary.