

Question 2

For the autonomous equation

$$y' = y(3 - y),$$

consider the initial condition $y(0) = 4$.

Tasks

1. Find every constant solution before dividing by any expression involving y .
2. Obtain the nonconstant solution family by separation and solve the given initial-value problem.
3. Find its maximal interval containing 0, its direction of motion, and its limiting values at the ends of that interval.
4. Find the exact positive time when $y = 7/2$. Explain why this solution never reaches or crosses $y = 3$ at a finite time.

Question 2 – Solution

Strategy. Preserve the two equilibrium solutions, then use partial fractions on a nonconstant branch.

Step 1: Separate without losing constants. The constant solutions are $y \equiv 0, y \equiv 3$. Away from those values,

$$\frac{1}{y(3-y)} = \frac{1}{3y} + \frac{1}{3(3-y)}, \quad \frac{1}{3} \ln \left| \frac{y}{3-y} \right| = x + C.$$

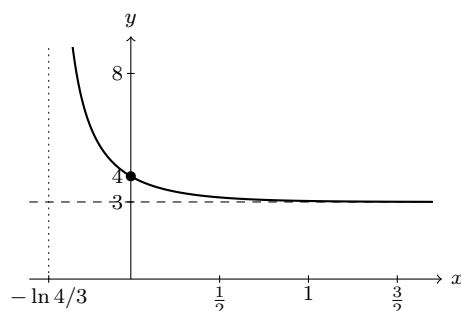
The nonconstant family can be written $y = 3/(1 + Ae^{-3x})$, $A \neq 0$, on intervals where the denominator is nonzero. The value $A = 0$ restores $y \equiv 3$; $y \equiv 0$ must still be listed separately.

Step 2: Apply the data and determine the domain. From $4 = 3/(1 + A)$, $A = -1/4$, so

$$y(x) = \frac{3}{1 - \frac{1}{4}e^{-3x}}, \quad I = (-\ln 4/3, \infty).$$

Here $y > 3$, so $y' < 0$. The limits are $+\infty$ at the left endpoint and 3 as $x \rightarrow \infty$. Writing $D = 1 - \frac{1}{4}e^{-3x}$ gives $y' = -9e^{-3x}/(4D^2) = y(3-y)$, verifying the equation.

Step 3: Compute the passage time. The equation $7/2 = 3/(1 - \frac{1}{4}e^{-3x})$ gives $x = \frac{1}{3} \ln(7/4)$. At any finite $x \in I$, the exponential is positive, so $y - 3 > 0$: the limiting level is never attained or crossed.



The decreasing solution stays above the dashed equilibrium $y = 3$.