

Question 1

Consider the initial-value problem

$$y' = 2x(1 + y^2), \quad y(0) = 1.$$

Tasks

1. Solve by separation, explaining how the initial condition selects the integration constant.
2. Find the maximal open interval containing 0 on which the solution is finite and differentiable.
3. Determine its monotonicity, minimum, and behavior at both interval endpoints.
4. A student says that the periodicity of tangent extends this same solution through each vertical asymptote. Explain the error and verify the valid solution directly.

Question 1 – Solution

Strategy. Separate using the everywhere-positive factor $1 + y^2$, then respect the range of the inverse tangent.

Step 1: Integrate and select the branch.

$$\frac{dy}{1+y^2} = 2x \, dx, \quad \arctan y = x^2 + C, \quad C = \frac{\pi}{4}.$$

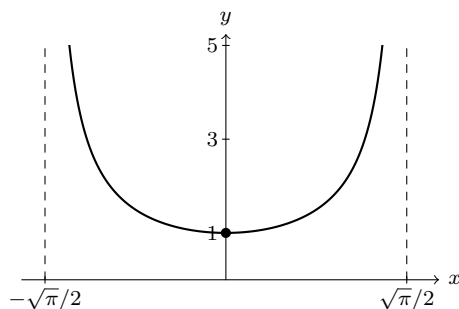
Thus $y(x) = \tan(x^2 + \pi/4)$ near 0. No constant solution was lost, since $1 + y^2$ never vanishes for real y .

Step 2: Identify the interval and shape. For the branch through 0, we need $-\pi/2 < x^2 + \pi/4 < \pi/2$. Hence

$$I = (-\sqrt{\pi}/2, \sqrt{\pi}/2).$$

The derivative has the sign of x : the solution decreases to its minimum $y(0) = 1$ and then increases. At either endpoint approached from inside I , the angle tends to $\pi/2$ from below, so $y \rightarrow +\infty$.

Step 3: Verify and explain the obstruction. Differentiation gives $y' = 2x \sec^2(x^2 + \pi/4) = 2x(1 + y^2)$, and $y(0) = 1$. A real differentiable extension through an endpoint would have a finite, continuous value there, contradicting blow-up. Other tangent branches lie on separate intervals; periodicity does not connect them across a pole.



The branch through $(0, 1)$ has a pole at each dashed boundary.