

Question 10

Let a be a real parameter in

$$y' + ay = e^{-x}, \quad y(0) = 1.$$

Tasks

1. Solve the IVP for every real a , handling separately any parameter value for which the usual antiderivative formula changes.
2. Verify the initial condition and equation, and state the solution interval for each parameter.
3. For fixed x , show that the formula for $a \neq 1$ tends to the $a = 1$ solution as $a \rightarrow 1$. Explain why an apparent denominator singularity in a parameter need not be a singularity of the solution.
4. Determine exactly which values of a produce a bounded solution on $[0, \infty)$, and compute the limit or divergence as $x \rightarrow \infty$ in each parameter range.

Question 10 – Solution

Strategy. Keep the parameter visible during integration, and isolate the exceptional exponent before dividing by $a - 1$.

Step 1: Integrate with the initial value. With $\mu = e^{ax}$,

$$(e^{ax}y)' = e^{(a-1)x}, \quad e^{ax}y = 1 + \int_0^x e^{(a-1)t} dt.$$

Consequently

$$y(x) = \begin{cases} \frac{e^{-x} + (a-2)e^{-ax}}{a-1}, & a \neq 1, \\ (1+x)e^{-x}, & a = 1. \end{cases}$$

Each formula is defined on $\mathbb{R}$.

Step 2: Verify. At zero the numerator for $a \neq 1$ is $a - 1$, so $y(0) = 1$; the exceptional formula also gives 1. Differentiating the definite-integral identity above gives $y' + ay = e^{-x}$. At $a = 1$, a direct check is $y' = -xe^{-x}$ and $y' + y = e^{-x}$.

Step 3: Remove the apparent parameter singularity. For $a \neq 1$, rewrite the result as

$$y = e^{-ax} \left[1 + \frac{e^{(a-1)x} - 1}{a-1} \right].$$

For fixed x , the quotient tends to x as $a \rightarrow 1$, by the derivative of the exponential (also at $x = 0$). Hence $y \rightarrow (1+x)e^{-x}$. The singularity in the divided parameter formula is removable.

Step 4: Long-time behavior. If $a > 0$ and $a \neq 1$, both exponentials decay, so $y \rightarrow 0$; the same holds at $a = 1$. If $a = 0$, $y = 2 - e^{-x} \rightarrow 2$. If $a < 0$, the coefficient $(a-2)/(a-1)$ is positive and e^{-ax} grows without bound, so $y \rightarrow +\infty$. Thus

$$y \text{ is bounded on } [0, \infty) \iff a \geq 0.$$