

Question 9

The equation

$$xy' - y = x^2$$

has a vanishing leading coefficient at $x = 0$. Initially impose $y(1) = 2$.

Tasks

1. Solve the IVP on $x > 0$ using an integrating factor after normalization.
2. Determine whether this solution extends to a differentiable solution of the original equation on all of $\mathbb{R}$. If so, prove that such an extension is unique.
3. Now discard $y(1) = 2$ and prescribe only $y(0) = 0$. Determine all differentiable solutions on $\mathbb{R}$ satisfying this new initial condition.
4. Decide whether a differentiable solution with $y(0) = 1$ can exist. Explain why dividing by x and invoking a regular standard-form IVP at 0 would be invalid.

Question 9 – Solution

Strategy. Normalize only where $x \neq 0$. To cross the exceptional point, return to the original equation and check value and derivative matching.

Step 1: Solve on the positive half-line. Dividing by $x > 0$ gives $y' - y/x = x$. The integrating factor is $e^{-\ln x} = 1/x$, so

$$(y/x)' = 1, \quad y = x^2 + Cx.$$

Since $y(1) = 2$, $C = 1$, giving $y = x^2 + x \quad (x > 0)$.

Step 2: Unique differentiable extension. The polynomial $x^2 + x$ satisfies $xy' - y = x^2$ everywhere, including $x = 0$. Any solution on $x < 0$ also satisfies $(y/x)' = 1$, hence has form $x^2 + Dx$. Continuity at 0 forces $y(0) = 0$, and differentiability requires the slopes at zero to match: $D = 1$. Thus

$$y = x^2 + x \text{ is the unique differentiable extension to } \mathbb{R}.$$

Step 3: Data only at the exceptional point. With only $y(0) = 0$, the same argument allows any common slope C on the two sides. All global differentiable solutions are

$$y = x^2 + Cx, \quad C \in \mathbb{R}.$$

Indeed $x(2x + C) - (x^2 + Cx) = x^2$, and every such polynomial has value zero at 0. This IVP is therefore nonunique.

Step 4: Incompatible data and normalization. At $x = 0$, the original equation requires $-y(0) = 0$, ruling out $y(0) = 1$. The normalized coefficient $-1/x$ is undefined at zero, so regular standard-form conclusions cannot be applied there. The algebraic condition at the exceptional point must be checked separately.