

Question 8

A switched forcing acts on a linear equation for $t \geq 0$:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ 0, & t > 1. \end{cases}$$

Seek a continuous function that is continuously differentiable on each side of $t = 1$ and satisfies the equation away from that switching time. No derivative at $t = 1$ is assumed.

Tasks

1. Solve on $0 \leq t < 1$ using an integrating factor, then use continuity to solve on $t > 1$.
2. Find the global maximum for $t \geq 0$, including its time, and the limit as $t \rightarrow \infty$.
3. Compute the left and right derivatives at $t = 1$. Is the resulting function a classical differentiable solution at that point?
4. Evaluate $\int_0^\infty y(t) dt$ and relate it to the total forcing $\int_0^\infty f(t) dt$. Sketch the continuous response in your solution, showing the change in slope at the switch.

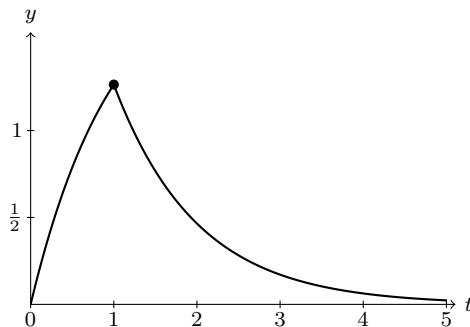
Question 8 – Solution

Strategy. Integrate separately on the two smooth pieces and match values at the switch; continuity need not imply a continuous derivative.

Step 1: Solve and match. Before the switch, $(e^t y)' = 2e^t$, so $y = 2 + Ce^{-t}$. Initial data give $C = -2$ and $y(1) = A := 2(1 - e^{-1})$. Afterward $(e^t y)' = 0$; matching this value yields

$$y(t) = \begin{cases} 2(1 - e^{-t}), & 0 \leq t \leq 1, \\ Ae^{-(t-1)}, & t > 1, \end{cases} \quad A = 2(1 - e^{-1}).$$

Differentiation on each open piece verifies its corresponding equation.



The maximum occurs at the switch $t = 1$, where the derivative jumps by -2 .

Step 2: Maximum and derivative jump. Before 1, $y' = 2e^{-t} > 0$; after 1, $y' = -Ae^{-(t-1)} < 0$. Hence the global maximum is A at $t = 1$, and the limit is zero. At the switch,

$$y'(1^-) = 2/e, \quad y'(1^+) = -A = 2/e - 2.$$

The jump is -2 , so the derivative does not exist there. The response meets the requested piecewise conditions, but is not a classical differentiable solution at the switch.

Step 3: Total response. Direct integration gives

$$\int_0^1 2(1 - e^{-t})dt = 2/e, \quad \int_1^\infty Ae^{-(t-1)}dt = A,$$

so $\int_0^\infty y(t)dt = 2 = \int_0^\infty f(t)dt$. Equivalently, integrate $y' + y = f$ on the two pieces: continuity cancels the joining values and $y(0) = \lim_{t \rightarrow \infty} y = 0$ cancels the endpoint contribution.