

Question 7

Consider the periodically forced equation

$$y' + 2y = 3 \cos x + \sin x,$$

with initial condition $y(0) = 0$.

Tasks

1. Solve the IVP using an integrating factor. Show how you evaluate the integral of the forcing times that factor.
2. Find the unique 2π -periodic solution of the equation, and verify both its periodicity and its differential equation.
3. Find the earliest $X \geq 0$ such that the IVP solution differs from the periodic solution by at most $1/100$ for every $x \geq X$.
4. Sketch the IVP solution and the periodic solution on $[0, 2\pi]$ in your solution. Explain which part of the general solution is the transient and why the prescribed IVP solution is not itself periodic.

Question 7 – Solution

Strategy. Integrate the forcing after multiplying by e^{2x} , then separate periodic and decaying terms.

Step 1: Integrate. The equation becomes

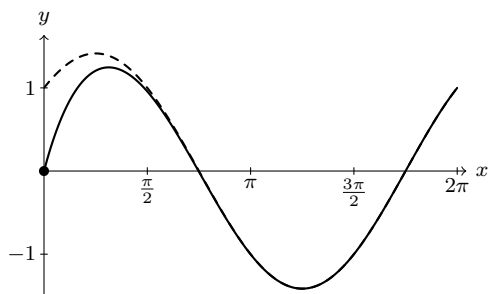
$$(e^{2x}y)' = e^{2x}(3\cos x + \sin x).$$

Seek an antiderivative $e^{2x}(A\cos x + B\sin x)$. Matching coefficients gives $2A + B = 3$ and $2B - A = 1$, hence $A = B = 1$. Indeed,

$$[e^{2x}(\cos x + \sin x)]' = e^{2x}(3\cos x + \sin x).$$

Therefore $y = \cos x + \sin x + Ce^{-2x}$. Since $y(0) = 0$, $C = -1$, giving

$$y = \cos x + \sin x - e^{-2x}.$$



Solid: IVP solution. Dashed: periodic solution $y_p = \cos x + \sin x$.

Step 2: The only periodic member. The function $y_p = \cos x + \sin x$ is 2π -periodic and satisfies $y_p' + 2y_p = 3\cos x + \sin x$. For the general solution,

$$y(x + 2\pi) - y(x) = Ce^{-2x}(e^{-4\pi} - 1).$$

This vanishes for every x only if $C = 0$. Hence $y_p = \cos x + \sin x$ is the unique periodic member; the IVP has $C = -1$ and is not periodic.

Step 3: Uniform accuracy after a time. The exact discrepancy is $|y - y_p| = e^{-2x}$, strictly decreasing on $x \geq 0$. Thus

$$e^{-2X} \leq \frac{1}{100} \iff X \geq \ln 10.$$

The earliest such time is $\ln 10$. The term Ce^{-2x} is transient because it vanishes as $x \rightarrow \infty$, leaving the periodic response.