

Question 6

For each real initial value a , consider

$$y' + 2xy = 1, \quad y(0) = a.$$

A definite integral is an acceptable exact expression; no elementary antiderivative is required.

Tasks

1. Use an integrating factor to obtain an exact solution based at $x = 0$, and state its interval of definition.
2. Verify the formula by differentiation and the initial condition directly.
3. For solutions with initial values a and b , determine their difference, its greatest absolute value on $\mathbb{R}$, and whether their curves can cross when $a \neq b$.
4. Show that the solution with $a = 0$ is odd and positive for $x > 0$. Sketch the curves for $a = -1, 0, 1$ in your solution.

Question 6 – Solution

Strategy. Use a definite integral to incorporate the initial value exactly. The homogeneous difference makes comparison easier than evaluating the integral.

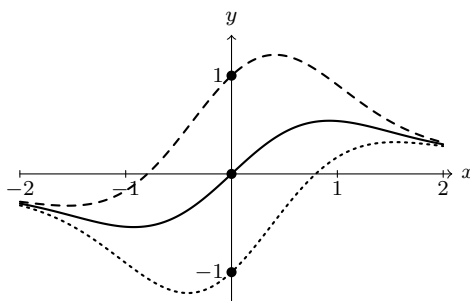
Step 1: Integrating factor and solution. Here

$$\mu = e^{\int 2x \, dx} = e^{x^2}, \quad (e^{x^2} y)' = e^{x^2}.$$

Integrating from 0 to x gives

$$y_a(x) = e^{-x^2} \left(a + \int_0^x e^{t^2} \, dt \right), \quad x \in \mathbb{R}.$$

The integral exists for every finite real x .



Solid: $a = 0$. Dashed: $a = 1$. Dotted: $a = -1$. The curves never cross.

Step 2: Direct verification. The product rule and Fundamental Theorem give $y'_a = -2xy_a + e^{-x^2}e^{x^2} = 1 - 2xy_a$. Also the integral is zero at 0, so $y_a(0) = a$.

Step 3: Compare initial values. The integral cancels in the difference:

$$y_a - y_b = (a - b)e^{-x^2}, \quad \max_{x \in \mathbb{R}} |y_a - y_b| = |a - b|.$$

For $a \neq b$, equality in the maximum occurs only at $x = 0$, and the difference never changes sign or vanishes at a finite point. The curves cannot cross.

Step 4: Symmetry of the zero-data solution. Since e^{t^2} is even, $\int_0^x e^{t^2} \, dt$ is odd. Multiplying by the even factor e^{-x^2} preserves oddness. For $x > 0$, both the integral and prefactor are positive, so $y_0(x) > 0$.