

Question 5

Two functions are known to solve the same first-order equation in standard linear form,

$$y' + p(x)y = q(x), \quad x \in \mathbb{R},$$

where p, q are continuous. The functions are

$$y_1 = x + e^{-x}, \quad y_2 = x - e^{-x}.$$

Tasks

1. Recover $p(x)$ and $q(x)$ from these two solutions. Explain why they are determined at every real x .
2. Solve the recovered equation using an integrating factor to obtain all its solutions.
3. Determine all real constants α, β for which $\alpha y_1 + \beta y_2$ is also a solution of the same nonhomogeneous equation.
4. Explain why arbitrary sums of solutions need not remain solutions, although the difference of two solutions always satisfies the associated homogeneous equation.

Question 5 – Solution

Strategy. Subtract the two equations to eliminate the forcing. A nonzero solution difference reveals the coefficient p .

Step 1: Recover the coefficients. Let $d = y_1 - y_2 = 2e^{-x}$. Subtraction gives $d' + pd = 0$, so

$$p = -\frac{d'}{d} = 1,$$

valid for every real x because d never vanishes. Using $y_1' = 1 - e^{-x}$,

$$q = y_1' + y_1 = 1 - e^{-x} + x + e^{-x} = x + 1.$$

Thus $\boxed{y' + y = x + 1}$ is uniquely recovered within the stated standard linear form.

Step 2: All solutions. The integrating factor is e^x , giving

$$(e^x y)' = (x + 1)e^x = (xe^x)'.$$

Integration yields $\boxed{y = x + Ce^{-x}, \quad C \in \mathbb{R}}$. Substitution confirms $y' + y = x + 1$ for every constant.

Step 3: Which combinations work? Write $L[v] = v' + v$. Linearity gives

$$L[\alpha y_1 + \beta y_2] = (\alpha + \beta)(x + 1).$$

For this to equal $x + 1$ throughout an interval, we require

$$\boxed{\alpha + \beta = 1.}$$

These combinations are $x + (\alpha - \beta)e^{-x}$ and generate every solution.

Step 4: Nonhomogeneous versus homogeneous. The sum $y_1 + y_2 = 2x$ produces $L[2x] = 2x + 2$, not $x + 1$. The difference instead satisfies $L[y_1 - y_2] = q - q = 0$. Thus differences solve the homogeneous equation; combinations with coefficients summing to one preserve the original forcing.