

Question 4

Consider the linear equation

$$y' - y = e^{-x}, \quad x \geq 0.$$

Instead of prescribing an initial value, require that the solution remain bounded on the entire half-line $[0, \infty)$.

Tasks

1. Find the general solution using an integrating factor.
2. Determine the unique bounded solution and the initial value it must have.
3. If that initial value is changed by a nonzero amount ε , find the difference from the bounded solution and describe its behavior as $x \rightarrow \infty$.
4. Sketch the bounded solution and the solutions for $\varepsilon = \pm 1/20$ in your solution. Explain why a forcing term tending to zero does not force every solution to tend to zero.

Question 4 – Solution

Strategy. Separate the forced response from the homogeneous term. A condition at infinity can select the constant just as an initial value can.

Step 1: General solution. With $\mu = e^{-x}$,

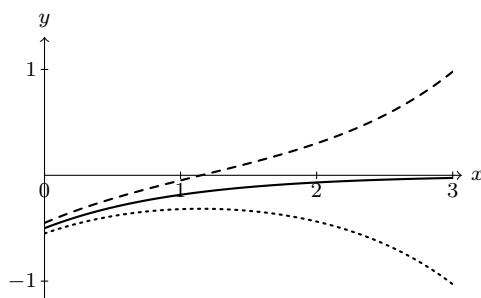
$$(e^{-x}y)' = e^{-2x}, \quad e^{-x}y = -\frac{1}{2}e^{-2x} + C.$$

Thus $y = Ce^x - \frac{1}{2}e^{-x}$; each formula is defined on $\mathbb{R}$.

Step 2: Impose boundedness. As $x \rightarrow \infty$, the second term tends to zero. If $C \neq 0$, the term Ce^x is unbounded and cannot be canceled by that decaying term. Therefore

$$y_b = -\frac{1}{2}e^{-x}, \quad y_b(0) = -\frac{1}{2}.$$

This solution is bounded and tends to zero, proving both existence and uniqueness under the stated boundedness condition.



Solid: bounded solution. Dashed: $\varepsilon = 1/20$. Dotted: $\varepsilon = -1/20$.

Step 3: Perturb the selected initial value. Since $y(0) = C - 1/2$, changing it to $-1/2 + \varepsilon$ gives $C = \varepsilon$. Consequently

$$y_\varepsilon - y_b = \varepsilon e^x.$$

For $\varepsilon > 0$ the perturbed solution tends to $+\infty$; for $\varepsilon < 0$ it tends to $-\infty$. Direct differentiation verifies $y' - y = e^{-x}$ for every C . The vanishing forcing does not remove a growing homogeneous component.