

Question 3

A student tries to solve

$$y' - 2y = xe^{2x}, \quad y(0) = 0,$$

and writes

$$\mu = e^{2x}, \quad (e^{2x}y)' = xe^{4x}.$$

Tasks

1. Identify the precise error by expanding the student's claimed product derivative.
2. Find a correct integrating factor and solve the IVP.
3. Verify the solution by direct substitution in $y' - 2y = xe^{2x}$.
4. Replace the initial condition by $y(0) = a$, with a any real number. Find the resulting family and prove directly that two different choices of a cannot produce intersecting solution curves.

Question 3 – Solution

Strategy. The integrating factor must match the signed coefficient of y . Check its product derivative before integrating.

Step 1: Diagnose the sign error. The student's derivative is

$$(e^{2x}y)' = e^{2x}(y' + 2y),$$

whereas multiplying the original equation by e^{2x} gives $e^{2x}(y' - 2y)$. These expressions differ in the sign of the y term. The required coefficient is $p = -2$, so $\mu = e^{-2x}$.

Step 2: Correct integration. Multiplication gives

$$(e^{-2x}y)' = x, \quad e^{-2x}y = \frac{x^2}{2} + C.$$

Since $y(0) = 0$, $C = 0$, and

$$y = \frac{x^2}{2}e^{2x}, \quad x \in \mathbb{R}.$$

Step 3: Verify. Differentiating explicitly,

$$y' = xe^{2x} + x^2e^{2x}, \quad y' - 2y = xe^{2x} + x^2e^{2x} - x^2e^{2x} = xe^{2x}.$$

The initial value is also zero.

Step 4: Vary the initial datum. The general initial value sets $C = a$, giving

$$y_a(x) = \left(a + \frac{x^2}{2}\right)e^{2x}.$$

For distinct a, b ,

$$y_a(x) - y_b(x) = (a - b)e^{2x} \neq 0$$

at every real x . Hence their graphs never intersect. The nonintersection follows from the explicit homogeneous difference, not merely from a sketch.