

Question 2

On the domain $x > 0$, consider

$$xy' + 2y = x^2 \ln x, \quad y(1) = 0.$$

Tasks

1. Solve the IVP using an integrating factor, showing the integration-by-parts step needed for the forcing term.
2. State the largest open interval containing 1 on which the solution solves this equation, and verify the initial value and differential equation.
3. Rewrite the solution using a definite integral based at 1. Use the sign of that integral to decide whether the solution can be negative.
4. Locate the global minimum on $x > 0$ and sketch the solution near $x = 1$ in your solution. Explain why positive forcing only for $x > 1$ does not make the solution negative for $0 < x < 1$.

Question 2 – Solution

Strategy. Use x^2 as integrating factor, then retain a definite-integral form to analyze signs on both sides of the initial point.

Step 1: Solve. Dividing by x gives $y' + 2y/x = x \ln x$. With $\mu = x^2$,

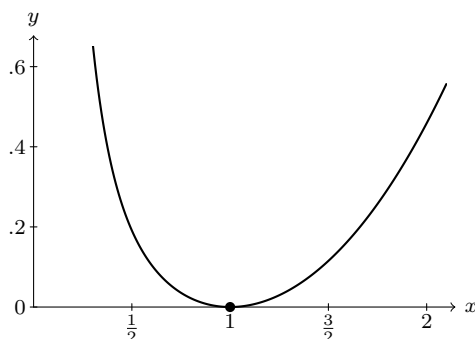
$$(x^2 y)' = x^3 \ln x.$$

Integration by parts, $u = \ln x$, $dv = x^3 dx$, yields

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C.$$

The initial value forces $C = 1/16$, hence

$$y = \frac{x^2}{4} \ln x - \frac{x^2}{16} + \frac{1}{16x^2}, \quad x > 0.$$



The solution is positive except at its global minimum $(1, 0)$.

Step 2: Verify. Substitution at 1 gives zero. Differentiating $x^2 y$ gives $x^3 \ln x$, or $x^2 y' + 2xy = x^3 \ln x$; division by $x > 0$ recovers the original equation. The logarithm and coefficient division restrict the interval to $(0, \infty)$.

Step 3: Sign and minimum. Equivalently,

$$x^2 y(x) = \int_1^x t^3 \ln t \, dt.$$

For $x > 1$ the integral is positive. For $0 < x < 1$, the integrand is negative between x and 1, but the reversed limits make the integral positive. Since $x^2 > 0$, $y(x) > 0$ for $x \neq 1$. Thus $\min_{x>0} y = 0$ only at $x = 1$. The sign of the forcing alone does not determine the sign of the solution.