

Question 1

Consider the initial value problem

$$(1 + x^2)y' + 2xy = 3x^2 + 1, \quad y(0) = 2.$$

Tasks

1. Put the equation in standard linear form $y' + p(x)y = q(x)$ and find an integrating factor. Explain why multiplying by that factor produces a product derivative.
2. Find the general solution and then the solution of the IVP.
3. Verify the IVP solution in the original, unnormalized equation and state its largest open interval.
4. Identify a straight-line particular solution and determine whether every solution approaches that line as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. State precisely what approaches zero.

Question 1 – Solution

Strategy. Normalize first. Choose μ with $\mu' = p\mu$, so the product rule gives $(\mu y)' = \mu(y' + py)$.

Step 1: Integrating factor. Since $1 + x^2 > 0$,

$$y' + \frac{2x}{1+x^2}y = \frac{3x^2+1}{1+x^2}, \quad \mu = e^{\int 2x/(1+x^2) dx} = 1+x^2.$$

Multiplication restores the original left side as $[(1+x^2)y]'$.

Step 2: Integrate and select the constant. We obtain

$$(1+x^2)y = \int (3x^2+1) dx = x^3 + x + C,$$

so

$$y = x + \frac{C}{1+x^2}.$$

At $x = 0$, $C = 2$. The IVP solution is therefore $y = x + 2/(1+x^2)$.

Step 3: Direct check and interval. Its derivative is $y' = 1 - 4x/(1+x^2)^2$, and

$$\begin{aligned}(1+x^2)y' + 2xy &= 1+x^2 - \frac{4x}{1+x^2} + 2x^2 + \frac{4x}{1+x^2} \\ &= 3x^2 + 1.\end{aligned}$$

Also $y(0) = 2$. No denominator vanishes for real x , so the largest open interval is $\mathbb{R}$.

Step 4: Approach to a line. Setting $C = 0$ gives the particular solution $y_p = x$. For every fixed real C ,

$$y(x) - x = \frac{C}{1+x^2} \longrightarrow 0 \quad \text{as } x \rightarrow \pm\infty.$$

It is the vertical difference from the line that tends to zero; the solution itself does not tend to a finite constant.