

Question 8

An implicit description of a function is proposed for $x \geq 0$:

$$y^3 + xy = 1, \quad y > 0.$$

The associated initial value problem is

$$y' = -\frac{y}{x + 3y^2}, \quad y(0) = 1.$$

You may use implicit differentiation where the coefficient of y' is nonzero.

Tasks

1. Show that for every fixed $x \geq 0$ the implicit relation has exactly one positive root y .
2. Differentiate the relation, verify the ODE and initial value, and calculate the initial slope. Explain why the denominator stays nonzero on this branch.
3. Prove that the branch decreases for $x \geq 0$ and tends to zero as $x \rightarrow \infty$.
4. Prove $2/3 < y(1) < 3/4$ and sketch the positive branch in your solution. Explain why an explicit formula obtained by solving the cubic is unnecessary for these conclusions.

Question 8 – Solution

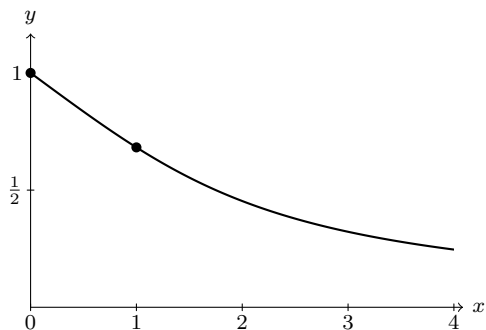
Strategy. Establish a unique branch before differentiating. Monotonicity and inequalities can describe an implicit solution without an explicit cubic formula.

Step 1: A well-defined positive branch. For fixed $x \geq 0$, let $H(y) = y^3 + xy$. It is continuous, starts at $H(0) = 0$, and tends to infinity as $y \rightarrow \infty$. For $y > 0$, $H'(y) = 3y^2 + x > 0$, so the Intermediate Value Theorem and strict increase give exactly one positive root of $H(y) = 1$.

Step 2: Verification. Differentiation yields

$$3y^2y' + y + xy' = 0, \quad \boxed{y' = -\frac{y}{x + 3y^2}}.$$

The denominator is positive on the branch, allowing local differentiable continuation, including near $(0, 1)$. At zero the relation gives $y^3 = 1$, so $y(0) = 1$ and $\boxed{y'(0) = -1/3}$.



The positive branch of $y^3 + xy = 1$; dots at $x = 0$ and $x = 1$.

Step 3: Decrease and limit. Positivity of y and the denominator implies $y' < 0$. Also $y^3 + xy = 1$ gives $0 < y < 1/x$ for $x > 0$, so the squeeze principle gives $\boxed{y(x) \rightarrow 0 \text{ as } x \rightarrow \infty}$.

Step 4: A rigorous bracket. At $x = 1$,

$$(2/3)^3 + 2/3 = 26/27 < 1, \quad (3/4)^3 + 3/4 = 75/64 > 1.$$

Strict increase in y proves $\boxed{2/3 < y(1) < 3/4}$. Root existence, differentiation and inequalities establish these results directly; an explicit cubic formula adds no needed information.