

Question 7

Consider the IVP $y' = 0$, $y(0) = 0$ and the functions

$$u(x) = 0, \quad v(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x^2}, & x > 0. \end{cases}$$

You may use the fact that v is infinitely differentiable on $\mathbb{R}$ and that $v^{(n)}(0) = 0$ for every integer $n \geq 0$.

Tasks

1. Determine the Taylor series at 0 for both functions, using the given derivative information.
2. Calculate $v'(x)$ for $x > 0$ and decide whether v solves the IVP on any open interval containing 0.
3. Prove that u is the only solution of the IVP on an interval containing 0.
4. Explain why agreement of every derivative at a single point does not contradict your conclusion. Distinguish a smooth function from one known to equal its Taylor series near that point.

Question 7 – Solution

Strategy. Use interval verification for the ODE. Infinite-order agreement at one point is still pointwise information unless convergence to the function is established.

Step 1: Identical Taylor coefficients. For both functions, every derivative at zero is zero, so both Taylor series are

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

This describes the series built from the derivatives; it does not yet assert that the series represents either function away from zero.

Step 2: The equation fails to the right. For $x > 0$, the chain rule gives

$$v'(x) = \frac{2}{x^3} e^{-1/x^2} > 0.$$

Every open interval containing 0 includes positive x , where the required identity $v' = 0$ fails. Thus v does not solve this IVP on such an interval, despite satisfying the initial value and having every derivative there equal to those of u .

Step 3: Uniqueness directly. If $y' = 0$ on an interval, the Mean Value Theorem implies that y is constant on that interval. The initial value forces that constant to be zero. Hence $y \equiv 0$ is the unique IVP solution.

Step 4: Smoothness versus a series representation. Smoothness means the derivatives of all orders exist and are continuous. A function is analytic at a point if it equals its Taylor series in some neighborhood of that point. The supplied v is smooth but not analytic at 0: its Taylor series is zero, whereas $v(x) > 0$ for every $x > 0$. Agreement of derivatives at a single point therefore does not replace the interval identity required by the ODE.