

Question 6

Compare an equation with its squared version:

$$(A) \quad y' = -\sqrt{1 - y^2}, \quad |y| \leq 1,$$

$$(B) \quad (y')^2 = 1 - y^2.$$

Both are assigned $y(0) = 0$. Consider $u(x) = \sin x$ and $v(x) = -\sin x$.

Tasks

1. Verify that both candidates satisfy (B) and the initial value. Which satisfies (A) on $(-\pi/2, \pi/2)$?
2. State the sign restriction lost by squaring. Give a condition that, together with (B), recovers (A) for real differentiable functions.
3. Explain why $v = -\sin x$ does not satisfy (A) on all of $\mathbb{R}$.
4. Extend the accepted middle branch to a C^1 solution of (A) on all of $\mathbb{R}$ by adjoining constant pieces outside $[-\pi/2, \pi/2]$. Verify the joins and sketch the extension in your solution.

Question 6 – Solution

Strategy. Squaring forgets a sign. Restore that sign when selecting a branch, and check differentiability when extending it.

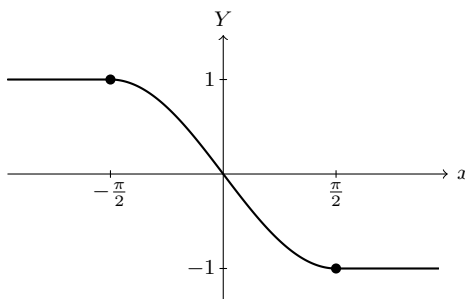
Step 1: Compare the candidates. Both satisfy $(y')^2 = \cos^2 x = 1 - y^2$ and have value zero at 0. On $(-\pi/2, \pi/2)$, $\cos x > 0$, so the right-hand side of (A) is $-|\cos x| = -\cos x$. Hence

$$v = -\sin x \text{ satisfies (A), but } u = \sin x \text{ does not.}$$

Step 2: Recover the lost information. Equation (A) forces $y' \leq 0$. Conversely, (B) and $y' \leq 0$ imply $|y| \leq 1$ and $y' = -\sqrt{1 - y^2}$. Outside the middle interval, $v' = -\cos x$ is positive wherever $\cos x < 0$, so v fails (A) there.

Step 3: A global extension. Define

$$Y(x) = \begin{cases} 1, & x \leq -\pi/2, \\ -\sin x, & -\pi/2 < x < \pi/2, \\ -1, & x \geq \pi/2. \end{cases}$$



The constant pieces meet $-\sin x$ with matching value and zero derivative.

At either join, the middle value matches the adjoining constant and its derivative $-\cos x$ tends to zero, the derivative of the constant piece. Thus Y is C^1 . On the constant pieces and at the joins, both sides of (A) are zero. On the middle piece, the verification above applies. Therefore Y solves the original IVP on $\mathbb{R}$; extension is possible here because both endpoint values and derivatives remain finite.