

Question 5

An experiment records the complete trajectory $p(x) = e^{-x}$ for all $x \geq 0$, with $p(0) = 1$. Two proposed models are

$$(A) \quad y' = -y, \quad (B) \quad y' = -y + (y - e^{-x})^2.$$

Tasks

1. Verify that the recorded trajectory satisfies both models.
2. Compare the slopes predicted by the models at $(0, 0)$ and $(0, 2)$. Are the right-hand sides the same function of (x, y) ?
3. Test whether $y \equiv 0$ is an equilibrium solution of each model.
4. Explain why even a complete, noiseless trajectory does not identify an unrestricted differential equation uniquely. Generalize the construction by replacing the added square with a term involving an arbitrary smooth function $H(x, y)$.

Question 5 – Solution

Strategy. A trajectory tests a model only along one curve in the (x, y) -plane. Compare predictions away from that curve.

Step 1: Agreement on the observed curve. Since $p' = -e^{-x} = -p$, model (A) holds. Along $y = p(x)$, the added term in (B) is $(p - e^{-x})^2 = 0$, so (B) also gives $p' = -p$. Both match the entire observation and its initial value exactly.

Step 2: Different off-curve predictions. At $x = 0$, $e^{-x} = 1$. Therefore

Point	Slope from (A)	Slope from (B)
$(0, 0)$	0	1
$(0, 2)$	-2	-1

Thus the models agree on p but are different laws.

Step 3: An equilibrium distinguishes them. For $y \equiv 0$, the derivative is zero. Model (A) assigns zero, so this is an equilibrium. Model (B) assigns $e^{-2x} > 0$, so the same constant function is not a solution of (B).

Step 4: General non-identifiability. For any smooth H , the model

$$y' = -y + H(x, y)(y - e^{-x})^2$$

admits $p = e^{-x}$, because the added term vanishes identically along that curve. Many choices of H change slopes elsewhere. Additional experiments with different initial data or restrictions on the allowed model class are needed to distinguish such laws. Complete data on one trajectory constrain a curve, not the whole plane.