

Question 4

For the initial value problem

$$y' = y^2, \quad y(0) = 1,$$

a candidate is

$$f(x) = \frac{1}{1-x}.$$

A report claims: “The right-hand side y^2 is smooth everywhere, so this formula gives a solution through $x = 1$ and for all real x .”

Tasks

1. Verify the candidate wherever it is defined. Find the largest open interval containing 0 on which it solves the IVP.
2. Calculate its one-sided limits at $x = 1$ and explain why assigning any finite value at 1 cannot extend this solution across that point.
3. Does the same formula solve the differential equation on $(1, \infty)$? Explain why that branch is not a continuation of the given IVP solution.
4. Sketch both branches in your solution and correct the report’s inference about smooth equations and global solutions.

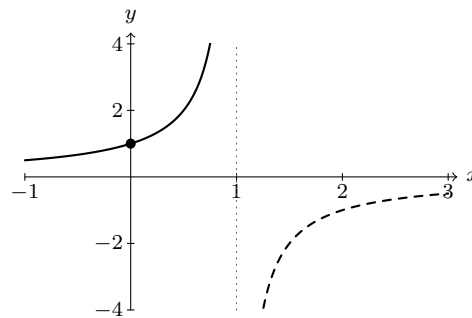
Question 4 – Solution

Strategy. Check the function's domain as well as the equation's right-hand side. A solution can become unbounded in finite time even for a smooth equation.

Step 1: Verification and interval. For $x \neq 1$,

$$f'(x) = \frac{1}{(1-x)^2} = f(x)^2, \quad f(0) = 1.$$

The largest open interval containing 0 that avoids 1 is $(-\infty, 1)$.



Solid: IVP branch. Dashed: separate solution branch. Dotted: $x = 1$.

Step 2: No finite extension. The limits are

$$\lim_{x \rightarrow 1^-} f(x) = +\infty, \quad \lim_{x \rightarrow 1^+} f(x) = -\infty.$$

A differentiable extension across 1 would be continuous there and would have a finite value. The left-hand divergence rules this out, independently of what formula is proposed after 1.

Step 3: The separate branch. On $(1, \infty)$, the same derivative calculation still proves $f' = f^2$. This branch is a solution of the differential equation, but its interval does not contain the initial point 0 and cannot be joined through $x = 1$ to the IVP branch.

Step 4: Corrected inference. Smoothness of a right-hand side is not a guarantee of existence for all time. Here the equation is defined at every finite pair (x, y) , but the IVP solution escapes to unbounded y as $x \rightarrow 1^-$. The failure belongs to global continuation of the solution, not to a singularity of y^2 at a finite point.