

Question 3

Two measurements of an initial value lead to the candidate functions

$$y(x) = e^{2x}, \quad \tilde{y}(x) = (1 + \varepsilon)e^{2x}, \quad \varepsilon = 10^{-3},$$

for the equation $y' = 2y$. Define absolute error and relative error by

$$A(x) = |\tilde{y}(x) - y(x)|, \quad R(x) = \frac{|\tilde{y}(x) - y(x)|}{|y(x)|}.$$

Tasks

1. Verify both candidates and identify their different initial values.
2. Determine $A(x)$ and $R(x)$ for $x \geq 0$. Find the first time the absolute error reaches 0.1.
3. For the equation $z' = -2z$, compare the candidates $z = e^{-2x}$ and $\tilde{z} = (1 + \varepsilon)e^{-2x}$ in the same way.
4. Plot the two absolute errors in your solution. Explain why increasing separation here is not evidence that one IVP has multiple solutions, and why absolute and relative error tell different stories.

Question 3 – Solution

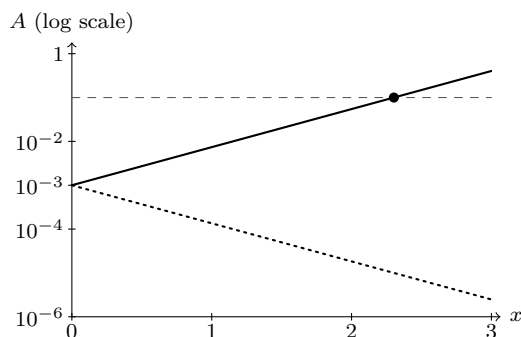
Strategy. Keep the initial conditions distinct and compute both error measures before interpreting the behavior.

Step 1: Verification. Differentiation gives $y' = 2y$ and $\tilde{y}' = 2\tilde{y}$. Their initial values are 1 and 1.001, respectively. They solve different IVPs for the same differential equation.

Step 2: Growth case. For $x \geq 0$,

$$A(x) = 10^{-3}e^{2x}, \quad R(x) = 10^{-3}.$$

The absolute error increases strictly. Its first value 0.1 occurs when $e^{2x} = 100$, giving $x = \ln 10$.



Solid: growth error. Dotted: decay error. Dashed: $A = 0.1$; dot at $x = \ln 10$.

Step 3: Decay case. Both decay candidates satisfy $z' = -2z$, again with initial values 1 and 1.001. Their errors are

$$|\tilde{z} - z| = 10^{-3}e^{-2x}, \quad \frac{|\tilde{z} - z|}{|z|} = 10^{-3}.$$

The absolute error decreases to zero and never reaches 0.1 for $x \geq 0$. The relative error stays fixed at 0.1% in both examples.

Step 4: Interpretation. Uniqueness concerns identical equations and identical initial data. Here the data differ. In the growth case the absolute discrepancy becomes large, yet it remains the same fraction of the growing reference solution. In the decay case both the solution and its discrepancy shrink proportionally.