

Question 2

The exact solution of $y' = y$, $y(0) = 1$, is supplied as $y = e^x$. A polynomial approximation is

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

You may use Taylor's theorem with a fourth-derivative remainder.

Tasks

1. Calculate the residual $P' - P$ and determine whether P solves the IVP exactly on any open interval containing 0.
2. Let $E(x) = e^x - P(x)$. Prove that for $0 \leq x \leq 1$,

$$\frac{x^4}{24} \leq E(x) \leq \frac{ex^4}{24}.$$

3. Give exact lower and upper bounds for $E(1)$, and identify whether P overestimates or underestimates the solution for $0 < x \leq 1$.
4. Plot the error and its two bounds in your solution. Explain why matching the initial value and several derivatives at 0 does not make the polynomial an exact interval solution.

Question 2 – Solution

Strategy. Use a residual to test exactness and Taylor's theorem to assess the approximation. These answer different questions.

Step 1: Exactness test. Since $P' = 1 + x + x^2/2$,

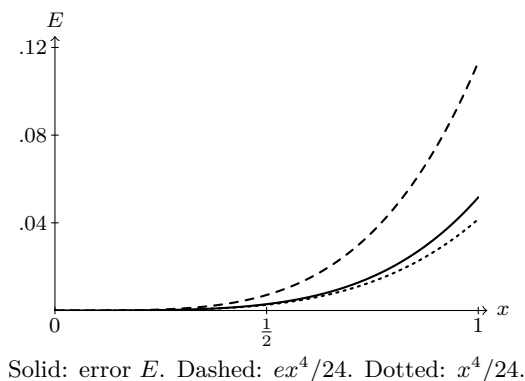
$$P' - P = -\frac{x^3}{6}.$$

The residual is nonzero whenever $x \neq 0$. Although $P(0) = 1$, the polynomial does not solve the equation on any open interval containing 0.

Step 2: Error bounds. For $0 < x \leq 1$, Taylor's theorem supplies a number ξ with $0 < \xi < x$ such that

$$E(x) = \frac{e^\xi x^4}{24}.$$

Because $1 \leq e^\xi \leq e$, the requested bounds follow; at $x = 0$ all three expressions are zero.



Step 3: Size and sign. At $x = 1$, $P(1) = 8/3$, so

$$\frac{1}{24} \leq E(1) = e - \frac{8}{3} \leq \frac{e}{24}.$$

For $x > 0$ the lower bound is positive, so P underestimates the exact solution. The two endpoint bounds are in fact strict at 1.

Step 4: Local agreement is limited evidence. The values of P and its first three derivatives agree with those of e^x at 0. That is local Taylor information. Solving the ODE requires $P'(x) = P(x)$ at every point of an interval, which the nonzero residual disproves.