

Question 1

Consider the initial value problem

$$y' = e^{-x^2}, \quad y(0) = 1.$$

A proposed answer is

$$F(x) = 1 + \int_0^x e^{-t^2} dt.$$

A student rejects this expression because the integral has no elementary antiderivative. You may take that non-elementarity as given; proving it is not part of this problem.

Tasks

1. Verify F as an exact solution and state its interval of definition.
2. Prove uniqueness directly: if G is another solution with $G(0) = 1$ on an interval containing 0, show that $G = F$ there.
3. Without evaluating the integral numerically, prove the bounds

$$1 + e^{-1} < F(1) < 2.$$

4. Explain the distinction between an exact representation, an elementary formula, and a numerical approximation. Does the lack of an elementary antiderivative imply that the IVP has no solution?

Question 1 – Solution

Strategy. A definite integral defines a function. The Fundamental Theorem of Calculus can verify that function even when elementary antidifferentiation is unavailable.

Step 1: Verification and domain. The integrand is continuous for every real t , so the integral exists for every finite real x . The Fundamental Theorem gives

$$F'(x) = e^{-x^2}, \quad F(0) = 1 + 0 = 1.$$

Thus F is an exact solution on $\mathbb{R}$. Negative x causes no problem: the integral is interpreted with reversed limits.

Step 2: Uniqueness without a general theorem. If G solves the same IVP on an interval I containing 0, then

$$(G - F)' = e^{-x^2} - e^{-x^2} = 0.$$

A differentiable function with zero derivative on an interval is constant there. At 0, $G - F = 0$, so

$$G(x) = F(x) \quad (x \in I).$$

Step 3: Strict integral bounds. For $0 < t < 1$,

$$e^{-1} < e^{-t^2} < 1.$$

These strict inequalities hold throughout an interval of positive length. Integrating from 0 to 1 therefore gives

$$e^{-1} < \int_0^1 e^{-t^2} dt < 1, \quad 1 + e^{-1} < F(1) < 2.$$

Step 4: What counts as a solution. The definite-integral expression specifies F exactly, with no rounding. An elementary formula is a more restricted type of expression, assembled from familiar elementary functions. A decimal obtained by numerical integration is an approximation to a value of F . Failure to find an elementary expression does not negate the exact solution already verified.