

Question 9

A plotting program samples slopes only at the five columns

$$x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$$

and at any chosen collection of y -values. Compare the two smooth equations

$$(A) \quad y' = 0, \quad (B) \quad y' = \sin(2\pi x).$$

Both have initial value $y(0) = 0$.

Tasks

1. Prove that the sampled direction fields are identical at every plotted point.
2. Add the column $x = \frac{1}{4}$. Calculate the slopes assigned by each law there and explain why it reveals the difference.
3. Verify the candidates $u(x) = 0$ and $v(x) = [1 - \cos(2\pi x)]/(2\pi)$ for the respective IVPs. Compare their values at $x = \frac{1}{2}$.
4. In your solution, display the shared coarse field and a more finely sampled field for (B), together with the corresponding curves. Explain why a finite sampled field cannot determine an arbitrary smooth differential equation uniquely.

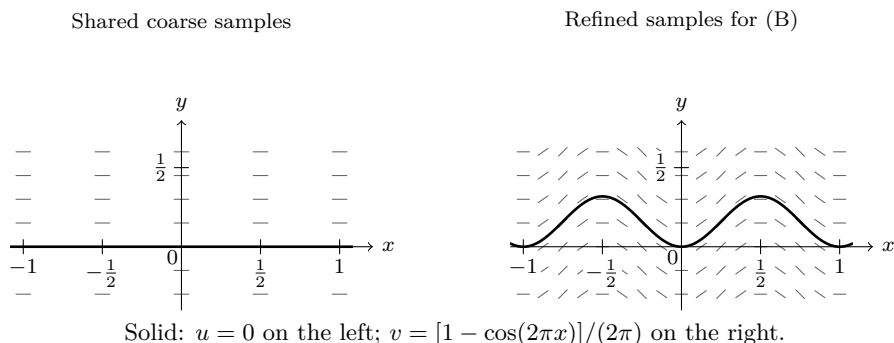
Question 9 – Solution

Strategy. Evaluate the law at the actual sampling locations before trusting a plot. A grid can systematically miss variation between columns.

Step 1: Hidden variation. Every sampled column has $x = j/2$ for an integer j . Therefore $\sin(2\pi x) = \sin(j\pi) = 0$. Both programs draw only horizontal segments there, regardless of y . At the added column,

$$f_A(\tfrac{1}{4}, y) = 0, \quad f_B(\tfrac{1}{4}, y) = 1,$$

so the fields visibly differ.



Step 2: Distinct solutions. Direct differentiation gives $u' = 0$ and

$$v'(x) = \frac{2\pi \sin(2\pi x)}{2\pi} = \sin(2\pi x), \quad u(0) = v(0) = 0.$$

At the half-unit point, $u(\frac{1}{2}) = 0, \quad v(\frac{1}{2}) = 1/\pi$. Equal sampled slopes do not imply equal solution values between or even at later grid columns.

Step 3: What a refined grid proves. The extra column disproves equality of these two laws. It does not make finite sampling sufficient for an unrestricted smooth law. For any finite set of sampled columns x_j , the nonzero smooth function $\prod_j (x - x_j)^2$ vanishes at every one of them. Adding it to a right-hand side preserves those sampled slopes while changing the field elsewhere.