

Question 8

Consider

$$y' = xy.$$

For any solution y on an interval I , define the reflected functions

$$g(x) = y(-x) \quad (x \in -I), \quad h(x) = -y(x) \quad (x \in I),$$

where $-I = \{-t : t \in I\}$. Also consider the candidate $p(x) = e^{x^2/2}$.

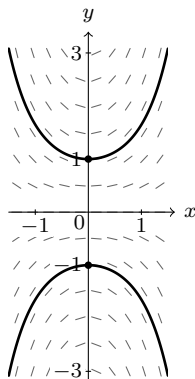
Tasks

1. Determine the zero-slope locations and the slope signs in the four quadrants.
2. Verify by differentiation that both reflected functions satisfy the same equation on their stated intervals. Explain why reflecting across the vertical axis requires a sign change in the derivative.
3. Verify p , find its minimum, and describe the corresponding extremum of $-p$.
4. Draw the field with p and $-p$ in your solution. Explain how the graph exhibits both reflection symmetries without treating a direction segment as an arrow giving a unique direction of travel.

Question 8 – Solution

Strategy. Check symmetry by transforming the differential equation itself, using the chain rule for reflection in the independent variable.

Step 1: Slopes. Since $xy = 0$ on both coordinate axes, all field segments there are horizontal. Slopes are positive in Quadrants I and III and negative in II and IV. The horizontal axis is an equilibrium solution; the vertical axis is not a function $y(x)$.



Solid: $p = e^{x^2/2}$ and $-p$. The field and curves respect both reflections.

Step 2: Transform the derivatives. For $x \in -I$, the chain rule gives

$$g'(x) = -y'(-x) = -[(-x)y(-x)] = xg(x).$$

The minus sign comes from differentiating $-x$. For $x \in I$,

$$h'(x) = -y'(x) = -xy(x) = xh(x).$$

Thus each reflection takes solutions to solutions on the stated reflected or original interval.

Step 3: Verify the displayed curves. We have $p' = xe^{x^2/2} = xp$ and $p(0) = 1$. Because $x^2/2 \geq 0$,

$$\boxed{\min_{\mathbb{R}} p = 1 \text{ at } x = 0, \quad \max_{\mathbb{R}}(-p) = -1 \text{ at } x = 0.}$$

The curve p is even, and $-p$ is its reflection across the horizontal axis. Field segments indicate tangent lines. When following a graph in the direction of increasing x , the derivative sign specifies whether y rises or falls; the drawn segments themselves need no arrowheads.