

Question 7

For the equation

$$y' = x^2 - y,$$

consider the supplied candidate through the origin

$$w(x) = x^2 - 2x + 2 - 2e^{-x}.$$

You may use the calculus inequality $e^t > 1 + t$ for $t \neq 0$.

Tasks

1. Find the zero-slope isocline and describe the slope signs above and below it.
2. Verify that w solves the equation and that $w(0) = w'(0) = 0$.
3. Decide whether w has a local extremum at 0. Determine its concavity on either side of 0 and classify the horizontal tangent there.
4. Draw the field, the isocline, and w near the origin in your solution. Explain why observing a horizontal field segment is insufficient to conclude that a solution has a maximum or minimum.

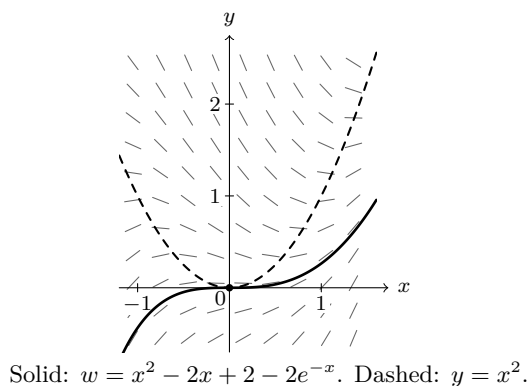
Question 7 – Solution

Strategy. A zero first derivative identifies a stationary point, not its type. Inspect nearby derivative signs and concavity.

Step 1: Isocline and verification. Zero slopes occur on $y = x^2$. Slopes are positive below this parabola and negative above it. For the candidate,

$$w' = 2x - 2 + 2e^{-x} = x^2 - w, \quad w(0) = w'(0) = 0.$$

It is therefore a solution on $\mathbb{R}$ with a horizontal tangent at the origin.



Step 2: No local extremum. For $x \neq 0$, the supplied inequality gives $e^{-x} > 1 - x$, so

$$w' = 2(x - 1 + e^{-x}) > 0.$$

The derivative is positive on both sides of 0. Thus w increases through the origin and has neither a local maximum nor a local minimum there. Moreover, $x^2 - w = w' > 0$ for $x \neq 0$, so the curve lies below its zero-slope isocline on both sides.

Step 3: Concavity. Since $w'' = 2 - 2e^{-x}$, the curve is concave down for $x < 0$ and concave up for $x > 0$. Hence

$$(0, 0) \text{ is an inflection point with a horizontal tangent.}$$

The field gives the tangent slope at a point; the surrounding behavior is needed to classify that point.