

Question 3

For the nonautonomous equation

$$y' = x + y,$$

a student claims that the zero-slope isocline is a solution because “every field segment on it is horizontal.” A candidate through the origin is supplied:

$$v(x) = e^x - x - 1.$$

Tasks

1. Find the zero-slope isocline and the regions where solutions rise or fall. Test the student’s claim.
2. Verify that v solves the equation and satisfies $v(0) = 0$.
3. Determine whether the horizontal tangent at the origin is a maximum, a minimum, or neither. Explain how v crosses the zero-slope isocline.
4. Draw the field, the isocline, and the verified solution near the origin in your solution. Explain why crossing an isocline does not contradict uniqueness of solution curves.

Question 3 – Solution

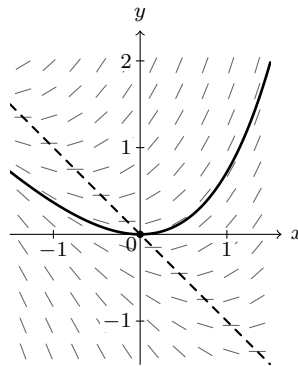
Strategy. Distinguish a line made of zero-slope locations from a curve whose own tangent agrees with the field everywhere.

Step 1: Zero slopes. The zero-slope isocline is $y = -x$. Slopes are positive above it and negative below it. The line itself has derivative -1 , while the equation assigns 0 along it, so it is not a solution.

Step 2: Verify the candidate. We have

$$v' = e^x - 1 = x + (e^x - x - 1) = x + v, \quad v(0) = 0.$$

Thus v is a solution on all of $\mathbb{R}$.



Solid: $v = e^x - x - 1$. Dashed: the zero-slope isocline $y = -x$.

Step 3: Classify the contact. Since $e^x - 1 < 0$ for $x < 0$ and $e^x - 1 > 0$ for $x > 0$, v decreases before 0 and increases after 0 . Therefore $(0, 0)$ is a strict minimum. Also $v'' = e^x > 0$ everywhere.

Step 4: Crossing the reference line. The signed vertical difference from $y = -x$ is

$$v(x) - (-x) = e^x - 1.$$

It changes from negative to positive at 0 , so the solution crosses the isocline from below to above. Nonintersection of distinct solutions is irrelevant: the isocline is not a solution curve.