

Question 10

Let $a \in \mathbb{R}$ be a fixed parameter in the equation

$$(a - 1)y'' + (a^2 - 1)(y')^2 + (a + 1)y' = x.$$

The proposed initial data are

$$y(0) = 0, \quad y'(0) = 0.$$

Tasks

1. Classify the equation's order and linearity for every real value of a . Simplify the exceptional cases before classifying them.
2. For each parameter value that makes the equation linear, find all solutions satisfying the proposed data using elementary integration.
3. For $a = 1$, decide whether $y'(0) = 0$ supplies independent information beyond the equation. What happens if this condition is replaced by $y'(0) = 1$?
4. Explain why the mere appearance of the symbol y'' in a parameterized formula does not guarantee that every member is a second-order equation.

Question 10 – Solution

Strategy. Determine when the leading coefficient or the nonlinear coefficient vanishes. Classify only after substituting those exceptional parameter values.

Step 1: Exhaustive classification. Factor $a^2 - 1 = (a - 1)(a + 1)$.

a	Simplified equation	Classification
1	$2y' = x$	First order, linear
-1	$-2y'' = x$	Second order, linear
$a \neq \pm 1$	Original equation	Second order, nonlinear

For $a \neq \pm 1$, both the y'' coefficient and the $(y')^2$ coefficient are nonzero. Both linear exceptional equations are nonhomogeneous because their right-hand sides are not identically zero.

Step 2: Solve when $a = 1$. From $y' = x/2$,

$$y = \frac{x^2}{4} + C, \quad y(0) = 0 \implies C = 0.$$

Thus $y = x^2/4$ on $\mathbb{R}$. Its derivative at 0 is 0, as required, and $2y' = x$ verifies the equation directly.

Step 3: Solve when $a = -1$. Two integrations of $y'' = -x/2$ give

$$y' = -\frac{x^2}{4} + C_1, \quad y = -\frac{x^3}{12} + C_1x + C_2.$$

The data yield $C_1 = C_2 = 0$, so $y = -x^3/12$ on $\mathbb{R}$. Indeed $-2y'' = x$ and both initial values are 0.

Step 4: Redundant or incompatible data. For $a = 1$, the equation at $x = 0$ already forces $2y'(0) = 0$. Thus the stated slope condition is redundant; replacing it with $y'(0) = 1$ makes the problem inconsistent. The coefficient $a - 1$ vanishes at $a = 1$, removing y'' entirely. Order is determined by derivatives with nonzero coefficients in the actual equation, not by unsimplified notation.