

Question 9

For a function $u(x, t)$, consider the partial differential equation and initial profile

$$u_t = 4u_{xx}, \quad u(x, 0) = 3 \cos(2x), \quad x \in \mathbb{R}.$$

Search only within the candidate family

$$u(x, t) = Ae^{-\lambda t} \cos(kx), \quad A \neq 0, \quad k > 0,$$

where A , λ , and k are real constants.

Tasks

1. State the independent and dependent variables, the order of the PDE, and whether it is linear and homogeneous.
2. Calculate u_t and u_{xx} and determine exactly which relation among the parameters makes the family satisfy the PDE everywhere.
3. Use the initial profile to determine A , k , and λ , and verify the resulting function.
4. Explain why a condition specifying $u(x, 0)$ supplies more information than one scalar condition such as $u(0, 0) = 3$. Distinguish uniqueness within the proposed family from uniqueness among all PDE solutions.

Question 9 – Solution

Strategy. Treat the other independent variable as constant in each partial derivative. Verify the PDE before imposing the entire initial profile.

Step 1: Definitions. The independent variables are x, t and the dependent variable is u . Written as $u_t - 4u_{xx} = 0$, the equation is second order, linear, and homogeneous.

Step 2: Parameter relation. Direct differentiation gives

$$u_t = -\lambda A e^{-\lambda t} \cos(kx), \quad u_{xx} = -k^2 A e^{-\lambda t} \cos(kx).$$

Thus the residual is

$$u_t - 4u_{xx} = (4k^2 - \lambda) A e^{-\lambda t} \cos(kx).$$

At $x = 0$ the cosine equals 1, while $A e^{-\lambda t} \neq 0$. The residual vanishes everywhere if and only if $\lambda = 4k^2$; no division by a possibly zero cosine is needed.

Step 3: Initial profile. At $t = 0$, the condition is $A \cos(kx) = 3 \cos(2x)$ for every real x . Evaluation at $x = 0$ gives $A = 3$. Taking two x -derivatives of this identity at 0 gives $-Ak^2 = -12$, so $k^2 = 4$. Since $k > 0$, $k = 2$, and $\lambda = 16$. Therefore

$$u(x, t) = 3e^{-16t} \cos(2x).$$

For this function $u_t = -48e^{-16t} \cos(2x) = 4u_{xx}$, and the initial profile is recovered at $t = 0$. The function is smooth for all real x, t ; in particular it works for forward time $t \geq 0$.

Step 4: Amount of data. The profile prescribes a value for every x along $t = 0$. The single condition $u(0, 0) = 3$ fixes only A , leaving any $k > 0$ with $\lambda = 4k^2$ possible. The full profile selects one candidate in the stated family; no uniqueness assertion about all possible PDE solutions has been proved.