

Question 8

For a real parameter a , define

$$z_a(x) = \begin{cases} x^2, & x \leq 0, \\ ax^2, & x > 0. \end{cases}$$

Compare the equations

$$(A) \quad xy' = 2y, \quad (B) \quad y'' = 2.$$

A classical solution must have all derivatives appearing in its equation at every point of its open interval of definition.

Tasks

1. Determine for which a the function z_a is C^1 on $\mathbb{R}$, and for which a its second derivative exists at 0.
2. Determine for which a it solves (A) on $\mathbb{R}$, checking $x = 0$ explicitly.
3. Determine for which a it solves (B) on $\mathbb{R}$.
4. Explain why replacing (A) by $y' = 2y/x$ can change the permitted solution intervals, even though the two equations agree whenever $x \neq 0$.

Question 8 – Solution

Strategy. Compute derivatives at the joining point using limits, then test each equation in its original form.

Step 1: First derivative. The pieces meet at $z_a(0) = 0$. The difference quotient at 0 is h for $h < 0$ and ah for $h > 0$, so $z'_a(0) = 0$ for every real a . Therefore

$$z'_a(x) = \begin{cases} 2x, & x < 0, \\ 0, & x = 0, \\ 2ax, & x > 0, \end{cases}$$

which is continuous at 0. Hence $z_a \in C^1(\mathbb{R})$ for every $a \in \mathbb{R}$.

Step 2: Second derivative at the join. The left-hand limit of $[z'_a(h) - z'_a(0)]/h$ is 2, and the right-hand limit is $2a$. Thus

$$z''_a(0) \text{ exists exactly when } a = 1.$$

Step 3: Check equation (A). For $x < 0$, $xz'_a = x(2x) = 2z_a$. For $x > 0$, $xz'_a = x(2ax) = 2z_a$. At $x = 0$, both sides are 0. Consequently all real a give solutions of (A) on $\mathbb{R}$.

Step 4: Check equation (B). On $x > 0$, $z''_a = 2a$, so (B) requires $a = 1$. For that value $z_a = x^2$ globally and $z''_a = 2$, including at 0. Thus $a = 1$ is necessary and sufficient for (B).

Step 5: Domain of the equation. The quotient $2y/x$ is undefined at $x = 0$, even when $y(0) = 0$. The normalized equation therefore cannot have a solution interval containing 0. It agrees with (A) on intervals contained in $(-\infty, 0)$ or $(0, \infty)$, but division removes the original equation's valid point $x = 0$.