

Question 7

You are given that every real solution of $y'' + y = 0$ on $\mathbb{R}$ has the form

$$y(x) = A \cos x + B \sin x.$$

Consider three sets of conditions:

- (I) $y(0) = 2, \quad y'(0) = -1,$
- (II) $y(0) = 2, \quad y(\pi) = -2,$
- (III) $y(0) = 2, \quad y(\pi) = 0.$

Tasks

1. Verify the given family by differentiation.
2. Identify each problem as an initial value problem or a boundary value problem.
3. Determine whether each has exactly one solution, infinitely many solutions, or no solution. Give the corresponding functions when they exist.
4. Explain why the statement “two conditions determine a unique solution of a second-order equation” needs additional qualifications.

Question 7 – Solution

Strategy. Translate each condition into an equation for A and B . Conditions may be independent, redundant, or inconsistent.

Step 1: Verify the family. We have

$$y' = -A \sin x + B \cos x, \quad y'' = -A \cos x - B \sin x = -y.$$

Thus every family member solves $y'' + y = 0$ on $\mathbb{R}$.

Step 2: Initial conditions. In (I), both values are prescribed at $x = 0$, so this is an IVP. Since $y(0) = A$ and $y'(0) = B$,

$$A = 2, \quad B = -1, \quad \boxed{y = 2 \cos x - \sin x.}$$

The supplied completeness of the family makes this the unique solution.

Step 3: Boundary conditions. In (II) and (III), values are prescribed at two different points, so these are BVPs. In both, $y(0) = 2$ gives $A = 2$. But

$$y(\pi) = A \cos \pi + B \sin \pi = -A = -2,$$

independently of B . Consequently,

$$\boxed{\text{(II): } y = 2 \cos x + B \sin x, \quad B \in \mathbb{R}}$$

has infinitely many solutions. In (III), the demand $y(\pi) = 0$ contradicts $y(\pi) = -2$, so $\boxed{\text{(III) has no solution.}}$

Step 4: Why counting fails. Two written conditions need not provide two independent, consistent constraints. In (II) the second repeats information already forced by the first; in (III) it conflicts with that information. Their number alone cannot establish uniqueness or even existence.