

Question 6

A one-parameter family of curves is

$$y(x) = (x - C)^2, \quad C \in \mathbb{R}.$$

Tasks

1. Differentiate the family and eliminate C to obtain a first-order differential equation involving only y and y' .
2. Verify that every family member satisfies the resulting equation on $\mathbb{R}$.
3. Find all family members satisfying $y(0) = 1$, and compare their slopes at 0.
4. Test $y \equiv 0$ in your equation. Is it a member of the displayed family on a nonempty open interval? Explain why finding one family of solutions need not describe every solution.

Question 6 – Solution

Strategy. Use the derivative to remove the constant, then investigate what information was lost and whether new solutions are admitted.

Step 1: Eliminate the parameter. Differentiating yields $y' = 2(x - C)$. Squaring and using $y = (x - C)^2$ gives

$$(y')^2 = 4y.$$

This is a first-order nonlinear ODE. The exponent on y' changes linearity, not derivative order.

Step 2: Direct verification. For any real C ,

$$(y')^2 = [2(x - C)]^2 = 4(x - C)^2 = 4y.$$

The family members are polynomials and therefore satisfy the equation on all of $\mathbb{R}$, including $x = C$.

Step 3: Apply the initial value. At $x = 0$ we require $C^2 = 1$, so $C = \pm 1$. The two selected family members are

$$y = (x - 1)^2 \quad \text{and} \quad y = (x + 1)^2.$$

Their slopes at 0 are -2 and 2 , respectively. The equation itself permits either sign because it specifies $(y')^2$, not y' .

Step 4: A solution outside the family. The function $y \equiv 0$ satisfies $(y')^2 = 4y$ identically. If $(x - C)^2$ were identically 0 on a nonempty open interval, every x in that interval would have to equal one fixed number C , which is impossible. Thus

$$y \equiv 0 \text{ is a solution not contained in the displayed family.}$$

Elimination proves that family members solve the equation; it does not prove the converse. In particular, a parameterized family cannot be called a complete description merely because substitution succeeds.