

Question 5

Consider the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0,$$

where the square root is the nonnegative real root. For each real parameter $a \geq 0$, define on $\mathbb{R}$

$$y_a(x) = \begin{cases} 0, & x \leq a, \\ (x - a)^2, & x > a. \end{cases}$$

Tasks

1. Prove that y_a is continuously differentiable at its joining point $x = a$.
2. Verify the differential equation everywhere, including at $x = a$, and verify the initial value.
3. Show that different parameters give distinct solutions of the same IVP. Identify an additional solution that never leaves 0.
4. Sketch y_0 , y_1 , and y_2 in your solution and explain what this example shows about uniqueness. Does it mean that every first-order IVP is nonunique?

Question 5 – Solution

Strategy. Check the joining point directly; solving on the two sides alone is insufficient. Then compare solutions with the same initial value.

Step 1: Smooth joining. Both pieces tend to 0 as $x \rightarrow a$. At a , the difference quotient is 0 for $h < 0$ and $h^2/h = h$ for $h > 0$. Hence $y'_a(a) = 0$ and

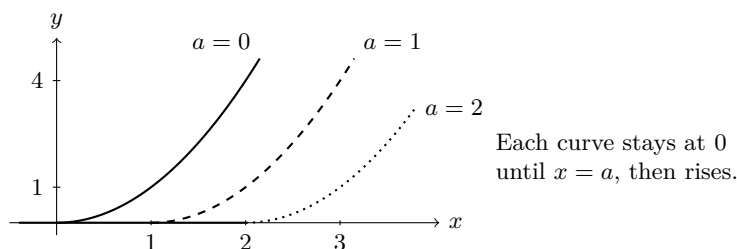
$$y'_a(x) = \begin{cases} 0, & x \leq a, \\ 2(x - a), & x > a. \end{cases}$$

This derivative is continuous at a , so y_a is C^1 on $\mathbb{R}$.

Step 2: Equation and initial value. For $x \leq a$, both y'_a and $2\sqrt{y_a}$ are 0. For $x > a$, $x - a > 0$, so

$$2\sqrt{(x - a)^2} = 2|x - a| = 2(x - a) = y'_a.$$

Since $a \geq 0$, $y_a(0) = 0$. Thus every y_a solves the IVP on $\mathbb{R}$.



Step 3: Distinctness. If $0 \leq a < b$, choose $a < x < b$. Then $y_a(x) > 0$ but $y_b(x) = 0$. Also $y \equiv 0$ is a solution distinct from every finite-parameter y_a .

Step 4: Meaning. This IVP has infinitely many solutions: the initial value does not determine when a solution leaves 0. It disproves automatic uniqueness for all first-order IVPs; it does not assert nonuniqueness for every IVP. No general uniqueness theorem is needed for this counterexample.