

Question 4

Consider

$$y' = y(1 - y).$$

A student divides by $y(1 - y)$ and claims that

$$\frac{y'}{y(1 - y)} = 1$$

is an equivalent equation for every solution. The following functions are available for testing:

$$y_0(x) = 0, \quad y_1(x) = 1, \quad p(x) = \frac{e^x}{1 + e^x}.$$

Tasks

1. Find all constant solutions of the original equation.
2. Verify p directly and state its domain and range.
3. Explain exactly when the division is valid and which listed solutions it excludes.
4. Match the initial values $y(0) = 0$, $y(0) = 1$, and $y(0) = \frac{1}{2}$ to the listed candidates. Explain why this matching alone is not a proof of uniqueness among all possible solutions.

Question 4 – Solution

Strategy. Check the original equation before transforming it. Division by an expression involving the unknown can discard legitimate solutions.

Step 1: Constant solutions. If $y(x) = k$, then $y' = 0$, so

$$0 = k(1 - k) \iff \boxed{k = 0 \text{ or } k = 1.}$$

Both constant functions solve the equation on $\mathbb{R}$; these are its equilibrium solutions.

Step 2: Nonconstant verification. The quotient rule gives

$$p'(x) = \frac{e^x(1 + e^x) - e^{2x}}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}.$$

Also

$$p(1 - p) = \frac{e^x}{1 + e^x} \frac{1}{1 + e^x} = p'.$$

The denominator is positive for every real x . Because e^x ranges over $(0, \infty)$,

$$\boxed{\text{domain}(p) = \mathbb{R}, \quad \text{range}(p) = (0, 1).}$$

Step 3: Restricted equivalence. Division is valid on an interval only if $y(x) \neq 0$ and $y(x) \neq 1$ everywhere on that interval. Thus it is valid for p , but is undefined for both equilibrium solutions. The equations are equivalent under the nonvanishing restriction, not for every original solution.

Step 4: Initial data and scope. Direct substitution yields

$$\boxed{y_0(0) = 0, \quad y_1(0) = 1, \quad p(0) = \frac{1}{2}.}$$

Each is therefore a verified solution of the corresponding IVP. Testing three candidates does not exclude every other function: existence of a matching candidate and uniqueness of an IVP are logically different claims.