

Question 2

For the initial value problem

$$y' + 2y = 6, \quad y(0) = 1,$$

three candidate functions are proposed:

$$f(x) = 3 - 2e^{-2x}, \quad g(x) = 3 - 2e^{2x}, \quad h(x) = 3.$$

Tasks

1. For each candidate, calculate its residual $R[v] = v' + 2v - 6$ and its value at $x = 0$.
2. Determine which candidates solve the differential equation and which solve the initial value problem. State an interval for each accepted solution.
3. Within the family $y = 3 + Ce^{-2x}$, determine the parameter selected by the initial condition.
4. Explain why matching one initial value, or satisfying an equation at a single point, does not verify a solution of an initial value problem.

Question 2 – Solution

Strategy. Verify two separate requirements: the differential equation must hold throughout an interval, and the initial value must hold at the specified point.

Step 1: Residuals. Direct differentiation gives

$$\begin{aligned}f' &= 4e^{-2x}, & R[f] &= 4e^{-2x} + 6 - 4e^{-2x} - 6 = 0, \\g' &= -4e^{2x}, & R[g] &= -4e^{2x} + 6 - 4e^{2x} - 6 = -8e^{2x}, \\h' &= 0, & R[h] &= 0 + 6 - 6 = 0.\end{aligned}$$

Also $f(0) = 1$, $g(0) = 1$, and $h(0) = 3$.

Step 2: Decisions. Since $e^{2x} > 0$, the residual for g never vanishes. Hence

f and h solve the differential equation on $\mathbb{R}$,
 f alone among these candidates solves the IVP on $\mathbb{R}$.

The function g passes the initial-value test but fails the equation; h does the reverse.

Step 3: Parameter selection. For $y = 3 + Ce^{-2x}$,

$$y' = -2Ce^{-2x}, \quad y' + 2y = 6.$$

Every real C gives an equation solution on $\mathbb{R}$. The initial condition requires $3 + C = 1$, so $C = -2$, recovering f .

Step 4: Logical distinction. An initial value is one pointwise constraint. A differential equation is an identity along the candidate function on an interval. For example, $v(x) = 3 + x^2$ satisfies $v' + 2v = 6$ at $x = 0$, but its residual $2x + 2x^2$ is not identically zero on any open interval. Neither pointwise test replaces interval verification.