Differential Equations

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Question 1

Consider the following equations. In (A)–(D), y is a function of x; in (E), u is a function of the independent variables x and t.

(A) $x^2y'' + \sin(x)y' - 3y = e^x$,

(B) $y'' + y\,y' = 0$,

(C) $(y')^2 + y = x$,

(D) $y'' + \sin(y) = 0$,

(E) $u_t = 4u_{xx} + x\,u$.

Tasks

1. Identify each equation as an ordinary or partial differential equation, and state its order.
2. Decide whether each equation is linear in its unknown function. For every nonlinear equation, identify the precise obstruction to linearity.
3. Among the linear equations, identify which are homogeneous after all unknown-function terms are moved to the left.
4. Explain why the factor x^2 in (A) and the factor x in (E) do not cause nonlinearity.

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Question 1 – Solution

Strategy. Count the highest derivative order, then examine how the unknown function and its derivatives occur. A linear equation has coefficients depending only on independent variables.

Step 1: Classification.

Equation	Type	Order	Linearity
(A)	ODE	2	Linear
(B)	ODE	2	Nonlinear: product $y\,y'$
(C)	ODE	1	Nonlinear: square $(y')^2$
(D)	ODE	2	Nonlinear: $\sin(y)$
(E)	PDE	2	Linear

In (E), u_t has order one and u_{xx} has order two; the highest order is therefore two. In (C), squaring a first derivative does not make it a second derivative.

Step 2: Homogeneity. The two linear equations can be written as

$$x^2y'' + \sin(x)y' - 3y = e^x, \qquad u_t - 4u_{xx} - xu = 0.$$

Thus (A) is **nonhomogeneous**, and (E) is **homogeneous**. Here "homogeneous" is being used in the linear-equation sense.

Step 3: Coefficients versus unknowns. The factors x^2, $\sin(x)$, and x are specified functions of independent variables, so they are allowed coefficients. By contrast, y multiplying y' in (B) is part of the unknown solution.

Check. Order and linearity are different properties: (C) is first order but nonlinear, whereas (A) is second order but linear. The vanishing of x^2 at 0 affects normalization of (A), not this linearity classification.

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Question 2

For the initial value problem

$$y' + 2y = 6, \qquad y(0) = 1,$$

three candidate functions are proposed:

$$f(x) = 3 - 2e^{-2x}, \qquad g(x) = 3 - 2e^{2x}, \qquad h(x) = 3.$$

Tasks

1. For each candidate, calculate its residual $R[v] = v' + 2v - 6$ and its value at $x = 0$.
2. Determine which candidates solve the differential equation and which solve the initial value problem. State an interval for each accepted solution.
3. Within the family $y = 3 + Ce^{-2x}$, determine the parameter selected by the initial condition.
4. Explain why matching one initial value, or satisfying an equation at a single point, does not verify a solution of an initial value problem.

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Question 2 – Solution

Strategy. Verify two separate requirements: the differential equation must hold throughout an interval, and the initial value must hold at the specified point.

Step 1: Residuals. Direct differentiation gives

$$\begin{aligned} f' &= 4e^{-2x}, \quad & R[f] &= 4e^{-2x} + 6 - 4e^{-2x} - 6 = 0, \\ g' &= -4e^{2x}, \quad & R[g] &= -4e^{2x} + 6 - 4e^{2x} - 6 = -8e^{2x}, \\ h' &= 0, & R[h] &= 0 + 6 - 6 = 0. \end{aligned}$$

Also $f(0) = 1$, $g(0) = 1$, and $h(0) = 3$.

Step 2: Decisions. Since $e^{2x} > 0$, the residual for g never vanishes. Hence

$$\boxed{\begin{array}{c} f \text{ and } h \text{ solve the differential equation on } \mathbb{R}, \\ f \text{ alone among these candidates solves the IVP on } \mathbb{R}. \end{array}}$$

The function g passes the initial-value test but fails the equation; h does the reverse.

Step 3: Parameter selection. For $y = 3 + Ce^{-2x}$,

$$y' = -2Ce^{-2x}, \qquad y' + 2y = 6.$$

Every real C gives an equation solution on $\mathbb{R}$. The initial condition requires $3 + C = 1$, so $\boxed{C = -2}$, recovering f.

Step 4: Logical distinction. An initial value is one pointwise constraint. A differential equation is an identity along the candidate function on an interval. For example, $v(x) = 3 + x^2$ satisfies $v' + 2v = 6$ at $x = 0$, but its residual $2x + 2x^2$ is not identically zero on any open interval. Neither pointwise test replaces interval verification.

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Question 3

An implicit relation and an initial value are given by

$$x^2 + y^2 = 9, \qquad y(0) = 3.$$

The proposed differential equation is

$$y' = -\frac{x}{y}.$$

Tasks

1. Identify the explicit branch selected by the initial value and verify the differential equation.
2. Find the largest open interval containing 0 on which that branch is a classical solution of the displayed equation.
3. Explain why the entire circle is not one solution function $y(x)$, and why adding the endpoints of the upper semicircle does not extend the solution interval.
4. If the initial value is changed to $y(0) = -3$, identify the corresponding branch and its interval. Sketch both branches in your solution.

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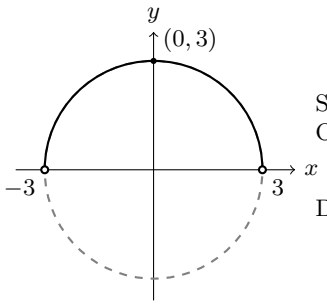
Question 3 – Solution

Strategy. An implicit curve can contain multiple function branches. Select the branch using the initial value, then check both differentiability and the equation's denominator.

Step 1: Select and verify. Solving for y gives $y = \pm\sqrt{9 - x^2}$. Since $y(0) = 3$, choose

$$\boxed{y(x) = \sqrt{9 - x^2}, \qquad -3 < x < 3.}$$

On this interval, $y' = -x/\sqrt{9 - x^2} = -x/y$, and $y(0) = 3$.



Solid: the branch selected by $y(0) = 3$.
Open endpoints are excluded.

Dashed: the remaining circle.

Step 2: Interval and endpoints. At $x = \pm 3$, the branch has $y = 0$, so $-x/y$ is undefined and the derivative is not finite. For $|x| > 3$, this branch is not real. Thus the largest open interval containing 0 is exactly $\boxed{(-3, 3)}$.

Step 3: Relation versus function. For each $-3 < x < 3$, the circle has two distinct y-values. The whole circle fails the single-valued requirement for a function $y(x)$. Even the undivided equation $yy' = -x$ cannot hold with a finite derivative at $(\pm 3, 0)$: it would require $0 = \mp 3$.

Step 4: Other initial value. For $y(0) = -3$,

$$\boxed{y(x) = -\sqrt{9 - x^2}, \qquad -3 < x < 3.}$$

Its derivative is $x/\sqrt{9 - x^2} = -x/y$, verifying the lower branch separately.

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Question 4

Consider

$$y' = y(1 - y).$$

A student divides by $y(1 - y)$ and claims that

$$\frac{y'}{y(1 - y)} = 1$$

is an equivalent equation for every solution. The following functions are available for testing:

$$y_0(x) = 0, \qquad y_1(x) = 1, \qquad p(x) = \frac{e^x}{1 + e^x}.$$

Tasks

1. Find all constant solutions of the original equation.
2. Verify p directly and state its domain and range.
3. Explain exactly when the division is valid and which listed solutions it excludes.
4. Match the initial values $y(0) = 0$, $y(0) = 1$, and $y(0) = \frac{1}{2}$ to the listed candidates. Explain why this matching alone is not a proof of uniqueness among all possible solutions.

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Question 4 – Solution

Strategy. Check the original equation before transforming it. Division by an expression involving the unknown can discard legitimate solutions.

Step 1: Constant solutions. If $y(x) = k$, then $y' = 0$, so

$$0 = k(1-k) \quad \Longleftrightarrow \quad \boxed{k = 0 \text{ or } k = 1.}$$

Both constant functions solve the equation on $\mathbb{R}$; these are its equilibrium solutions.

Step 2: Nonconstant verification. The quotient rule gives

$$p'(x) = \frac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}.$$

Also

$$p(1-p) = \frac{e^x}{1+e^x}\frac{1}{1+e^x} = p'.$$

The denominator is positive for every real x. Because e^x ranges over $(0,\infty)$,

$$\boxed{\text{domain}(p) = \mathbb{R}, \qquad \text{range}(p) = (0,1).}$$

Step 3: Restricted equivalence. Division is valid on an interval only if $y(x) \neq 0$ and $y(x) \neq 1$ everywhere on that interval. Thus it is valid for p, but is undefined for both equilibrium solutions. The equations are equivalent under the nonvanishing restriction, not for every original solution.

Step 4: Initial data and scope. Direct substitution yields

$$\boxed{y_0(0) = 0, \qquad y_1(0) = 1, \qquad p(0) = \tfrac{1}{2}.}$$

Each is therefore a verified solution of the corresponding IVP. Testing three candidates does not exclude every other function: existence of a matching candidate and uniqueness of an IVP are logically different claims.

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Question 5

Consider the initial value problem

$$y' = 2\sqrt{y}, \qquad y(0) = 0,$$

where the square root is the nonnegative real root. For each real parameter $a \geq 0$, define on $\mathbb{R}$

$$y_a(x) = \begin{cases} 0, & x \leq a, \\ (x-a)^2, & x > a. \end{cases}$$

Tasks

1. Prove that y_a is continuously differentiable at its joining point $x = a$.
2. Verify the differential equation everywhere, including at $x = a$, and verify the initial value.
3. Show that different parameters give distinct solutions of the same IVP. Identify an additional solution that never leaves 0.
4. Sketch y_0, y_1, and y_2 in your solution and explain what this example shows about uniqueness. Does it mean that every first-order IVP is nonunique?

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Question 5 – Solution

Strategy. Check the joining point directly; solving on the two sides alone is insufficient. Then compare solutions with the same initial value.

Step 1: Smooth joining. Both pieces tend to 0 as $x \to a$. At a, the difference quotient is 0 for $h < 0$ and $h^2/h = h$ for $h > 0$. Hence $y_a'(a) = 0$ and

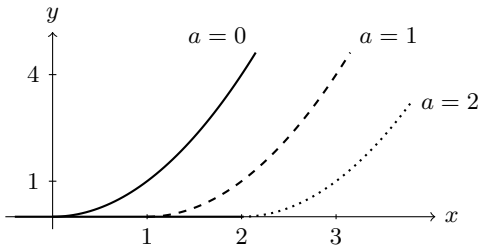
$$y_a'(x) = \begin{cases} 0, & x \le a, \\ 2(x-a), & x > a. \end{cases}$$

This derivative is continuous at a, so y_a is C^1 on $\mathbb{R}$.

Step 2: Equation and initial value. For $x \le a$, both y_a' and $2\sqrt{y_a}$ are 0. For $x > a$, $x - a > 0$, so

$$2\sqrt{(x-a)^2} = 2|x-a| = 2(x-a) = y_a'.$$

Since $a \ge 0$, $y_a(0) = 0$. Thus $\boxed{\text{every } y_a \text{ solves the IVP on } \mathbb{R}.}$



Each curve stays at 0 until $x = a$, then rises.

Step 3: Distinctness. If $0 \le a < b$, choose $a < x < b$. Then $y_a(x) > 0$ but $y_b(x) = 0$. Also $y \equiv 0$ is a solution distinct from every finite-parameter y_a.

Step 4: Meaning. This IVP has infinitely many solutions: the initial value does not determine when a solution leaves 0. It disproves automatic uniqueness for all first-order IVPs; it does not assert nonuniqueness for every IVP. No general uniqueness theorem is needed for this counterexample.

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Question 6

A one-parameter family of curves is

$$y(x) = (x - C)^2, \qquad C \in \mathbb{R}.$$

Tasks

1. Differentiate the family and eliminate C to obtain a first-order differential equation involving only y and y'.
2. Verify that every family member satisfies the resulting equation on $\mathbb{R}$.
3. Find all family members satisfying $y(0) = 1$, and compare their slopes at 0.
4. Test $y \equiv 0$ in your equation. Is it a member of the displayed family on a nonempty open interval? Explain why finding one family of solutions need not describe every solution.

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Question 6 – Solution

Strategy. Use the derivative to remove the constant, then investigate what information was lost and whether new solutions are admitted.

Step 1: Eliminate the parameter. Differentiating yields $y' = 2(x-C)$. Squaring and using $y = (x-C)^2$ gives

$$\boxed{(y')^2 = 4y.}$$

This is a first-order nonlinear ODE. The exponent on y' changes linearity, not derivative order.

Step 2: Direct verification. For any real C,

$$(y')^2 = [2(x-C)]^2 = 4(x-C)^2 = 4y.$$

The family members are polynomials and therefore satisfy the equation on all of $\mathbb{R}$, including $x = C$.

Step 3: Apply the initial value. At $x = 0$ we require $C^2 = 1$, so $C = \pm 1$. The two selected family members are

$$\boxed{y = (x-1)^2 \quad \text{and} \quad y = (x+1)^2.}$$

Their slopes at 0 are -2 and 2, respectively. The equation itself permits either sign because it specifies $(y')^2$, not y'.

Step 4: A solution outside the family. The function $y \equiv 0$ satisfies $(y')^2 = 4y$ identically. If $(x-C)^2$ were identically 0 on a nonempty open interval, every x in that interval would have to equal one fixed number C, which is impossible. Thus

$$\boxed{y \equiv 0 \text{ is a solution not contained in the displayed family.}}$$

Elimination proves that family members solve the equation; it does not prove the converse. In particular, a parameterized family cannot be called a complete description merely because substitution succeeds.

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Question 7

You are given that every real solution of $y'' + y = 0$ on $\mathbb{R}$ has the form

$$y(x) = A\cos x + B\sin x.$$

Consider three sets of conditions:

$$\begin{aligned} &\text{(I)} && y(0) = 2, && y'(0) = -1, \\ &\text{(II)} && y(0) = 2, && y(\pi) = -2, \\ &\text{(III)} && y(0) = 2, && y(\pi) = 0. \end{aligned}$$

Tasks

1. Verify the given family by differentiation.
2. Identify each problem as an initial value problem or a boundary value problem.
3. Determine whether each has exactly one solution, infinitely many solutions, or no solution. Give the corresponding functions when they exist.
4. Explain why the statement "two conditions determine a unique solution of a second-order equation" needs additional qualifications.

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Question 7 – Solution

Strategy. Translate each condition into an equation for A and B. Conditions may be independent, redundant, or inconsistent.

Step 1: Verify the family. We have

$$y' = -A\sin x + B\cos x, \qquad y'' = -A\cos x - B\sin x = -y.$$

Thus every family member solves $y'' + y = 0$ on $\mathbb{R}$.

Step 2: Initial conditions. In (I), both values are prescribed at $x = 0$, so this is an IVP. Since $y(0) = A$ and $y'(0) = B$,

$$A = 2, \quad B = -1, \qquad \boxed{y = 2\cos x - \sin x.}$$

The supplied completeness of the family makes this the unique solution.

Step 3: Boundary conditions. In (II) and (III), values are prescribed at two different points, so these are BVPs. In both, $y(0) = 2$ gives $A = 2$. But

$$y(\pi) = A\cos\pi + B\sin\pi = -A = -2,$$

independently of B. Consequently,

$$\boxed{\text{(II): } y = 2\cos x + B\sin x, \quad B \in \mathbb{R}}$$

has infinitely many solutions. In (III), the demand $y(\pi) = 0$ contradicts $y(\pi) = -2$, so $\boxed{\text{(III) has no solution.}}$

Step 4: Why counting fails. Two written conditions need not provide two independent, consistent constraints. In (II) the second repeats information already forced by the first; in (III) it conflicts with that information. Their number alone cannot establish uniqueness or even existence.

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Question 8

For a real parameter a, define

$$z_a(x) = \begin{cases} x^2, & x \le 0, \\ ax^2, & x > 0. \end{cases}$$

Compare the equations

$$\text{(A)} \quad xy' = 2y, \qquad \text{(B)} \quad y'' = 2.$$

A classical solution must have all derivatives appearing in its equation at every point of its open interval of definition.

Tasks

1. Determine for which a the function z_a is C^1 on $\mathbb{R}$, and for which a its second derivative exists at 0.
2. Determine for which a it solves (A) on $\mathbb{R}$, checking $x = 0$ explicitly.
3. Determine for which a it solves (B) on $\mathbb{R}$.
4. Explain why replacing (A) by $y' = 2y/x$ can change the permitted solution intervals, even though the two equations agree whenever $x \neq 0$.

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Question 8 – Solution

Strategy. Compute derivatives at the joining point using limits, then test each equation in its original form.

Step 1: First derivative. The pieces meet at $z_a(0) = 0$. The difference quotient at 0 is h for $h < 0$ and ah for $h > 0$, so $z_a'(0) = 0$ for every real a. Therefore

$$z_a'(x) = \begin{cases} 2x, & x < 0, \\ 0, & x = 0, \\ 2ax, & x > 0, \end{cases}$$

which is continuous at 0. Hence $\boxed{z_a \in C^1(\mathbb{R}) \text{ for every } a \in \mathbb{R}.}$

Step 2: Second derivative at the join. The left-hand limit of $[z_a'(h) - z_a'(0)]/h$ is 2, and the right-hand limit is $2a$. Thus

$$\boxed{z_a''(0) \text{ exists exactly when } a = 1.}$$

Step 3: Check equation (A). For $x < 0$, $xz_a' = x(2x) = 2z_a$. For $x > 0$, $xz_a' = x(2ax) = 2z_a$. At $x = 0$, both sides are 0. Consequently $\boxed{\text{all real } a \text{ give solutions of (A) on } \mathbb{R}.}$

Step 4: Check equation (B). On $x > 0$, $z_a'' = 2a$, so (B) requires $a = 1$. For that value $z_a = x^2$ globally and $z_a'' = 2$, including at 0. Thus $\boxed{a = 1 \text{ is necessary and sufficient for (B).}}$

Step 5: Domain of the equation. The quotient $2y/x$ is undefined at $x = 0$, even when $y(0) = 0$. The normalized equation therefore cannot have a solution interval containing 0. It agrees with (A) on intervals contained in $(-\infty, 0)$ or $(0, \infty)$, but division removes the original equation's valid point $x = 0$.

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Question 9

For a function $u(x,t)$, consider the partial differential equation and initial profile

$$u_t = 4u_{xx}, \qquad u(x,0) = 3\cos(2x), \qquad x \in \mathbb{R}.$$

Search only within the candidate family

$$u(x,t) = Ae^{-\lambda t}\cos(kx), \qquad A \neq 0, \quad k > 0,$$

where A, λ, and k are real constants.

Tasks

1. State the independent and dependent variables, the order of the PDE, and whether it is linear and homogeneous.
2. Calculate u_t and u_{xx} and determine exactly which relation among the parameters makes the family satisfy the PDE everywhere.
3. Use the initial profile to determine A, k, and λ, and verify the resulting function.
4. Explain why a condition specifying $u(x,0)$ supplies more information than one scalar condition such as $u(0,0) = 3$. Distinguish uniqueness within the proposed family from uniqueness among all PDE solutions.

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Question 9 – Solution

Strategy. Treat the other independent variable as constant in each partial derivative. Verify the PDE before imposing the entire initial profile.

Step 1: Definitions. The independent variables are x, t and the dependent variable is u. Written as $u_t - 4u_{xx} = 0$, the equation is $\boxed{\text{second order, linear, and homogeneous.}}$

Step 2: Parameter relation. Direct differentiation gives

$$u_t = -\lambda A e^{-\lambda t}\cos(kx), \qquad u_{xx} = -k^2 A e^{-\lambda t}\cos(kx).$$

Thus the residual is

$$u_t - 4u_{xx} = (4k^2 - \lambda)Ae^{-\lambda t}\cos(kx).$$

At $x = 0$ the cosine equals 1, while $Ae^{-\lambda t} \neq 0$. The residual vanishes everywhere if and only if $\boxed{\lambda = 4k^2}$; no division by a possibly zero cosine is needed.

Step 3: Initial profile. At $t = 0$, the condition is $A\cos(kx) = 3\cos(2x)$ for every real x. Evaluation at $x = 0$ gives $A = 3$. Taking two x-derivatives of this identity at 0 gives $-Ak^2 = -12$, so $k^2 = 4$. Since $k > 0$, $k = 2$, and $\lambda = 16$. Therefore

$$\boxed{u(x,t) = 3e^{-16t}\cos(2x).}$$

For this function $u_t = -48e^{-16t}\cos(2x) = 4u_{xx}$, and the initial profile is recovered at $t = 0$. The function is smooth for all real x, t; in particular it works for forward time $t \geq 0$.

Step 4: Amount of data. The profile prescribes a value for every x along $t = 0$. The single condition $u(0, 0) = 3$ fixes only A, leaving any $k > 0$ with $\lambda = 4k^2$ possible. The full profile selects one candidate in the stated family; no uniqueness assertion about all possible PDE solutions has been proved.

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Question 10

Let $a \in \mathbb{R}$ be a fixed parameter in the equation

$$(a-1)y'' + (a^2-1)(y')^2 + (a+1)y' = x.$$

The proposed initial data are

$$y(0) = 0, \qquad y'(0) = 0.$$

Tasks

1. Classify the equation's order and linearity for every real value of a. Simplify the exceptional cases before classifying them.
2. For each parameter value that makes the equation linear, find all solutions satisfying the proposed data using elementary integration.
3. For $a = 1$, decide whether $y'(0) = 0$ supplies independent information beyond the equation. What happens if this condition is replaced by $y'(0) = 1$?
4. Explain why the mere appearance of the symbol y'' in a parameterized formula does not guarantee that every member is a second-order equation.

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Question 10 – Solution

Strategy. Determine when the leading coefficient or the nonlinear coefficient vanishes. Classify only after substituting those exceptional parameter values.

Step 1: Exhaustive classification. Factor $a^2 - 1 = (a-1)(a+1)$.

a	Simplified equation	Classification
1	$2y' = x$	First order, linear
-1	$-2y'' = x$	Second order, linear
$a \neq \pm 1$	Original equation	Second order, nonlinear

For $a \neq \pm 1$, both the y'' coefficient and the $(y')^2$ coefficient are nonzero. Both linear exceptional equations are nonhomogeneous because their right-hand sides are not identically zero.

Step 2: Solve when $a = 1$. From $y' = x/2$,

$$y = \frac{x^2}{4} + C, \qquad y(0) = 0 \Longrightarrow C = 0.$$

Thus $\boxed{y = x^2/4 \text{ on } \mathbb{R}}$. Its derivative at 0 is 0, as required, and $2y' = x$ verifies the equation directly.

Step 3: Solve when $a = -1$. Two integrations of $y'' = -x/2$ give

$$y' = -\frac{x^2}{4} + C_1, \qquad y = -\frac{x^3}{12} + C_1 x + C_2.$$

The data yield $C_1 = C_2 = 0$, so $\boxed{y = -x^3/12 \text{ on } \mathbb{R}}$. Indeed $-2y'' = x$ and both initial values are 0.

Step 4: Redundant or incompatible data. For $a = 1$, the equation at $x = 0$ already forces $2y'(0) = 0$. Thus the stated slope condition is redundant; replacing it with $y'(0) = 1$ makes the problem inconsistent. The coefficient $a - 1$ vanishes at $a = 1$, removing y'' entirely. Order is determined by derivatives with nonzero coefficients in the actual equation, not by unsimplified notation.

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Question 1

Consider the direction field for

$$y' = x - y.$$

An *isocline of slope k* is the set of points where the equation assigns slope k. A direction field consists of short line segments with the prescribed slopes; the segments are not themselves whole solution curves.

Tasks

1. Calculate the slopes at $(-1, 1)$, $(0, 1)$, $(1, 1)$, and $(1, -1)$, and describe the orientation of each segment.
2. Find the isoclines of slopes -1, 0, and 1. Determine the regions of positive and negative slope.
3. Draw a direction field on $-2 \le x \le 2$, $-2 \le y \le 2$ in your solution. Mark the three isoclines and distinguish them from solution curves.
4. A solution passes through $(0, 1)$. Find its tangent line there and decide whether it initially rises or falls as x increases. Does the tangent line itself solve the equation on an interval?

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Question 1 – Solution

Strategy. Evaluate $x - y$ at points, then hold its value fixed to find isoclines. Test a proposed curve by comparing its derivative with the assigned slope.

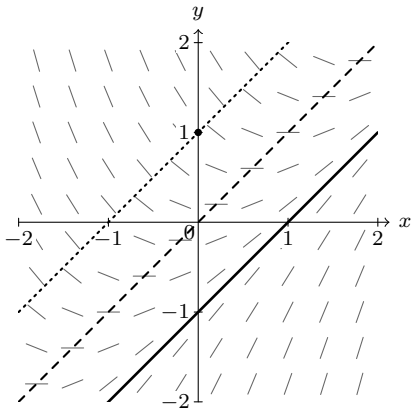
Step 1: Slopes and regions. The four slopes, in the listed order, are

$$\boxed{-2, \quad -1, \quad 0, \quad 2.}$$

Thus the first two segments fall to the right, the third is horizontal, and the fourth rises. The isocline equation is $x - y = k$, or $y = x - k$. Hence

$$k = -1 : \ y = x + 1, \qquad k = 0 : \ y = x, \qquad k = 1 : \ y = x - 1.$$

Below $y = x$ the slopes are positive; above it they are negative.



Solid: $y = x - 1$; dashed: $y = x$; dotted: $y = x + 1$. Dot: $(0, 1)$.

Step 2: Isoclines are not automatically solutions. Each displayed isocline has geometric slope 1. On $y = x + 1$ and $y = x$, the field instead prescribes -1 and 0, so these lines are not solutions. On $y = x - 1$, the prescribed slope is 1, so that particular isocline also happens to be a solution.

Step 3: Tangent versus solution. At $(0, 1)$ the slope is -1, giving $\boxed{y = 1 - x}$ as the tangent line. The solution initially falls. Along this line, however, $x - y = 2x - 1$, which equals the line's derivative -1 only at $x = 0$. Thus the tangent line does not solve the equation on any open interval.

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Question 2

Consider the autonomous equation

$$y' = y(2 - y).$$

You may use the following fact: for this smooth right-hand side, two solution curves cannot intersect at a common point unless they coincide on their common interval. Assume the solution with $y(0) = 1$ exists for all $x \geq 0$.

Tasks

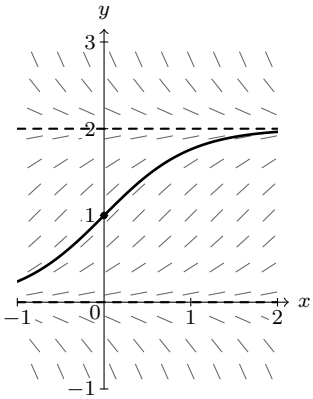
1. Explain why the direction field repeats horizontally, and find all horizontal solution lines.
2. Determine the slope sign in each region separated by those lines. Draw the field for $-1 \leq x \leq 2$ and $-1 \leq y \leq 3$ in your solution.
3. Prove that the solution with $y(0) = 1$ remains between 0 and 2 and is increasing for $x \geq 0$.
4. Determine its limit as $x \to \infty$. Justify why a limiting value strictly below 2 would contradict the differential equation; do not solve the equation explicitly.

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Question 2 – Solution

Strategy. Read the signs between horizontal zero-slope lines, then use nonintersection and a bounded-monotone argument.

Step 1: Field structure. The slope depends only on y, so it is the same along every horizontal row. Constant solutions require $y(2-y)=0$, giving $\boxed{y=0 \text{ and } y=2}$. The sign is negative for $y<0$, positive for $0<y<2$, and negative for $y>2$.



Solid: solution through $(0,1)$. Dashed: the equilibrium lines $y=0,2$.

Step 2: Confinement and monotonicity. Starting at 1, a continuous solution could leave $(0,2)$ only by meeting one of the equilibrium solutions. The stated nonintersection fact excludes this. Therefore

$$\boxed{0<y(x)<2, \qquad y'(x)>0 \quad (x\geq 0).}$$

Since $y(0)=1$, the function is bounded below by 1 on this half-line.

Step 3: Limit without an explicit formula. An increasing function bounded above by 2 has a limit L with $1\leq L\leq 2$. If $L<2$, continuity of $g(y)=y(2-y)$ gives $g(L)>0$ and, eventually, $y'(x)\geq g(L)/2=:c>0$. Integrating from a sufficiently large X yields

$$y(x)\geq y(X)+c(x-X),$$

contradicting the upper bound 2. Thus $\boxed{\lim_{x\to\infty} y(x)=2}$. The limit is approached from below, without touching the equilibrium at any finite x.

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Question 3

For the nonautonomous equation

$$y' = x + y,$$

a student claims that the zero-slope isocline is a solution because "every field segment on it is horizontal." A candidate through the origin is supplied:

$$v(x) = e^x - x - 1.$$

Tasks

1. Find the zero-slope isocline and the regions where solutions rise or fall. Test the student's claim.
2. Verify that v solves the equation and satisfies $v(0) = 0$.
3. Determine whether the horizontal tangent at the origin is a maximum, a minimum, or neither. Explain how v crosses the zero-slope isocline.
4. Draw the field, the isocline, and the verified solution near the origin in your solution. Explain why crossing an isocline does not contradict uniqueness of solution curves.

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Question 3 – Solution

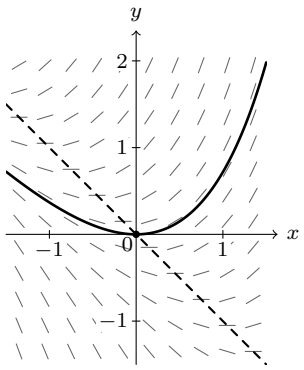
Strategy. Distinguish a line made of zero-slope locations from a curve whose own tangent agrees with the field everywhere.

Step 1: Zero slopes. The zero-slope isocline is $\boxed{y = -x}$. Slopes are positive above it and negative below it. The line itself has derivative -1, while the equation assigns 0 along it, so it is not a solution.

Step 2: Verify the candidate. We have

$$v' = e^x - 1 = x + (e^x - x - 1) = x + v, \qquad v(0) = 0.$$

Thus v is a solution on all of $\mathbb{R}$.



Solid: $v = e^x - x - 1$. Dashed: the zero-slope isocline $y = -x$.

Step 3: Classify the contact. Since $e^x - 1 < 0$ for $x < 0$ and $e^x - 1 > 0$ for $x > 0$, v decreases before 0 and increases after 0. Therefore $\boxed{(0, 0) \text{ is a strict minimum.}}$ Also $v'' = e^x > 0$ everywhere.

Step 4: Crossing the reference line. The signed vertical difference from $y = -x$ is

$$v(x) - (-x) = e^x - 1.$$

It changes from negative to positive at 0, so the solution crosses the isocline from below to above. Nonintersection of distinct solutions is irrelevant: the isocline is not a solution curve.

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Question 4

An unknown direction field is known to come from an affine law

$$y' = ax + by + c,$$

with fixed real constants a, b, c. Three exact slope measurements are

(x, y)	$(-1, 0)$	$(0, 1)$	$(1, 1)$
slope	-1	0	$2.$

A fourth report claims that the slope at $(2, 0)$ is 4.

Tasks

1. Recover a, b, c and show that the first three measurements determine them uniquely.
2. Decide whether the fourth report is consistent with the model.
3. Find the zero-slope isocline. Then determine every straight-line solution $y = mx + d$ of the recovered equation.
4. Draw the recovered field, its zero-slope isocline, and its straight-line solution in your solution. Explain why the affine-model assumption matters when inferring a whole field from three points.

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Question 4 – Solution

Strategy. Recover the coefficients from independent data, then compare the geometric slope of a line with the field along it.

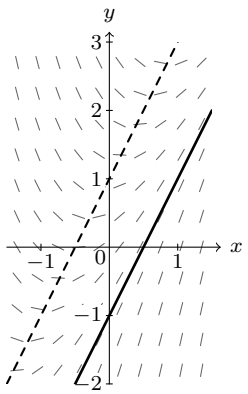
Step 1: Recover the law. The measurements give

$$-a + c = -1, \qquad b + c = 0, \qquad a + b + c = 2.$$

Subtracting the second equation from the third gives $a = 2$. The first then gives $c = 1$, and the second gives $b = -1$. These forced values prove uniqueness within the affine class:

$$\boxed{y' = 2x - y + 1.}$$

At $(2, 0)$ the predicted slope is 5, so the reported value 4 is inconsistent.



Solid: solution $y = 2x - 1$. Dashed: zero-slope isocline $y = 2x + 1$.

Step 2: Reference line and solution line. Zero slopes occur on $y = 2x + 1$, which is not a solution: its derivative is 2, not 0. Substituting $y = mx + d$ into the equation requires

$$m = (2 - m)x + 1 - d$$

for every x in an interval. Hence $m = 2$ and $d = -1$. The only straight-line solution is $\boxed{y = 2x - 1}$, verified by $2x - (2x - 1) + 1 = 2$.

Step 3: Limits of inference. Three data points determine the three affine coefficients here. Without the affine assumption, many nonlinear functions could agree at those points while assigning different slopes elsewhere. Finite measurements alone do not determine an unrestricted field.

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Question 5

Consider the equation

$$y' = \frac{x}{y}, \qquad y \neq 0,$$

and a solution through $(0, 1)$.

Tasks

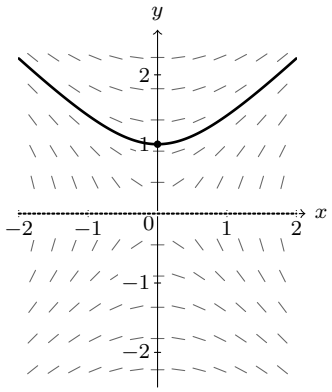
1. Determine the slope sign in each open quadrant. Find every point in the domain with a horizontal field segment.
2. Explain why no segment should be placed at $(1, 0)$, even though nearby segments become steep.
3. Along a solution, differentiate $y^2 - x^2$. Use the initial point to identify an explicit solution branch and verify it directly. State its largest open interval.
4. Draw the field and the selected branch in your solution, marking the excluded line $y = 0$. Explain why that excluded line does not prevent this particular solution from existing for every real x.

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Question 5 – Solution

Strategy. Respect the equation's domain before drawing. Use a constant quantity along a solution to identify the curve through the given point.

Step 1: Signs and missing points. Slopes are positive in Quadrants I and III, and negative in II and IV. Horizontal segments occur exactly where $x = 0$ and $y \neq 0$. The expression x/y is undefined on all of $y = 0$; a vertical mark at $(1, 0)$ would falsely assign a direction to a point outside the domain.



Solid: $y = \sqrt{1 + x^2}$. Dotted horizontal line: excluded $y = 0$.

Step 2: Identify and verify the branch. Wherever a solution is defined,

$$\frac{d}{dx}(y^2 - x^2) = 2yy' - 2x = 2y\frac{x}{y} - 2x = 0.$$

Thus $y^2 - x^2$ is constant on its interval. At $(0, 1)$ it equals 1. Continuity and the positive initial value select

$$\boxed{y = \sqrt{1 + x^2}, \qquad -\infty < x < \infty.}$$

Indeed $y' = x/\sqrt{1 + x^2} = x/y$ and $y(0) = 1$.

Step 3: Interval and geometry. This branch is smooth and satisfies $y \geq 1$ for every real x, so it never reaches the excluded axis. The entire real line is therefore its largest open interval. It decreases for $x < 0$ and increases for $x > 0$, with a minimum at $(0, 1)$, matching the field.

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Question 6

Compare two equations on the same coordinate scale:

$$\text{(A)} \quad y' = y, \qquad \text{(B)} \quad y' = 3y.$$

Both are assigned the initial value $y(0) = 1$. Candidate solutions are supplied:

$$u(x) = e^x, \qquad v(x) = e^{3x}.$$

Tasks

1. Determine the zero-slope locations and slope signs for both fields. Are these facts enough to make the two fields identical?
2. At height $y = 1$, compute each slope and its angle with the positive x-axis. If the two fields use equal-length segments, explain how steepness should be represented.
3. Verify the two candidates and calculate when each first reaches $y = 2$ for $x > 0$.
4. Draw the two fields and their respective verified curves on identical axes in your solution. State precisely which geometric information is lost by recording only slope signs.

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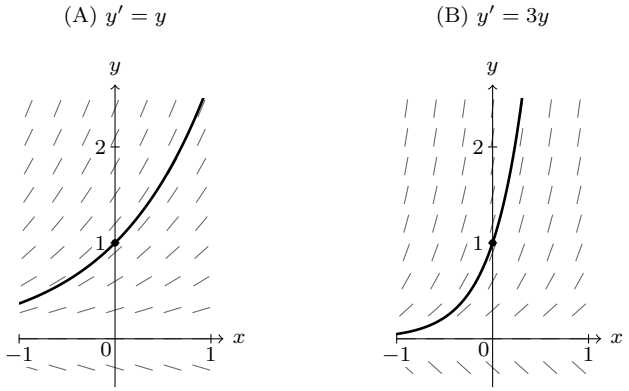
Question 6 – Solution

Strategy. Separate slope sign from slope magnitude. Equal-length field segments encode magnitude through their angle, not their length.

Step 1: Shared and different information. Both fields have horizontal segments on $y = 0$, positive slopes above it, and negative slopes below it. But at $y = 1$ their slopes are 1 and 3, so the fields are different. The corresponding angles are

$$\boxed{\theta_A = \arctan(1) = 45^\circ, \qquad \theta_B = \arctan(3) \approx 71.6^\circ.}$$

A segment of total length ℓ and slope m can use the displacement $\ell(1, m)/\sqrt{1 + m^2}$. This preserves its slope while controlling its length.



Solid curves: e^x on the left and e^{3x} on the right; identical axis scales.

Step 2: Verify and compare growth. Directly,

$$u' = e^x = u, \qquad v' = 3e^{3x} = 3v, \qquad u(0) = v(0) = 1.$$

Both increase strictly. Solving $e^x = 2$ and $e^{3x} = 2$ gives their first hitting times

$$\boxed{x_A = \ln 2, \qquad x_B = \frac{\ln 2}{3}.}$$

Thus (B) reaches the same height in one-third the time.

Step 3: What signs omit. Signs indicate rising, falling, or horizontal tangents. They do not specify tangent angles, rates of change, or travel times between heights. Identical sign regions and the same equilibrium line do not determine the same solution curves.

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Question 7

For the equation

$$y' = x^2 - y,$$

consider the supplied candidate through the origin

$$w(x) = x^2 - 2x + 2 - 2e^{-x}.$$

You may use the calculus inequality $e^t > 1 + t$ for $t \neq 0$.

Tasks

1. Find the zero-slope isocline and describe the slope signs above and below it.
2. Verify that w solves the equation and that $w(0) = w'(0) = 0$.
3. Decide whether w has a local extremum at 0. Determine its concavity on either side of 0 and classify the horizontal tangent there.
4. Draw the field, the isocline, and w near the origin in your solution. Explain why observing a horizontal field segment is insufficient to conclude that a solution has a maximum or minimum.

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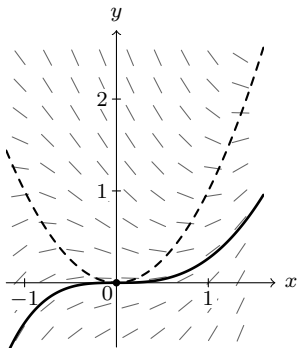
Question 7 – Solution

Strategy. A zero first derivative identifies a stationary point, not its type. Inspect nearby derivative signs and concavity.

Step 1: Isocline and verification. Zero slopes occur on $\boxed{y = x^2}$. Slopes are positive below this parabola and negative above it. For the candidate,

$$w' = 2x - 2 + 2e^{-x} = x^2 - w, \qquad w(0) = w'(0) = 0.$$

It is therefore a solution on $\mathbb{R}$ with a horizontal tangent at the origin.



Solid: $w = x^2 - 2x + 2 - 2e^{-x}$. Dashed: $y = x^2$.

Step 2: No local extremum. For $x \neq 0$, the supplied inequality gives $e^{-x} > 1 - x$, so

$$w' = 2(x - 1 + e^{-x}) > 0.$$

The derivative is positive on both sides of 0. Thus w increases through the origin and has neither a local maximum nor a local minimum there. Moreover, $x^2 - w = w' > 0$ for $x \neq 0$, so the curve lies below its zero-slope isocline on both sides.

Step 3: Concavity. Since $w'' = 2 - 2e^{-x}$, the curve is concave down for $x < 0$ and concave up for $x > 0$. Hence

$$\boxed{(0, 0) \text{ is an inflection point with a horizontal tangent.}}$$

The field gives the tangent slope at a point; the surrounding behavior is needed to classify that point.

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Question 8

Consider

$$y' = xy.$$

For any solution y on an interval I, define the reflected functions

$$g(x) = y(-x) \quad (x \in -I), \qquad h(x) = -y(x) \quad (x \in I),$$

where $-I = \{-t : t \in I\}$. Also consider the candidate $p(x) = e^{x^2/2}$.

Tasks

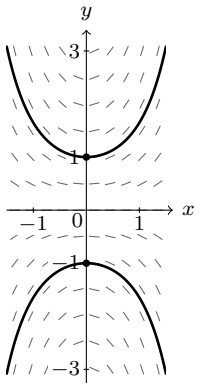
1. Determine the zero-slope locations and the slope signs in the four quadrants.
2. Verify by differentiation that both reflected functions satisfy the same equation on their stated intervals. Explain why reflecting across the vertical axis requires a sign change in the derivative.
3. Verify p, find its minimum, and describe the corresponding extremum of $-p$.
4. Draw the field with p and $-p$ in your solution. Explain how the graph exhibits both reflection symmetries without treating a direction segment as an arrow giving a unique direction of travel.

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Question 8 – Solution

Strategy. Check symmetry by transforming the differential equation itself, using the chain rule for reflection in the independent variable.

Step 1: Slopes. Since $xy = 0$ on both coordinate axes, all field segments there are horizontal. Slopes are positive in Quadrants I and III and negative in II and IV. The horizontal axis is an equilibrium solution; the vertical axis is not a function $y(x)$.



Solid: $p = e^{x^2/2}$ and $-p$. The field and curves respect both reflections.

Step 2: Transform the derivatives. For $x \in -I$, the chain rule gives

$$g'(x) = -y'(-x) = -[(-x)y(-x)] = xg(x).$$

The minus sign comes from differentiating $-x$. For $x \in I$,

$$h'(x) = -y'(x) = -xy(x) = xh(x).$$

Thus each reflection takes solutions to solutions on the stated reflected or original interval.

Step 3: Verify the displayed curves. We have $p' = xe^{x^2/2} = xp$ and $p(0) = 1$. Because $x^2/2 \geq 0$,

$$\boxed{\min_{\mathbb{R}} p = 1 \text{ at } x = 0, \qquad \max_{\mathbb{R}}(-p) = -1 \text{ at } x = 0.}$$

The curve p is even, and $-p$ is its reflection across the horizontal axis. Field segments indicate tangent lines. When following a graph in the direction of increasing x, the derivative sign specifies whether y rises or falls; the drawn segments themselves need no arrowheads.

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Question 9

A plotting program samples slopes only at the five columns

$$x = -1,\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 1$$

and at any chosen collection of y-values. Compare the two smooth equations

$$\text{(A)}\quad y' = 0, \qquad \text{(B)}\quad y' = \sin(2\pi x).$$

Both have initial value $y(0) = 0$.

Tasks

1. Prove that the sampled direction fields are identical at every plotted point.
2. Add the column $x = \frac{1}{4}$. Calculate the slopes assigned by each law there and explain why it reveals the difference.
3. Verify the candidates $u(x) = 0$ and $v(x) = [1 - \cos(2\pi x)]/(2\pi)$ for the respective IVPs. Compare their values at $x = \frac{1}{2}$.
4. In your solution, display the shared coarse field and a more finely sampled field for (B), together with the corresponding curves. Explain why a finite sampled field cannot determine an arbitrary smooth differential equation uniquely.

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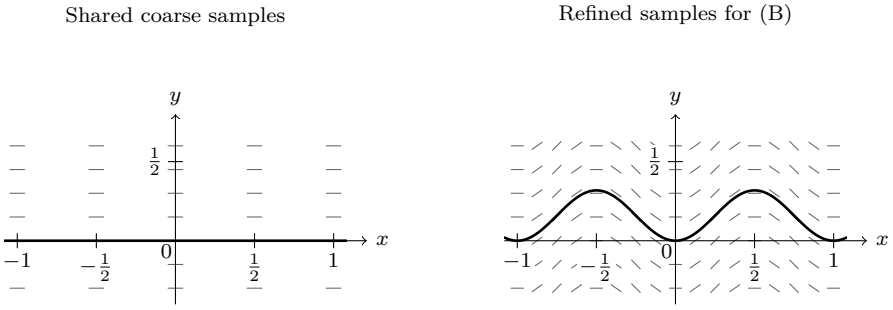
Question 9 – Solution

Strategy. Evaluate the law at the actual sampling locations before trusting a plot. A grid can systematically miss variation between columns.

Step 1: Hidden variation. Every sampled column has $x = j/2$ for an integer j. Therefore $\sin(2\pi x) = \sin(j\pi) = 0$. Both programs draw only horizontal segments there, regardless of y. At the added column,

$$\boxed{f_A(\tfrac{1}{4}, y) = 0, \qquad f_B(\tfrac{1}{4}, y) = 1,}$$

so the fields visibly differ.



Solid: $u = 0$ on the left; $v = [1 - \cos(2\pi x)]/(2\pi)$ on the right.

Step 2: Distinct solutions. Direct differentiation gives $u' = 0$ and

$$v'(x) = \frac{2\pi \sin(2\pi x)}{2\pi} = \sin(2\pi x), \qquad u(0) = v(0) = 0.$$

At the half-unit point, $\boxed{u(\tfrac{1}{2}) = 0, \quad v(\tfrac{1}{2}) = 1/\pi}$. Equal sampled slopes do not imply equal solution values between or even at later grid columns.

Step 3: What a refined grid proves. The extra column disproves equality of these two laws. It does not make finite sampling sufficient for an unrestricted smooth law. For any finite set of sampled columns x_j, the nonzero smooth function $\prod_j (x - x_j)^2$ vanishes at every one of them. Adding it to a right-hand side preserves those sampled slopes while changing the field elsewhere.

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Question 10

A scaled temperature model uses time x and temperature y:

$$y' = T(x) - y, \qquad T(x) = 2 + \sin x.$$

The prescribed ambient temperature T varies in time. A periodic response is proposed:

$$r(x) = 2 + \frac{\sin x - \cos x}{2}.$$

Tasks

1. Interpret the slope signs above and below $y = T(x)$. Compute the slopes at $(0, 1)$, $(0, 3)$, and $(\pi/2, 2)$.
2. Test whether the ambient-temperature curve itself solves the equation. Verify that r does solve it.
3. On $0 \le x \le 2\pi$, find the maximum and minimum of r, including their times. Compare the time of its maximum with the time of the ambient maximum.
4. Draw the field, the ambient curve, and the response in your solution. Explain why a response extremum occurs where the response meets the zero-slope curve, although that curve is not itself a solution.

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Question 10 – Solution

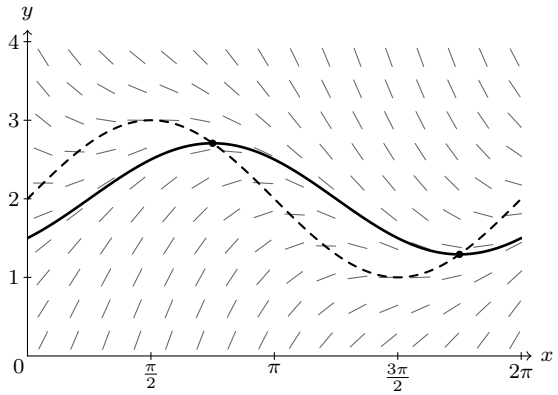
Strategy. Interpret the moving zero-slope curve as a temperature comparison, then verify the response and locate its horizontal tangents.

Step 1: Heating and cooling. Below $y = T(x)$, $y' > 0$ and the object warms; above it, $y' < 0$ and it cools. At the three listed points, the slopes are $\boxed{1, -1, 1}$.

Step 2: Check both curves. Along $y = T(x)$ the right-hand side is zero, but $T'(x) = \cos x$ is not identically zero on any interval. Thus T is not a solution. By contrast,

$$r' = \frac{\cos x + \sin x}{2} = 2 + \sin x - r,$$

so r is a solution for all real x.



Solid: response r. Dashed: ambient temperature T. Dots: response extrema.

Step 3: Extrema and delay. Rewrite

$$r(x) = 2 + \frac{1}{\sqrt{2}} \sin(x - \pi/4).$$

On the stated interval, its extreme values are

$$\boxed{r_{\max} = 2 + \frac{1}{\sqrt{2}} \text{ at } x = \frac{3\pi}{4}, \qquad r_{\min} = 2 - \frac{1}{\sqrt{2}} \text{ at } x = \frac{7\pi}{4}.}$$

The endpoint values both equal $3/2$ and lie between these extremes. The ambient maximum occurs at $x = \pi/2$, so the response maximum follows it by $\boxed{\pi/4}$ time units.

Step 4: Field interpretation. At a response extremum, $r' = T - r = 0$, so the response meets the zero-slope curve. Elsewhere it need not equal T. A locus of zero assigned slopes can move with time without being a solution curve itself.

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Question 1

Consider the initial value problem

$$y' = e^{-x^2}, \qquad y(0) = 1.$$

A proposed answer is

$$F(x) = 1 + \int_0^x e^{-t^2}\, dt.$$

A student rejects this expression because the integral has no elementary antiderivative. You may take that non-elementarity as given; proving it is not part of this problem.

Tasks

1. Verify F as an exact solution and state its interval of definition.
2. Prove uniqueness directly: if G is another solution with $G(0) = 1$ on an interval containing 0, show that $G = F$ there.
3. Without evaluating the integral numerically, prove the bounds

$$1 + e^{-1} < F(1) < 2.$$

4. Explain the distinction between an exact representation, an elementary formula, and a numerical approximation. Does the lack of an elementary antiderivative imply that the IVP has no solution?

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Question 1 – Solution

Strategy. A definite integral defines a function. The Fundamental Theorem of Calculus can verify that function even when elementary antidifferentiation is unavailable.

Step 1: Verification and domain. The integrand is continuous for every real t, so the integral exists for every finite real x. The Fundamental Theorem gives

$$F'(x) = e^{-x^2}, \qquad F(0) = 1 + 0 = 1.$$

Thus $\boxed{F \text{ is an exact solution on } \mathbb{R}.}$ Negative x causes no problem: the integral is interpreted with reversed limits.

Step 2: Uniqueness without a general theorem. If G solves the same IVP on an interval I containing 0, then

$$(G - F)' = e^{-x^2} - e^{-x^2} = 0.$$

A differentiable function with zero derivative on an interval is constant there. At 0, $G - F = 0$, so $\boxed{G(x) = F(x) \quad (x \in I).}$

Step 3: Strict integral bounds. For $0 < t < 1$,

$$e^{-1} < e^{-t^2} < 1.$$

These strict inequalities hold throughout an interval of positive length. Integrating from 0 to 1 therefore gives

$$e^{-1} < \int_0^1 e^{-t^2}\, dt < 1, \qquad \boxed{1 + e^{-1} < F(1) < 2.}$$

Step 4: What counts as a solution. The definite-integral expression specifies F exactly, with no rounding. An elementary formula is a more restricted type of expression, assembled from familiar elementary functions. A decimal obtained by numerical integration is an approximation to a value of F. Failure to find an elementary expression does not negate the exact solution already verified.

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Question 2

The exact solution of $y' = y$, $y(0) = 1$, is supplied as $y = e^x$. A polynomial approximation is

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

You may use Taylor's theorem with a fourth-derivative remainder.

Tasks

1. Calculate the residual $P' - P$ and determine whether P solves the IVP exactly on any open interval containing 0.
2. Let $E(x) = e^x - P(x)$. Prove that for $0 \le x \le 1$,
$$\frac{x^4}{24} \le E(x) \le \frac{ex^4}{24}.$$
3. Give exact lower and upper bounds for $E(1)$, and identify whether P overestimates or underestimates the solution for $0 < x \le 1$.
4. Plot the error and its two bounds in your solution. Explain why matching the initial value and several derivatives at 0 does not make the polynomial an exact interval solution.

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Question 2 – Solution

Strategy. Use a residual to test exactness and Taylor's theorem to assess the approximation. These answer different questions.

Step 1: Exactness test. Since $P' = 1 + x + x^2/2$,

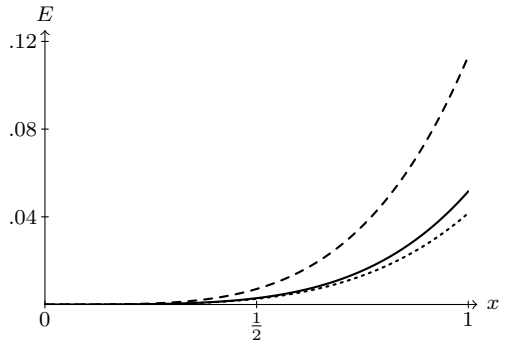
$$\boxed{P' - P = -\frac{x^3}{6}.}$$

The residual is nonzero whenever $x \neq 0$. Although $P(0) = 1$, the polynomial does not solve the equation on any open interval containing 0.

Step 2: Error bounds. For $0 < x \leq 1$, Taylor's theorem supplies a number ξ with $0 < \xi < x$ such that

$$E(x) = \frac{e^{\xi} x^4}{24}.$$

Because $1 \leq e^{\xi} \leq e$, the requested bounds follow; at $x = 0$ all three expressions are zero.



Solid: error E. Dashed: $ex^4/24$. Dotted: $x^4/24$.

Step 3: Size and sign. At $x = 1$, $P(1) = 8/3$, so

$$\boxed{\frac{1}{24} \leq E(1) = e - \frac{8}{3} \leq \frac{e}{24}.}$$

For $x > 0$ the lower bound is positive, so P underestimates the exact solution. The two endpoint bounds are in fact strict at 1.

Step 4: Local agreement is limited evidence. The values of P and its first three derivatives agree with those of e^x at 0. That is local Taylor information. Solving the ODE requires $P'(x) = P(x)$ at every point of an interval, which the nonzero residual disproves.

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Question 3

Two measurements of an initial value lead to the candidate functions

$$y(x) = e^{2x}, \qquad \widetilde{y}(x) = (1+\varepsilon)e^{2x}, \qquad \varepsilon = 10^{-3},$$

for the equation $y' = 2y$. Define absolute error and relative error by

$$A(x) = |\widetilde{y}(x) - y(x)|, \qquad R(x) = \frac{|\widetilde{y}(x) - y(x)|}{|y(x)|}.$$

Tasks

1. Verify both candidates and identify their different initial values.
2. Determine $A(x)$ and $R(x)$ for $x \geq 0$. Find the first time the absolute error reaches 0.1.
3. For the equation $z' = -2z$, compare the candidates $z = e^{-2x}$ and $\widetilde{z} = (1+\varepsilon)e^{-2x}$ in the same way.
4. Plot the two absolute errors in your solution. Explain why increasing separation here is not evidence that one IVP has multiple solutions, and why absolute and relative error tell different stories.

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Question 3 – Solution

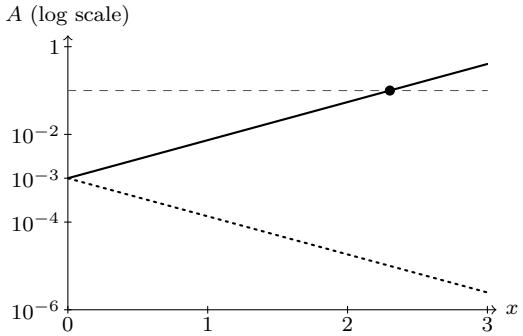
Strategy. Keep the initial conditions distinct and compute both error measures before interpreting the behavior.

Step 1: Verification. Differentiation gives $y' = 2y$ and $\widetilde{y}' = 2\widetilde{y}$. Their initial values are 1 and 1.001, respectively. They solve different IVPs for the same differential equation.

Step 2: Growth case. For $x \geq 0$,

$$\boxed{A(x) = 10^{-3}e^{2x}, \qquad R(x) = 10^{-3}.}$$

The absolute error increases strictly. Its first value 0.1 occurs when $e^{2x} = 100$, giving $\boxed{x = \ln 10}$.



Solid: growth error. Dotted: decay error. Dashed: $A = 0.1$; dot at $x = \ln 10$.

Step 3: Decay case. Both decay candidates satisfy $z' = -2z$, again with initial values 1 and 1.001. Their errors are

$$\boxed{|\widetilde{z} - z| = 10^{-3}e^{-2x}, \qquad \frac{|\widetilde{z} - z|}{|z|} = 10^{-3}.}$$

The absolute error decreases to zero and never reaches 0.1 for $x \geq 0$. The relative error stays fixed at 0.1% in both examples.

Step 4: Interpretation. Uniqueness concerns identical equations and identical initial data. Here the data differ. In the growth case the absolute discrepancy becomes large, yet it remains the same fraction of the growing reference solution. In the decay case both the solution and its discrepancy shrink proportionally.

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Question 4

For the initial value problem

$$y' = y^2, \qquad y(0) = 1,$$

a candidate is

$$f(x) = \frac{1}{1-x}.$$

A report claims: "The right-hand side y^2 is smooth everywhere, so this formula gives a solution through $x = 1$ and for all real x."

Tasks

1. Verify the candidate wherever it is defined. Find the largest open interval containing 0 on which it solves the IVP.
2. Calculate its one-sided limits at $x = 1$ and explain why assigning any finite value at 1 cannot extend this solution across that point.
3. Does the same formula solve the differential equation on $(1, \infty)$? Explain why that branch is not a continuation of the given IVP solution.
4. Sketch both branches in your solution and correct the report's inference about smooth equations and global solutions.

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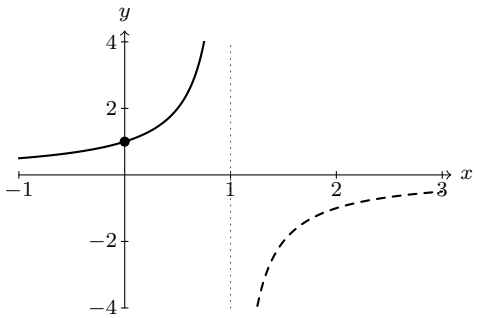
Question 4 – Solution

Strategy. Check the function's domain as well as the equation's right-hand side. A solution can become unbounded in finite time even for a smooth equation.

Step 1: Verification and interval. For $x \neq 1$,

$$f'(x) = \frac{1}{(1-x)^2} = f(x)^2, \qquad f(0) = 1.$$

The largest open interval containing 0 that avoids 1 is $\boxed{(-\infty, 1)}$.



Solid: IVP branch. Dashed: separate solution branch. Dotted: $x = 1$.

Step 2: No finite extension. The limits are

$$\boxed{\lim_{x \to 1^-} f(x) = +\infty, \qquad \lim_{x \to 1^+} f(x) = -\infty.}$$

A differentiable extension across 1 would be continuous there and would have a finite value. The left-hand divergence rules this out, independently of what formula is proposed after 1.

Step 3: The separate branch. On $(1, \infty)$, the same derivative calculation still proves $f' = f^2$. This branch is a solution of the differential equation, but its interval does not contain the initial point 0 and cannot be joined through $x = 1$ to the IVP branch.

Step 4: Corrected inference. Smoothness of a right-hand side is not a guarantee of existence for all time. Here the equation is defined at every finite pair (x, y), but the IVP solution escapes to unbounded y as $x \to 1^-$. The failure belongs to global continuation of the solution, not to a singularity of y^2 at a finite point.

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Question 5

An experiment records the complete trajectory $p(x) = e^{-x}$ for all $x \geq 0$, with $p(0) = 1$. Two proposed models are

$$\text{(A)} \quad y' = -y, \qquad \text{(B)} \quad y' = -y + (y - e^{-x})^2.$$

Tasks

1. Verify that the recorded trajectory satisfies both models.
2. Compare the slopes predicted by the models at $(0, 0)$ and $(0, 2)$. Are the right-hand sides the same function of (x, y)?
3. Test whether $y \equiv 0$ is an equilibrium solution of each model.
4. Explain why even a complete, noiseless trajectory does not identify an unrestricted differential equation uniquely. Generalize the construction by replacing the added square with a term involving an arbitrary smooth function $H(x, y)$.

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Question 5 – Solution

Strategy. A trajectory tests a model only along one curve in the (x, y)-plane. Compare predictions away from that curve.

Step 1: Agreement on the observed curve. Since $p' = -e^{-x} = -p$, model (A) holds. Along $y = p(x)$, the added term in (B) is $(p - e^{-x})^2 = 0$, so (B) also gives $p' = -p$. Both match the entire observation and its initial value exactly.

Step 2: Different off-curve predictions. At $x = 0$, $e^{-x} = 1$. Therefore

Point	Slope from (A)	Slope from (B)
$(0, 0)$	0	1
$(0, 2)$	-2	-1

Thus $\boxed{\text{the models agree on } p \text{ but are different laws.}}$

Step 3: An equilibrium distinguishes them. For $y \equiv 0$, the derivative is zero. Model (A) assigns zero, so this is an equilibrium. Model (B) assigns $e^{-2x} > 0$, so the same constant function is not a solution of (B).

Step 4: General non-identifiability. For any smooth H, the model

$$\boxed{y' = -y + H(x, y)(y - e^{-x})^2}$$

admits $p = e^{-x}$, because the added term vanishes identically along that curve. Many choices of H change slopes elsewhere. Additional experiments with different initial data or restrictions on the allowed model class are needed to distinguish such laws. Complete data on one trajectory constrain a curve, not the whole plane.

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Question 6

Compare an equation with its squared version:

$$\text{(A)} \quad y' = -\sqrt{1-y^2}, \qquad |y| \le 1,$$

$$\text{(B)} \quad (y')^2 = 1 - y^2.$$

Both are assigned $y(0) = 0$. Consider $u(x) = \sin x$ and $v(x) = -\sin x$.

Tasks

1. Verify that both candidates satisfy (B) and the initial value. Which satisfies (A) on $(-\pi/2, \pi/2)$?
2. State the sign restriction lost by squaring. Give a condition that, together with (B), recovers (A) for real differentiable functions.
3. Explain why $v = -\sin x$ does not satisfy (A) on all of $\mathbb{R}$.
4. Extend the accepted middle branch to a C^1 solution of (A) on all of $\mathbb{R}$ by adjoining constant pieces outside $[-\pi/2, \pi/2]$. Verify the joins and sketch the extension in your solution.

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Question 6 – Solution

Strategy. Squaring forgets a sign. Restore that sign when selecting a branch, and check differentiability when extending it.

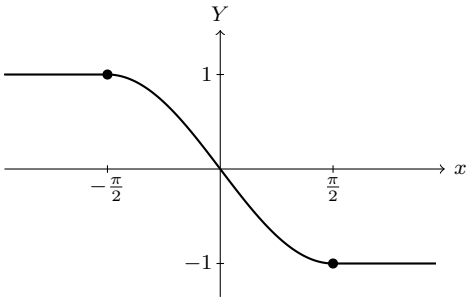
Step 1: Compare the candidates. Both satisfy $(y')^2 = \cos^2 x = 1 - y^2$ and have value zero at 0. On $(-\pi/2, \pi/2)$, $\cos x > 0$, so the right-hand side of (A) is $-|\cos x| = -\cos x$. Hence

$$\boxed{v = -\sin x \text{ satisfies (A), but } u = \sin x \text{ does not.}}$$

Step 2: Recover the lost information. Equation (A) forces $y' \le 0$. Conversely, (B) and $y' \le 0$ imply $|y| \le 1$ and $y' = -\sqrt{1 - y^2}$. Outside the middle interval, $v' = -\cos x$ is positive wherever $\cos x < 0$, so v fails (A) there.

Step 3: A global extension. Define

$$\boxed{Y(x) = \begin{cases} 1, & x \le -\pi/2, \\ -\sin x, & -\pi/2 < x < \pi/2, \\ -1, & x \ge \pi/2. \end{cases}}$$



The constant pieces meet $-\sin x$ with matching value and zero derivative.

At either join, the middle value matches the adjoining constant and its derivative $-\cos x$ tends to zero, the derivative of the constant piece. Thus Y is C^1. On the constant pieces and at the joins, both sides of (A) are zero. On the middle piece, the verification above applies. Therefore Y solves the original IVP on $\mathbb{R}$; extension is possible here because both endpoint values and derivatives remain finite.

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Question 7

Consider the IVP $y' = 0$, $y(0) = 0$ and the functions

$$u(x) = 0, \qquad v(x) = \begin{cases} 0, & x \le 0, \\ e^{-1/x^2}, & x > 0. \end{cases}$$

You may use the fact that v is infinitely differentiable on $\mathbb{R}$ and that $v^{(n)}(0) = 0$ for every integer $n \ge 0$.

Tasks

1. Determine the Taylor series at 0 for both functions, using the given derivative information.
2. Calculate $v'(x)$ for $x > 0$ and decide whether v solves the IVP on any open interval containing 0.
3. Prove that u is the only solution of the IVP on an interval containing 0.
4. Explain why agreement of every derivative at a single point does not contradict your conclusion. Distinguish a smooth function from one known to equal its Taylor series near that point.

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Question 7 – Solution

Strategy. Use interval verification for the ODE. Infinite-order agreement at one point is still pointwise information unless convergence to the function is established.

Step 1: Identical Taylor coefficients. For both functions, every derivative at zero is zero, so both Taylor series are

$$\boxed{\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.}$$

This describes the series built from the derivatives; it does not yet assert that the series represents either function away from zero.

Step 2: The equation fails to the right. For $x > 0$, the chain rule gives

$$\boxed{v'(x) = \frac{2}{x^3} e^{-1/x^2} > 0.}$$

Every open interval containing 0 includes positive x, where the required identity $v' = 0$ fails. Thus v does not solve this IVP on such an interval, despite satisfying the initial value and having every derivative there equal to those of u.

Step 3: Uniqueness directly. If $y' = 0$ on an interval, the Mean Value Theorem implies that y is constant on that interval. The initial value forces that constant to be zero. Hence $\boxed{y \equiv 0 \text{ is the unique IVP solution.}}$

Step 4: Smoothness versus a series representation. Smoothness means the derivatives of all orders exist and are continuous. A function is analytic at a point if it equals its Taylor series in some neighborhood of that point. The supplied v is smooth but not analytic at 0: its Taylor series is zero, whereas $v(x) > 0$ for every $x > 0$. Agreement of derivatives at a single point therefore does not replace the interval identity required by the ODE.

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Question 8

An implicit description of a function is proposed for $x \geq 0$:

$$y^3 + xy = 1, \qquad y > 0.$$

The associated initial value problem is

$$y' = -\frac{y}{x + 3y^2}, \qquad y(0) = 1.$$

You may use implicit differentiation where the coefficient of y' is nonzero.

Tasks

1. Show that for every fixed $x \geq 0$ the implicit relation has exactly one positive root y.
2. Differentiate the relation, verify the ODE and initial value, and calculate the initial slope. Explain why the denominator stays nonzero on this branch.
3. Prove that the branch decreases for $x \geq 0$ and tends to zero as $x \to \infty$.
4. Prove $2/3 < y(1) < 3/4$ and sketch the positive branch in your solution. Explain why an explicit formula obtained by solving the cubic is unnecessary for these conclusions.

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Question 8 – Solution

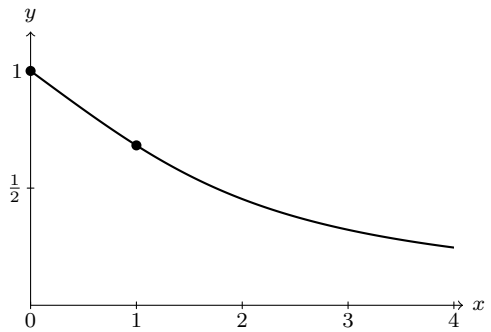
Strategy. Establish a unique branch before differentiating. Monotonicity and inequalities can describe an implicit solution without an explicit cubic formula.

Step 1: A well-defined positive branch. For fixed $x \geq 0$, let $H(y) = y^3 + xy$. It is continuous, starts at $H(0) = 0$, and tends to infinity as $y \to \infty$. For $y > 0$, $H'(y) = 3y^2 + x > 0$, so the Intermediate Value Theorem and strict increase give exactly one positive root of $H(y) = 1$.

Step 2: Verification. Differentiation yields

$$3y^2y' + y + xy' = 0, \qquad \boxed{y' = -\frac{y}{x + 3y^2}.}$$

The denominator is positive on the branch, allowing local differentiable continuation, including near $(0, 1)$. At zero the relation gives $y^3 = 1$, so $y(0) = 1$ and $\boxed{y'(0) = -1/3}$.



The positive branch of $y^3 + xy = 1$; dots at $x = 0$ and $x = 1$.

Step 3: Decrease and limit. Positivity of y and the denominator implies $y' < 0$. Also $y^3 + xy = 1$ gives $0 < y < 1/x$ for $x > 0$, so the squeeze principle gives $\boxed{y(x) \to 0 \text{ as } x \to \infty}$.

Step 4: A rigorous bracket. At $x = 1$,

$$(2/3)^3 + 2/3 = 26/27 < 1, \qquad (3/4)^3 + 3/4 = 75/64 > 1.$$

Strict increase in y proves $\boxed{2/3 < y(1) < 3/4}$. Root existence, differentiation and inequalities establish these results directly; an explicit cubic formula adds no needed information.

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Question 9

A tank initially holds 10 liters of water containing 30 grams of salt. Fresh water enters at 2 liters per minute, and a well-mixed solution leaves at 4 liters per minute. Before the tank empties, the model is

$$V(t) = 10 - 2t, \qquad S'(t) = -\frac{4S(t)}{10 - 2t}, \qquad S(0) = 30,$$

where S is salt mass in grams. A candidate is

$$S(t) = 30(1 - t/5)^2.$$

Tasks

1. Explain the units and physical meaning of the right-hand side of the salt equation. Verify the candidate wherever that equation is defined.
2. State the largest mathematical solution interval containing 0, and the physical time interval for the stated tank model.
3. Find the concentration $C(t) = S(t)/V(t)$ before emptying and the limits of S and C as the emptying time is approached.
4. Plot salt mass on the physical interval in your solution. Explain why the polynomial formula does not validate continued operation of this model after the tank empties, and distinguish a concentration limit from a concentration assigned to an empty tank.

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Question 9 – Solution

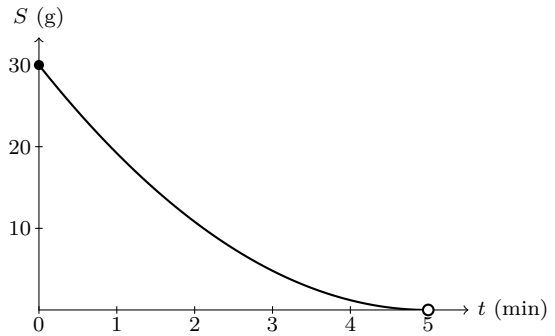
Strategy. Separate algebraic verification from the equation's domain and from the physical assumptions behind it.

Step 1: Units and substitution. The concentration S/V has units g/L; multiplying by the outflow 4 L/min gives salt loss in g/min. Fresh inflow adds no salt. The candidate satisfies

$$S' = -12(1 - t/5), \qquad -\frac{4S}{10 - 2t} = -12(1 - t/5) \quad (t \neq 5),$$

and $S(0) = 30$.

Step 2: Two different intervals. The differential equation is undefined at $t = 5$. The largest open mathematical interval containing the initial time is $\boxed{(-\infty, 5)}$. The stated experiment starts at 0 and requires positive volume, so its physical interval is $\boxed{0 \leq t < 5}$ minutes.



Solid: salt mass for $0 \leq t < 5$. Open endpoint: emptying time is excluded.

Step 3: Concentration and endpoint limits. Before emptying,

$$\boxed{C(t) = 3(1 - t/5) \text{ g/L}, \qquad \lim_{t \to 5^-} S(t) = 0, \quad \lim_{t \to 5^-} C(t) = 0.}$$

At $t = 5$, both S and V have zero limits. The quotient S/V is not defined for the empty tank, even though the pre-emptying concentration has a limit.

Step 4: Why a polynomial is not a license to extrapolate. The polynomial can be evaluated after 5, but $V(t)$ would then be negative and the assumed outflow could not be maintained from this empty tank. The algebraic formula is not sufficient to preserve the model's physical meaning or extend the original ODE across its undefined point.

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Question 10

A population model in the original units is

$$\frac{dP}{dt} = \frac{1}{5}P(1 - P/50), \qquad P(0) = 10,$$

where t is measured in days and 50 has the same population units as P. Define dimensionless variables

$$\tau = t/5, \qquad u(\tau) = P(5\tau)/50.$$

A candidate for a normalized model is

$$u(\tau) = \frac{1}{1 + 4e^{-\tau}}.$$

Tasks

1. Use the chain rule to derive the equation and initial value for u.
2. Verify the candidate in the normalized variables, then express the corresponding $P(t)$ in the original variables.
3. Find when the population first reaches 25, giving both dimensionless time and time in days.
4. Explain why changing units alters numerical slopes and time labels without producing a different physical prediction. Identify what would go wrong if the factor from $t = 5\tau$ were omitted in differentiation.

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Question 10 – Solution

Strategy. Transform both the dependent variable and time. The chain-rule factor is essential when comparing numerical rates in different units.

Step 1: Transform the IVP. Since $u(\tau) = P(5\tau)/50$,

$$\frac{du}{d\tau} = \frac{5}{50}P'(5\tau) = \frac{5}{50} \cdot \frac{1}{5}(50u)(1-u) = u(1-u).$$

The initial value is $u(0) = 10/50 = 1/5$. Hence

$$\boxed{u' = u(1-u), \qquad u(0) = 1/5.}$$

Step 2: Verify and return to original units. The supplied function satisfies

$$u' = \frac{4e^{-\tau}}{(1+4e^{-\tau})^2} = u(1-u), \qquad u(0) = 1/5.$$

Substituting $\tau = t/5$ gives

$$\boxed{P(t) = \frac{50}{1+4e^{-t/5}}.}$$

As a direct check, $P' = 40e^{-t/5}/(1+4e^{-t/5})^2$ agrees with the original right-hand side and $P(0) = 10$.

Step 3: First passage to half capacity. The candidate increases strictly and reaches $P = 25$ when $u = 1/2$. Therefore

$$1 + 4e^{-\tau} = 2 \quad \Longrightarrow \quad \boxed{\tau = \ln 4, \qquad t = 5\ln 4 \text{ days.}}$$

Step 4: Same prediction, different labels. One unit of τ is five days, and one unit of u is fifty population units. Omitting the factor 5 in $du/d\tau = (5/50)P'(5\tau)$ would incorrectly give $u' = u(1-u)/5$ and misrepresent the normalized time scale. A change of units must transform rates as well as values.

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Question 1

Consider the initial value problem

$$(1 + x^2)y' + 2xy = 3x^2 + 1, \qquad y(0) = 2.$$

Tasks

1. Put the equation in standard linear form $y' + p(x)y = q(x)$ and find an integrating factor. Explain why multiplying by that factor produces a product derivative.
2. Find the general solution and then the solution of the IVP.
3. Verify the IVP solution in the original, unnormalized equation and state its largest open interval.
4. Identify a straight-line particular solution and determine whether every solution approaches that line as $x \to +\infty$ and as $x \to -\infty$. State precisely what approaches zero.

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Question 1 – Solution

Strategy. Normalize first. Choose μ with $\mu' = p\mu$, so the product rule gives $(\mu y)' = \mu(y' + py)$.

Step 1: Integrating factor. Since $1 + x^2 > 0$,

$$y' + \frac{2x}{1+x^2}y = \frac{3x^2+1}{1+x^2}, \qquad \mu = e^{\int 2x/(1+x^2)\,dx} = 1 + x^2.$$

Multiplication restores the original left side as $[(1 + x^2)y]'$.

Step 2: Integrate and select the constant. We obtain

$$(1 + x^2)y = \int (3x^2 + 1)\,dx = x^3 + x + C,$$

so

$$\boxed{y = x + \frac{C}{1+x^2}.}$$

At $x = 0$, $C = 2$. The IVP solution is therefore $\boxed{y = x + 2/(1 + x^2)}$.

Step 3: Direct check and interval. Its derivative is $y' = 1 - 4x/(1 + x^2)^2$, and

$$\begin{aligned}(1 + x^2)y' + 2xy &= 1 + x^2 - \frac{4x}{1+x^2} + 2x^2 + \frac{4x}{1+x^2} \\ &= 3x^2 + 1.\end{aligned}$$

Also $y(0) = 2$. No denominator vanishes for real x, so the largest open interval is $\boxed{\mathbb{R}}$.

Step 4: Approach to a line. Setting $C = 0$ gives the particular solution $y_p = x$. For every fixed real C,

$$\boxed{y(x) - x = \frac{C}{1+x^2} \longrightarrow 0 \quad \text{as } x \to \pm\infty.}$$

It is the vertical difference from the line that tends to zero; the solution itself does not tend to a finite constant.

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Question 2

On the domain $x > 0$, consider

$$xy' + 2y = x^2 \ln x, \qquad y(1) = 0.$$

Tasks

1. Solve the IVP using an integrating factor, showing the integration-by-parts step needed for the forcing term.
2. State the largest open interval containing 1 on which the solution solves this equation, and verify the initial value and differential equation.
3. Rewrite the solution using a definite integral based at 1. Use the sign of that integral to decide whether the solution can be negative.
4. Locate the global minimum on $x > 0$ and sketch the solution near $x = 1$ in your solution. Explain why positive forcing only for $x > 1$ does not make the solution negative for $0 < x < 1$.

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Question 2 – Solution

Strategy. Use x^2 as integrating factor, then retain a definite-integral form to analyze signs on both sides of the initial point.

Step 1: Solve. Dividing by x gives $y' + 2y/x = x \ln x$. With $\mu = x^2$,

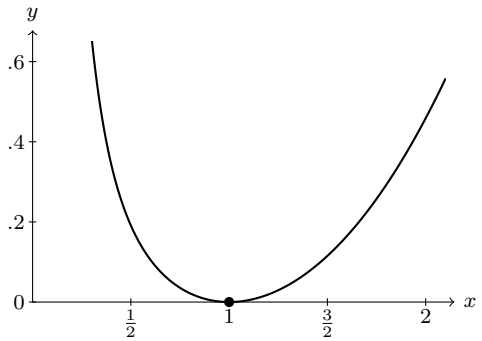
$$(x^2 y)' = x^3 \ln x.$$

Integration by parts, $u = \ln x$, $dv = x^3 dx$, yields

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C.$$

The initial value forces $C = 1/16$, hence

$$\boxed{y = \frac{x^2}{4} \ln x - \frac{x^2}{16} + \frac{1}{16x^2}, \qquad x > 0.}$$



The solution is positive except at its global minimum $(1, 0)$.

Step 2: Verify. Substitution at 1 gives zero. Differentiating $x^2 y$ gives $x^3 \ln x$, or $x^2 y' + 2xy = x^3 \ln x$; division by $x > 0$ recovers the original equation. The logarithm and coefficient division restrict the interval to $\boxed{(0, \infty)}$.

Step 3: Sign and minimum. Equivalently,

$$x^2 y(x) = \int_1^x t^3 \ln t \, dt.$$

For $x > 1$ the integral is positive. For $0 < x < 1$, the integrand is negative between x and 1, but the reversed limits make the integral positive. Since $x^2 > 0$, $y(x) > 0$ for $x \neq 1$. Thus $\boxed{\min_{x>0} y = 0 \text{ only at } x = 1}$. The sign of the forcing alone does not determine the sign of the solution.

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Question 3

A student tries to solve

$$y' - 2y = xe^{2x}, \qquad y(0) = 0,$$

and writes

$$\mu = e^{2x}, \qquad (e^{2x}y)' = xe^{4x}.$$

Tasks

1. Identify the precise error by expanding the student's claimed product derivative.
2. Find a correct integrating factor and solve the IVP.
3. Verify the solution by direct substitution in $y' - 2y = xe^{2x}$.
4. Replace the initial condition by $y(0) = a$, with a any real number. Find the resulting family and prove directly that two different choices of a cannot produce intersecting solution curves.

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Question 3 – Solution

Strategy. The integrating factor must match the signed coefficient of y. Check its product derivative before integrating.

Step 1: Diagnose the sign error. The student's derivative is

$$(e^{2x}y)' = e^{2x}(y' + 2y),$$

whereas multiplying the original equation by e^{2x} gives $e^{2x}(y' - 2y)$. These expressions differ in the sign of the y term. The required coefficient is $p = -2$, so $\boxed{\mu = e^{-2x}}$.

Step 2: Correct integration. Multiplication gives

$$(e^{-2x}y)' = x, \qquad e^{-2x}y = \frac{x^2}{2} + C.$$

Since $y(0) = 0$, $C = 0$, and

$$\boxed{y = \frac{x^2}{2}e^{2x}, \qquad x \in \mathbb{R}.}$$

Step 3: Verify. Differentiating explicitly,

$$y' = xe^{2x} + x^2e^{2x}, \qquad y' - 2y = xe^{2x} + x^2e^{2x} - x^2e^{2x} = xe^{2x}.$$

The initial value is also zero.

Step 4: Vary the initial datum. The general initial value sets $C = a$, giving

$$\boxed{y_a(x) = \left(a + \frac{x^2}{2}\right)e^{2x}.}$$

For distinct a, b,

$$y_a(x) - y_b(x) = (a - b)e^{2x} \neq 0$$

at every real x. Hence their graphs never intersect. The nonintersection follows from the explicit homogeneous difference, not merely from a sketch.

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Question 4

Consider the linear equation

$$y' - y = e^{-x}, \qquad x \geq 0.$$

Instead of prescribing an initial value, require that the solution remain bounded on the entire half-line $[0, \infty)$.

Tasks

1. Find the general solution using an integrating factor.
2. Determine the unique bounded solution and the initial value it must have.
3. If that initial value is changed by a nonzero amount ε, find the difference from the bounded solution and describe its behavior as $x \to \infty$.
4. Sketch the bounded solution and the solutions for $\varepsilon = \pm 1/20$ in your solution. Explain why a forcing term tending to zero does not force every solution to tend to zero.

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Question 4 – Solution

Strategy. Separate the forced response from the homogeneous term. A condition at infinity can select the constant just as an initial value can.

Step 1: General solution. With $\mu = e^{-x}$,

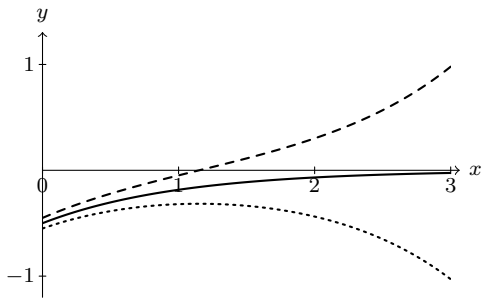
$$(e^{-x}y)' = e^{-2x}, \qquad e^{-x}y = -\frac{1}{2}e^{-2x} + C.$$

Thus $\boxed{y = Ce^{x} - \frac{1}{2}e^{-x}}$; each formula is defined on $\mathbb{R}$.

Step 2: Impose boundedness. As $x \to \infty$, the second term tends to zero. If $C \neq 0$, the term Ce^{x} is unbounded and cannot be canceled by that decaying term. Therefore

$$\boxed{y_b = -\frac{1}{2}e^{-x}, \qquad y_b(0) = -\frac{1}{2}.}$$

This solution is bounded and tends to zero, proving both existence and uniqueness under the stated boundedness condition.



Solid: bounded solution. Dashed: $\varepsilon = 1/20$. Dotted: $\varepsilon = -1/20$.

Step 3: Perturb the selected initial value. Since $y(0) = C - 1/2$, changing it to $-1/2 + \varepsilon$ gives $C = \varepsilon$. Consequently

$$\boxed{y_\varepsilon - y_b = \varepsilon e^{x}.}$$

For $\varepsilon > 0$ the perturbed solution tends to $+\infty$; for $\varepsilon < 0$ it tends to $-\infty$. Direct differentiation verifies $y' - y = e^{-x}$ for every C. The vanishing forcing does not remove a growing homogeneous component.

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Question 5

Two functions are known to solve the same first-order equation in standard linear form,

$$y' + p(x)y = q(x), \qquad x \in \mathbb{R},$$

where p, q are continuous. The functions are

$$y_1 = x + e^{-x}, \qquad y_2 = x - e^{-x}.$$

Tasks

1. Recover $p(x)$ and $q(x)$ from these two solutions. Explain why they are determined at every real x.
2. Solve the recovered equation using an integrating factor to obtain all its solutions.
3. Determine all real constants α, β for which $\alpha y_1 + \beta y_2$ is also a solution of the same nonhomogeneous equation.
4. Explain why arbitrary sums of solutions need not remain solutions, although the difference of two solutions always satisfies the associated homogeneous equation.

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Question 5 – Solution

Strategy. Subtract the two equations to eliminate the forcing. A nonzero solution difference reveals the coefficient p.

Step 1: Recover the coefficients. Let $d = y_1 - y_2 = 2e^{-x}$. Subtraction gives $d' + pd = 0$, so

$$p = -\frac{d'}{d} = 1,$$

valid for every real x because d never vanishes. Using $y_1' = 1 - e^{-x}$,

$$q = y_1' + y_1 = 1 - e^{-x} + x + e^{-x} = x + 1.$$

Thus $\boxed{y' + y = x + 1}$ is uniquely recovered within the stated standard linear form.

Step 2: All solutions. The integrating factor is e^x, giving

$$(e^x y)' = (x+1)e^x = (xe^x)'.$$

Integration yields $\boxed{y = x + Ce^{-x}, \quad C \in \mathbb{R}}$. Substitution confirms $y' + y = x + 1$ for every constant.

Step 3: Which combinations work? Write $L[v] = v' + v$. Linearity gives

$$L[\alpha y_1 + \beta y_2] = (\alpha + \beta)(x + 1).$$

For this to equal $x + 1$ throughout an interval, we require

$$\boxed{\alpha + \beta = 1.}$$

These combinations are $x + (\alpha - \beta)e^{-x}$ and generate every solution.

Step 4: Nonhomogeneous versus homogeneous. The sum $y_1 + y_2 = 2x$ produces $L[2x] = 2x + 2$, not $x + 1$. The difference instead satisfies $L[y_1 - y_2] = q - q = 0$. Thus differences solve the homogeneous equation; combinations with coefficients summing to one preserve the original forcing.

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Question 6

For each real initial value a, consider

$$y' + 2xy = 1, \qquad y(0) = a.$$

A definite integral is an acceptable exact expression; no elementary antiderivative is required.

Tasks

1. Use an integrating factor to obtain an exact solution based at $x = 0$, and state its interval of definition.
2. Verify the formula by differentiation and the initial condition directly.
3. For solutions with initial values a and b, determine their difference, its greatest absolute value on $\mathbb{R}$, and whether their curves can cross when $a \neq b$.
4. Show that the solution with $a = 0$ is odd and positive for $x > 0$. Sketch the curves for $a = -1, 0, 1$ in your solution.

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Question 6 – Solution

Strategy. Use a definite integral to incorporate the initial value exactly. The homogeneous difference makes comparison easier than evaluating the integral.

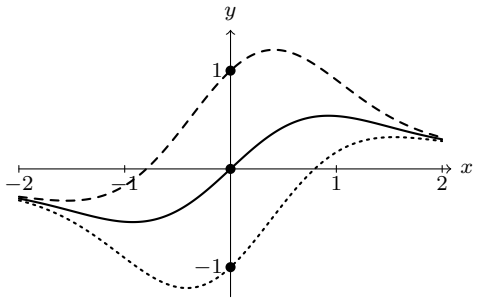
Step 1: Integrating factor and solution. Here

$$\mu = e^{\int 2x\,dx} = e^{x^2}, \qquad (e^{x^2}y)' = e^{x^2}.$$

Integrating from 0 to x gives

$$\boxed{y_a(x) = e^{-x^2}\left(a + \int_0^x e^{t^2}\,dt\right), \qquad x \in \mathbb{R}.}$$

The integral exists for every finite real x.



Solid: $a = 0$. Dashed: $a = 1$. Dotted: $a = -1$. The curves never cross.

Step 2: Direct verification. The product rule and Fundamental Theorem give $y_a' = -2xy_a + e^{-x^2}e^{x^2} = 1 - 2xy_a$. Also the integral is zero at 0, so $y_a(0) = a$.

Step 3: Compare initial values. The integral cancels in the difference:

$$\boxed{y_a - y_b = (a-b)e^{-x^2}, \qquad \max_{x\in\mathbb{R}} |y_a - y_b| = |a-b|.}$$

For $a \neq b$, equality in the maximum occurs only at $x = 0$, and the difference never changes sign or vanishes at a finite point. The curves cannot cross.

Step 4: Symmetry of the zero-data solution. Since e^{t^2} is even, $\int_0^x e^{t^2}dt$ is odd. Multiplying by the even factor e^{-x^2} preserves oddness. For $x > 0$, both the integral and prefactor are positive, so $y_0(x) > 0$.

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Question 7

Consider the periodically forced equation

$$y' + 2y = 3\cos x + \sin x,$$

with initial condition $y(0) = 0$.

Tasks

1. Solve the IVP using an integrating factor. Show how you evaluate the integral of the forcing times that factor.
2. Find the unique 2π-periodic solution of the equation, and verify both its periodicity and its differential equation.
3. Find the earliest $X \geq 0$ such that the IVP solution differs from the periodic solution by at most $1/100$ for every $x \geq X$.
4. Sketch the IVP solution and the periodic solution on $[0, 2\pi]$ in your solution. Explain which part of the general solution is the transient and why the prescribed IVP solution is not itself periodic.

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Question 7 – Solution

Strategy. Integrate the forcing after multiplying by e^{2x}, then separate periodic and decaying terms.

Step 1: Integrate. The equation becomes

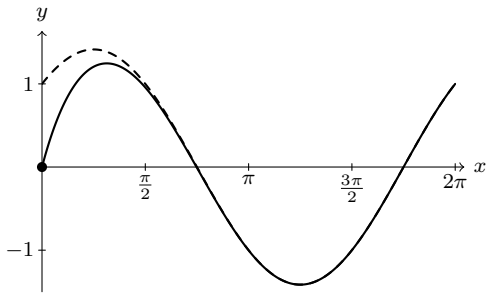
$$(e^{2x}y)' = e^{2x}(3\cos x + \sin x).$$

Seek an antiderivative $e^{2x}(A\cos x + B\sin x)$. Matching coefficients gives $2A + B = 3$ and $2B - A = 1$, hence $A = B = 1$. Indeed,

$$[e^{2x}(\cos x + \sin x)]' = e^{2x}(3\cos x + \sin x).$$

Therefore $y = \cos x + \sin x + Ce^{-2x}$. Since $y(0) = 0$, $C = -1$, giving

$$\boxed{y = \cos x + \sin x - e^{-2x}.}$$



Solid: IVP solution. Dashed: periodic solution $y_p = \cos x + \sin x$.

Step 2: The only periodic member. The function $y_p = \cos x + \sin x$ is 2π-periodic and satisfies $y_p' + 2y_p = 3\cos x + \sin x$. For the general solution,

$$y(x + 2\pi) - y(x) = Ce^{-2x}(e^{-4\pi} - 1).$$

This vanishes for every x only if $C = 0$. Hence $\boxed{y_p = \cos x + \sin x}$ is the unique periodic member; the IVP has $C = -1$ and is not periodic.

Step 3: Uniform accuracy after a time. The exact discrepancy is $|y - y_p| = e^{-2x}$, strictly decreasing on $x \geq 0$. Thus

$$e^{-2X} \leq \frac{1}{100} \quad \Longleftrightarrow \quad \boxed{X \geq \ln 10.}$$

The earliest such time is $\ln 10$. The term Ce^{-2x} is transient because it vanishes as $x \to \infty$, leaving the periodic response.

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Question 8

A switched forcing acts on a linear equation for $t \geq 0$:

$$y' + y = f(t), \qquad y(0) = 0, \qquad f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ 0, & t > 1. \end{cases}$$

Seek a continuous function that is continuously differentiable on each side of $t = 1$ and satisfies the equation away from that switching time. No derivative at $t = 1$ is assumed.

Tasks

1. Solve on $0 \leq t < 1$ using an integrating factor, then use continuity to solve on $t > 1$.
2. Find the global maximum for $t \geq 0$, including its time, and the limit as $t \to \infty$.
3. Compute the left and right derivatives at $t = 1$. Is the resulting function a classical differentiable solution at that point?
4. Evaluate $\int_0^\infty y(t)\,dt$ and relate it to the total forcing $\int_0^\infty f(t)\,dt$. Sketch the continuous response in your solution, showing the change in slope at the switch.

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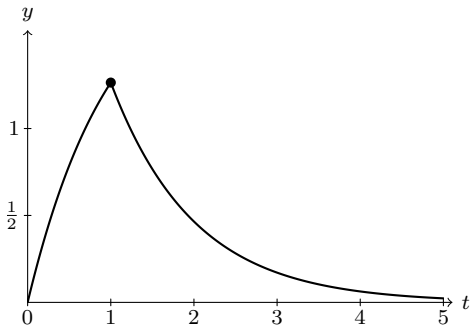
Question 8 – Solution

Strategy. Integrate separately on the two smooth pieces and match values at the switch; continuity need not imply a continuous derivative.

Step 1: Solve and match. Before the switch, $(e^t y)' = 2e^t$, so $y = 2 + Ce^{-t}$. Initial data give $C = -2$ and $y(1) = A := 2(1 - e^{-1})$. Afterward $(e^t y)' = 0$; matching this value yields

$$\boxed{y(t) = \begin{cases} 2(1 - e^{-t}), & 0 \le t \le 1, \\ Ae^{-(t-1)}, & t > 1, \end{cases} \qquad A = 2(1 - e^{-1}).}$$

Differentiation on each open piece verifies its corresponding equation.



The maximum occurs at the switch $t = 1$, where the derivative jumps by -2.

Step 2: Maximum and derivative jump. Before 1, $y' = 2e^{-t} > 0$; after 1, $y' = -Ae^{-(t-1)} < 0$. Hence the global maximum is $\boxed{A \text{ at } t = 1}$, and the limit is zero. At the switch,

$$y'(1^-) = 2/e, \qquad y'(1^+) = -A = 2/e - 2.$$

The jump is -2, so the derivative does not exist there. The response meets the requested piecewise conditions, but is not a classical differentiable solution at the switch.

Step 3: Total response. Direct integration gives

$$\int_0^1 2(1 - e^{-t})dt = 2/e, \qquad \int_1^\infty Ae^{-(t-1)}dt = A,$$

so $\boxed{\int_0^\infty y(t)dt = 2 = \int_0^\infty f(t)dt}$. Equivalently, integrate $y' + y = f$ on the two pieces: continuity cancels the joining values and $y(0) = \lim y = 0$ cancels the endpoint contribution.

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Question 9

The equation

$$xy' - y = x^2$$

has a vanishing leading coefficient at $x = 0$. Initially impose $y(1) = 2$.

Tasks

1. Solve the IVP on $x > 0$ using an integrating factor after normalization.
2. Determine whether this solution extends to a differentiable solution of the original equation on all of $\mathbb{R}$. If so, prove that such an extension is unique.
3. Now discard $y(1) = 2$ and prescribe only $y(0) = 0$. Determine all differentiable solutions on $\mathbb{R}$ satisfying this new initial condition.
4. Decide whether a differentiable solution with $y(0) = 1$ can exist. Explain why dividing by x and invoking a regular standard-form IVP at 0 would be invalid.

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Question 9 – Solution

Strategy. Normalize only where $x \neq 0$. To cross the exceptional point, return to the original equation and check value and derivative matching.

Step 1: Solve on the positive half-line. Dividing by $x > 0$ gives $y' - y/x = x$. The integrating factor is $e^{-\ln x} = 1/x$, so

$$(y/x)' = 1, \qquad y = x^2 + Cx.$$

Since $y(1) = 2$, $C = 1$, giving $\boxed{y = x^2 + x \quad (x > 0)}$.

Step 2: Unique differentiable extension. The polynomial $x^2 + x$ satisfies $xy' - y = x^2$ everywhere, including $x = 0$. Any solution on $x < 0$ also satisfies $(y/x)' = 1$, hence has form $x^2 + Dx$. Continuity at 0 forces $y(0) = 0$, and differentiability requires the slopes at zero to match: $D = 1$. Thus

$$\boxed{y = x^2 + x \text{ is the unique differentiable extension to } \mathbb{R}.}$$

Step 3: Data only at the exceptional point. With only $y(0) = 0$, the same argument allows any common slope C on the two sides. All global differentiable solutions are

$$\boxed{y = x^2 + Cx, \qquad C \in \mathbb{R}.}$$

Indeed $x(2x + C) - (x^2 + Cx) = x^2$, and every such polynomial has value zero at 0. This IVP is therefore nonunique.

Step 4: Incompatible data and normalization. At $x = 0$, the original equation requires $-y(0) = 0$, ruling out $y(0) = 1$. The normalized coefficient $-1/x$ is undefined at zero, so regular standard-form conclusions cannot be applied there. The algebraic condition at the exceptional point must be checked separately.

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Question 10

Let a be a real parameter in

$$y' + ay = e^{-x}, \qquad y(0) = 1.$$

Tasks

1. Solve the IVP for every real a, handling separately any parameter value for which the usual antiderivative formula changes.
2. Verify the initial condition and equation, and state the solution interval for each parameter.
3. For fixed x, show that the formula for $a \neq 1$ tends to the $a = 1$ solution as $a \to 1$. Explain why an apparent denominator singularity in a parameter need not be a singularity of the solution.
4. Determine exactly which values of a produce a bounded solution on $[0, \infty)$, and compute the limit or divergence as $x \to \infty$ in each parameter range.

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Question 10 – Solution

Strategy. Keep the parameter visible during integration, and isolate the exceptional exponent before dividing by $a-1$.

Step 1: Integrate with the initial value. With $\mu = e^{ax}$,

$$(e^{ax}y)' = e^{(a-1)x}, \qquad e^{ax}y = 1 + \int_0^x e^{(a-1)t}\,dt.$$

Consequently

$$\boxed{y(x) = \begin{cases} \dfrac{e^{-x} + (a-2)e^{-ax}}{a-1}, & a \neq 1, \\ (1+x)e^{-x}, & a = 1. \end{cases}}$$

Each formula is defined on $\mathbb{R}$.

Step 2: Verify. At zero the numerator for $a \neq 1$ is $a-1$, so $y(0) = 1$; the exceptional formula also gives 1. Differentiating the definite-integral identity above gives $y' + ay = e^{-x}$. At $a = 1$, a direct check is $y' = -xe^{-x}$ and $y' + y = e^{-x}$.

Step 3: Remove the apparent parameter singularity. For $a \neq 1$, rewrite the result as

$$y = e^{-ax}\left[1 + \frac{e^{(a-1)x} - 1}{a-1}\right].$$

For fixed x, the quotient tends to x as $a \to 1$, by the derivative of the exponential (also at $x = 0$). Hence $y \to (1+x)e^{-x}$. The singularity in the divided parameter formula is removable.

Step 4: Long-time behavior. If $a > 0$ and $a \neq 1$, both exponentials decay, so $y \to 0$; the same holds at $a = 1$. If $a = 0$, $y = 2 - e^{-x} \to 2$. If $a < 0$, the coefficient $(a-2)/(a-1)$ is positive and e^{-ax} grows without bound, so $y \to +\infty$. Thus

$$\boxed{y \text{ is bounded on } [0,\infty) \quad \Longleftrightarrow \quad a \geq 0.}$$

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Question 1

Consider the initial-value problem

$$y' = 2x(1 + y^2), \qquad y(0) = 1.$$

Tasks

1. Solve by separation, explaining how the initial condition selects the integration constant.
2. Find the maximal open interval containing 0 on which the solution is finite and differentiable.
3. Determine its monotonicity, minimum, and behavior at both interval endpoints.
4. A student says that the periodicity of tangent extends this same solution through each vertical asymptote. Explain the error and verify the valid solution directly.

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Question 1 – Solution

Strategy. Separate using the everywhere-positive factor $1 + y^2$, then respect the range of the inverse tangent.

Step 1: Integrate and select the branch.

$$\frac{dy}{1+y^2} = 2x\,dx, \qquad \arctan y = x^2 + C, \qquad C = \frac{\pi}{4}.$$

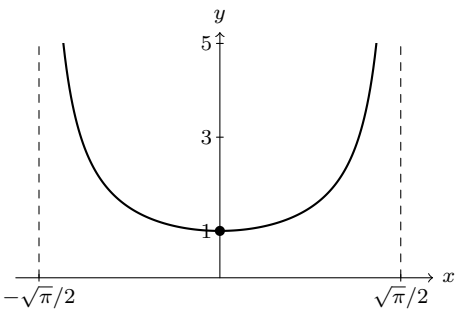
Thus $\boxed{y(x) = \tan(x^2 + \pi/4)}$ near 0. No constant solution was lost, since $1 + y^2$ never vanishes for real y.

Step 2: Identify the interval and shape. For the branch through 0, we need $-\pi/2 < x^2 + \pi/4 < \pi/2$. Hence

$$\boxed{I = (-\sqrt{\pi}/2, \sqrt{\pi}/2)}.$$

The derivative has the sign of x: the solution decreases to its minimum $y(0) = 1$ and then increases. At either endpoint approached from inside I, the angle tends to $\pi/2$ from below, so $y \to +\infty$.

Step 3: Verify and explain the obstruction. Differentiation gives $y' = 2x\sec^2(x^2 + \pi/4) = 2x(1 + y^2)$, and $y(0) = 1$. A real differentiable extension through an endpoint would have a finite, continuous value there, contradicting blow-up. Other tangent branches lie on separate intervals; periodicity does not connect them across a pole.



The branch through $(0, 1)$ has a pole at each dashed boundary.

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Question 2

For the autonomous equation

$$y' = y(3 - y),$$

consider the initial condition $y(0) = 4$.

Tasks

1. Find every constant solution before dividing by any expression involving y.
2. Obtain the nonconstant solution family by separation and solve the given initial-value problem.
3. Find its maximal interval containing 0, its direction of motion, and its limiting values at the ends of that interval.
4. Find the exact positive time when $y = 7/2$. Explain why this solution never reaches or crosses $y = 3$ at a finite time.

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Question 2 – Solution

Strategy. Preserve the two equilibrium solutions, then use partial fractions on a nonconstant branch.

Step 1: Separate without losing constants. The constant solutions are $\boxed{y \equiv 0,\ y \equiv 3}$. Away from those values,

$$\frac{1}{y(3-y)} = \frac{1}{3y} + \frac{1}{3(3-y)}, \qquad \frac{1}{3}\ln\left|\frac{y}{3-y}\right| = x + C.$$

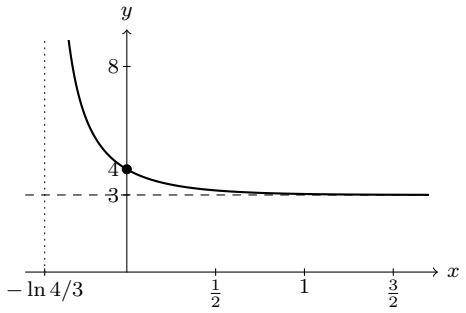
The nonconstant family can be written $y = 3/(1 + Ae^{-3x})$, $A \neq 0$, on intervals where the denominator is nonzero. The value $A = 0$ restores $y \equiv 3$; $y \equiv 0$ must still be listed separately.

Step 2: Apply the data and determine the domain. From $4 = 3/(1 + A)$, $A = -1/4$, so

$$\boxed{y(x) = \frac{3}{1 - \frac{1}{4}e^{-3x}}, \qquad I = (-\ln 4/3, \infty)}.$$

Here $y > 3$, so $y' < 0$. The limits are $+\infty$ at the left endpoint and 3 as $x \to \infty$. Writing $D = 1 - \frac{1}{4}e^{-3x}$ gives $y' = -9e^{-3x}/(4D^2) = y(3-y)$, verifying the equation.

Step 3: Compute the passage time. The equation $7/2 = 3/(1 - \frac{1}{4}e^{-3x})$ gives $\boxed{x = \frac{1}{3}\ln(7/4)}$. At any finite $x \in I$, the exponential is positive, so $y - 3 > 0$: the limiting level is never attained or crossed.



The decreasing solution stays above the dashed equilibrium $y = 3$.

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Question 3

Consider an equation whose right-hand side is undefined on the line $y = 2$:

$$y' = -\frac{e^x}{y-2}, \qquad y(0) = 1.$$

Tasks

1. Separate the equation and obtain an implicit relation.
2. Select the correct explicit branch and determine its maximal interval containing 0.
3. Find the limiting value of y and the behavior of y' at the finite endpoint. Decide whether that endpoint can be included.
4. Another student takes the opposite square-root sign, then proposes switching signs at the endpoint. Test both claims against the equation and initial condition.

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Question 3 – Solution

Strategy. Integration gives a squared relation; the initial condition and the excluded line determine which part is a solution.

Step 1: Integrate and choose the sign.

$$(y-2)\,dy = -e^x\,dx, \qquad \frac{1}{2}(y-2)^2 = -e^x + C.$$

At $(0,1)$, $C = 3/2$, so $(y-2)^2 = 3 - 2e^x$. Since $y(0) - 2 = -1$, continuity selects

$$\boxed{y = 2 - \sqrt{3-2e^x}, \qquad I = (-\infty, \ln(3/2))}.$$

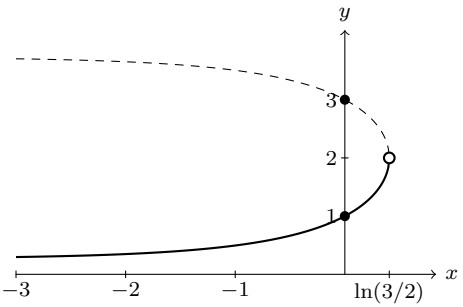
The radicand must be strictly positive: zero would give the forbidden value $y = 2$.

Step 2: Check behavior and verify. Throughout I,

$$y' = \frac{e^x}{\sqrt{3-2e^x}} = -\frac{e^x}{y-2} > 0.$$

As $x \to -\infty$, $y \to 2 - \sqrt{3}$. As $x \uparrow \ln(3/2)$, $y \to 2$ but $y' \to +\infty$. A finite limit of y alone does not permit extension: the equation is undefined at the endpoint and a finite derivative is impossible there.

Step 3: Audit the other branch. The branch $2 + \sqrt{3-2e^x}$ satisfies the differential equation on I, but has $y(0) = 3$, not 1. Switching signs would still pass through $y = 2$, and for larger x the radicand is negative. Neither proposal extends the given real solution.



Solid: $y(0) = 1$. Dashed: $y(0) = 3$. The common limiting point is excluded.

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Question 4

An explicit elementary formula is not required in this problem:

$$y' = \frac{x}{1 + e^y}, \qquad y(0) = 0.$$

Tasks

1. Use separation to find an implicit equation for the solution.
2. Prove that it determines exactly one real value of y for every real x, and that the resulting function is differentiable.
3. Determine its symmetry, monotonicity, and minimum. Prove the bound $0 \le y(x) \le x^2/4$ and identify every equality case.
4. Verify the initial condition and differential equation, and explain why failure to isolate y in elementary functions is not a failure to solve the problem.

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Question 4 – Solution

Strategy. Treat the left side of the integrated equation as an invertible function rather than forcing an elementary inverse.

Step 1: Integrate. Multiplication and integration give

$$(1+e^y)\,dy = x\,dx, \qquad \boxed{y + e^y = 1 + \frac{x^2}{2}}.$$

The constant is 1 because $y(0) = 0$.

Step 2: Prove global existence and uniqueness of the implicit value. Let $F(u) = u + e^u$. It is continuous and strictly increasing because $F'(u) = 1 + e^u > 0$. Also $F(u) \to -\infty$ as $u \to -\infty$ and $F(u) \to \infty$ as $u \to \infty$. It therefore has an inverse on all of $\mathbb{R}$. Since its derivative never vanishes, the inverse is differentiable. Thus

$$y(x) = F^{-1}(1 + x^2/2)$$

defines a unique differentiable solution for every real x. Any solution through $(0, 0)$ must satisfy the same integrated identity, establishing uniqueness for the initial-value problem.

Step 3: Extract shape and an exact bound. Uniqueness of the implicit value and the dependence on x^2 show $y(-x) = y(x)$. Since $F(0) = 1$, $y \geq 0$, with equality only at $x = 0$. The differential equation makes y decreasing for $x < 0$ and increasing for $x > 0$, so its unique minimum is 0.

Using $e^y \geq 1 + y$, with equality only at $y = 0$, gives

$$1 + \frac{x^2}{2} = y + e^y \geq 1 + 2y, \qquad \boxed{0 \leq y \leq \frac{x^2}{4}}.$$

Equality in either bound occurs only at $x = 0$.

Step 4: Verify and interpret. Implicit differentiation yields $(1 + e^y)y' = x$, exactly the given equation. At $x = 0$, strict monotonicity of F forces $y = 0$. The implicit formula uniquely specifies every solution value, its full domain, and its derivative; an elementary inverse is unnecessary.

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Question 5

Consider these two equations for real x and y:

$$\text{(A)} \quad y' = x + y, \qquad \text{(B)} \quad y' = xy + x + y + 1.$$

Here "directly separable" means that the right-hand side can be written $g(x)h(y)$ on the region in question, without a change of variables.

Tasks

1. Prove that any product $f(x, y) = g(x)h(y)$ satisfies
$$f(a, c)f(b, d) = f(a, d)f(b, c).$$
2. Use $a = c = 1$ and $b = d = 2$ to show that (A) is not directly separable on the rectangle $[1, 2] \times [1, 2]$.
3. Factor the right-hand side of (B), then find all its real solutions on intervals and list any constant solution separately.
4. Solve (B) with $y(0) = -2$, verify it, and explain why division by $x + 1$ is unnecessary even at $x = -1$.

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Question 5 – Solution

Strategy. Test whether variables actually factor before performing separation; a sum of terms can still conceal a product.

Step 1: Establish the necessary product test. If $f(x, y) = g(x)h(y)$, then both products in the stated identity equal $g(a)g(b)h(c)h(d)$. This argument does not divide by any factor and remains valid if some factors vanish.

Step 2: Reject direct separation for (A). For $f(x, y) = x + y$, the proposed rectangle gives

$$f(1,1)f(2,2) = 2 \cdot 4 = 8, \qquad f(1,2)f(2,1) = 3 \cdot 3 = 9.$$

The necessary identity fails, so there is no product representation on this rectangle. This conclusion concerns direct separation on that region; it makes no claim about other methods of solving (A).

Step 3: Solve the factored equation (B). Here $xy + x + y + 1 = (x + 1)(y + 1)$. First, $y \equiv -1$ is a constant solution. For $y \neq -1$,

$$\frac{dy}{y+1} = (x+1)\,dx, \qquad \ln|y+1| = \frac{x^2}{2} + x + C.$$

Absorbing the fixed sign into a nonzero constant gives

$$\boxed{y = -1 + Ae^{x^2/2+x}}.$$

Allowing $A = 0$ includes the constant solution as well. These are all solutions: for any differentiable solution, the product rule applied to $(y + 1)e^{-x^2/2-x}$ gives derivative zero, even at $y = -1$.

Step 4: Apply and check the data. The initial condition requires $A = -1$, so $\boxed{y = -1 - e^{x^2/2+x}}$ on all of $\mathbb{R}$. Its derivative is $-(x + 1)e^{x^2/2+x} = (x + 1)(y + 1)$, and $y(0) = -2$. Separation integrates the factor $x + 1$; it never requires its reciprocal. The point $x = -1$ is an ordinary point with $y' = 0$.

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Question 6

A nonnegative quantity satisfies

$$y' = -3y^{2/3}, \qquad y(0) = 8, \qquad t \geq 0,$$

where the prime means d/dt. For real y, define $y^{2/3} = (\sqrt[3]{y})^2$.

Tasks

1. Solve while $y > 0$ and find the first time the quantity reaches zero.
2. Find and justify the unique continuation for all $t \geq 0$ if the solution must remain nonnegative.
3. If negative values are allowed, classify all continuously differentiable continuations after the first zero, including solutions that wait at zero before becoming negative.
4. Check the equation at every joining time and identify exactly what is lost by dividing by $y^{2/3}$.

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Question 6 – Solution

Strategy. Separate only where $y \neq 0$, then use the sign of y' and check any joins at zero directly.

Step 1: Reach zero. While $y > 0$, integration gives $3y^{1/3} = -3t + C$. The data yield $y = (2-t)^3$ for $0 \leq t < 2$, so the first zero is $\boxed{t = 2}$.

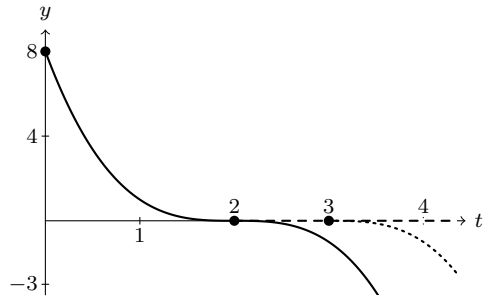
Step 2: Enforce nonnegativity. The equation always gives $y' \leq 0$. Once a nonnegative solution reaches zero it cannot increase or decrease without violating that constraint. Thus the unique nonnegative continuation is $(2-t)^3$ up to 2, followed by 0 forever.

Step 3: Classify real continuations. For any finite $b \geq 2$, a real continuation is

$$\boxed{y_b(t) = \begin{cases} (2-t)^3, & 0 \leq t \leq 2, \\ 0, & 2 \leq t \leq b, \\ -(t-b)^3, & t \geq b. \end{cases}}$$

Also allow $b = \infty$, meaning the solution stays zero forever. These exhaust the possibilities: monotonicity makes the zero set after 2 an interval; once negative, separation gives $\sqrt[3]{y} = b - t$, where continuity fixes the constant at the departure time b.

Step 4: Verify the joins and the lost solutions. Each cubic piece has $y' = -3(\sqrt[3]{y})^2$. At $t = 2$ and any finite b, both one-sided derivatives are 0, matching the equation at $y = 0$. The pieces are therefore C^1. Division excludes zero itself, so it misses the constant zero solution and all zero waiting intervals; using one integration constant through zero is unjustified.



After $t = 2$: solid $b = 2$, dotted $b = 3$, dashed $b = \infty$ (zero forever).

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Question 7

A positive concentration obeys a power-law decay model

$$\frac{dy}{dt} = -ky^p, \qquad k > 0, \quad p \in \mathbb{R}.$$

Use fixed units for time and concentration. Exact observations are

$$y(0) = 16, \qquad y(1) = 8, \qquad y(2) = 4.$$

Assume one pair of constants (k, p) describes the entire observation interval.

Tasks

1. Express each of the two observed one-unit decay times as a separated definite integral, without assuming $p = 1$.
2. Prove that these observations determine p uniquely. Treat the case $p = 1$ without dividing by $1 - p$.
3. Find k and the resulting solution, and predict when the concentration first reaches 1.
4. Explain why the first two observations alone would not determine p.

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Question 7 – Solution

Strategy. Compare passage-time integrals on two concentration intervals related by scaling; the unknown constant k then cancels.

Step 1: Write the two passage times. Since $y > 0$ and $k > 0$, the concentration decreases strictly. Separation gives

$$1 = \frac{1}{k}\int_8^{16} s^{-p}\,ds, \qquad 1 = \frac{1}{k}\int_4^{8} s^{-p}\,ds.$$

These integrals are positive and finite for every real p.

Step 2: Recover the exponent. In the first integral put $s = 2u$:

$$\int_8^{16} s^{-p}\,ds = 2^{1-p}\int_4^{8} u^{-p}\,du.$$

The observed equal passage times force $2^{1-p} = 1$, hence $\boxed{p = 1}$. Conversely, $p = 1$ makes the two integrals equal. No division by $1 - p$ was used, so the logarithmic case was retained throughout.

Step 3: Determine the rate and prediction. For $p = 1$,

$$k = \int_8^{16} \frac{ds}{s} = \ln 2, \qquad \ln\frac{y}{16} = -t\ln 2.$$

Therefore $\boxed{y(t) = 16\,2^{-t}}$. It gives exactly $16, 8, 4$ at the measured times, and differentiating gives $y' = -(\ln 2)y$. It is positive for every real t. The value 1 is first reached at $\boxed{t = 4}$ because the solution is strictly decreasing.

Step 4: Identify the information supplied by the third observation. With only $y(0) = 16$ and $y(1) = 8$, each real p could be paired with

$$k(p) = \int_8^{16} s^{-p}\,ds > 0.$$

The corresponding separated solution decreases from 16 to 8 in one unit of time and stays positive during that interval. Thus infinitely many exponents fit the first two data; the second measured halving time identifies the exponent within this model family.

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Question 8

Consider the parameterized initial-value problem

$$y' = (\sin x)y^2, \qquad y(0) = a, \qquad a > 0.$$

Tasks

1. Solve by separation and determine the maximal open interval containing 0 for every positive a.
2. Find the precise threshold separating global bounded solutions from finite-endpoint blow-up. Include the threshold case.
3. For the global solutions, find the minimum, maximum, and a period.
4. A student argues that $\int_0^{2\pi} \sin x\, dx = 0$ guarantees $y(2\pi) = a$ for every $a > 0$. Explain exactly when this reasoning is valid and why it can fail.

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Question 8 – Solution

Strategy. Track the denominator of the separated formula throughout an interval, not merely at its endpoints.

Step 1: Integrate. Since $a > 0$, separation is valid near 0:

$$-\frac{1}{y} = -\cos x + C, \qquad \boxed{y = \frac{a}{1 - a + a\cos x}}.$$

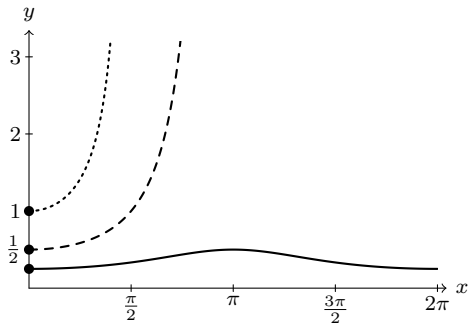
Its derivative is $a^2 \sin x/(1 - a + a\cos x)^2 = (\sin x)y^2$, and $y(0) = a$.

Step 2: Classify all positive parameters. The denominator $D = 1 - a + a\cos x$ ranges from $1 - 2a$ to 1.

- If $0 < a < 1/2$, then $D > 0$ everywhere: $I = \mathbb{R}$ and y is bounded.
- If $a = 1/2$, then $y = 1/(1 + \cos x)$ and $I = (-\pi, \pi)$.
- If $a > 1/2$, let $\alpha = \arccos(1 - 1/a) \in (0, \pi)$. Then $I = (-\alpha, \alpha)$.

In the last two cases $D \downarrow 0$ at either endpoint from inside I, so $y \to +\infty$. Thus the threshold itself already blows up.

Step 3: Determine extrema and audit the return claim. For $0 < a < 1/2$, $\min y = a$ at $x = 2n\pi$ and $\max y = a/(1 - 2a)$ at $x = (2n + 1)\pi$, $n \in \mathbb{Z}$; 2π is a period. Only this parameter range gives a solution through the full cycle. For $a \geq 1/2$, a pole occurs before reaching 2π. Substituting 2π into the algebraic formula gives a value on a disconnected branch, not a continuation of the initial solution.



Solid: $a = 1/4$. Dashed: $a = 1/2$. Dotted: $a = 1$. Only the solid curve completes a cycle.

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Question 9

Let A and B be prescribed real numbers. Seek a real function y that is continuously differentiable on $[-1, 1]$ and satisfies

$$y' = 2xy^2, \qquad y(-1) = A, \qquad y(1) = B.$$

At the endpoints, derivatives are understood as one-sided derivatives.

Tasks

1. For $A \neq 0$, separate the equation and express the candidate solution in terms of A. Also treat $A = 0$.
2. Determine all pairs (A, B) for which a solution exists on the entire interval, and prove your conditions are sufficient as well as necessary.
3. Show why matching the two endpoint values in a formula does not by itself prove existence. Use $A = B = -2$ as a test case.
4. For every admissible pair, find $y(0)$ and decide whether the solution is unique.

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Question 9 – Solution

Strategy. Separate from the left endpoint, then check the denominator on the whole prescribed interval before accepting the right endpoint.

Step 1: Obtain the candidate. For a nonzero solution,

$$-\frac{1}{y} = x^2 + C, \qquad \frac{1}{y} = \frac{1}{A} + 1 - x^2, \qquad \boxed{y = \frac{A}{1 + A(1 - x^2)}}.$$

For $A = 0$, $y \equiv 0$ is a solution. No solution can leave or cross zero: the right-hand side $2xy^2$ has a continuous y-derivative, $4xy$, so the usual local uniqueness theorem applies at every finite point. Consequently the separated formula applies to any solution with $A \neq 0$ throughout its interval.

Step 2: Test the entire interval. On $[-1, 1]$, the quantity $1 - x^2$ ranges over $[0, 1]$. If $A \geq 0$, the denominator is at least 1. If $-1 < A < 0$, it is at least $1 + A > 0$. For $A = -1$ it vanishes at $x = 0$; for $A < -1$ it vanishes at

$$x = \pm\sqrt{1 + 1/A} \in (-1, 1).$$

The nonzero numerator cannot cancel these zeros. Any admissible formula gives $y(1) = A$, so the necessary conditions are

$$\boxed{B = A \quad \text{and} \quad A > -1}.$$

They are also sufficient: the displayed formula, including $A = 0$, is smooth on a neighborhood of $[-1, 1]$. Its derivative is $2xA^2/[1 + A(1 - x^2)]^2 = 2xy^2$, and it gives both specified endpoint values.

Step 3: Expose the false endpoint test. For $A = B = -2$, the candidate is $y = -2/(2x^2 - 1)$. It has the desired values at $x = \pm 1$ but poles at $x = \pm 1/\sqrt{2}$. Hence it is not a solution on the entire interval.

Step 4: Give the center value and uniqueness. For every admissible pair, $\boxed{y(0) = A/(1 + A)}$. The initial value at -1 already fixes the candidate uniquely; for $A = 0$, local uniqueness also forbids any departure from zero. The second endpoint is a compatibility condition, not an extra free integration constant.

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Question 10

Design an affine coefficient $g(x) = \alpha + \beta x$ in the separable equation

$$y' = g(x)(1 - y^2)$$

so that one solution satisfies

$$y(0) = 0, \qquad y(1) = \frac{1}{2}, \qquad y'(1) = 0.$$

The constants α and β are real.

Tasks

1. Find α and β and justify why the three requirements determine them uniquely.
2. Find an explicit formula for the selected solution. You may use exponentials instead of hyperbolic functions.
3. Determine its maximal interval, every zero, its global maximum, and its limits as $x \to \pm\infty$.
4. Verify the three requirements directly and list the constant solutions omitted when separating by $1-y^2$.

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Question 10 – Solution

Strategy. Use the zero terminal slope to constrain the coefficient, then use the separated integral to enforce the target height.

Step 1: Determine the coefficient. Since $1 - y(1)^2 = 3/4$, $y'(1) = 0$ forces $\alpha + \beta = 0$. The smooth right-hand side gives local uniqueness, so the solution starting at 0 cannot meet either constant solution $y = \pm 1$. On its branch $-1 < y < 1$,

$$\frac{1}{2} \ln \frac{1+y}{1-y} = \alpha x + \frac{\beta x^2}{2}.$$

At $x = 1$, this becomes $\frac{1}{2} \ln 3 = \alpha + \beta/2$. Hence

$$\boxed{\alpha = \ln 3, \qquad \beta = -\ln 3}.$$

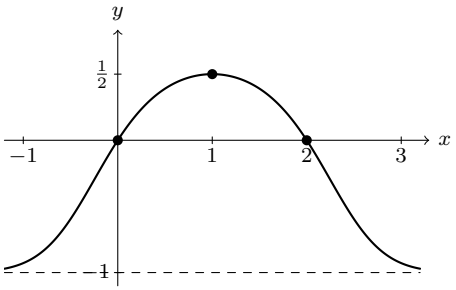
These two independent linear conditions have exactly one pair of coefficients.

Step 2: Solve and interpret the curve. Set $E(x) = 3^{\,2x - x^2} > 0$. Exponentiating and isolating y gives

$$\boxed{y(x) = \frac{E(x) - 1}{E(x) + 1}, \qquad I = \mathbb{R}}.$$

Since $E + 1 > 0$, the formula is globally smooth and lies strictly between -1 and 1. Its zeros are $x = 0, 2$. The exponent $2x - x^2 = 1 - (x-1)^2$ has its unique maximum at 1, and $(E-1)/(E+1)$ increases with E. Thus the unique global maximum is $y(1) = 1/2$, while both limits at infinity are -1.

Step 3: Verify and restore constants. Here $E' = 2(\ln 3)(1-x)E$ and $1 - y^2 = 4E/(E+1)^2$, so $y' = (\ln 3)(1-x)(1-y^2)$. Also $E(0) = 1$ and $E(1) = 3$ verify both values and the zero slope. The omitted constants are $\boxed{y \equiv 1,\ y \equiv -1}$; neither satisfies $y(0) = 0$.



The unique maximum is $(1, 1/2)$; both tails approach the dashed level $y = -1$.

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Question 1

Consider the differential equation

$$(2xy + 1)\,dx + (x^2 + y^2)\,dy = 0, \qquad y(1) = 1.$$

A potential for $M\,dx + N\,dy$ is a function F satisfying $F_x = M$ and $F_y = N$; its level curves $F = C$ give implicit solutions.

Tasks

1. Check exactness and construct a potential, retaining the unknown function that arises when integrating with respect to x.
2. Apply the initial condition and verify the resulting implicit solution by differentiation.
3. Find the slope and tangent line at $(1, 1)$.
4. Prove that the selected level determines a unique differentiable function $y(x)$ on all of $\mathbb{R}$, despite the vanishing of $x^2 + y^2$ at the origin.

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Question 1 – Solution

Strategy. Recover a potential from both coefficients, then examine its derivative with respect to y on the selected level.

Step 1: Test and integrate. Here $M_y = 2x = N_x$, so the form is exact on $\mathbb{R}^2$. Integrating $F_x = M$ gives

$$F = x^2y + x + h(y), \qquad F_y = x^2 + h'(y).$$

Matching N requires $h'(y) = y^2$, so we may choose $F = x^2y + x + y^3/3$. At $(1, 1)$, $F = 7/3$. Thus

$$\boxed{x^2y + x + \frac{y^3}{3} = \frac{7}{3}}.$$

Step 2: Verify and find the tangent. Differentiating this identity gives

$$2xy + 1 + (x^2 + y^2)y' = 0,$$

exactly the original equation for a graph. At $(1, 1)$ this yields $y' = -3/2$, hence $\boxed{y - 1 = -\frac{3}{2}(x - 1)}$.

Step 3: Establish a global graph. For fixed x, the function $u \mapsto x^2u + u^3/3$ is strictly increasing and ranges over all real numbers. This remains true at $x = 0$, where the cubic alone is strictly increasing. Therefore the equation

$$x^2u + \frac{u^3}{3} = \frac{7}{3} - x$$

has exactly one real root for every real x.

The derivative $F_y = x^2 + y^2$ can vanish only at $(0, 0)$, but that point has $F = 0$, not $7/3$. It is consequently nonzero everywhere on the selected level. The implicit function theorem gives a differentiable local graph at each point; the uniqueness of each root makes those graphs agree. They form one differentiable solution on all of $\mathbb{R}$. The isolated degeneracy on a different level does not obstruct this IVP.

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Question 2

A student studies

$$(e^y + y^2)\,dx + (xe^y + 2xy + \cos y)\,dy = 0, \qquad y(0) = 0.$$

After integrating the coefficient of dx, the student writes

$$F_*(x, y) = xe^y + xy^2$$

and claims that $F_* = 0$ solves the initial-value problem.

Tasks

1. Check exactness and identify precisely what is missing from the student's integration.
2. Construct a correct potential and select the correct level using the data.
3. Compute the local slope at $(0, 0)$ and explain why the student's proposed level cannot be the required solution graph.
4. Integrate the coefficient of dy first as an independent check. Explain why the two correct potentials can differ by a constant.

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Question 2 – Solution

Strategy. A partial antiderivative permits an arbitrary function of the other variable; matching the second coefficient determines it.

Step 1: Locate the missing term. Both cross partials equal $e^y + 2y$, so the form is exact on $\mathbb{R}^2$. Integration with respect to x must give

$$F = xe^y + xy^2 + h(y).$$

Then $F_y = xe^y + 2xy + h'(y)$, so $h'(y) = \cos y$. We may take $h(y) = \sin y$; treating it as merely a numerical constant would omit a required derivative.

Step 2: Apply and test the initial condition. The correct potential is $F = xe^y + xy^2 + \sin y$, and $F(0,0) = 0$. Thus

$$\boxed{x(e^y + y^2) + \sin y = 0}.$$

Its derivative along a graph is exactly the given equation. At the initial point, $M = 1$ and $N = 1$, so the local graph exists and $\boxed{y'(0) = -1}$.

In contrast, $F_* = 0$ says $x(e^y + y^2) = 0$. Since $e^y + y^2 > 0$, this is the vertical line $x = 0$, which cannot define $y(x)$ on an open interval. Also $(F_*)_y = N - \cos y$, so its differential is not the stated form.

Step 3: Check by reversing the integration order. Integrating N with respect to y gives

$$F = xe^y + xy^2 + \sin y + g(x).$$

Now $F_x = e^y + y^2 + g'(x) = M$ requires $g'(x) = 0$, leaving only an additive constant. Any two potentials for the same form have difference with both partial derivatives zero; on the connected plane that difference is constant. Changing this constant only relabels the same solution levels.

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Question 3

Let $a, b \in \mathbb{R}$. Consider

$$(3x^2y^2 + 2bxy + e^y)\,dx + (2ax^3y + 5x^2 + xe^y + 4y^3)\,dy = 0.$$

Tasks

1. Determine every pair (a, b) for which the form is exact on $\mathbb{R}^2$. Explain why testing equality at a single point is insufficient.
2. For those parameters, construct a potential and verify both of its first partial derivatives.
3. Find the implicit solution through $(0, 1)$ and its tangent line there.
4. A student checks $M_y = N_x$ only at $(0, 0)$ and accepts every pair (a, b). Show why that check cannot distinguish any of the parameters.

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Question 3 – Solution

Strategy. Exactness requires an identity on a region, so compare the variable-dependent terms before constructing a potential.

Step 1: Determine the parameters. The cross partials are

$$M_y = 6x^2y + 2bx + e^y, \qquad N_x = 6ax^2y + 10x + e^y.$$

Their difference is $6(1-a)x^2y + (2b-10)x$. For this to vanish for all (x, y), first put $y = 0$ and $x = 1$, obtaining $b = 5$. Then put $x = y = 1$, obtaining $a = 1$. Conversely, those choices make the difference identically zero. Hence

$$\boxed{a = 1, \qquad b = 5}.$$

Step 2: Construct and verify the potential. With these values, integrating M in x yields

$$F = x^3y^2 + 5x^2y + xe^y + h(y).$$

Matching F_y with N gives $h'(y) = 4y^3$. Thus choose

$$F = x^3y^2 + 5x^2y + xe^y + y^4.$$

Direct differentiation gives $F_x = 3x^2y^2 + 10xy + e^y = M$ and $F_y = 2x^3y + 5x^2 + xe^y + 4y^3 = N$.

Step 3: Apply the data and find the tangent. Since $F(0, 1) = 1$, the selected level is $\boxed{F(x, y) = 1}$. At that point, $N = 4 \neq 0$, so it determines a local solution graph. Its slope is $-M/N = -e/4$, giving

$$\boxed{y - 1 = -\frac{e}{4}x}.$$

Step 4: Explain the failed one-point test. At $(0, 0)$, both cross partials equal 1 for every a, b: all parameter-dependent terms vanish there. This agreement is necessary at that point but says nothing about agreement on a neighborhood. Exactness cannot be certified from that single sample.

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Question 4

The coefficient $N(x, y)$ is unknown in

$$(2x + y)\,dx + N(x, y)\,dy = 0.$$

Assume N is continuously differentiable on $\mathbb{R}^2$, the form is exact there, and the calibration data are $N(0, y) = 2y$ for every real y.

Tasks

1. Recover N uniquely and construct a potential.
2. Select the solution through $(0, 1)$ and find an explicit formula for its graph near that point.
3. Determine the maximal open interval of this graph and its slope at $x = 0$.
4. Find the vertical-tangent points of the full selected level curve. Explain why the full curve is not a single differentiable function $y(x)$.

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Question 4 – Solution

Strategy. Use exactness as an equation for the missing coefficient, then distinguish a complete level curve from its graph branches.

Step 1: Recover the form. Exactness gives $N_x = M_y = 1$, so $N = x + h(y)$. The data force $h(y) = 2y$, uniquely. A potential is

$$\boxed{N = x + 2y, \qquad F = x^2 + xy + y^2}.$$

Its derivatives are $2x + y$ and $x + 2y$, as required.

Step 2: Select a branch. The initial point gives $F = 1$. Solving the quadratic in y and selecting $y(0) = 1$ yields

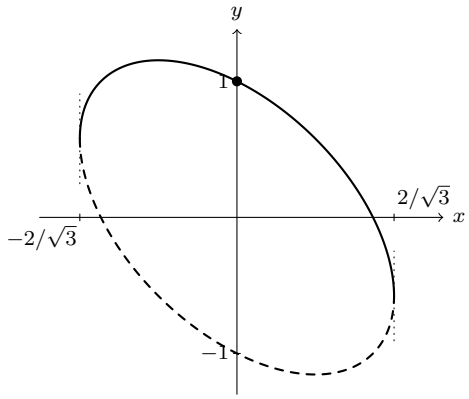
$$\boxed{y = \frac{-x + \sqrt{4 - 3x^2}}{2}, \qquad I = (-2/\sqrt{3}, 2/\sqrt{3})}.$$

At $(0, 1)$, $y' = -(2x + y)/(x + 2y) = -1/2$.

Step 3: Interpret the endpoints and full curve. On this branch, $N = \sqrt{4 - 3x^2} > 0$. At the two endpoints, $N = 0$, while $M = 3x/2 \neq 0$. Thus a finite y' cannot satisfy $M + Ny' = 0$ there. The full ellipse has vertical tangents at

$$\boxed{(2/\sqrt{3}, -1/\sqrt{3}), \quad (-2/\sqrt{3}, 1/\sqrt{3})}.$$

It remains a smooth curve at these points, but cannot be continued through either as a differentiable graph over x. Most vertical lines through the ellipse meet both quadratic branches, so the complete ellipse is not one function $y(x)$.



Solid: IVP branch, excluding its vertical-tangent endpoints. Dashed: the other half.

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Question 5

Consider the exact equation

$$-2x\,dx + 2y\,dy = 0, \qquad y(0) = 0.$$

In this problem a solution means a real, continuously differentiable function $y(x)$ on an open interval containing 0, satisfying $-2x + 2yy' = 0$ at every point, including 0.

Tasks

1. Find a potential and the level selected by the initial condition.
2. Classify all solution graphs on such an interval. Justify why no additional sign-switching choices are differentiable at 0.
3. Check $y = x$, $y = -x$, $y = |x|$ and $y = -|x|$ against the stated definition.
4. Explain why neither division by y nor the usual nonzero-F_y implicit-function criterion settles the initial point.

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Question 5 – Solution

Strategy. The selected level has crossing branches. Analyze differentiability at their intersection directly in the original equation.

Step 1: Find the level. Since $M_y = N_x = 0$, a potential is $F = y^2 - x^2$. Along any solution, $dF/dx = 0$, and the initial condition gives

$$\boxed{y^2 - x^2 = 0}.$$

Thus for each $x \neq 0$, y/x is either 1 or -1.

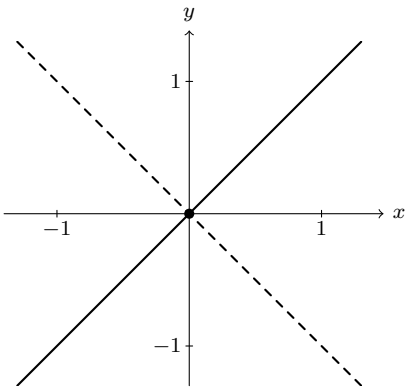
Step 2: Classify differentiable choices. Continuity makes y/x constant on each of the two connected pieces $x < 0$ and $x > 0$: a continuous function taking only the values ± 1 cannot switch between them. Write those constants as s_- and s_+. At 0, the left and right difference quotients are s_- and s_+. Differentiability requires them to agree. Therefore the only possibilities are

$$\boxed{y = x \quad \text{or} \quad y = -x}.$$

Both extend to all of $\mathbb{R}$.

Step 3: Verify and identify the failures. For either line, $yy' = x$ everywhere, including 0. The choices $|x|$ and $-|x|$ satisfy the level identity and the equation away from 0, but their one-sided derivatives disagree at 0. They fail the required definition.

Step 4: Explain the degenerate point. Dividing gives $y' = x/y$, which is undefined at the initial point and discards information about solutions of the original form there. Also $F_y = 2y = 0$ at $(0, 0)$, so the nonzero-F_y criterion does not apply. Its failure does not prove nonexistence: here two valid graph branches pass through the point.



Two differentiable solutions cross at $(0, 0)$: solid $y = x$, dashed $y = -x$.

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Question 6

On the half-plane $x > 0$, consider

$$(2y + x)\,dx + x\,dy = 0, \qquad y(1) = 2.$$

An integrating factor is a nonzero function that makes a differential form exact when multiplied into both coefficients.

Tasks

1. Show that the original form is not exact. Derive, using the product rule, the condition on an integrating factor $\mu(x)$ depending only on x.
2. Find every such nonzero factor on $x > 0$, choose one, and construct a potential for the multiplied form.
3. Solve and verify the initial-value problem on $x > 0$.
4. Now consider the original equation on the whole plane. Decide whether the selected solution can extend through $x = 0$, and identify which member of the implicit solution family does extend differentiably through that point.

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Question 6 – Solution

Strategy. Derive the integrating-factor condition from exactness rather than importing a formula, and return to the original equation at a zero of the chosen factor.

Step 1: Derive the factor. Originally $M_y = 2$ while $N_x = 1$. For $\mu = \mu(x)$, exactness requires

$$(\mu M)_y = 2\mu = (\mu x)_x = x\mu' + \mu.$$

Thus $x\mu' = \mu$. On $x > 0$, its nonzero solutions are $\mu = Kx$, $K \neq 0$. Choose $\boxed{\mu = x}$, which is nonzero throughout the stated half-plane.

Step 2: Integrate the exact form. The multiplied coefficients are $2xy + x^2$ and x^2. Integrating in y gives

$$F = x^2y + g(x), \qquad F_x = 2xy + g'(x) = 2xy + x^2.$$

Hence $g = x^3/3$ is a choice, giving the family $\boxed{x^2y + x^3/3 = C}$.

Step 3: Apply and verify the data. At $(1, 2)$, $C = 7/3$, so

$$\boxed{y = \frac{7}{3x^2} - \frac{x}{3}, \qquad x > 0}.$$

Its derivative is $-14/(3x^3) - 1/3$, and substitution gives

$$2y + x + xy' = \frac{14}{3x^2} - \frac{2x}{3} + x - \frac{14}{3x^2} - \frac{x}{3} = 0.$$

The initial value is also 2.

Step 4: Check extension in the original equation. The selected solution tends to $+\infty$ as $x \downarrow 0$, so it cannot extend continuously through 0. In the general family, $y = C/x^2 - x/3$, a finite extension requires $C = 0$. This member extends as $\boxed{y = -x/3}$ on all of $\mathbb{R}$ and satisfies the original equation even at 0, where the equation requires $2y(0) = 0$. The multiplied form alone gives $0 = 0$ there because its factor vanishes; that weakened condition must not replace the original one.

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Question 7

Compare the following forms on $\mathbb{R}^2$:

$$\begin{aligned}&\text{(A)}\quad x\,dx + y\,dy = 0,\\ &\text{(B)}\quad e^x x\,dx + e^x y\,dy = 0,\\ &\text{(C)}\quad (1+x^2+y^2)x\,dx + (1+x^2+y^2)y\,dy = 0.\end{aligned}$$

Tasks

1. Determine which forms are exact and give a potential for each exact form.
2. Prove that all three nevertheless have the same differentiable solution graphs.
3. Find their common solution graph with $y(0) = 2$ and its maximal open interval containing 0.
4. Let $dF = M\,dx + N\,dy$. Explain why multiplication by a continuous function $h(F)$ produces another exact form, and state an additional condition that guarantees equivalence of the graph equations.

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Question 7 – Solution

Strategy. Separate two different questions: whether the displayed coefficients are partial derivatives of a potential, and whether multiplying the equation changes its solutions.

Step 1: Test each form. For (A), $M_y = N_x = 0$, with potential

$$F = \frac{x^2 + y^2}{2}.$$

For (B), $M_y = 0$ and $N_x = e^x y$, which are not equal on the plane, so (B) is not exact there. For (C), both cross partials are $2xy$, and a potential is

$$G = \frac{x^2 + y^2}{2} + \frac{(x^2 + y^2)^2}{4}.$$

Differentiating G gives the two coefficients in (C).

Step 2: Establish equivalence. The equations for a graph are respectively

$$x + yy' = 0, \qquad e^x(x + yy') = 0, \qquad (1 + x^2 + y^2)(x + yy') = 0.$$

Both multiplying factors are strictly positive at every real point. Each equation therefore holds exactly when the first does. Exactness of a particular form is not required for it to share the same solutions as an exact form.

Step 3: Select the common IVP solution. The first potential gives $x^2 + y^2 = 4$. Continuity from $y(0) = 2$ selects

$$\boxed{y = \sqrt{4 - x^2}, \qquad I = (-2, 2)}.$$

Here $y' = -x/\sqrt{4 - x^2}$ verifies all three equations. At either endpoint, $y = 0$ but $x \neq 0$, so no finite derivative can satisfy $x + yy' = 0$; a differentiable graph cannot extend through it.

Step 4: Generalize using the chain rule. Choose an antiderivative H of the continuous function h. Then

$$d(H(F)) = H'(F)\, dF = h(F)(M\, dx + N\, dy).$$

This gives a potential for the new form. If $h(F)$ is nonzero throughout the region considered, division is valid there and the graph equations are equivalent. Allowing a zero factor can introduce extra solutions, so equivalence cannot be concluded from exactness alone.

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Question 8

On the domain $xy \neq 0$, consider

$$\left(\frac{1}{x} + y\right)\,dx + \left(\frac{1}{y} + x\right)\,dy = 0, \qquad y(1) = 2.$$

Tasks

1. Find a potential valid on this domain, using absolute values wherever required by logarithmic integration.
2. Use the initial condition to identify the correct level. On the positive quadrant, prove that the implicit relation reduces to an explicit solution.
3. Find its maximal interval containing 1. Explain why the branch of the same algebraic hyperbola in the negative quadrant is not an extension of this IVP.
4. Analyze the different initial condition $y(1) = -1$. Determine its selected level and solution on $x > 0$, even though both coefficients vanish along that solution.

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Question 8 – Solution

Strategy. Combine the logarithms using the product $u = xy$, but keep the connected domain and the sign of u in view.

Step 1: Construct the potential. Both cross partials are 1. Integration gives

$$\boxed{F = \ln|x| + \ln|y| + xy = \ln|xy| + xy}.$$

Indeed $F_x = 1/x + y$ and $F_y = 1/y + x$ wherever $xy \neq 0$.

Step 2: Select the positive branch. The data $(1, 2)$ give $F = \ln 2 + 2$. For $u > 0$, the function $\phi(u) = \ln u + u$ is strictly increasing, since $\phi'(u) = 1/u + 1 > 0$. Thus $\phi(u) = \phi(2)$ forces $u = 2$. Consequently

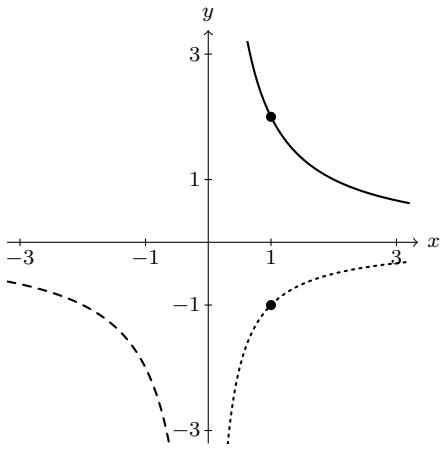
$$\boxed{y = 2/x, \qquad I = (0, \infty)}.$$

Substitution gives $M = 3/x$, $N = 3x/2$ and $y' = -2/x^2$, so $M + Ny' = 0$. The branch cannot cross $x = 0$: the equation is undefined there and $y \to +\infty$ from the right. The negative-quadrant branch $y = 2/x$, $x < 0$, is disconnected from the initial point.

Step 3: Analyze the degenerate negative level. For $(1, -1)$, $F = -1$. On $u < 0$, $\phi(u) = \ln(-u) + u$ has derivative $1/u + 1$, positive for $u < -1$ and negative for $-1 < u < 0$. Its unique maximum is $\phi(-1) = -1$. Therefore this level forces $xy = -1$, giving

$$\boxed{y = -1/x \quad (x > 0)}.$$

Along this graph both M and N are zero, so the original equation is satisfied directly. Every solution through the point must conserve F and remain in the same quadrant, which proves this branch is unique there despite the vanishing coefficients.



Solid: $y(1) = 2$. Dashed: its disconnected hyperbola branch. Dotted: $y(1) = -1$.

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Question 9

Consider

$$e^x\,dx + (1+y^2)\,dy = 0, \qquad y(0) = 0.$$

An approximate calculation claims that the selected solution passes through $(\ln 2, -1)$.

Tasks

1. Find the exact implicit solution and prove that it defines a unique differentiable graph for all real x.
2. Test the claimed point using the potential, without explicitly solving a cubic.
3. Let y_* be the true value at $x = \ln 2$. Prove $-1 < y_* < 0$ and decide whether the claimed value is too high or too low.
4. Use the mean value theorem to prove the quantitative error bound

   $$\frac{1}{6} \le |y_* - (-1)| \le \frac{1}{3}.$$

   Explain why a potential residual can provide more information than simply declaring an approximate point incorrect.

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Question 9 – Solution

Strategy. Use the conserved potential to test the point, then convert its residual into an error bound using a derivative estimate.

Step 1: Find the exact relation. The cross partials are both zero. A potential is $F = e^x + y + y^3/3$, and $F(0,0) = 1$, so

$$\boxed{e^x + y + \frac{y^3}{3} = 1}.$$

The function $H(u) = u+u^3/3$ is strictly increasing, onto $\mathbb{R}$, and has $H'(u) = 1+u^2 > 0$. Thus $H(y) = 1-e^x$ gives exactly one differentiable value for every x. Differentiation recovers the original graph equation.

Step 2: Compute the residual and locate the true value. At the claimed point, $F = 2-1-1/3 = 2/3$, so the potential residual $F - 1$ is $-1/3$. The point is not on the selected level. At $x = \ln 2$, the true value satisfies $H(y_*) = -1$. Since $H(-1) = -4/3 < -1 < H(0) = 0$, strict monotonicity gives $-1 < y_* < 0$. The claimed value -1 is too low.

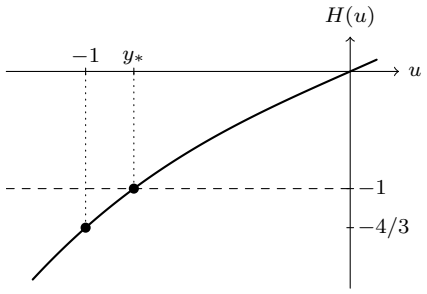
Step 3: Bound the actual error. The mean value theorem gives, for some $\xi \in (-1, y_*)$,

$$\frac{1}{3} = H(y_*) - H(-1) = H'(\xi)(y_* + 1).$$

On $[-1, 0]$, $1 \le H' \le 2$, so

$$\boxed{\frac{1}{6} \le y_* + 1 = |y_* - (-1)| \le \frac{1}{3}}.$$

The residual, together with monotonicity and derivative bounds, certifies both the direction and the size range of the error; no cubic formula is needed.



The true value y_* lies to the right of -1 because H is strictly increasing.

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Question 10

For the exact form

$$\omega = (2xy + e^x)\,dx + (x^2 + 2y)\,dy,$$

compare two routes from $A = (0, 0)$ to $B = (1, 1)$. Route 1 goes horizontally to $(1, 0)$ and then vertically to B; Route 2 goes vertically to $(0, 1)$ and then horizontally to B.

For an axis-aligned segment, define its accumulated value as the integral of the coefficient of dx along a horizontal segment, or of the coefficient of dy along a vertical segment, with the stated orientation.

Tasks

1. Construct a potential F normalized by $F(0, 0) = 0$.
2. Compute both route totals directly as sums of ordinary single-variable integrals.
3. Explain their equality using the chain rule and find the total around the closed route that follows Route 1 forward and Route 2 backward.
4. For the differential equation $\omega = 0$, decide whether one solution curve could pass through both A and B. Distinguish this question from the equality of the two route totals.

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Question 10 – Solution

Strategy. A potential measures accumulated change along a route; a solution of the equation $\omega = 0$ must instead keep that potential constant.

Step 1: Construct a normalized potential. The cross partials are $2x$. Integrating M gives $F = x^2y + e^x + h(y)$; matching N requires $h' = 2y$. Normalization yields

$$\boxed{F = x^2y + e^x + y^2 - 1}, \qquad F(B) = e + 1.$$

Step 2: Compute the routes directly. On Route 1, the horizontal and vertical contributions are

$$\int_0^1 e^x\,dx = e - 1, \qquad \int_0^1 (1 + 2y)\,dy = 2,$$

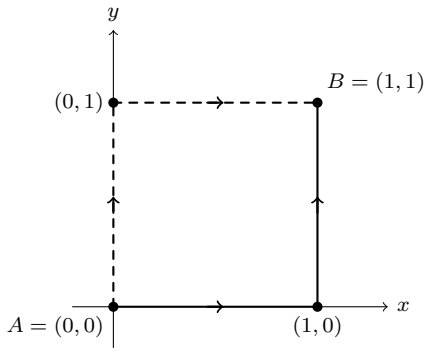
so its total is $e + 1$. On Route 2 they are

$$\int_0^1 2y\,dy = 1, \qquad \int_0^1 (2x + e^x)\,dx = e,$$

again totaling $\boxed{e + 1}$.

Step 3: Explain equality and the closed route. Along a differentiable parametrized segment, $dF/dt = F_x\,x'(t) + F_y\,y'(t) = Mx' + Ny'$. Integration gives the endpoint difference of F; these differences telescope across joined segments. Both routes therefore give $F(B) - F(A) = e + 1$. Reversing a route negates its integrals, so the closed-route total is $\boxed{0}$.

Step 4: Distinguish routes from solutions. A solution curve of $\omega = 0$ has $dF/dt = 0$ and stays on one level. Since $F(A) = 0$ and $F(B) = e + 1 \neq 0$, no such curve can pass through both points. Equal route totals express endpoint dependence; they do not make either route a solution of the differential equation.



Solid: Route 1. Dashed: Route 2. Arrows indicate travel from A toward B.

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Question 1

A Bernoulli equation has the form $y' + p(x)y = q(x)y^n$. Consider the positive initial-value problem

$$y' + \frac{2}{x}y = x^3y^2, \qquad y(1) = 1, \qquad x > 0.$$

Tasks

1. For positive y and $n \neq 1$, derive the linear equation satisfied by $v = y^{1-n}$ using the chain rule.
2. Apply that reduction to the given IVP and solve the resulting linear equation with an integrating factor.
3. Find the maximal open interval containing 1, and determine the minimum of the selected solution on that interval.
4. Verify the answer in the original equation and identify a solution omitted by the substitution.

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Question 1 – Solution

Strategy. Derive the reduction first, then retain only the interval where its inverse gives the selected positive solution.

Step 1: Reduce to a linear equation. Multiplying by y^{-n} and using $v' = (1-n)y^{-n}y'$ gives

$$\boxed{v' + (1-n)pv = (1-n)q}.$$

Here $n = 2$, so $v = 1/y$ obeys $v' - 2v/x = -x^3$. The integrating factor is x^{-2}, hence

$$(x^{-2}v)' = -x, \qquad v = Cx^2 - \frac{x^4}{2}.$$

Since $v(1) = 1$, $C = 3/2$.

Step 2: Invert and analyze. Thus

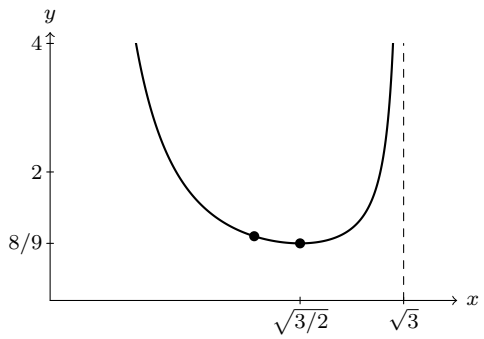
$$\boxed{y = \frac{2}{x^2(3-x^2)}, \qquad I = (0, \sqrt{3})}.$$

The denominator $3x^2 - x^4$ is positive on I, vanishes at its endpoints, and has its unique maximum $9/4$ at $x = \sqrt{3/2}$. The solution therefore has minimum $\boxed{8/9}$ there and tends to $+\infty$ at both ends of I.

Step 3: Verify and restore the lost solution. With $D = 3x^2 - x^4$, $y' = -2(6x - 4x^3)/D^2$, so

$$y' + 2y/x = 4x^3/D^2 = x^3y^2, \qquad y(1) = 1.$$

Also $\boxed{y \equiv 0}$ satisfies the original equation on $x > 0$ but is excluded by $v = 1/y$; it does not satisfy this IVP.



The selected branch has minimum 8/9 and diverges at both interval endpoints.

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Question 2

For the IVP

$$y' + y = e^{2x}y^3, \qquad y(0) = -1,$$

a student sets $v = y^{-2}$ but writes $v' + 2v = 2e^{2x}$ and then plans to use $y = 1/\sqrt{v}$.

Tasks

1. Find both errors in this procedure and derive the correct equation for v.
2. Solve the linear equation and select the original solution using the initial condition.
3. Find its maximal interval containing 0, explaining why a positive transformed variable does not determine the sign of y.
4. Verify the original IVP, and state whether the opposite sign and the zero function solve the same differential equation.

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Question 2 – Solution

Strategy. The derivative of y^{-2} carries a negative factor, and the inverse transformation needs a separately chosen sign.

Step 1: Correct the transformed equation. For $y \neq 0$, divide by y^3 and use $v' = -2y^{-3}y'$. This gives

$$-\frac{1}{2}v' + v = e^{2x}, \qquad \boxed{v' - 2v = -2e^{2x}}.$$

The student's two coefficient signs are reversed. Moreover, $v = 1/y^2$ is the same for y and $-y$, so selecting the positive inverse would contradict the negative initial value.

Step 2: Solve and choose the branch. The integrating factor is e^{-2x}:

$$(e^{-2x}v)' = -2, \qquad v = e^{2x}(C - 2x).$$

The condition $v(0) = 1$ gives $C = 1$. Continuity and $y(0) = -1$ select

$$\boxed{y = -\frac{e^{-x}}{\sqrt{1-2x}}, \qquad I = (-\infty, 1/2)}.$$

Here $v > 0$. At $x \uparrow 1/2$, the solution tends to $-\infty$, so there is no finite extension; for $x > 1/2$, this v cannot equal $1/y^2$ for real y.

Step 3: Verify and distinguish other solutions. On I, logarithmic differentiation of $|y|$ gives

$$\frac{y'}{y} = -1 + \frac{1}{1-2x}.$$

Consequently $y' + y = y/(1-2x) = e^{2x}y^3$, and the initial value is -1. The opposite branch $+e^{-x}/\sqrt{1-2x}$ also solves the differential equation, with initial value 1. The zero function solves it as well but is omitted by y^{-2}. Neither of those alternatives solves the stated IVP.

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Question 3

Seek nonnegative, continuously differentiable solutions for $t \geq 0$ of

$$y' + 2y = 2\sqrt{y}, \qquad y(0) = 0,$$

where the prime denotes d/dt and $\sqrt{y}$ is the nonnegative square root.

Tasks

1. On an interval where $y > 0$, derive and solve the equation for $w = \sqrt{y}$.
2. Find every solution that waits at zero until a time $b \geq 0$ and then becomes positive. Include the possibility of staying zero forever.
3. Check differentiability and the original equation at the joining time, and justify that your list is exhaustive for $t \geq 0$.
4. Explain why squaring a linear formula with $w < 0$ does not give a valid solution of the original equation on that interval.

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Question 3 – Solution

Strategy. The linearized equation applies only on positive pieces; restore zero intervals and enforce the sign of the square root.

Step 1: Solve a positive piece. Writing $y = w^2$ with $w > 0$ gives $2ww' + 2w^2 = 2w$, hence $w' + w = 1$. Thus $w = 1 + Ce^{-t}$. If positivity starts at b with $w(b) = 0$, then $w = 1 - e^{-(t-b)}$, positive precisely for $t > b$.

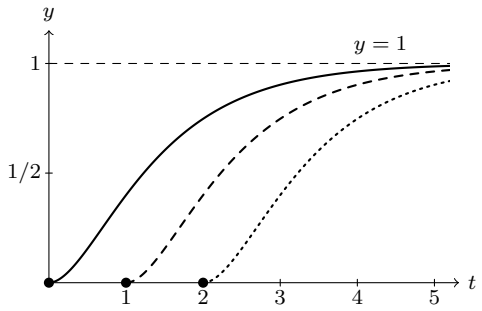
Step 2: Restore the zero intervals. For each finite $b \geq 0$, define

$$\boxed{y_b(t) = \begin{cases} 0, & 0 \leq t \leq b, \\ (1 - e^{-(t-b)})^2, & t \geq b. \end{cases}}$$

Also include $y \equiv 0$, denoted by $b = \infty$. On the positive piece let $r = e^{-(t-b)}$; then $y' = 2r(1 - r)$ and $y' + 2y = 2(1 - r) = 2\sqrt{y}$. At a positive joining time, both one-sided derivatives are 0; if $b = 0$, the right derivative is 0. Thus the solution is C^1 on its stated domain and satisfies the equation at the join.

Step 3: Prove completeness and reject a false extension. Any positive component must start from zero at its finite left endpoint, including $b = 0$, and the linear formula above fixes that component uniquely. It stays positive forever after b, so it cannot end at another zero. Hence there is at most one positive component and the list is exhaustive.

For $t < b$, the same linear expression has $w < 0$. If it is squared, $\sqrt{w^2} = |w| = -w$, while the computed left side is $2w$. Thus it fails the original equation there. Squaring cannot remove this sign restriction.



Solid: $b = 0$. Dashed: $b = 1$. Dotted: $b = 2$. Each is zero before its departure.

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Question 4

Consider the negative-power Bernoulli equation

$$y' + y = \frac{e^{-x}}{y}, \qquad y(0) = -1.$$

The original equation is defined only when $y \neq 0$.

Tasks

1. Derive the linear equation for $v = y^2$ and solve it using the initial condition.
2. Select the real branch for y and find its maximal interval containing 0.
3. Determine what happens to y and y' at its finite endpoint. Decide whether reaching the value zero permits continuation.
4. Verify the equation, and explain why the zero function cannot be restored as an omitted solution in this example.

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Question 4 – Solution

Strategy. The squared variable must be strictly positive, and the original reciprocal term remains undefined at zero.

Step 1: Linearize and integrate. Multiplication by $2y$ yields $v' + 2v = 2e^{-x}$, where $v = y^2$. Using the integrating factor e^{2x},

$$(e^{2x}v)' = 2e^{x}, \qquad v = 2e^{-x} + Ce^{-2x}.$$

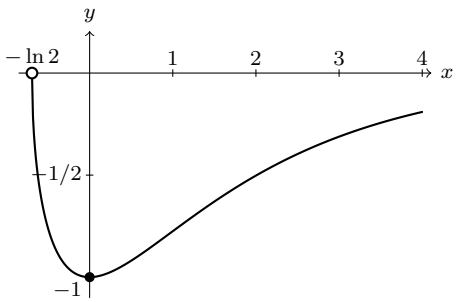
Since $v(0) = 1$, $C = -1$.

Step 2: Invert on the correct interval. The condition $v = e^{-2x}(2e^{x} - 1) > 0$ requires $x > -\ln 2$. The initial sign gives

$$\boxed{y = -e^{-x}\sqrt{2e^{x} - 1}, \qquad I = (-\ln 2, \infty)}.$$

As $x \downarrow -\ln 2$, $y \to 0$ from below. The equation gives $y' = e^{-x}/y - y \to -\infty$, so no differentiable extension exists at that endpoint. Also the original right-hand side is undefined there. As $x \to \infty$, $y \to 0$ from below, but it never reaches zero at a finite point of I.

Step 3: Verify and check the excluded value. Differentiating $y^2 = v$ gives $2yy' = v'$. Since $v' + 2v = 2e^{-x}$ and $y \neq 0$ on I, division by $2y$ recovers $y' + y = e^{-x}/y$. The selected formula gives $y(0) = -1$. Unlike positive-power examples, $\boxed{y \equiv 0 \text{ is not a solution}}$: the original equation has no value at zero.



The finite endpoint is excluded; the negative branch approaches zero again as $x \to \infty$.

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Question 5

For each real initial value a, consider forward solutions of

$$y' + y = y^3, \qquad y(0) = a, \qquad t \geq 0,$$

where the prime denotes d/dt.

Tasks

1. Solve using $v = y^{-2}$ when $a \neq 0$, retaining the sign of a. Treat $a = 0$ separately.
2. Classify all a according to whether the forward solution tends to zero, is constant, or blows up in finite time.
3. In the blow-up cases, find the exact lifetime and the sign of the divergence.
4. Verify the solution and explain why the threshold initial values are not included in either neighboring regime.

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Question 5 – Solution

Strategy. A linear denominator exposes the threshold, while the sign of the original solution must be carried separately.

Step 1: Solve the transformed IVP. For $a \neq 0$, $v = y^{-2}$ gives $v' - 2v = -2$, so

$$v = 1 + (a^{-2} - 1)e^{2t}, \qquad \boxed{y = \frac{a}{\sqrt{a^2 + (1 - a^2)e^{2t}}}}.$$

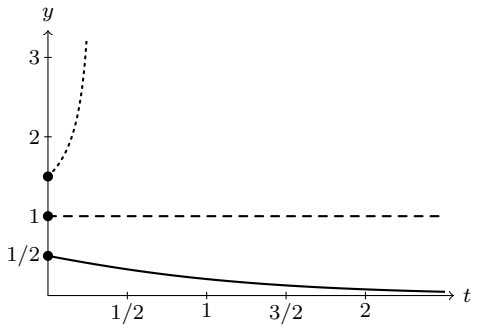
The denominator is the positive square root on the interval through 0. For $a = 0$, the unique solution is $y \equiv 0$; the smooth right-hand side $y^3 - y$ ensures local uniqueness.

Step 2: Classify the thresholds. If $|a| < 1$, the denominator is positive for every $t \geq 0$ and $y \to 0$. This includes $a = 0$. If $a = \pm 1$, the formula is constant: $\boxed{y \equiv \pm 1}$. If $|a| > 1$, the denominator first vanishes at

$$\boxed{T = \frac{1}{2} \ln \frac{a^2}{a^2 - 1}}.$$

The maximal forward lifetime is $0 \leq t < T$. As $t \uparrow T$, $y \to +\infty$ for $a > 1$ and $y \to -\infty$ for $a < -1$.

Step 3: Verify and interpret. For nonzero branches, $v' = -2y^{-3}y' = 2v - 2$ implies $y' = -y + y^3$, and the formula gives $y(0) = a$. The three constants $0, 1, -1$ also check directly. At $a = \pm 1$ the exponential term in v is exactly zero; these solutions neither decay nor blow up. A threshold case must be substituted into the equation, not inferred from neighboring values.



Solid: $a = 1/2$. Dashed: $a = 1$. Dotted: $a = 3/2$. Negative solutions are reflections.

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Question 6

On $x > 0$, two positive solutions of one unknown Bernoulli equation

$$y' + p(x)y = q(x)y^2$$

are known exactly:

$$y_1(x) = \frac{1}{1+x}, \qquad y_2(x) = \frac{1}{1+2x}.$$

Assume p and q are continuous there.

Tasks

1. Transform both solutions using $v = 1/y$ and recover $p(x)$ and $q(x)$ uniquely.
2. Verify both supplied solutions directly in the recovered nonlinear equation.
3. Solve that equation with the different condition $y(1) = 2$, and find its maximal interval within $x > 0$.
4. Explain why subtracting the original nonlinear solutions is less useful here than subtracting their transformed equations, and identify the solution lost by taking reciprocals.

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Question 6 – Solution

Strategy. The reciprocal transformation turns coefficient recovery into two linear identities; their difference eliminates the forcing term.

Step 1: Recover the coefficients. Each transformed solution satisfies $v' - pv = -q$. Here $v_1 = 1 + x$ and $v_2 = 1 + 2x$, so

$$1 - p(1 + x) = -q, \qquad 2 - p(1 + 2x) = -q.$$

Subtracting gives $1 - px = 0$. Since $x > 0$, this fixes $p = 1/x$ uniquely. Substituting back gives $q = 1/x$. Thus

$$\boxed{y' + \frac{y}{x} = \frac{y^2}{x}}.$$

Step 2: Verify the given data. For $y = 1/(1 + kx)$, with $k = 1$ or 2,

$$y' + \frac{y}{x} = -\frac{k}{(1 + kx)^2} + \frac{1}{x(1 + kx)} = \frac{1}{x(1 + kx)^2} = \frac{y^2}{x}.$$

Both functions are positive and defined throughout $x > 0$.

Step 3: Construct the new solution. The transformed equation is $v' - v/x = -1/x$. Multiplying by $1/x$ gives $(v/x)' = -1/x^2$, hence $v = 1 + Cx$. Since $y(1) = 2$, $v(1) = 1/2$, so $C = -1/2$. Therefore

$$\boxed{y = \frac{1}{1 - x/2}, \qquad I = (0, 2)}.$$

This is the maximal interval within the specified domain through 1: the solution has a pole at 2, and the coefficients are undefined at 0.

Step 4: Explain the advantage and restore zero. Subtracting the transformed equations gives the homogeneous linear relation $\Delta v' - p\Delta v = 0$. Subtracting the original equations leaves $q(y_2^2 - y_1^2)$ and does not give that linear elimination. The function $\boxed{y \equiv 0}$ satisfies the recovered equation but is excluded by the reciprocal transformation.

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Question 7

Suppose y_1, y_2 are positive solutions on an interval of

$$y' + p(x)y = q(x)y^3.$$

Let $0 < \alpha < 1$.

Tasks

1. Prove that $v = \alpha y_1^{-2} + (1-\alpha)y_2^{-2}$ solves the transformed linear equation and gives a positive solution $y = v^{-1/2}$.
2. Show that this new solution lies between y_1 and y_2 pointwise.
3. Apply the construction with $\alpha = 1/4$ to the equation $y' + xy = xy^3$ and the solutions $y_1 = 1$, $y_2 = (1 + e^{x^2})^{-1/2}$. Verify the supplied solutions through their transforms.
4. Prove that the ordinary arithmetic mean $m = (y_1 + y_2)/2$ of these two supplied solutions fails the nonlinear equation at $x = 1$.

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Question 7 – Solution

Strategy. Affine combinations preserve the common forcing in the transformed linear equation; they do not generally preserve the original cubic equation.

Step 1: Combine the transformed solutions. Set $v_i = y_i^{-2}$. Both satisfy $v_i' - 2pv_i = -2q$. Multiplying by α and $1-\alpha$ and adding yields $v' - 2pv = -2q$. Positivity of v_1, v_2 implies $v > 0$, so $y = v^{-1/2}$ is well defined. Differentiating this inverse recovers $y' + py = qy^3$.

Step 2: Prove the pointwise bound. At each point, v lies between v_1 and v_2. The function $u \mapsto u^{-1/2}$ is decreasing on $u > 0$, so reversing their order still puts y between y_1 and y_2. Equality occurs when the supplied values agree; otherwise the bounds are strict.

Step 3: Apply the construction. For the given equation, the transformed equation is $v' - 2xv = -2x$. The supplied transforms are $v_1 = 1$ and $v_2 = 1 + e^{x^2}$; direct differentiation verifies both. Their weighted combination gives

$$v = 1 + \frac{3}{4}e^{x^2}, \qquad \boxed{y = \frac{1}{\sqrt{1 + \frac{3}{4}e^{x^2}}}} \quad (x \in \mathbb{R}).$$

It is positive and lies strictly between the two supplied solutions.

Step 4: Reject ordinary superposition. Because the original functions solve the equation,

$$m' + xm = \frac{x}{2}(y_1^3 + y_2^3).$$

At $x = 1$, put $a = y_1(1)$ and $b = y_2(1)$. They are distinct and positive. The residual is

$$m' + m - m^3 = \frac{a^3 + b^3}{2} - \frac{(a+b)^3}{8} = \frac{3}{8}(a+b)(a-b)^2 > 0.$$

Thus the arithmetic mean fails the nonlinear equation. The valid averaging occurs after the Bernoulli transformation.

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Question 8

Consider

$$y' + xy = e^{x^2} y^2, \qquad y(0) = a, \qquad a \in \mathbb{R}.$$

Define $J(x) = \int_0^x e^{t^2/2}\, dt$; an elementary antiderivative is not required.

Tasks

1. Solve the IVP for $a \neq 0$ by the Bernoulli substitution, expressing the answer in terms of J. Treat $a = 0$ separately.
2. Prove that J is strictly increasing onto $\mathbb{R}$ and that the equation $J(T) = 1/a$ has exactly one real solution.
3. Determine the maximal interval containing 0 for each sign of a, and the sign of the divergence at its finite endpoint.
4. Verify the formula using the fundamental theorem of calculus and explain why an exact definite-integral expression is a complete answer.

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Question 8 – Solution

Strategy. Keep the forcing integral exact, then use its monotonicity to determine the entire admissible denominator interval.

Step 1: Linearize and integrate. For $a \neq 0$, set $v = 1/y$. Then $v' - xv = -e^{x^2}$, with integrating factor $e^{-x^2/2}$. Thus

$$(e^{-x^2/2}v)' = -e^{x^2/2}, \qquad v = e^{x^2/2}\left(\frac{1}{a} - J(x)\right).$$

Inverting gives

$$\boxed{y(x) = \frac{ae^{-x^2/2}}{1 - aJ(x)}}.$$

For $a = 0$, $y \equiv 0$ is the unique solution, since the original right-hand side is smooth in y.

Step 2: Locate the unique obstruction. By the fundamental theorem, $J' = e^{x^2/2} > 0$. The integrand is even, so J is odd. Also $J(x) \geq x$ for $x \geq 0$, so its limits at the two infinities are $\pm\infty$. Therefore it is one-to-one and onto, and there is exactly one T with $J(T) = 1/a$.

Step 3: Select the interval and sign. If $a > 0$, then $T > 0$, and the maximal interval is $\boxed{(-\infty, T)}$. The denominator approaches zero from above as $x \uparrow T$, so $y \to +\infty$. If $a < 0$, then $T < 0$, and the maximal interval is $\boxed{(T, \infty)}$; as $x \downarrow T$, the denominator approaches zero from above and $y \to -\infty$. A pole cannot be crossed by a finite differentiable continuation.

Step 4: Verify and interpret. With $D = 1 - aJ$, differentiation using $J' = e^{x^2/2}$ gives

$$y' = -xy + \frac{a^2}{D^2} = -xy + e^{x^2}y^2.$$

Also $J(0) = 0$ gives $y(0) = a$. The definite integral specifies an exact differentiable function and uniquely determines the endpoint; elementary notation is unnecessary for either the solution or its interval.

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Question 9

Seek strictly positive, 2π-periodic solutions, defined for every real x, of

$$y' + y = (2 + \sin x)y^2.$$

Tasks

1. Use the reciprocal substitution to find the general transformed solution.
2. Prove that there is exactly one strictly positive 2π-periodic solution and find its initial value at $x = 0$.
3. Determine its minimum and maximum over one period.
4. Show that changing the initial value to $y(0) = 1/2$ causes a finite positive-time blow-up, even though the periodic solution remains bounded. Verify the original equation for the periodic solution.

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Question 9 – Solution

Strategy. Periodicity removes the exponentially growing homogeneous term in the reciprocal variable; positivity must then be checked before inversion.

Step 1: Solve the linear equation. With $v = 1/y$, the equation becomes $v' - v = -(2 + \sin x)$. A particular solution and the general solution are

$$v_p = 2 + \frac{\sin x + \cos x}{2}, \qquad v = v_p + Ce^x.$$

Indeed $v_p' - v_p = -2 - \sin x$.

Step 2: Select and bound the periodic solution. If positive y is 2π-periodic, then so is $v = 1/y$. The term Ce^x is periodic only if $C = 0$. Since

$$2 - \frac{\sqrt{2}}{2} \le v_p \le 2 + \frac{\sqrt{2}}{2},$$

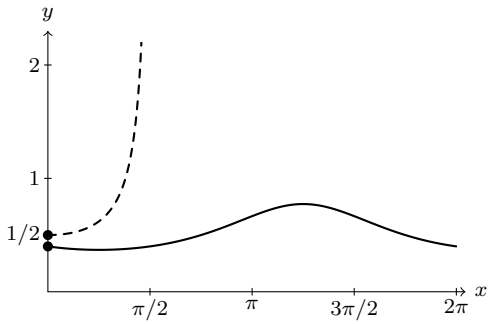
its reciprocal exists globally and is positive. Therefore

$$\boxed{y_p = \frac{1}{2 + (\sin x + \cos x)/2}, \qquad y_p(0) = \frac{2}{5}}.$$

Its minimum is $1/(2 + \sqrt{2}/2)$ at $x = \pi/4$, and its maximum is $1/(2 - \sqrt{2}/2)$ at $x = 5\pi/4$, modulo 2π.

Step 3: Check the perturbed initial condition. For $y(0) = 1/2$, $v(0) = 2$, so $C = -1/2$. The function $v = v_p - e^x/2$ starts positive, but $v(2\pi) = 5/2 - e^{2\pi}/2 < 0$. It therefore has a first zero $T \in (0, 2\pi)$. At this zero, $v'(T) = -(2 + \sin T) < 0$. Hence $v \downarrow 0$ from the positive side and $y = 1/v \to +\infty$ as $x \uparrow T$.

Step 4: Verify the periodic branch. Since $v_p' = v_p - (2 + \sin x)$, differentiation gives $y_p' = -y_p + (2 + \sin x)y_p^2$, as required.



Solid: the positive periodic solution. Dashed: $y(0) = 1/2$, which blows up before a full cycle.

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Question 10

Consider positive solutions, near $x = 0$, of the exponent-dependent IVP

$$y' + y = y^n, \qquad y(0) = 2, \qquad n \in \mathbb{R}.$$

For $y > 0$, interpret $y^n = \exp(n \ln y)$.

Tasks

1. Solve the cases $n = 0$ and $n = 1$ directly. Explain why they do not require the usual nonlinear Bernoulli reduction.
2. For $n \neq 1$, derive the solution formula and state the positivity condition needed to interpret its real power.
3. Check that the general formula agrees with the direct answer at $n = 0$ and verifies the initial condition for every admissible n.
4. For each fixed real x, prove that the formula tends to the $n = 1$ solution as $n \to 1$. Do not substitute $n = 1$ into an expression with exponent $1/(1 - n)$.

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Question 10 – Solution

Strategy. Handle the exceptional equations directly, then analyze the apparent singularity of the general expression through its logarithm.

Step 1: Solve the elementary exceptional cases. If $n = 0$, the equation is $y' + y = 1$, giving $\boxed{y = 1 + e^{-x}}$. If $n = 1$, the two y terms cancel, giving $\boxed{y = 2}$. Both are positive on $\mathbb{R}$; the first is linear and the second has zero derivative.

Step 2: Derive the parameterized formula. For $n \neq 1$, set $h = 1 - n \neq 0$ and $v = y^h$. Then

$$v' + hv = h, \qquad v(0) = 2^h,$$

so $v = 1 + (2^h - 1)e^{-hx}$. Therefore

$$\boxed{y_n(x) = \left[1 + (2^{1-n} - 1)e^{-(1-n)x}\right]^{1/(1-n)}}.$$

Use the connected interval through 0 on which the bracket is strictly positive. This is necessary because the original positive solution has $v = y^h > 0$; a negative base cannot be accepted by choosing a special rational exponent.

Step 3: Check the consistency conditions. At $n = 0$, the formula becomes $1 + e^{-x}$. At $x = 0$, its bracket is 2^{1-n}, and the positive-real power gives 2. Conversely, differentiating $v = y^h$ on the stated interval recovers the original equation, so the inversion is valid there.

Step 4: Take the limit without dividing by zero. For fixed x, let $B(h) = 1 + (e^{h \ln 2} - 1)e^{-hx}$. Since $B(0) = 1$, it is positive for all sufficiently small $|h|$. Now

$$\ln y_n = \frac{\ln B(h)}{h}, \qquad B'(0) = \ln 2.$$

The derivative definition gives $\lim_{h\to 0} \ln B(h)/h = \ln 2$. Exponentiating yields $\boxed{\lim_{n\to 1} y_n(x) = 2}$, exactly the direct $n = 1$ solution. Joint continuity on the compact segment between 0 and the fixed x keeps the bracket positive there for small $|h|$, so these values belong to the IVP branch.

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Question 1

Consider the homogeneous first-order equation

$$y' = \frac{y}{x} + \sqrt{1 + (y/x)^2}, \qquad y(1) = 0, \qquad x > 0.$$

Here "homogeneous" means that the right-hand side depends only on y/x; it does not mean a homogeneous linear equation.

Tasks

1. Set $v = y/x$ and derive the equation for v, showing the product-rule term in y'.
2. Solve the transformed IVP and recover an explicit formula for y.
3. Verify the original equation, taking care with the nonnegative square root, and find the maximal interval within $x > 0$.
4. A student uses $y' = xv'$ when $y = xv$. Identify the omitted term and explain why that error changes the transformed equation.

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Question 1 – Solution

Strategy. The ratio substitution removes the repeated y/x, but differentiating $y = xv(x)$ requires both product-rule terms.

Step 1: Transform the equation. Since $y' = v + xv'$, substitution gives

$$v + xv' = v + \sqrt{1+v^2}, \qquad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}.$$

The initial value is $v(1) = 0$. An antiderivative on the left is $\ln(v + \sqrt{1+v^2})$, whose logarithm has a strictly positive argument for every real v. Hence

$$\ln(v + \sqrt{1+v^2}) = \ln x.$$

Step 2: Recover the original unknown. Exponentiating gives $v + \sqrt{1+v^2} = x$. Squaring $\sqrt{1+v^2} = x - v$ and simplifying gives

$$v = \frac{x^2-1}{2x}, \qquad \boxed{y = \frac{x^2-1}{2}}.$$

This step is valid: for the recovered expression, $x - v = (x^2+1)/(2x) > 0$ on $x > 0$, so the unsquared relation also holds.

Step 3: Verify the branch and interval. For this y,

$$\sqrt{1+(y/x)^2} = \frac{x^2+1}{2x}, \qquad \frac{y}{x} + \sqrt{1+(y/x)^2} = x = y'.$$

Also $y(1) = 0$. The formula satisfies the original equation for every $x > 0$, so its maximal interval there is $\boxed{(0,\infty)}$. The polynomial has a value at 0, but the given equation does not; an algebraic extension alone does not extend this solution through the excluded point.

Step 4: Diagnose the missing derivative term. The omitted term is v. Writing only xv' would produce $xv' = v + \sqrt{1+v^2}$ instead of $xv' = \sqrt{1+v^2}$. The two equations describe different ratio functions, so the product rule is essential to the reduction.

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Question 2

On $x > 0$, consider

$$y' = \frac{y}{x} + (y/x - 1)(y/x - 2), \qquad y(1) = 3.$$

Tasks

1. Use $v = y/x$ to derive a separable equation. Find every constant value of v before dividing by a factor involving v.
2. Solve the transformed equation, and give the resulting original solution family together with any missing straight-line solution.
3. Select the IVP solution and its maximal interval containing 1.
4. Verify the IVP solution and explain why it cannot cross either straight-line solution within that interval.

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Question 2 – Solution

Strategy. Constant ratios correspond to straight lines, so preserve them before separating the nonconstant ratio equation.

Step 1: Transform and preserve the constants. Writing $y = xv$ gives $xv' = (v-1)(v-2)$. The constant ratios $v = 1, 2$ yield the solutions $\boxed{y = x,\ y = 2x}$. For other ratios,

$$\int \frac{dv}{(v-1)(v-2)} = \int \frac{dx}{x}, \qquad \ln\left|\frac{v-2}{v-1}\right| = \ln x + C_0.$$

Thus $(v-2)/(v-1) = Cx$, with nonzero C for nonconstant v, and

$$y = \frac{x(2-Cx)}{1-Cx}.$$

Allowing $C = 0$ restores $y = 2x$; $y = x$ remains a separate solution. Each formula is used only on intervals where its denominator is nonzero.

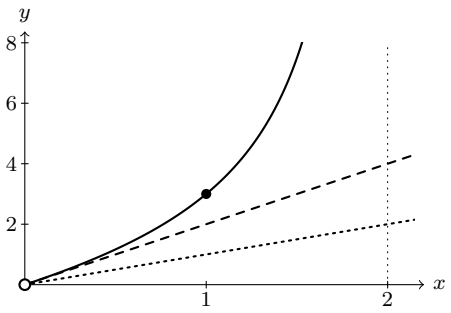
Step 2: Select the IVP branch. The condition $v(1) = 3$ gives $C = 1/2$, hence

$$\boxed{y = \frac{x(2-x/2)}{1-x/2}, \qquad I = (0,2)}.$$

The coefficient domain excludes 0, while $y \to +\infty$ as $x \uparrow 2$.

Step 3: Verify and compare. For $v = (2-Cx)/(1-Cx)$, $v' = C/(1-Cx)^2$, and $(v-1)(v-2) = Cx/(1-Cx)^2 = xv'$. Thus $y' = v + xv'$ recovers the original equation, and the selected initial value is 3.

On $0 < x < 2$, $v = 1 + 1/(1-x/2) > 2$. Therefore $y > 2x > x$ throughout the interval: neither line is crossed.



Solid: IVP solution. Dashed: $y = 2x$. Dotted sloping line: $y = x$. The domain is $x > 0$.

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Question 3

Consider an IVP on the negative half-line:

$$y' = \frac{y}{x} + \frac{x}{y}, \qquad y(-1) = 1, \qquad x < 0,$$

with the original equation defined only when $y \neq 0$.

Tasks

1. Use $v = y/x$ and separate the equation, keeping the correct logarithmic absolute value.
2. Select both the constant and the sign of the transformed solution from the initial condition.
3. Recover y and determine its maximal interval containing -1 within $x < 0$.
4. Verify the equation and describe the finite endpoint. Explain why replacing $\ln|x|$ by $\ln x$, or choosing the positive sign for v, would fail here.

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Question 3 – Solution

Strategy. The substitution is valid for negative x too, but its initial ratio is negative and the logarithm must respect the domain.

Step 1: Transform and integrate. From $y = xv$ and $y' = v + xv'$,

$$xv' = \frac{1}{v}, \qquad v\,dv = \frac{dx}{x}, \qquad v^2 = 2\ln|x| + C.$$

At $x = -1$, $v = y/x = -1$, so $C = 1$. Continuity and the nonzero ratio select

$$v = -\sqrt{1 + 2\ln|x|}.$$

Step 2: Recover the branch and domain. The radicand must be strictly positive, since zero would give the excluded value $y = 0$. Thus $|x| > e^{-1/2}$. On the negative half-line this gives

$$\boxed{y = -x\sqrt{1 + 2\ln|x|}, \qquad I = (-\infty, -e^{-1/2})}.$$

It is positive and gives $y(-1) = 1$.

Step 3: Verify and check the endpoint. Let $D = 1 + 2\ln|x|$. Since $D' = 2/x$,

$$y' = -\sqrt{D} - \frac{1}{\sqrt{D}}, \qquad \frac{y}{x} + \frac{x}{y} = -\sqrt{D} - \frac{1}{\sqrt{D}}.$$

At the finite endpoint approached from the left, $D \downarrow 0$, so $y \to 0$ from above and $y' \to -\infty$. The original equation is undefined there, and no differentiable extension through it is possible.

Step 4: Identify the two domain mistakes. The real logarithm $\ln x$ is not defined for these negative x; the antiderivative is $\ln|x|$. Also $v > 0$ would give $y = xv < 0$ on this half-line, contradicting $y(-1) = 1$. The sign must be selected for the transformed ratio, not guessed from the sign of y alone.

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Question 4

Consider the nonseparable-looking equation

$$y' = (x + y)^2 - 1, \qquad y(0) = 1.$$

Tasks

1. Set $u = x + y$. Derive its differential equation, including the derivative of x.
2. Solve the transformed IVP, recover y, and find its maximal interval containing 0.
3. Determine the minimum of the selected solution and verify the original equation.
4. Find the original solution represented by the constant transformed value $u = 0$. Explain why dividing by u^2 would miss it and why it is not a constant function y.

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Question 4 – Solution

Strategy. Replace the repeated linear combination by one variable. The extra derivative of x cancels the constant term in the equation.

Step 1: Reduce to a separable IVP. For $u = x + y$, $u' = 1 + y' = u^2$, and $u(0) = 1$. Separating on the nonzero branch gives

$$-\frac{1}{u} = x + C, \qquad C = -1, \qquad u = \frac{1}{1-x}.$$

Thus

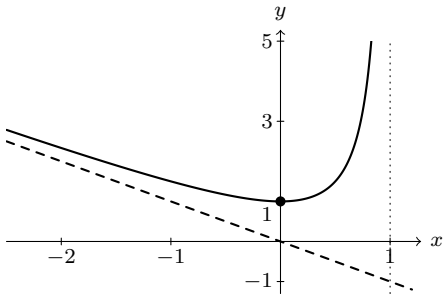
$$\boxed{y = -x + \frac{1}{1-x}, \qquad I = (-\infty, 1)}.$$

The only finite obstruction is the pole at $x = 1$, where $y \to +\infty$ from the left.

Step 2: Find the minimum and verify. Differentiation gives $y' = -1 + (1-x)^{-2}$. On I this is negative for $x < 0$, zero at 0, and positive for $0 < x < 1$. Hence the unique global minimum on I is $\boxed{y(0) = 1}$.

Also $x + y = 1/(1-x)$, so $(x+y)^2 - 1 = (1-x)^{-2} - 1 = y'$, and the initial value checks directly.

Step 3: Restore the constant transformed branch. The equation $u' = u^2$ also has $u \equiv 0$. In the original variables it gives $\boxed{y = -x}$ on all of $\mathbb{R}$. Its derivative is -1 and its right-hand side is $0^2 - 1 = -1$, so it is valid. Division by u^2 excludes it. A constant value of a combination of x and y need not represent a constant value of y.



Solid: IVP solution, with minimum $(0, 1)$. Dashed: the restored solution $y = -x$.

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Question 5

Consider

$$y' = \frac{x + y - 3}{x - y + 1}, \qquad y(2) = 2,$$

on the domain $x - y + 1 \neq 0$.

Tasks

1. Find constants h, k such that $X = x - h$, $Y = y - k$ remove the constant terms in both affine expressions. Write the shifted equation and initial point.
2. On a neighborhood of that point with $X > 0$, use $v = Y/X$ to obtain an implicit solution in X, Y.
3. Verify the local implicit relation and find the slope at the original initial point.
4. Is $X = 0$ automatically a singular line of the original differential equation? Identify the actual excluded line and explain the difference between these restrictions.

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Question 5 – Solution

Strategy. Translate the intersection of the two affine lines to the origin, then apply the homogeneous ratio substitution locally.

Step 1: Choose the translation. The equations $h + k - 3 = 0$ and $h - k + 1 = 0$ give $\boxed{h = 1,\ k = 2}$. Hence

$$X = x - 1, \quad Y = y - 2, \quad \frac{dY}{dX} = \frac{X + Y}{X - Y}, \qquad (X, Y) = (1, 0) \text{ initially.}$$

A translation does not change the derivative because $dX/dx = 1$.

Step 2: Substitute and integrate. With $Y = Xv$,

$$v + Xv' = \frac{1 + v}{1 - v}, \qquad \frac{1 - v}{1 + v^2}\, dv = \frac{dX}{X}.$$

On the local region $X > 0$, integration gives

$$\arctan v - \frac{1}{2} \ln(1 + v^2) = \ln X + C.$$

At $(1, 0)$, $C = 0$. Substituting $v = Y/X$ and simplifying yields

$$\boxed{\arctan(Y/X) - \frac{1}{2} \ln(X^2 + Y^2) = 0}$$

for the local branch through $(1, 0)$ with $X > 0$.

Step 3: Verify and find the slope. For the displayed left side F, differentiation gives

$$F_X = -\frac{X + Y}{X^2 + Y^2}, \qquad F_Y = \frac{X - Y}{X^2 + Y^2}.$$

Thus $F_X + F_Y Y' = 0$ recovers the shifted equation wherever $X - Y \neq 0$. At $(1, 0)$, $F_Y = 1$, so a local graph exists and $\boxed{y'(2) = 1}$.

Step 4: Separate coordinate and equation restrictions. The ratio Y/X excludes $X = 0$, namely $x = 1$, but the original equation is defined there whenever $y \neq 2$. Its actual excluded line is $\boxed{x - y + 1 = 0}$, or $y = x + 1$. An alternative local coordinate or angle branch can be needed when the ratio chart fails; that failure alone does not prove that an original solution ends. No global branch claim is made by this local arctangent formula.

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Question 6

Consider an equation with parallel affine expressions:

$$y' = \frac{2x + y + 1}{4x + 2y + 3}, \qquad y(0) = 0,$$

where $4x + 2y + 3 \neq 0$.

Tasks

1. Explain why no translation of x, y removes the constants in both numerator and denominator simultaneously.
2. Use $u = 2x + y + 1$ to derive a separable equation, then obtain an implicit formula for the IVP.
3. Find the solution corresponding to a constant u that would be lost during separation, and verify it in the original equation.
4. Prove that the selected implicit formula defines the IVP solution for every real x, while remaining inside the original coefficient domain.

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Question 6 – Solution

Strategy. When the affine forms cannot be centered together, use their shared linear combination directly.

Step 1: Reject a simultaneous centering. A translation by (h, k) would require $2h + k + 1 = 0$ and $4h + 2k + 3 = 0$. Doubling the first equation gives $4h + 2k + 2 = 0$, contradicting the second. Their zero lines are parallel and distinct.

Step 2: Reduce and integrate. Set $u = 2x + y + 1$. Then the denominator is $2u + 1$, and

$$u' = 2 + y' = \frac{5u + 2}{2u + 1}, \qquad \frac{2u + 1}{5u + 2}\, du = dx.$$

The original domain excludes $u = -1/2$. For $u \neq -2/5$, an antiderivative is

$$H(u) = \frac{2}{5}u + \frac{1}{25}\ln|5u + 2|.$$

Since $u(0) = 1$, the IVP relation is

$$\boxed{H(u) = x + \frac{2}{5} + \frac{1}{25}\ln 7, \qquad y = u - 2x - 1}.$$

Step 3: Restore the lost constant combination. The transformed equation admits $u \equiv -2/5$, giving $\boxed{y = -2x - 7/5}$. Its numerator is $-2/5$ and its denominator is $1/5$, so the right side is $-2 = y'$. It is a global solution, but its initial value is not 0.

Step 4: Prove the selected branch is global. The initial value lies in $u > -2/5$. On this interval,

$$H'(u) = \frac{2u + 1}{5u + 2} > 0,$$

and $H(u)$ tends to $-\infty$ at the left endpoint and to $+\infty$ as $u \to \infty$. Thus the implicit relation gives exactly one differentiable u for every real x. Also $2u + 1 > 1/5$, so the original denominator never vanishes. Differentiating $H(u) = x + H(1)$ gives the transformed equation and then the original equation; this verifies the unique global IVP branch.

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Question 7

Consider

$$y' = xe^{-y} - 1, \qquad y(0) = 0.$$

Tasks

1. Set $u = e^y$ and derive a linear differential equation for u.
2. Solve it using the initial condition and recover y.
3. Prove that the selected transformed solution stays positive for every real x. Find the minimum value of y and where it occurs.
4. The same linear equation has the solution $u = x - 1$. Determine precisely where it produces a real original solution, and explain why a valid transformed solution need not be invertible everywhere.

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Question 7 – Solution

Strategy. Exponentiating the unknown makes the reciprocal exponential linear, but the inverse logarithm requires a positive transformed value.

Step 1: Transform and solve. For $u = e^y$, $u' = e^y y' = x - u$. The initial value is $u(0) = 1$, so

$$u' + u = x, \qquad (e^x u)' = xe^x,$$

which gives $u = x - 1 + Ce^{-x}$. The data select $C = 2$, hence

$$\boxed{y(x) = \ln(x - 1 + 2e^{-x})}.$$

Step 2: Establish the inverse domain and minimum. Let $U(x) = x - 1 + 2e^{-x}$. Then $U' = 1 - 2e^{-x}$ and $U'' = 2e^{-x} > 0$. Its unique global minimum occurs at $x = \ln 2$ and equals $\ln 2 > 0$. Therefore the logarithm is defined and smooth for all real x, so the IVP interval is $\boxed{\mathbb{R}}$.

Since the logarithm is increasing, the solution has unique minimum

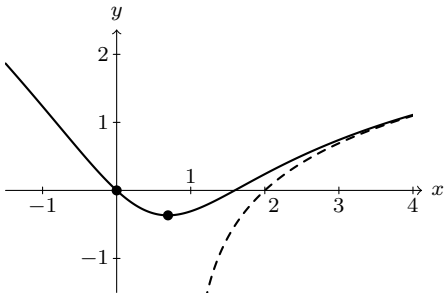
$$\boxed{y(\ln 2) = \ln(\ln 2)}.$$

Verification is immediate from $U' = x - U$: $y' = U'/U = x/U - 1 = xe^{-y} - 1$, and $y(0) = \ln 1 = 0$.

Step 3: Test another transformed solution. The linear solution $u = x - 1$ is defined everywhere, but $u > 0$ only for $x > 1$. It gives

$$\boxed{y = \ln(x - 1), \qquad x > 1}.$$

On that interval, $y' = 1/(x - 1) = x/(x - 1) - 1$, verifying the original equation. At $x \downarrow 1$, $y \to -\infty$, and for $x \leq 1$ no real logarithm of u exists. The map $y \mapsto e^y$ has range $(0, \infty)$; solving the transformed linear equation does not remove that range restriction.



Solid: the global IVP solution. Dashed: $\ln(x - 1)$, which is valid only for $x > 1$.

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Question 8

On $x > 0$, consider

$$x^2y' + y = e^{-1/x}.$$

We seek a solution that remains bounded as $x \downarrow 0$.

Tasks

1. Change the independent variable to $t = 1/x$ and define $Y(t) = y(1/t)$. Derive the transformed equation carefully using the chain rule.
2. Solve the transformed equation and recover the full original solution family on $x > 0$.
3. Select the unique solution bounded as $x \downarrow 0$. Find its limit at 0, its limit as $x \to \infty$, and its value at $x = 1$.
4. Verify it in the original equation and explain how the substitution changes the endpoint at which the boundedness condition is imposed.

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Question 8 – Solution

Strategy. A reciprocal change of the independent variable absorbs x^2, while converting a finite endpoint into an infinite one.

Step 1: Transform the derivative. Since $t = 1/x$, $dt/dx = -1/x^2$. Writing $y(x) = Y(t)$ gives

$$x^2 y' = -Y'(t), \qquad -Y' + Y = e^{-t}.$$

Thus $\boxed{Y' - Y = -e^{-t}}$ for $t > 0$. The minus sign comes from the reversal of the independent variable.

Step 2: Solve and return to x. The integrating factor is e^{-t}:

$$(e^{-t}Y)' = -e^{-2t}, \qquad Y = \frac{1}{2}e^{-t} + Ce^{t}.$$

Therefore all solutions on the stated half-line are

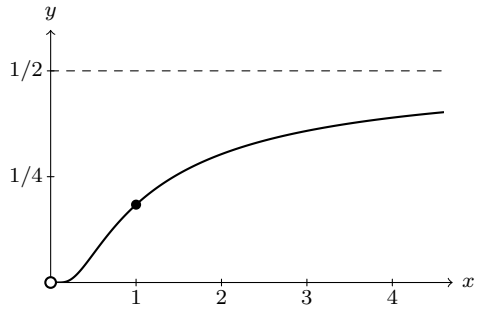
$$\boxed{y(x) = \frac{1}{2}e^{-1/x} + Ce^{1/x}}.$$

Step 3: Apply the endpoint condition. As $x \downarrow 0$, $t \to +\infty$. The first term tends to zero, whereas any nonzero C makes the second term unbounded. Consequently the unique bounded choice is

$$\boxed{y_b(x) = \frac{1}{2}e^{-1/x}}.$$

Its limits are 0 as $x \downarrow 0$ and $1/2$ as $x \to \infty$, and $\boxed{y_b(1) = 1/(2e)}$.

Step 4: Verify and interpret the reversal. Differentiation gives $y_b' = e^{-1/x}/(2x^2)$, so $x^2 y_b' + y_b = e^{-1/x}$. The condition near $x = 0$ becomes a condition at $t = \infty$, not at $t = 0$. Conversely, $x \to \infty$ corresponds to $t \downarrow 0$. The limit at $x = 0$ does not make that point part of the stated domain of the equation.



The bounded solution approaches the excluded point $(0, 0)$ and tends to $1/2$ as $x \to \infty$.

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Question 9

Consider the equation

$$y' = y^2 + (1 - 2x)y + x^2 - x + 1, \qquad y(0) = 2.$$

A particular solution is suggested: $y_p(x) = x$.

Tasks

1. Verify the suggested solution and derive the equation for $z = y - y_p$.
2. On a branch with $z \neq 0$, set $u = 1/z$ and derive a linear equation for u.
3. Solve the IVP and find its maximal interval containing 0.
4. Verify the recovered solution through the substitutions, restore the solution excluded by $u = 1/z$, and explain why this reciprocal is applied to the difference from y_p rather than directly to y.

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Question 9 – Solution

Strategy. Subtract a known particular solution so that the constant term cancels, then use a reciprocal on the remaining quadratic equation.

Step 1: Verify and subtract the particular solution. For $y_p = x$, the right-hand side is $x^2 + (1 - 2x)x + x^2 - x + 1 = 1 = y_p'$. Now write $y = x + z$. Substitution and cancellation give

$$1 + z' = 1 + z + z^2, \qquad \boxed{z' = z + z^2}, \qquad z(0) = 2.$$

Step 2: Make the remaining equation linear. For $z \neq 0$, $u = 1/z$ gives

$$u' = -\frac{z'}{z^2} = -u - 1, \qquad u(0) = \frac{1}{2}.$$

Solving $u' + u = -1$ yields $u = Ce^{-x} - 1$, and the initial value requires $C = 3/2$.

Step 3: Recover the IVP and its interval. Thus

$$\boxed{y = x + \frac{1}{\frac{3}{2}e^{-x} - 1}, \qquad I = (-\infty, \ln(3/2))}.$$

The denominator is positive before its unique zero at $\ln(3/2)$ and gives $y(0) = 2$. At that endpoint $y \to +\infty$, preventing a finite differentiable extension.

Step 4: Verify and account for the missing solution. Where $u \neq 0$, $z = 1/u$ satisfies $z' = -u'/u^2 = z + z^2$. Thus $y' = 1 + z + z^2$, which is exactly the original right side after $y = x + z$. The excluded case $z \equiv 0$ restores $\boxed{y = x}$, a global solution with a different initial value.

The original equation contains a nonzero term independent of y. A direct reciprocal $w = 1/y$ would retain a quadratic term in w and would not be linear. Subtracting the known solution first cancels that obstruction. This two-step substitution is the useful structure, not the reciprocal alone.

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Question 10

For a homogeneous equation $y' = F(y/x)$ on $x > 0$, introduce the logarithmic independent variable $t = \ln x$ and the ratio $v(t) = y(e^t)/e^t$.

We want F to be a polynomial of degree at most two such that, in the transformed equation, $v = 1$ and $v = 3$ are equilibrium values and $dv/dt = -1$ when $v = 2$.

Tasks

1. Derive the equation for dv/dt in terms of F and v.
2. Determine the unique polynomial F satisfying the design requirements.
3. Solve the resulting original IVP with $y(1) = 2$, showing the transformed integration and recovery of y.
4. Find the limits of y/x as $x \downarrow 0$ and as $x \to \infty$. Identify the original solutions represented by the equilibrium ratios and explain why they are not constant functions y. Decide whether the selected original $y(x)$ increases or decreases.

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Question 10 – Solution

Strategy. The logarithmic independent variable turns multiplication of x into translation of t and removes the factor x from the ratio equation.

Step 1: Derive and design the transformed equation. Writing $y = xv(\ln x)$ gives $y' = v + dv/dt$, so

$$\frac{dv}{dt} = F(v) - v.$$

This polynomial has roots $1, 3$ and degree at most two, hence equals $k(v-1)(v-3)$. Its value -1 at $v = 2$ forces $k = 1$. Therefore

$$\boxed{F(v) = v^2 - 3v + 3}, \qquad v' = (v-1)(v-3).$$

Step 2: Solve the selected trajectory. The data give $v(0) = 2$, between the equilibria. Separation yields

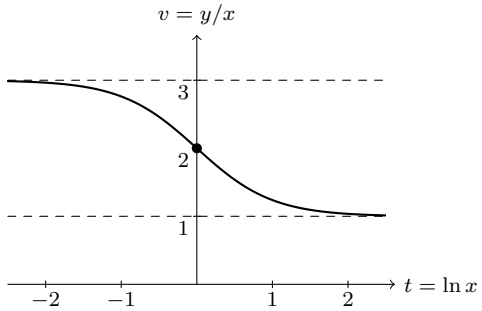
$$\frac{1}{2}\ln\left|\frac{v-3}{v-1}\right| = t + C, \qquad C = 0.$$

The ratio is negative on the selected branch, so $(v-3)/(v-1) = -e^{2t}$. Thus

$$v = 1 + \frac{2}{1+e^{2t}}, \qquad \boxed{y = x + \frac{2x}{1+x^2}, \quad x > 0}.$$

Directly $dv/dt = -4e^{2t}/(1+e^{2t})^2 = (v-1)(v-3)$, so the chain rule verifies the original equation and $y(1) = 2$.

Step 3: Interpret the limits and equilibria. As $x \downarrow 0$, $t \to -\infty$ and $y/x \to 3$; as $x \to \infty$, $t \to \infty$ and $y/x \to 1$. The constant ratios give $\boxed{y = x,\ y = 3x}$, both valid on $x > 0$. They are straight lines, not constant original functions: constancy applies to the ratio in the new coordinates. Moreover, $y' = F(v) = (v - 3/2)^2 + 3/4 > 0$, so the original y increases even though its ratio v decreases.



The transformed ratio decreases from 3 to 1; the dashed levels represent $y = 3x$ and $y = x$.

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Question 1

Consider the initial-value problem

$$y' = y^2, \qquad y(0) = 2.$$

A student argues: "The right-hand side and its derivative with respect to y are continuous everywhere, so the solution must exist for every real x."

Tasks

1. State what local existence and uniqueness actually guarantee for this IVP, and identify the gap in the student's argument.
2. Solve the IVP and verify the initial value and differential equation.
3. Find its maximal open interval of validity containing 0, and prove that a classical solution cannot extend it past its finite endpoint.
4. The same rational expression is defined on another open interval. Explain why that branch is not a continuation of this IVP solution.

Here a classical solution is a real, continuously differentiable function satisfying the equation at every point of its interval. A maximal interval cannot be enlarged while extending the same solution of the given equation.

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Question 1 – Solution

Strategy. Smoothness of the differential equation gives a local theorem; the explicit solution reveals whether its values remain finite.

Step 1: Separate local and global conclusions. Here $f(x, y) = y^2$ and $f_y(x, y) = 2y$ are continuous on the whole plane. The local theorem guarantees a solution on some open interval around 0 and uniqueness of solutions through $(0, 2)$ on their common interval. It does not guarantee a solution on the entire x-axis: growth in y can produce a finite-time blow-up.

Step 2: Solve and verify. Near the initial point $y \neq 0$, so separation gives

$$-\frac{1}{y} = x + C, \qquad C = -\frac{1}{2}, \qquad \boxed{y(x) = \frac{2}{1 - 2x}}.$$

Directly, $y(0) = 2$ and $y' = 4/(1 - 2x)^2 = y^2$. The expression therefore defines the selected solution at every $x < 1/2$.

Step 3: Establish maximality. The component of the formula's domain containing 0 is

$$\boxed{I = (-\infty, 1/2)}.$$

As $x \uparrow 1/2$, $y(x) \to +\infty$. Any extension to a larger interval would include $1/2$ as an interior point and would have a finite, continuous value there. That contradicts this limit. There is no finite obstruction to the left.

Step 4: Distinguish the disconnected branch. The expression also solves the differential equation on $(1/2, \infty)$, where it is negative. That interval does not contain the initial point. The union of the two components is not an interval, and no classical solution can connect them through the pole. An algebraic formula on both sides of a singularity is not a continuation across it.

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Question 2

For a real parameter a, consider the equation in the form written:

$$y' + \frac{2x}{x^2 - 9}y = \frac{1}{x^2 - 9}, \qquad y(0) = a.$$

The original equation is defined only for $x \neq -3, 3$.

Tasks

1. Use the linear existence theorem to identify an interval on which a unique solution is guaranteed for every a.
2. Find and verify the solution formula in terms of a.
3. Determine which values of a make a singularity of that formula removable, and compute the corresponding finite limit.
4. Determine the maximal interval for the original IVP for every a. Explain whether multiplying the equation by $x^2 - 9$ changes what it means to extend a solution through $x = \pm 3$.

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Question 2 – Solution

Strategy. Keep the domain of the given equation separate from the domain of a simplified formula or a multiplied equation.

Step 1: Locate the coefficient interval. Both coefficients are continuous on $(-3, 3)$, the largest coefficient interval containing 0. The linear theorem guarantees a unique solution throughout that interval for every real a.

Step 2: Integrate and verify. On this interval, multiplication and the product rule yield

$$((x^2 - 9)y)' = 1, \qquad (x^2 - 9)y = x - 9a.$$

Thus

$$\boxed{y(x) = \frac{x - 9a}{x^2 - 9}}.$$

The value at 0 is a, and differentiating $(x^2 - 9)y = x - 9a$ verifies the differential equation whenever $x \neq \pm 3$.

Step 3: Examine cancellation. At $x = 3$, the numerator vanishes exactly when $a = 1/3$. Then

$$y = \frac{1}{x + 3} \quad (x \neq \pm 3), \qquad \lim_{x \to 3} y = \frac{1}{6}.$$

At $x = -3$, cancellation occurs exactly when $a = -1/3$. Then

$$y = \frac{1}{x - 3} \quad (x \neq \pm 3), \qquad \lim_{x \to -3} y = -\frac{1}{6}.$$

No value of a cancels both factors. For other parameter values the corresponding endpoint is a pole.

Step 4: Use the original domain. For every a, the answer remains $\boxed{I = (-3, 3)}$. Even a finite removable limit does not define the original coefficients at an excluded point.

The multiplied equation $(x^2 - 9)y' + 2xy = 1$ is defined at $x = \pm 3$ and imposes algebraic conditions there. It is equivalent to the original equation only away from those points. A canceled formula can extend through one endpoint as a solution of the multiplied equation, but that is an extension for a different domain of definition.

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Question 3

Consider

$$y' = \frac{e^{-y}}{x-1}, \qquad y(2) = \ln 2,$$

with $x \neq 1$ and real y.

Tasks

1. Use $u = e^y$ to solve the IVP, explicitly enforcing the range of this substitution.
2. Determine the maximal interval containing 2; compare its left endpoint with the singular point of the coefficient $1/(x-1)$.
3. Verify the solution and calculate its limiting behavior at both ends of its maximal interval.
4. Explain why neither continuing the linear transformed solution through a zero nor replacing $\ln u$ by $\ln|u|$ produces a continuation of the original solution.

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Question 3 – Solution

Strategy. The exponential substitution is invertible only for $u > 0$. Its range can restrict the interval before a coefficient singularity is reached.

Step 1: Transform and invert. On the component $x > 1$ containing 2,

$$u' = e^y y' = \frac{1}{x-1}, \qquad u = \ln(x-1) + C.$$

Since $u(2) = 2$, $C = 2$. Therefore

$$\boxed{y = \ln\big(2 + \ln(x-1)\big)}$$

is a real solution only where $2 + \ln(x-1) > 0$.

Step 2: Find the interval. The positivity condition is $x > 1 + e^{-2}$. Let $\alpha = 1 + e^{-2}$; then

$$\boxed{I = (\alpha, \infty)}.$$

In particular, $1 < \alpha < 2$. The solution fails before reaching $x = 1$, even though the original right-hand side is smooth at every finite point (α, y).

Step 3: Verify and analyze the ends. Writing $u = 2 + \ln(x-1) > 0$ gives

$$y' = \frac{1}{(x-1)u} = \frac{e^{-y}}{x-1}, \qquad y(2) = \ln 2.$$

As $x \downarrow \alpha$, $u \downarrow 0$ and $y \to -\infty$, preventing a finite continuous extension. As $x \to \infty$, $u \to \infty$ and $y \to \infty$, but this occurs only at an infinite endpoint and does not shorten the interval.

Step 4: Reject an invalid inverse. The transformed function continues on $(1, \infty)$ but becomes negative on $(1, \alpha)$. No real y can have $e^y = u \le 0$.

On that negative portion, $z = \ln|u|$ would satisfy $z' = 1/((x-1)u) < 0$, whereas $e^{-z}/(x-1) = 1/((x-1)|u|) > 0$. Thus the absolute-value replacement fails the original equation and cannot repair the lost exponential range.

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Question 4

Consider the real IVP

$$y' = -\frac{x}{y}, \qquad y(0) = 2,$$

whose equation is defined only when $y \neq 0$.

Tasks

1. Obtain an implicit relation and select the explicit branch determined by the initial condition.
2. Find the maximal interval and the limits of both y and y' at its finite endpoints.
3. Decide whether either endpoint can be included, or whether switching to the other square-root branch gives a continuation as a classical solution $y(x)$.
4. Sketch the selected solution and mark its excluded endpoints. Explain why bounded values of y do not guarantee continuation in this example.

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Question 4 – Solution

Strategy. Track the forbidden value $y = 0$ and the slope, rather than checking only whether the solution stays bounded.

Step 1: Select the branch. Multiplying by y and integrating gives

$$yy' = -x, \qquad x^2 + y^2 = 4.$$

Continuity and $y(0) = 2$ select

$$\boxed{y = \sqrt{4 - x^2}}.$$

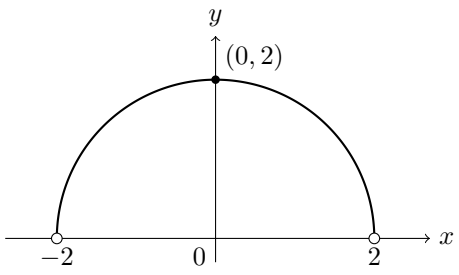
Its positive branch has $y' = -x/\sqrt{4 - x^2}$ and directly satisfies the IVP for $-2 < x < 2$.

Step 2: Test both endpoints. Thus $\boxed{I = (-2, 2)}$, and

	$x \downarrow -2$	$x \uparrow 2$
$y(x)$	0^+	0^+
$y'(x)$	$+\infty$	$-\infty$

At either endpoint the original right-hand side is undefined, and a continuously differentiable extension would also require a finite derivative.

Step 3: Exclude branch switching. The negative square root fails the initial condition. Joining the two semicircles at an endpoint still passes through $y = 0$, and the circle has no real points with $|x| > 2$. Following the circle as a parametrized curve is not an extension as a differentiable graph $y(x)$ on a larger interval.



Step 4: Identify the obstruction. Although $0 < y \leq 2$, the graph approaches the boundary of the equation's domain. Boundedness alone does not keep it inside a region where the differential equation is defined and regular. The open circles indicate excluded endpoints, not additional solution values.

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Question 5

Consider

$$y' = \frac{1}{3y^2 - 1}, \qquad y(0) = 0,$$

with $y \neq \pm 1/\sqrt{3}$.

Tasks

1. Derive an implicit equation and identify the portion that defines the IVP's branch as a function of x.
2. Determine the exact maximal interval containing 0, using monotonicity rather than a cubic-root formula.
3. Find the one-sided limits of y and y' at both endpoints and justify maximality.
4. Sketch the selected branch, and explain why other roots of the same cubic cannot be spliced onto it to extend this classical solution.

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Question 5 – Solution

Strategy. Regard x as a function of y. Its turning points identify where the selected inverse branch stops being a regular graph.

Step 1: Integrate without solving the cubic. The chain rule gives

$$\frac{d}{dx}(y^3 - y) = (3y^2 - 1)y' = 1, \qquad \boxed{x = y^3 - y}.$$

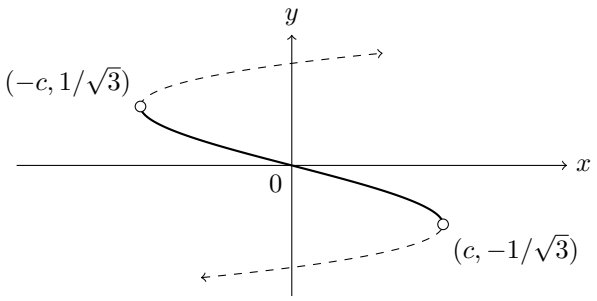
The initial value sets the integration constant to zero. Put $h(y) = y^3 - y$. On $-1/\sqrt{3} < y < 1/\sqrt{3}$, $h'(y) < 0$, so h is strictly decreasing and has a differentiable inverse there.

Step 2: Map the branch endpoints. Let $c = 2/(3\sqrt{3})$. Then $h(1/\sqrt{3}) = -c$ and $h(-1/\sqrt{3}) = c$. Hence the inverse through $(0, 0)$ exists exactly on

$$\boxed{I = (-c, c) = \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)}.$$

Differentiating its defining relation recovers $y' = 1/(3y^2 - 1)$ and verifies the original equation on this branch.

Step 3: Analyze the folds. As $x \downarrow -c$, $y \to 1/\sqrt{3}$; as $x \uparrow c$, $y \to -1/\sqrt{3}$. In both cases $3y^2 - 1 \to 0^-$, so $y' \to -\infty$. Both finite limiting values are excluded by the original equation.



$c = 2/(3\sqrt{3})$; solid: selected branch

Step 4: Reject a change of root. The dashed pieces show other parts of the algebraic curve. A continuous switch at a fold must include an excluded value of y with unbounded slope; a switch to a different root at that x would be discontinuous. Neither gives a classical extension. The cubic relation alone does not specify the IVP's interval or branch.

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Question 6

Consider the IVP

$$y' = 3|y|^{2/3}, \qquad y(0) = 0.$$

For finite real numbers $a \leq 0 \leq b$, define

$$Y_{a,b}(x) = \begin{cases} (x-a)^3, & x < a, \\ 0, & a \leq x \leq b, \\ (x-b)^3, & x > b. \end{cases}$$

Tasks

1. Verify the equation and continuous differentiability of $Y_{a,b}$, including both joining points.
2. Give two distinct global solutions satisfying the initial condition and determine their maximal intervals.
3. Explain why a restriction of $Y_{-1,1}$ to $(-1/2, 1/2)$ is not maximal, even though it solves the IVP there.
4. Sketch $Y_{-1,1}$ and identify precisely which local uniqueness hypothesis fails. Does a maximal interval imply a unique solution?

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Question 6 – Solution

Strategy. Check continuation at the joins directly. The length of a solution's interval and uniqueness of the solution are different questions.

Step 1: Verify all three pieces and the joins. For $x < a$,

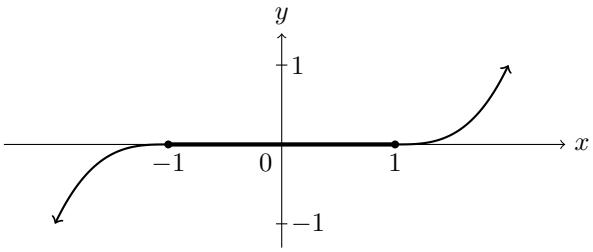
$$Y'_{a,b} = 3(x-a)^2 = 3|(x-a)^3|^{2/3},$$

and the same computation holds with b for $x > b$. On the middle piece both sides vanish.

At $x = a$, the values from both sides are 0, and the one-sided derivatives tend to 0. The derivative at the join itself is 0 by the difference quotient. The same holds at b, including the case $a = b = 0$. Thus $Y_{a,b}$ is continuously differentiable on $\mathbb{R}$, satisfies the equation everywhere, and has $Y_{a,b}(0) = 0$.

Step 2: Exhibit distinct maximal solutions. Two examples are $y(x) = 0$ for all x and $y(x) = x^3 = Y_{0,0}(x)$. Their maximal intervals are both $\boxed{\mathbb{R}}$: no larger real interval exists. The function $Y_{-1,1}$ is a third global solution.

Step 3: Separate a restriction from a maximal solution. On $(-1/2, 1/2)$, $Y_{-1,1} = 0$, but it extends as $Y_{-1,1}$ to the whole real line. Its restriction is therefore not maximal. That restriction can also be extended as the identically zero solution; an extension need not be unique here.



Solid graph: $Y_{-1,1}$; thick segment: $y = 0$

Step 4: Locate the failed uniqueness condition. The right-hand side is continuous, but it is not locally Lipschitz in y at 0:

$$\frac{|3|y|^{2/3} - 0|}{|y - 0|} = 3|y|^{-1/3} \longrightarrow \infty.$$

Consequently the usual local uniqueness hypothesis is unavailable at the initial point. The explicit examples prove nonuniqueness, even though every exhibited global solution has the same maximal interval $\mathbb{R}$.

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Question 7

Let p be any real number. Consider the family of IVPs

$$y' = y(1 - y), \qquad y(0) = p.$$

Focus on intervals of validity; a full equilibrium or phase-line analysis is not required.

Tasks

1. Derive a solution formula in terms of p, preserving the cases lost by division in separation.
2. Classify the maximal interval containing 0 for every real p. Give any finite endpoint explicitly.
3. Determine the direction and sign of blow-up whenever such an endpoint exists.
4. Verify the formula and explain why the interval depends on p even though the differential equation is smooth on the whole (x, y)-plane.

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Question 7 – Solution

Strategy. A denominator zero is relevant only in the connected component containing the initial point. Preserve constant solutions before separating.

Step 1: Obtain a parameter formula. The constant solutions are $y = 0, 1$. Otherwise separation gives $\ln|y/(1-y)| = x + C$. Applying the initial value yields a formula that also includes both constants:

$$\boxed{y(x) = \frac{pe^x}{1 - p + pe^x}}.$$

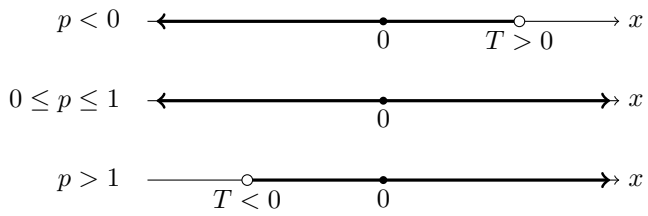
Its denominator $D(x)$ satisfies $D(0) = 1$. A zero requires $e^x = (p-1)/p > 0$, which occurs exactly when $p < 0$ or $p > 1$.

Step 2: Select the component containing zero. When it exists, set $T = \ln((p-1)/p)$. The complete classification is

initial value	endpoint location	maximal interval
$p < 0$	$T > 0$	$(-\infty, T)$
$0 \le p \le 1$	no finite endpoint	$\mathbb{R}$
$p > 1$	$T < 0$	(T, ∞)

For $0 < p < 1$, both terms of $D = 1 - p + pe^x$ are positive. The cases $p = 0, 1$ are global constants.

Step 3: Determine the endpoint behavior. If $p < 0$, then $D \downarrow 0^+$ as $x \uparrow T$, while $pe^x \to p - 1 < 0$, so $y \to -\infty$. If $p > 1$, then $D \downarrow 0^+$ as $x \downarrow T$, while $pe^x \to p - 1 > 0$, so $y \to +\infty$. Neither pole is removable.



Interval diagrams; endpoint positions schematic

Step 4: Verify and interpret. With $N = pe^x$, $N' = D' = N$, and

$$y' = \frac{N(D-N)}{D^2} = y(1-y), \qquad y(0) = p.$$

Smoothness gives local existence and uniqueness. It does not prevent the finite-time blow-up found for $p \notin [0, 1]$; the initial value determines whether the selected solution encounters a pole.

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Question 8

Consider the equation in its original, undivided form:

$$xy' = 2y, \qquad y(1) = 1.$$

It is defined for all real x, including $x = 0$. A classical solution must be continuously differentiable and satisfy this equation at every point of its open interval.

Tasks

1. Solve on $x > 0$ and explain why dividing by x cannot by itself settle continuation through 0 for the original equation.
2. Find every global classical solution of the original IVP.
3. Check both the differential equation and continuous differentiability at 0 for your full family.
4. Determine the maximal interval of each such solution. Explain how uniqueness on $x > 0$ can coexist with nonunique global continuation, and why this differs from an equation explicitly defined only for $x \neq 0$.

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Question 8 – Solution

Strategy. The original equation remains meaningful at the vanishing leading coefficient. Solve on the two half-lines and test the join in that original equation.

Step 1: Solve where division is allowed. For $x \neq 0$, the equation gives $(y/x^2)' = 0$. The initial condition therefore forces $y = x^2$ throughout $x > 0$. On $x < 0$, the general solution is $y = Cx^2$, with an independent real constant C.

The divided equation $y' = 2y/x$ has no meaning at 0, but the original equation does: it requires $y(0) = 0$. A singularity introduced by division is not automatically an obstruction for the original problem.

Step 2: List all global solutions. Every global candidate must have the form

$$\boxed{y_C(x) = \begin{cases} Cx^2, & x < 0, \\ 0, & x = 0, \\ x^2, & x > 0, \end{cases} \qquad C \in \mathbb{R}.}$$

The half-line solutions and the equation at 0 leave no other possible global classical solutions.

Step 3: Verify the join. For every finite C, y_C is continuous at 0. The difference quotient there is Ch for $h < 0$ and h for $h > 0$, so $y_C'(0) = 0$. Moreover,

$$y_C'(x) = 2Cx \ (x < 0), \qquad y_C'(x) = 2x \ (x > 0)$$

tends to 0 from both sides. Thus y_C is continuously differentiable. The identity $xy_C' = 2y_C$ holds on each half-line and reduces to $0 = 0$ at 0. Also $y_C(1) = 1$.

Step 4: Resolve maximality and uniqueness. Each y_C has maximal interval $\boxed{\mathbb{R}}$. The unique positive-half-line solution extends through 0 in infinitely many ways because the original equation loses its derivative term there. The usual normal-form uniqueness theorem cannot cross that degeneracy.

If the problem had instead specified $y' = 2y/x$ only for $x \neq 0$, its IVP interval containing 1 would be $(0, \infty)$, even for the polynomial formula. Which equation and domain were actually given matters.

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Question 9

Define a real function on the entire real line by

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Consider $y' = f(x)$ with $y(-1) = 0$. Require a classical solution: y is continuously differentiable on an open interval and the equation holds at every point.

Tasks

1. Find the IVP solution on $x < 0$ and its maximal interval containing -1.
2. Compute the finite left-hand limits of y and y' at 0.
3. Find the continuous function on $\mathbb{R}$ that agrees with this solution on $x < 0$ and solves the differential equation on $x > 0$. Decide whether it is a classical extension through 0.
4. Explain why this example does not contradict a continuation theorem requiring a continuous right-hand side. Would changing only $f(0)$ repair the obstruction?

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Question 9 – Solution

Strategy. A formula obtained by integrating on either side of a jump need not be differentiable at the joining point.

Step 1: Solve on the initial component. For $x < 0$, $y' = -1$, so the initial condition gives

$$\boxed{y = -x - 1}.$$

It solves the equation at every negative x. Its maximal classical interval containing -1 is $\boxed{(-\infty, 0)}$, as the join test below shows.

Step 2: Compute the finite endpoint limits. As $x \uparrow 0$, $y(x) \to -1$ and $y'(x) \to -1$. Neither the solution nor its derivative blows up. The obstruction will be the change of slope imposed immediately to the right.

Step 3: Test the unique continuous gluing. On $x > 0$, any solution is $x + C$. Continuity with the left branch forces $C = -1$ and the value $y(0) = -1$. The only continuous gluing is therefore

$$Y(x) = |x| - 1.$$

Its left difference quotient at 0 is -1 and its right difference quotient is 1. Consequently $Y'(0)$ does not exist. It is not a classical solution of the equation on any interval containing 0.

Any larger interval extending the negative-half-line solution would include points on both sides of 0 and would require this same gluing. Thus no classical extension is possible, proving the claimed maximality despite the finite limits.

Step 4: Identify the missing hypothesis. Here f is defined everywhere but is discontinuous at 0. A continuation result that requires continuity of the right-hand side near the endpoint cannot be applied. Domain membership and bounded values alone are insufficient when that hypothesis fails.

Changing $f(0)$ to any single real value cannot reconcile the unequal one-sided derivatives forced by $f = -1$ to the left and $f = 1$ to the right. The same corner remains. The function Y solves the equation away from the join, but the question requires the equation at every point of a classical interval.

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Question 10

Design an initial-value problem of the form

$$y' = (Ax + B)y^2, \qquad y(0) = 1,$$

where A, B are real constants, so that its solution has exactly the maximal interval $(-2, 3)$ and tends to $+\infty$ at both finite endpoints.

Tasks

1. Use the reciprocal variable $u = 1/y$ to determine A and B from the prescribed endpoints.
2. Recover the solution and prove that it is positive and has exactly the requested maximal interval.
3. Verify the IVP and prove that your coefficient pair is the only affine choice with these properties.
4. Find the solution's minimum on its interval and sketch the curve with its endpoint asymptotes. Explain why smooth coefficients do not force a global solution here.

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Question 10 – Solution

Strategy. For a quadratic-growth equation, the reciprocal solution is a quadratic polynomial. Put its zeros at the desired endpoints.

Step 1: Determine the reciprocal polynomial. While $y \neq 0$,

$$u' = -(Ax + B), \qquad u(0) = 1, \qquad u = 1 - Bx - \frac{A}{2}x^2.$$

Blow-up at both prescribed endpoints requires $u(-2) = u(3) = 0$. A quadratic with these roots and value 1 at 0 must be

$$u = \frac{(x+2)(3-x)}{6} = 1 + \frac{x}{6} - \frac{x^2}{6}.$$

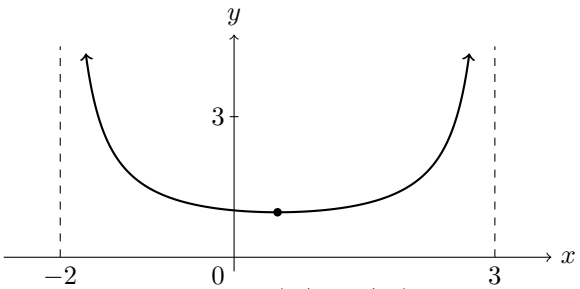
Thus $\boxed{A = 1/3,\ B = -1/6}$ and

$$\boxed{y = \frac{6}{(x+2)(3-x)}, \qquad I = (-2, 3)}.$$

Step 2: Check the interval and uniqueness of the design. Both denominator factors are positive on I. At either endpoint the denominator tends to 0^+, so $y \to +\infty$ and no finite continuous extension exists.

Also $y' = -u'/u^2 = ((2x-1)/6)y^2$ and $y(0) = 1$. For any proposed affine pair, a solution through 1 cannot cross the zero solution, by local uniqueness for the smooth right-hand side. Hence its reciprocal on the proposed interval has exactly the polynomial form above. The two endpoint zeros and $u(0) = 1$ determine it uniquely, proving uniqueness of the coefficient pair.

Step 3: Locate the minimum. The denominator is $25/4 - (x - 1/2)^2$, so it is largest at $x = 1/2$. Therefore $\boxed{y_{\min} = 24/25}$ there. Equivalently, y' changes from negative to positive at $1/2$.



Minimum: $(1/2, 24/25)$

Step 4: Interpret the construction. The coefficients and the right-hand side are smooth everywhere, but the solution grows without bound in both finite endpoint directions. Smoothness ensures local solvability and uniqueness, not global existence for this nonlinear equation.

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Question 1

A tank initially holds 100 L of well-mixed solution containing 10 kg of salt. Solution containing 0.20 kg/L enters at 3 L/min, and the mixture leaves at 1 L/min. The tank has capacity 200 L. Assume instantaneous mixing, additive liquid volumes, and no salt precipitation or other losses. Let t be minutes after the flows begin.

Tasks

1. Derive initial-value models for the liquid volume $V(t)$ and salt mass $S(t)$, including units for all rate terms and the operating time before overflow.
2. Derive and solve an equation for the concentration $c(t) = S(t)/V(t)$. Explain why c' is not simply S'/V.
3. Find the first time the concentration reaches 0.15 kg/L and decide whether this happens before the tank reaches capacity.
4. Find the salt mass when the tank first becomes full. Sketch the concentration over the modeled operation and explain why continuing the same volume formula beyond that time changes the physical problem.

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Question 1 – Solution

Strategy. Write separate liquid and salt balances, then use the quotient rule to model concentration in a changing volume.

Step 1: Balance volume and salt. The volume rate is $V' = 3 - 1 = 2$ L/min, so $V = 100 + 2t$ L. Capacity is reached at $t = 50$ min. Up to that time,

$$S' = 0.60 - \frac{S}{100 + 2t}, \qquad S(0) = 10.$$

Both terms have units kg/min: salt enters at $(3)(0.20)$ and leaves at $(1)(S/V)$. The pre-overflow model applies for $0 \le t < 50$, with a continuous value at 50.

Step 2: Derive the concentration equation. Since $S = cV$, $S' = Vc' + cV'$. Hence

$$(100 + 2t)c' = 0.60 - 3c, \qquad c(0) = 0.10.$$

Using the integrating factor $(100 + 2t)^{3/2}$ and $c(0) = 0.10$ gives

$$\boxed{c(t) = 0.20 - 0.10(100/(100 + 2t))^{3/2}}.$$

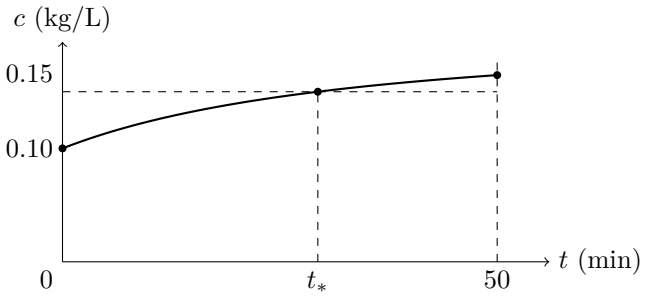
Indeed $c' = 3(0.20 - c)/V$, verifying the concentration balance. Omitting cV' would ignore the dilution caused by increasing volume.

Step 3: Locate the target and capacity values. The concentration increases strictly. Setting $c = 0.15$ gives

$$\boxed{t_* = 50(2^{2/3} - 1) \approx 29.37 \text{ min} < 50 \text{ min}}.$$

At capacity, $c(50) = 0.20 - 0.10/2^{3/2}$, so

$$\boxed{S(50) = 200c(50) = 40 - 5\sqrt{2} \approx 32.93 \text{ kg}}.$$



$t_* \approx 29.37$ min; final dot: first full tank

Step 4: Respect the operating boundary. The final point is the instant the tank becomes full. Beyond it, continued inflow would require overflow or a changed flow rate. Either changes the volume and salt balances, so the formula is not a model of the same operation after 50 min.

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Question 2

An object is placed in a controlled chamber. Its temperature is initially 80°C, while the chamber temperature is $T_a(t) = 20 + 2t$ degrees Celsius, with t in minutes. Assume the object has a uniform temperature and obeys Newton's law of cooling with constant coefficient $k = 0.10$ min^{-1} throughout the experiment. The chamber warming is prescribed and unaffected by the object.

Tasks

1. Formulate the temperature IVP, including the sign of the heat-transfer term.
2. Solve it and verify the initial temperature and differential equation.
3. Find when the object stops cooling and determine its minimum temperature. Explain how the object can subsequently warm under the same law.
4. Sketch the object and chamber temperatures. Find their limiting difference and explain why their temperatures do not become equal permanently.

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Question 2 – Solution

Strategy. Model the difference between the object and its time-dependent surroundings; the ambient temperature is not a constant equilibrium value.

Step 1: Form and solve the IVP. Newton's law gives

$$T' = -0.10(T - 20 - 2t), \qquad T(0) = 80.$$

The right side is in degrees Celsius per minute. The object cools when it is hotter than the chamber and warms when it is colder. An integrating factor $e^{t/10}$ yields

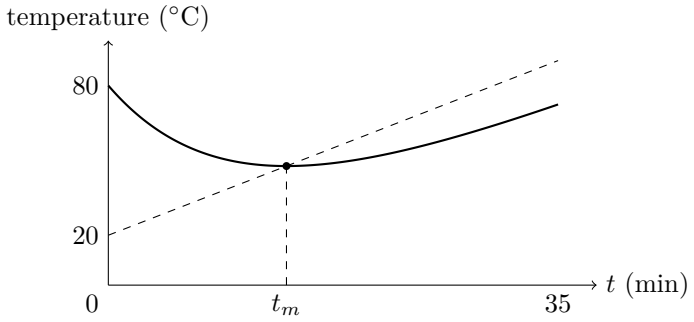
$$\boxed{T(t) = 2t + 80e^{-t/10}}, \qquad t \geq 0.$$

Its derivative is $2 - 8e^{-t/10}$, which equals $-0.10(T - T_a)$, and its initial value is 80.

Step 2: Identify the turning time. The derivative vanishes when $e^{-t/10} = 1/4$, giving

$$\boxed{t_m = 10\ln 4 \approx 13.86 \text{ min}}, \qquad \boxed{T_{\min} = 20 + 20\ln 4 \approx 47.73^\circ\text{C}}.$$

Since $T'' = 0.8e^{-t/10} > 0$, the derivative changes from negative to positive exactly once. At this time $T = T_a$; afterward the rising chamber is warmer than the object, so the heat-transfer term becomes positive.



Solid: object; dashed: chamber; $t_m = 10\ln 4$

Step 3: Interpret the long-time lag. The signed temperature difference is

$$T - T_a = 80e^{-t/10} - 20 \longrightarrow -20^\circ\text{C}.$$

The object eventually warms at almost 2°C/min. A temperature deficit of 20°C supplies that rate because $-0.10(-20) = 2$. Permanent equality would instead give zero heat-transfer rate, inconsistent with following an ambient temperature that keeps rising. The long-time conclusion is conditional on the stated linear ambient schedule and constant cooling coefficient remaining valid.

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Question 3

A specimen contains a stable material of unknown mass B and an active material of mass $R(t)$ that disappears from the measured specimen at a rate proportional to R. Assume the stable material is unaffected and all disappearance products leave the measured mass. Thus the measured mass is $M(t) = B + R(t)$. Measurements, treated as exact, are

$$M(0) = 100 \text{ g}, \qquad M(1) = 70 \text{ g}, \qquad M(2) = 55 \text{ g},$$

where t is in days. Assume $B \geq 0$, $R(0) > 0$, and a positive constant disappearance coefficient.

Tasks

1. Formulate the active-mass IVP and the resulting measurement model.
2. Determine uniquely the stable mass, initial active mass, and disappearance coefficient from the data.
3. Find the active material's half-life and the time at which only 5 g of active material remain.
4. A technician fits exponential decay directly to the total mass using the first two measurements. Test that model against the third measurement and explain the modeling error.

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Question 3 – Solution

Strategy. Separate the unchanging baseline from the decaying component. Differences of equally spaced measurements remove the baseline.

Step 1: Write the component and observation models. The active mass satisfies $R' = -kR$, where $k > 0$ has units day^{-1}. If $R(0) = R_0$, then

$$R = R_0 e^{-kt}, \qquad M = B + R_0 e^{-kt}.$$

The stable material contributes to the measurement but not to the loss rate.

Step 2: Identify the parameters. Let $r = e^{-k}$, where one day is the time step, so $0 < r < 1$. The measured successive differences satisfy

$$-30 = R_0(r-1), \qquad -15 = R_0 r(r-1).$$

Their ratio gives $r = 1/2$. Therefore

$$\boxed{R_0 = 60 \text{ g}, \quad B = 40 \text{ g}, \quad k = \ln 2 \text{ day}^{-1}}, \qquad \boxed{M(t) = 40 + 60\,2^{-t}}.$$

These values reproduce $100, 70, 55$ g. They are unique under the assumptions because the nonzero first difference fixes r, then R_0, then B. Also $R' = -kR$ verifies the rate law.

Step 3: Compute active-mass times. The active half-life is $\ln 2/k = \boxed{1 \text{ day}}$. The condition $R = 5$ gives $60\,2^{-t} = 5$, hence

$$\boxed{t = \frac{\ln 12}{\ln 2} \approx 3.585 \text{ days}}.$$

The active mass is below 5 g after that time. The total mass then is 45 g, not 5 g.

Step 4: Test the incorrect observation model. A single exponential through the first two total-mass measurements is $\widetilde{M}(t) = 100(0.7)^t$. It predicts $\widetilde{M}(2) = 49$ g, contradicting the measured 55 g. It wrongly assigns the stable component the same disappearance law as the active material and predicts a zero limiting total mass instead of 40 g.

The model is intended for $t \geq 0$. Its inference depends on exact data, constant k, and a truly constant baseline; noisy data would require an estimation method rather than exact identification from three values.

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Question 4

A 2 kg test package is released from rest. Take downward as positive and use $g = 10$ m/s^2. Model drag as a force $-bv$ opposing its downward velocity v. Before a parachute opens at $t = 4$ s, $b = 2$ kg/s; afterward, $b = 10$ kg/s. Assume the opening changes the drag coefficient instantly but delivers no impulse, and the package remains airborne through $t = 6$ s.

Tasks

1. Derive the velocity equation in each phase from Newton's second law. State the initial and matching conditions.
2. Solve for $v(t)$ through 6 s and calculate the velocity immediately before and after opening.
3. Find the total downward distance traveled during the first 6 s.
4. Find the one-sided accelerations at opening and sketch the velocity. Explain why velocity is continuous although acceleration jumps, and state where a classical phase equation applies.

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Question 4 – Solution

Strategy. Balance forces in each phase and carry velocity continuously across a coefficient change; integrate velocity to obtain displacement.

Step 1: Model and match the phases. Newton's law gives $2v' = 20 - bv$, with forces in newtons. Thus $v' = 10 - v$ before opening and $v' = 10 - 5v$ afterward. The initial value is $v(0) = 0$, and no impulse means $v(4^-) = v(4^+) = v_4$.

Step 2: Solve both initial-value problems. The solutions are

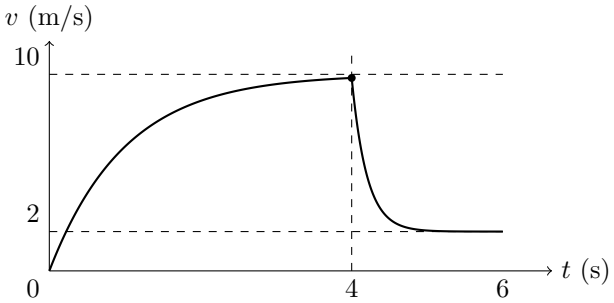
$$\boxed{v(t) = \begin{cases} 10(1 - e^{-t}), & 0 \le t \le 4, \\ 2 + (v_4 - 2)e^{-5(t-4)}, & 4 \le t \le 6, \end{cases} \qquad v_4 = 10(1 - e^{-4}) \approx 9.817 \text{ m/s}.}$$

Each expression satisfies its phase equation, and both give v_4 at the switch. The phase terminal speeds are 10 and 2 m/s, respectively.

Step 3: Integrate the distance. Velocity stays positive, so downward displacement equals distance traveled:

$$\boxed{d = 10(3 + e^{-4}) + 4 + \frac{8 - 10e^{-4}}{5}(1 - e^{-10}) \approx 35.75 \text{ m}}.$$

The first term integrates the pre-opening velocity over $[0, 4]$; the remaining terms integrate the post-opening velocity over $[4, 6]$.



Parachute opens at 4 s; velocity stays continuous

Step 4: Examine the acceleration jump. The one-sided accelerations are

$$v'(4^-) = 10e^{-4} \approx 0.183 \text{ m/s}^2, \qquad v'(4^+) = -40 + 50e^{-4} \approx -39.08 \text{ m/s}^2.$$

Changing a finite force changes acceleration without an instantaneous velocity jump. The velocity is continuous and piecewise differentiable, but is not differentiable at the idealized switch. Each classical phase equation applies away from $t = 4$; the switch is handled by continuity, not by requiring a single derivative there.

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Question 5

A series resistor-capacitor circuit has resistance $R = 1000\ \Omega$ and capacitance $C = 0.001$ F. Its capacitor voltage is $U(t)$, charge is $q = CU$, and current is $I = q'$. The source voltage $E(t)$ obeys the loop relation $E = RI + U$. Initially the capacitor is uncharged. Time is in seconds.

A designer wants the exact voltage trajectory

$$U_d(t) = 6(1 - e^{-t/2})\ \text{V}, \qquad t \geq 0,$$

but the available source must satisfy $0 \leq E(t) \leq 5$ V. Ideal circuit laws are assumed; capacitor energy is $W = \frac{1}{2}CU^2$.

Tasks

1. Derive a first-order IVP for the capacitor voltage and identify the time constant, with units.
2. Find the unique source voltage and the current required to produce the desired trajectory exactly.
3. Determine the longest interval starting at 0 on which this trajectory respects the source bound. Find the capacitor voltage at the limiting time.
4. Calculate the stored energy at that time. Explain why simply fixing the source at 5 V afterward cannot continue the same prescribed trajectory.

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Question 5 – Solution

Strategy. Use the circuit balance backward: a specified capacitor trajectory determines its current and therefore the required source.

Step 1: Derive the voltage model. Since $I = CU'$, the loop equation gives

$$RC\,U' + U = E(t), \qquad U(0) = 0.$$

Here $RC = (1000)(0.001) = 1$ s is the time constant. With time numerically measured in seconds, the model is $U' + U = E$; the derivative coefficient carries the one-second time constant.

Step 2: Find the required input and current. The desired derivative is $U_d' = 3e^{-t/2}$ V/s, so

$$\boxed{E(t) = 6 - 3e^{-t/2}\ \text{V}}, \qquad \boxed{I(t) = 0.003e^{-t/2}\ \text{A}}.$$

Substitution gives $RC\,U_d' + U_d = E$ and $U_d(0) = 0$. The prescribed U_d fixes its derivative, so the loop relation leaves no other possible source trajectory. In particular, the initial current is 3 mA and the required initial source is 3 V.

Step 3: Apply the source constraint. The required source increases strictly from 3 V toward 6 V. It first reaches 5 V when $e^{-t/2} = 1/3$. Therefore the full feasible initial interval is

$$\boxed{0 \le t \le t_* = 2\ln 3 \approx 2.197\ \text{s}}, \qquad \boxed{U_d(t_*) = 4\ \text{V}}.$$

Equality at the limiting time is allowed by the given source bound. Beyond it the unique required source exceeds 5 V.

Step 4: Compute energy and interpret saturation. At the limiting time,

$$\boxed{W = \tfrac{1}{2}(0.001)(4)^2 = 0.008\ \text{J}}.$$

A source fixed at 5 V afterward would instead produce $U = 5 - e^{-(t-t_*)}$ V, starting from 4 V. It tends to 5 V, whereas the desired trajectory tends to 6 V. Thus the circuit can continue operating, but it cannot continue this exact target under the source constraint. The feasible interval is a hardware limitation, not a singularity of the ideal differential equation.

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Question 6

An inverted conical tank has height 4 m and top radius 2 m. It initially contains water to the brim. Let $h(t)$ be water depth above the outlet, with t in seconds. Assume horizontal water surfaces and similar conical cross-sections. The outflow rate is prescribed by

$$Q(h) = \kappa\sqrt{h}, \qquad \kappa = 0.05 \text{ m}^{5/2}/\text{s},$$

while $h > 0$. There is no inflow. Ignore changes in the discharge coefficient and stop outflow once the tank is empty.

Tasks

1. Derive the volume as a function of depth and use volume conservation to form an IVP for h.
2. Solve for the depth until emptying and find the emptying time.
3. Find the fractions of the emptying time needed to reach half the original depth and half the original volume. Explain why these are different.
4. Sketch the physical depth, including the empty state. Decide whether depth and volume have continuously differentiable extensions through emptying, and explain the scope of the divided depth equation.

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Question 6 – Solution

Strategy. The water surface area changes with depth. Convert a volume-loss law into a depth equation before separating variables.

Step 1: Use the conical geometry. Similarity gives $r = h/2$, so $V = \pi h^3/12$ and

$$\frac{\pi h^2}{4} h' = -0.05\sqrt{h}, \qquad h' = -\frac{0.20}{\pi} h^{-3/2}, \qquad h(0) = 4.$$

The first equality has units m^3/s. Division by h^2 is valid only while water remains.

Step 2: Integrate to emptying. Integrating $h^{3/2}\,dh = -(0.20/\pi)\,dt$ gives $h^{5/2} = 32 - t/(2\pi)$, hence

$$\boxed{h(t) = (32 - t/(2\pi))^{2/5}}, \qquad 0 \le t < t_e, \qquad \boxed{t_e = 64\pi \approx 201.06 \text{ s}}.$$

Equivalently, $h = 4(1 - t/t_e)^{2/5}$. Differentiation verifies the depth equation; the physical depth is set to zero for $t \ge t_e$.

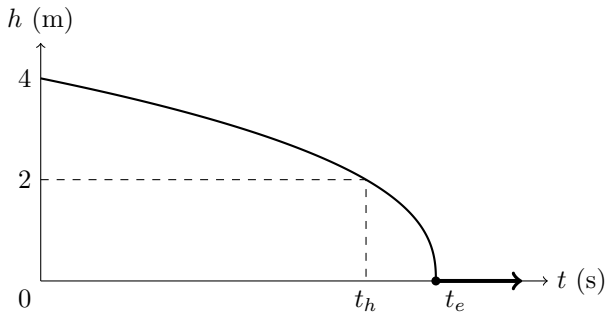
Step 3: Compare depth and volume targets. At $h = 2$ m, $h/4 = 1/2$, giving

$$\boxed{t_h/t_e = 1 - 2^{-5/2} \approx 0.8232}.$$

At half volume, $(h/4)^3 = 1/2$, so

$$\boxed{t_V/t_e = 1 - 2^{-5/6} \approx 0.4388}.$$

Half depth leaves only one eighth of the original volume, so it takes substantially longer than losing half the volume.



$t_e = 64\pi$ s; thick segment: physically empty

Step 4: Interpret the endpoint. As $t \uparrow t_e$, $h' \to -\infty$, so the zero continuation of depth is continuous but not differentiable at emptying. However, $V' = -0.05\sqrt{h} \to 0$, agreeing with $V' = 0$ afterward; volume has a continuously differentiable zero continuation. The divided depth equation is a wet-tank model and is undefined at $h = 0$. It must not be continued algebraically into a fictitious refilling branch.

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Question 7

A well-mixed pond maintains volume 1000 m^3. Water enters and leaves at 100 m^3/day. The entering water has a constant pollutant concentration $u \geq 0$ kg/m^3, which is to be selected. Pollutant also disappears inside the pond at a rate equal to 0.10 times its current mass per day. Initially the pond concentration is 0.08 kg/m^3.

For this hypothetical design problem, require the pond concentration to stay at or below 0.10 kg/m^3 throughout the first five days. Assume perfect mixing and that disappearance does not change the liquid volume.

Tasks

1. Derive the pollutant mass balance and the concentration IVP, keeping flushing and internal disappearance distinct.
2. Solve for the concentration in terms of u and verify the initial condition.
3. Determine the largest constant incoming concentration satisfying the five-day requirement, and justify checking the entire time interval.
4. Find the largest constant incoming concentration that would satisfy the same bound for all future time. Explain why it differs from the finite-horizon answer.

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Question 7 – Solution

Strategy. Separate input, outflow and internal loss, then use the solution's monotonicity to turn a whole-interval constraint into a justified endpoint test.

Step 1: Write the balances. If M is pollutant mass in kg and $c = M/1000$, then

$$M' = 100u - 100\frac{M}{1000} - 0.10M, \qquad M(0) = 80.$$

Each term is in kg/day. Dividing by the constant volume gives

$$c' = 0.10u - 0.20c, \qquad c(0) = 0.08.$$

The total loss coefficient combines flushing at 0.10 day^{-1} with internal disappearance at the same rate.

Step 2: Solve and identify monotonicity. An integrating factor $e^{0.20t}$ gives

$$\boxed{c(t) = \frac{u}{2} + (0.08 - u/2)e^{-0.20t}}, \qquad t \geq 0.$$

Its initial value is 0.08, and differentiating verifies $c' = 0.10u - 0.20c$. It is decreasing when $u < 0.16$, constant when $u = 0.16$, and increasing when $u > 0.16$.

Step 3: Enforce the five-day bound. For $u \leq 0.16$, the maximum is the acceptable initial concentration. For $u > 0.16$, the maximum on $[0, 5]$ is $c(5)$. Therefore the largest allowed input is

$$\boxed{u_5 = \frac{2(0.10 - 0.08e^{-1})}{1 - e^{-1}} \approx 0.2233 \text{ kg/m}^3}.$$

This value exceeds 0.16, so its concentration rises monotonically from 0.08 to exactly 0.10. Any larger u fails at day five, proving optimality rather than just feasibility.

Step 4: Compare a permanent operating requirement. The limiting pond concentration is $u/2$. A bound valid for all $t \geq 0$ requires and is guaranteed by

$$\boxed{u \leq 0.20 \text{ kg/m}^3}.$$

For these inputs, $c(t)$ lies between its initial value and $u/2$. If $u > 0.20$, the limit exceeds the bound, so it must eventually be violated. The larger finite-horizon input uses the initially cleaner pond as temporary dilution capacity; it cannot be continued indefinitely under the same requirement.

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Question 8

An isolated laboratory population is modeled by a per-capita growth rate that decreases linearly with population size:

$$P' = P(a - bP), \qquad a > 0,\ b > 0.$$

Treat P as a continuous number of individuals, with time in days. Under unchanged laboratory conditions, the instantaneous total growth rate is 20 individuals/day both when $P = 100$ and when $P = 200$. A new culture begins with $P(0) = 100$.

Tasks

1. Identify a and b, including units, from the two instantaneous rate observations.
2. Solve the new culture's IVP and verify the result. State the assumptions supporting use of the fitted parameters in this culture.
3. Find when the population first reaches 250 individuals and explain why this is a unique future time.
4. An exponential model is fitted only to the initial population and its initial growth rate. Compare its rate prediction at $P = 200$ with the second observation, and explain what that discrepancy says about the model.

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Question 8 – Solution

Strategy. Convert total growth observations to per-capita rates before estimating the crowding coefficient.

Step 1: Calibrate the rate law. The observations give

$$a - 100b = 0.20, \qquad a - 200b = 0.10.$$

Subtracting yields

$$\boxed{a = 0.30 \text{ day}^{-1}, \qquad b = 0.001 \text{ individual}^{-1}\text{day}^{-1}}.$$

The model becomes $P' = 0.30P(1 - P/300)$. Both specified populations give total rate 20 individuals/day.

Step 2: Solve the IVP. Separation for the branch $0 < P < 300$ gives

$$\ln \frac{P}{300 - P} = 0.30t + C.$$

Using $P(0) = 100$ gives $C = -\ln 2$ and

$$\boxed{P(t) = \frac{300}{1 + 2e^{-0.30t}}}, \qquad t \geq 0.$$

If $z = 2e^{-0.30t}$, then $P' = 90z/(1 + z)^2 = 0.30P(1 - P/300)$, and the initial value is 100. Applying the calibration assumes the same fixed resources and crowding law, no migration, and a population scale for which the continuous approximation is appropriate.

Step 3: Locate the target. Setting $P = 250$ yields $e^{-0.30t} = 1/10$, so

$$\boxed{t = \frac{\ln 10}{0.30} \approx 7.675 \text{ days}}.$$

The formula increases strictly from 100 toward 300 for $t \geq 0$, so 250 is attained exactly once. Its denominator remains positive throughout this physical time range.

Step 4: Test the competing exponential model. For $P' = rP$, the initial observation forces $r = 20/100 = 0.20$ day^{-1}. At $P = 200$, that model predicts 40 individuals/day, double the observed rate. Matching one population and one instantaneous slope does not validate a constant per-capita growth law.

The second observation distinguishes the two proposed laws. It supports the calibrated crowding model over this exponential alternative under the stated exact-data assumptions, but two rate observations alone do not establish that the model remains accurate at every population size.

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Question 9

A spherical droplet has constant density ρ and radius $r(t)$. Assume its mass disappears at a rate proportional to its current surface area, with a constant coefficient $k > 0$:

$$M = \frac{4}{3}\pi\rho r^3, \qquad M' = -k(4\pi r^2)$$

while the droplet exists. Radius decreases from 3 mm to 2 mm during the first 5 minutes. Ignore changes in shape, density and environmental conditions.

Tasks

1. Derive the radius IVP from the mass balance and identify k/ρ and its units from the observations. Can k itself be found without knowing ρ?
2. Find the extinction time and the fraction of original mass remaining at any time before extinction.
3. Determine when half the original mass remains; compare this with the time at which the radius is halved.
4. Fit an exponential mass-loss model with the same initial mass and initial mass-loss rate. Compare the two predictions at 10 minutes and discuss their extinction predictions.

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Question 9 – Solution

Strategy. Differentiate the geometric mass formula before applying the surface-loss law. A constant radius-loss rate need not imply exponential mass loss.

Step 1: Reduce the balance to radius. For $r > 0$, the chain rule gives

$$4\pi\rho r^2 r' = -4\pi k r^2, \qquad r' = -k/\rho, \qquad r(0) = 3 \text{ mm}.$$

The observed radius loss is 1 mm in 5 min, hence

$$\boxed{k/\rho = 0.20 \text{ mm/min}}, \qquad \boxed{r(t) = 3 - 0.20t \text{ mm}}.$$

If density is in mass/mm^3, k has units mass/(mm^2 min). The observations identify only the ratio; a density value is needed to determine k itself.

Step 2: Determine mass and extinction. The radius reaches zero at $\boxed{t_e = 15 \text{ min}}$. Since mass is proportional to r^3,

$$\boxed{\frac{M(t)}{M(0)} = (1 - t/15)^3}, \qquad 0 \le t \le 15.$$

The physical mass and radius remain zero afterward. Extending the linear radius formula past 15 would produce negative radius and is not part of the droplet model. Differentiating the cubic mass formula recovers the original surface-loss balance while $r > 0$.

Step 3: Compare fractional targets. Half mass occurs when $(1 - t/15)^3 = 1/2$, giving

$$\boxed{t_{1/2,M} = 15(1 - 2^{-1/3}) \approx 3.094 \text{ min}}.$$

Half radius occurs at $t = 7.5$ min, when only $(1/2)^3 = 1/8$ of the initial mass remains. Geometric scaling makes these distinct events.

Step 4: Test an exponential alternative. The initial relative mass slope from the cubic formula is $M'(0)/M(0) = -1/5 \text{ min}^{-1}$. The matching exponential is therefore

$$M_{\exp}(t) = M(0)e^{-t/5}.$$

At 10 min the surface-loss model predicts $M/M(0) = 1/27 \approx 0.0370$, whereas the exponential predicts $e^{-2} \approx 0.1353$. The exponential never reaches zero at finite time; the geometric model empties at 15 min. Agreement of initial value and slope does not imply agreement of the physical loss mechanism or its later predictions.

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Question 10

A reservoir initially contains 10 L of usable liquid. During a four-minute operation, a pump supplies a constant rate $u \geq 0$ L/min, scheduled demand removes liquid at rate $d(t) = 2 + t$ L/min, and leakage removes liquid at a rate equal to 0.50 times the current volume per minute. Here t is in minutes; the slope in $d(t)$ is 1 L/min^2.

The reservoir must retain at least 4 L at every time $0 \leq t \leq 4$. Assume no capacity restriction, no other flows, and that the stated demand and leakage laws apply while liquid remains.

Tasks

1. Derive and solve the volume IVP in terms of u.
2. Find the smallest constant pump rate that satisfies the reserve requirement throughout the operation. Justify feasibility over the whole interval, not only at its endpoint.
3. For this rate, calculate the cumulative leaked volume by $t = 4$ using an overall volume balance.
4. Compare the answer with the minimum pump rate obtained by ignoring leakage. Sketch the optimal volume and reserve level, and explain why the no-leakage design cannot meet the original requirement.

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Question 10 – Solution

Strategy. Solve the storage balance, obtain a necessary pump rate from the final reserve, and then verify that this candidate satisfies the full time constraint.

Step 1: Model and solve the storage balance. Input minus demand minus leakage gives

$$S' = u - (2 + t) - 0.50S, \qquad S(0) = 10,$$

with all rates in L/min. A particular solution is $2u - 2t$; adding $Ae^{-t/2}$ and applying $S(0) = 10$ gives

$$\boxed{S(t) = 2u - 2t + (10 - 2u)e^{-t/2}}.$$

It has initial value 10, and differentiation gives the required balance on any time interval on which the physical operation remains valid.

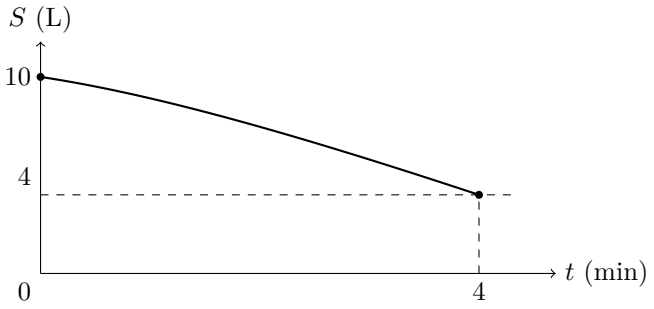
Step 2: Find and verify the minimum rate. Since $S(4) \geq 4$ is necessary,

$$\boxed{u \geq u_* = \frac{12 - 10e^{-2}}{2(1 - e^{-2})} = 6 + \frac{1}{e^2 - 1} \approx 6.157 \text{ L/min}}.$$

For this candidate, $6 < u_* < 7$ and

$$S'(t) = -2 + (u_* - 5)e^{-t/2} < 0 \qquad (0 \leq t \leq 4).$$

Thus S decreases from 10 to exactly 4 L and never violates the reserve. The candidate is feasible, and every smaller rate fails the necessary final condition, proving optimality.



Solid: optimal storage; dashed: required reserve

Step 3: Recover cumulative leakage. The scheduled demand totals $\int_0^4 (2 + t)\, dt = 16$ L. Initial storage plus pumped volume equals final storage plus demand plus leakage, so

$$\boxed{L_{\text{leak}} = 10 + 4u_* - 16 - 4 = 4u_* - 10 \approx 14.63 \text{ L}}.$$

Step 4: Compare the incomplete model. Ignoring leakage gives $S_0 = 10 + (u - 2)t - t^2/2$. This concave function has its minimum on $[0, 4]$ at an endpoint; $S_0(4) \geq 4$ gives the no-leakage minimum $u = 2.5$ L/min. That rate is below u_* and fails the actual reserve constraint. It may eventually exhaust the actual reservoir, so negative values of its unconstrained storage formula would not describe physical stored liquid. The optimal design above stays positive and avoids that limitation.

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Question 1

Consider the autonomous equation

$$y' = y(1-y)(y-3), \qquad y(0) = a \in \mathbb{R}.$$

An equilibrium is a constant solution. Call it stable if sufficiently small initial perturbations stay as small as prescribed for all future time, and asymptotically stable if it is also locally attracting. A basin of attraction is the set of initial values whose forward solutions exist for all $t \geq 0$ and tend to that equilibrium.

Tasks

1. Find all equilibria and construct a phase line, showing the sign of y' on every interval between them.
2. Classify the equilibria as asymptotically stable or unstable, explaining the role of the arrows.
3. Determine the forward limit for every real initial value a and the exact basins of the attracting equilibria.
4. Justify global forward existence and explain why a nonconstant solution cannot cross an equilibrium. Do not rely on an explicit solution formula.

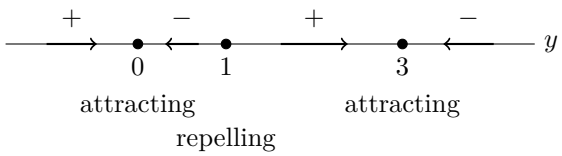
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Question 1 – Solution

Strategy. Use sign intervals and uniqueness to trap solutions; bounded monotone trajectories reveal their limits.

Step 1: Find the zeros and arrows. The only zeros of $f(y) = y(1-y)(y-3)$ are $0, 1, 3$. Its signs are

y	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
$f(y)$	$+$	$-$	$+$	$-$



Step 2: Classify the equilibria. Arrows point toward 0 and 3 on both sides, so both are asymptotically stable. The arrows on either side of 1 point away, so 1 is unstable. Nearby solutions remain between their initial value and the attracting zero, which supplies stability as well as attraction.

Step 3: Determine every forward limit. The classification is

$$\boxed{\lim_{t\to\infty} y(t) = \begin{cases} 0, & a < 1, \\ 1, & a = 1, \\ 3, & a > 1. \end{cases}}$$

In particular, the basins are $\boxed{(-\infty, 1)}$ for 0 and $\boxed{(1, \infty)}$ for 3. These include the attracting equilibria themselves. The value $a = 1$ stays exactly at the unstable equilibrium.

Step 4: Justify the global conclusions. Since f is a polynomial, local solutions are unique. A nonconstant solution cannot meet an equilibrium at a finite time: local uniqueness there, in both time directions, would force the two solutions to agree.

Every trajectory is trapped in a bounded interval between its initial value and adjacent equilibria. A smooth right-hand side permits continuation while the solution remains bounded, so all forward solutions are global. Each nonconstant trajectory is monotone and bounded. If its limit L had $f(L) \neq 0$, continuity would keep its speed bounded away from zero near L, contradicting convergence. The accessible zero in its direction of motion is therefore its limit.

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Question 2

Consider

$$y' = -(y+1)(y-2)^2.$$

Call an equilibrium semistable if it attracts from one side and repels from the other; such an equilibrium is not stable under unrestricted perturbations from both sides.

Tasks

1. Find the equilibria and construct a complete phase line, including signs on both sides of a repeated zero.
2. Classify each equilibrium and specify the attracting side of any semistable equilibrium.
3. Find the forward limits for initial values $y(0) = -2$, 1, and 3, with a justification of forward existence.
4. Evaluate f' at each equilibrium. Explain why a zero derivative is not itself a stability classification and why the repeated factor does not reverse the sign of f.

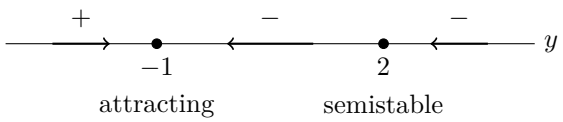
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Question 2 – Solution

Strategy. Preserve the sign across a squared factor; use one-sided attraction and repulsion when the derivative test is inconclusive.

Step 1: Build the sign chart. The equilibria are -1 and 2. Away from them, $(y-2)^2 > 0$, so the sign of f is the opposite of the sign of $y+1$:

y	$(-\infty,-1)$	$(-1,2)$	$(2,\infty)$
$f(y)$	$+$	$-$	$-$



Step 2: Classify with the sides stated. The equilibrium $\boxed{-1 \text{ is asymptotically stable}}$. At 2, solutions above it decrease toward it, while those just below decrease away toward -1. Hence

$$\boxed{2 \text{ is semistable: attracting from above, repelling from below.}}$$

It is unstable in the two-sided stability sense: however close an initial value is below 2, its trajectory eventually leaves, for example, the interval $(1, 3)$.

Step 3: Track the three initial values. Uniqueness prevents crossing either equilibrium. The solution starting at -2 increases toward -1; that starting at 1 decreases toward -1; that starting at 3 decreases toward 2. Each remains in a bounded closed interval, so the smooth equation has a global forward solution in each case. Monotone convergence and the nonzero-speed contradiction identify the stated limits.

Step 4: Interpret the derivative test. Differentiating gives $f'(y) = -(y-2)^2 - 2(y+1)(y-2)$, so

$$f'(-1) = -9 < 0, \qquad f'(2) = 0.$$

The negative derivative confirms attraction at -1. At 2, the linearized equation contains no nonzero first-order term and makes no decision. The sign chart supplies the missing information. A squared factor is positive on both sides of its zero, so it touches zero without a sign reversal.

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Question 3

Two proposed relaxation laws are

$$\text{A: } y' = -y^3, \qquad \text{B: } y' = y^3.$$

Both have an equilibrium at 0 and derivative of the right-hand side equal to zero there.

Tasks

1. Classify 0 for each law using the sign of the right-hand side. Explain why their identical derivative-test data do not imply identical stability.
2. Solve both IVPs with $y(0) = a$, including $a = 0$, and specify their maximal forward time ranges.
3. For law A, prove stability directly from the solution and determine its large-time decay for $a \neq 0$.
4. For law B, use the solution to prove instability and find the blow-up time for $a \neq 0$. Distinguish leaving a small neighborhood from blowing up.

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Question 3 – Solution

Strategy. When the linear term vanishes, retain the nonlinear term. Separation gives both a stability proof and the actual time scales.

Step 1: Classify from signs. For A, $-y^3$ is positive below 0 and negative above, so trajectories move toward 0. For B the signs reverse, so trajectories move away. Thus 0 is asymptotically stable for A and unstable for B. The shared value $f'(0) = 0$ is an inconclusive test, not evidence of neutral stability.

Step 2: Solve with the initial sign preserved. For $a \neq 0$, differentiating y^{-2} gives 2 for A and -2 for B. Applying the initial value yields

$$\boxed{y_A(t) = \frac{a}{\sqrt{1 + 2a^2 t}}, \qquad t \geq 0},$$

$$\boxed{y_B(t) = \frac{a}{\sqrt{1 - 2a^2 t}}, \qquad 0 \leq t < \frac{1}{2a^2}}.$$

The full maximal intervals containing 0 are $(-1/(2a^2), \infty)$ and $(-\infty, 1/(2a^2))$, respectively. Direct differentiation verifies the equations. If $a = 0$, the unique solution of either smooth equation is zero for all real time.

Step 3: Prove stability and attraction for A. For $t \geq 0$, $|y_A(t)| \leq |a|$. Given $\varepsilon > 0$, choosing $\delta = \varepsilon$ ensures $|a| < \delta$ implies $|y_A(t)| < \varepsilon$ for all future time. Also $y_A(t) \to 0$ for every a, so the attraction is global. More precisely,

$$\sqrt{2t}\,|y_A(t)| \longrightarrow 1 \quad (a \neq 0).$$

The decay is algebraic rather than exponential.

Step 4: Prove instability for B. Choose any fixed $\varepsilon > 0$ and any nonzero $|a| < \varepsilon$. The magnitude first reaches ε at

$$t_\varepsilon = \frac{1}{2a^2} - \frac{1}{2\varepsilon^2} > 0,$$

then exceeds it. Arbitrarily small nonzero perturbations therefore leave that neighborhood. Blow-up occurs later, at $1/(2a^2)$, with the sign of a. Instability only requires departure from a prescribed neighborhood; it does not require an infinite solution value.

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Question 4

Construct a real polynomial autonomous equation $y' = f(y)$ with exactly the equilibria $-2, 0, 3$ and these properties:

- -2 is asymptotically stable;
- 0 attracts from above and repels from below;
- 3 is unstable;
- $f(1) = -6$.

Tasks

1. Determine the smallest possible degree, using the sign changes required at the three zeros.
2. Construct the polynomial of that degree and prove uniqueness among polynomials of that degree satisfying all the requirements.
3. Draw its phase line and verify each requested stability property.
4. Find the forward limits for $y(0) = -1$ and $y(0) = 1$. Explain why knowing the equilibrium locations alone would not determine these limits.

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Question 4 – Solution

Strategy. Translate attraction and repulsion into signs, then into the parity of each root's multiplicity.

Step 1: Establish the minimum degree. At a two-sided attracting or repelling equilibrium, the sign must reverse, requiring an odd root multiplicity. At the specified semistable zero, the sign must be negative on both sides, requiring even multiplicity. The smallest multiplicities at $-2, 0, 3$ are therefore $1, 2, 1$, so the degree is at least 4.

Step 2: Determine the coefficient. Every degree-four candidate must have the form

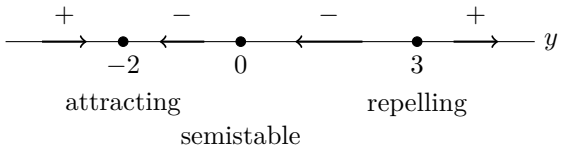
$$f(y) = K(y+2)y^2(y-3), \qquad K \neq 0.$$

At $y = 1$ this gives $f(1) = -6K$. The prescribed rate forces $K = 1$, hence

$$\boxed{y' = (y+2)y^2(y-3)}.$$

This has exactly the three requested real zeros. The forced multiplicities exhaust degree four, and the rate condition fixes the only remaining coefficient, proving uniqueness at the minimum degree.

Step 3: Verify the phase line. The signs are $+, -, -, +$ on the four successive intervals cut by $-2, 0, 3$.



Thus arrows approach -2 from both sides; near 0 they point left on both sides, attracting from above and repelling from below; and they point away from 3 on both sides. These are precisely the requested behaviors.

Step 4: Infer the selected limits. Starting at -1, the solution decreases within $(-2, 0)$ and tends to $\boxed{-2}$. Starting at 1, it decreases within $(0, 3)$ and tends to $\boxed{0}$. Smoothness and bounded trapping ensure global forward existence for both, and uniqueness prevents crossing the bounding equilibria.

Equilibrium locations only identify zeros. Changing root multiplicities or the overall sign can change the arrows and therefore the limiting behavior, even when the zero set is unchanged. The stability and rate requirements supply information that the locations alone do not.

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Question 5

Consider the one-parameter family

$$y' = \mu - y^2, \qquad \mu \in \mathbb{R}.$$

Tasks

1. Find and classify all equilibria for $\mu < 0$, $\mu = 0$, and $\mu > 0$. State the one-sided behavior at any semistable case.
2. Draw the equilibrium branches in the (μ, y)-plane, distinguishing attracting and repelling branches and marking the exceptional parameter value.
3. Solve the IVP $y(0) = 0$ for each parameter regime and determine its forward behavior and maximal forward time range.
4. Explain how the number and stability of equilibria change at $\mu = 0$, and why the derivative test alone does not settle that parameter value.

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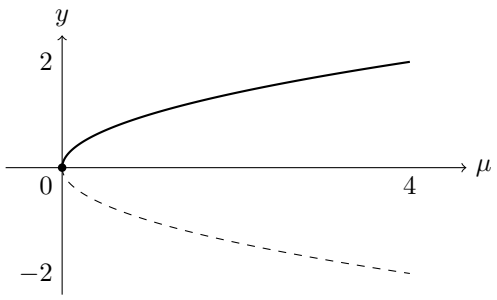
Question 5 – Solution

Strategy. Compare a horizontal parameter with y^2, and treat the double zero separately from the two simple roots.

Step 1: Classify the parameter regimes. If $\mu < 0$, there are no equilibria and $y' < 0$ everywhere. If $\mu = 0$, the only equilibrium is 0: $-y^2 < 0$ on both sides, so it attracts from above and repels from below. If $\mu > 0$, the equilibria are

$$\boxed{y = -\sqrt{\mu}\ \text{(unstable)}, \qquad y = \sqrt{\mu}\ \text{(asymptotically stable)}}.$$

Indeed $\mu - y^2$ is positive between the roots and negative outside them. The derivative $f_y = -2y$ confirms the classifications at the simple roots.



Solid: attracting; dashed: repelling; origin: semistable

Step 2: Solve the selected IVP. Separation and $y(0) = 0$ give the following cases, with $a > 0$ as indicated:

parameter	$y(t)$	forward time range
$\mu > 0,\ a = \sqrt{\mu}$	$a(1 - e^{-2at})/(1 + e^{-2at})$	$0 \le t < \infty$
$\mu = 0$	0	$0 \le t < \infty$
$\mu < 0,\ a = \sqrt{-\mu}$	$-a\tan(at)$	$0 \le t < \pi/(2a)$

For $\mu > 0$, integration on $-a < y < a$ gives $\ln((a + y)/(a - y)) = 2at$. For the last case, integration gives $\arctan(y/a) = -at$ on the initial branch. Differentiating the displayed expressions verifies $y' = \mu - y^2$ and their zero initial values.

Step 3: Describe the future. For $\mu > 0$, the selected solution increases to $\sqrt{\mu}$; at $\mu = 0$ it stays at zero; for $\mu < 0$ it decreases to $-\infty$ as $t \uparrow \pi/(2a)$, so no finite continuous extension exists.

Step 4: Interpret the branch merger. As μ decreases to zero, the attracting and repelling equilibria merge into a semistable double zero; for negative μ neither remains. At the merger $f_y(0) = 0$, so the derivative test is inconclusive. The sign of $-y^2$ supplies the one-sided classification.

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Question 6

For a fixed real constant L, compare the two equations on $t \geq 0$:

$$\text{A: } y' = -\frac{y-L}{(1+t)^2}, \qquad \text{B: } y' = -\frac{y-L}{1+t}.$$

Their right-hand sides depend on time. Nevertheless, at every finite time each points toward L whenever $y \neq L$. Use stability relative to the initial time $t = 0$.

Tasks

1. Find every constant solution of each equation and explain why these are equilibria despite the explicit time dependence.
2. Solve both IVPs with $y(0) = a$ and verify the initial values and differential equations.
3. Decide whether L is stable and whether it is asymptotically stable for each equation. Prove the claims directly from the solutions.
4. Explain why arrows pointing toward L at every finite time do not guarantee attraction in both models. Relate the distinction to the accumulated coefficient integral.

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Question 6 – Solution

Strategy. Measure the displacement from L. Its reduction depends on the total accumulated damping, not just the sign at each instant.

Step 1: Identify the constant solutions. A constant $y = c$ must make the right-hand side zero for every $t \geq 0$. Both coefficients are strictly positive, so the only choice in either equation is $\boxed{c = L}$. Explicit time dependence does not preclude a constant solution.

Step 2: Solve the displacement equations. Let $z = y - L$, so $z(0) = a - L$. For A,

$$\int_0^t \frac{ds}{(1+s)^2} = 1 - \frac{1}{1+t} = \frac{t}{1+t}, \qquad \boxed{y_A = L + (a-L)e^{-t/(1+t)}}.$$

For B, the corresponding integral is $\ln(1+t)$, giving

$$\boxed{y_B = L + \frac{a-L}{1+t}}.$$

Both formulas have value a at zero. Their displacement derivatives are exactly the prescribed negative coefficient times their displacement, verifying the equations. They exist for all $t \geq 0$.

Step 3: Separate stability from attraction. For both models, $|y(t) - L| \leq |a - L|$. Given $\varepsilon > 0$, $\delta = \varepsilon$ proves stability at initial time zero. But

$$\lim_{t\to\infty} y_A = L + (a-L)e^{-1}, \qquad \lim_{t\to\infty} y_B = L.$$

Thus L is **stable but not asymptotically stable for A**; no other initial value converges to L. For B, L is **globally asymptotically stable**.

Step 4: Explain the limitation of the arrow argument. In a scalar equation $z' = -b(t)z$, the solution is $z(0)\exp(-\int_0^t b(s)\,ds)$. A has positive but integrable damping, with total integral 1; a fixed fraction of any nonzero initial displacement remains. B has divergent accumulated damping and removes the displacement completely.

For a continuous autonomous equation, a non-equilibrium limiting value would have nonzero limiting speed. Here the coefficient can tend to zero with time, so that autonomous argument does not apply. Instantaneous inward arrows alone are insufficient for a time-dependent equation.

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Question 7

Consider the continuous, non-Lipschitz equation

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0, \qquad t \geq 0.$$

When solutions are not unique, call an equilibrium stable only if every forward solution from every sufficiently close initial value remains within any prescribed neighborhood.

Tasks

1. Find all equilibria and determine the sign of the right-hand side away from them.
2. For each $c \geq 0$, construct a solution that stays at zero through time c and then becomes positive. Verify differentiability at the joining time.
3. Construct a continuously differentiable solution with $y(0) = -1$ that crosses the equilibrium level at $t = 1$, and sketch it.
4. Prove that the equilibrium is unstable under the stated convention. Explain precisely why the usual prohibition against crossing an equilibrium does not apply.

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Question 7 – Solution

Strategy. Verify explicit joined solutions instead of invoking a uniqueness theorem whose hypothesis fails.

Step 1: Find the zero and signs. The only equilibrium is $y = 0$. The right-hand side is positive for every $y \neq 0$, so motion is upward on either side. Below zero the direction is toward the equilibrium level, but arrival need not force the solution to stay there.

Step 2: Verify delayed departures. For any $c \geq 0$, define

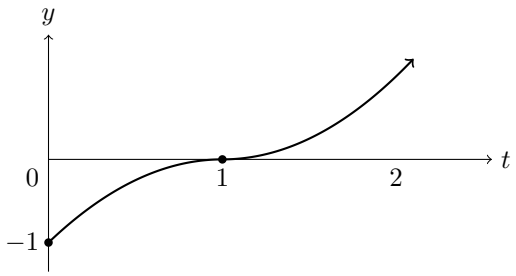
$$\boxed{y_c(t) = \begin{cases} 0, & 0 \leq t \leq c, \\ (t-c)^2, & t \geq c. \end{cases}}$$

Before c both sides of the equation vanish; afterward $y_c' = 2(t-c) = 2\sqrt{|y_c|}$. At c, both values and both one-sided derivatives are zero. The difference quotient also gives derivative zero, so the joined solution is continuously differentiable. The identically zero function is another solution.

Step 3: Cross from below. A crossing solution is

$$\boxed{y(t) = \begin{cases} -(1-t)^2, & 0 \leq t \leq 1, \\ (t-1)^2, & t \geq 1. \end{cases}}$$

Before 1, $y' = 2(1-t) = 2\sqrt{|y|}$; afterward the same check as above applies. Both derivatives at the join are zero, and $y(0) = -1$.



A classical solution crosses $y = 0$ at $t = 1$

Step 4: Diagnose instability and failed uniqueness. For any $\delta > 0$, initial value 0 is within δ of the equilibrium, yet the solution $y_0(t) = t^2$ leaves the fixed neighborhood $(-1, 1)$ after $t = 1$. This violates the stated requirement for every forward solution, so the equilibrium is unstable.

Local Lipschitz regularity fails since $|f(y) - f(0)|/|y| = 2/\sqrt{|y|}$ is unbounded near zero. The crossing and constant solutions agree at $(1, 0)$ but have different continuations. The usual noncrossing proof requires local uniqueness; continuity alone does not provide it.

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Question 8

Consider the autonomous equation with its stated domain:

$$y' = \frac{y-1}{y+2}, \qquad y \neq -2.$$

Tasks

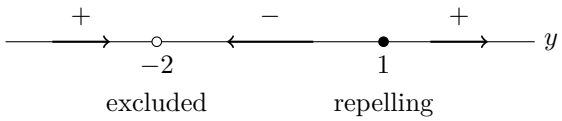
1. Identify all equilibria and construct a phase line that distinguishes an excluded level from a zero of the right-hand side.
2. Classify each equilibrium and determine the direction of motion in every component cut by the equilibrium and the excluded level.
3. For $y(0) = 0$, derive an implicit solution relation and find the exact maximal forward time range.
4. Determine the limiting value and slope at its finite endpoint. Explain why approaching a finite level does not make that level an attracting equilibrium and why multiplying by $y + 2$ does not remove the obstruction for this original equation.

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Question 8 – Solution

Strategy. A denominator zero splits the domain but is not an equilibrium. Test whether the selected trajectory reaches that boundary in finite time.

Step 1: Separate zeros from exclusions. The only equilibrium is $\boxed{y = 1}$. The value $y = -2$ is outside the equation's domain. The signs of f are positive on $(-\infty, -2)$, negative on $(-2, 1)$, and positive on $(1, \infty)$.



Thus 1 is unstable: arrows point away on both sides. Also $f'(y) = 3/(y+2)^2$, so $f'(1) = 1/3 > 0$ confirms the classification.

Step 2: Integrate the selected branch. Starting at 0, the solution decreases within $(-2, 1)$. Separation gives

$$dt = \frac{y+2}{y-1}\,dy, \qquad \boxed{t = y + 3\ln(1-y)}.$$

The constant is zero from $y(0) = 0$. On $-2 < y \le 0$, the derivative of the right side with respect to y is $(y+2)/(y-1) < 0$. It is therefore invertible along this branch, and implicit differentiation verifies the original equation.

Step 3: Determine the endpoint. As $y \downarrow -2$, the time tends to

$$\boxed{T = -2 + 3\ln 3 \approx 1.296}, \qquad \boxed{0 \le t < T}.$$

This is the maximal forward range: the inverse exists until T, but at T its limiting value is excluded. Within the branch,

$$y(t) \to -2^+, \qquad y'(t) = \frac{y-1}{y+2} \to -\infty.$$

Step 4: Interpret the domain boundary. An attracting equilibrium must itself define a constant solution of the given equation. Here -2 does not, and the selected solution reaches the boundary in finite time with unbounded slope rather than converging to a valid equilibrium as $t \to \infty$.

Multiplication gives $(y+2)y' = y-1$, equivalent to the original only when $y \neq -2$. At $y = -2$ it would demand $0 = -3$ for any finite derivative. Thus even the multiplied relation supplies no differentiable continuation through this boundary.

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Question 9

Let

$$V(y) = \frac{1}{4}(y^2 - 1)^2, \qquad y' = -V'(y).$$

Think of V as an energy function; no physical units are assumed.

Tasks

1. Derive the autonomous equation explicitly, find its equilibria, and classify them using the signs of the right-hand side.
2. Compute $dV(y(t))/dt$ along a solution. Identify exactly when this derivative vanishes.
3. Sketch V and relate its local minima and maximum to equilibrium stability. Explain why nonincreasing energy alone does not imply convergence to a global minimum for every initial value.
4. Determine the forward limit for every real initial value and justify global forward existence. State the basins of the attracting equilibria.

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Question 9 – Solution

Strategy. Combine the energy identity with the phase line; energy decrease is informative but does not by itself choose a limiting equilibrium.

Step 1: Derive the flow and classify its zeros. Since $V'(y) = y(y^2 - 1)$,

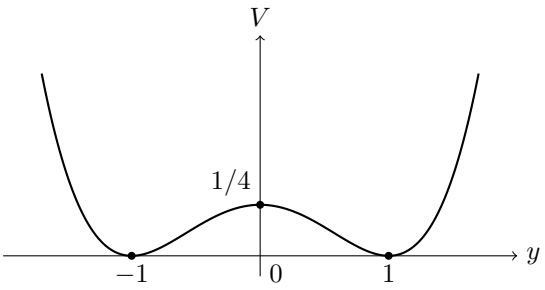
$$\boxed{y' = y - y^3 = y(1 - y^2)}.$$

Its zeros are $-1, 0, 1$. Its signs on the successive intervals are $+, -, +, -$, so -1 and 1 are asymptotically stable and 0 is unstable.

Step 2: Differentiate the energy. The chain rule gives

$$\boxed{\frac{dV}{dt} = V'(y)y' = -[V'(y)]^2 = -(y - y^3)^2 \leq 0}.$$

The derivative vanishes exactly at $y = -1, 0, 1$. On any nonconstant trajectory it is strictly negative, since uniqueness prevents meeting an equilibrium at a finite time.



Energy minima: ± 1; local maximum: 0

Step 3: Interpret the energy landscape. The points ± 1 are minima with $V = 0$; 0 is a local maximum with $V = 1/4$. Since the flow is down the energy gradient, the minima attract and the maximum repels. Nevertheless, the initial value $y(0) = 0$ stays at that maximum forever, with constant energy. Nonincreasing energy does not force every solution to reach a global minimum, nor distinguish which minimum will be selected.

Step 4: Determine the basins and future limits. The decreasing-energy bound confines a trajectory to $\{y : V(y) \leq V(y(0))\}$, a bounded set because $V(y) \to \infty$ as $|y| \to \infty$. Smoothness then ensures global forward continuation. Phase-line trapping gives bounded monotonic motion for every nonconstant solution. Its limit must be a zero of $y - y^3$, and it cannot cross zero. Therefore

$$\boxed{\lim_{t\to\infty} y(t) = \begin{cases} -1, & y(0) < 0, \\ 0, & y(0) = 0, \\ 1, & y(0) > 0. \end{cases}}$$

The basins are $\boxed{(-\infty, 0)}$ for -1 and $\boxed{(0, \infty)}$ for 1.

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Question 10

Compare two autonomous equations on the real line:

$$\text{A: } y' = -y, \qquad \text{B: } y' = -\frac{y}{1+y^2}.$$

For the quantitative comparison, use the same initial value $y(0) = a > 0$.

Tasks

1. Compare the equilibrium sets, phase-line arrows and derivatives of the right-hand sides at the equilibria. Prove the stability classification for both equations.
2. Find the solution of A and an implicit relation for B. Justify global forward existence and convergence for B.
3. Derive the time required to decrease from a to $a/2$ for each equation. Explain how the difference depends on a.
4. For $a = 2$, sketch both trajectories and determine which is larger for every $t > 0$. Explain what identical equilibrium and local derivative data fail to specify.

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Question 10 – Solution

Strategy. Multiplication by a positive state-dependent factor preserves the direction of motion, but changes the time spent along the same phase-line path.

Step 1: Compare equilibrium information. Both equations have only the equilibrium 0, with positive velocity below zero and negative velocity above. Both right-hand-side derivatives at zero equal -1. In each model, a solution is trapped between its initial value and zero, which proves stability. Smoothness, boundedness and monotonicity give global forward existence and convergence to the only accessible zero. Thus both equilibria are globally asymptotically stable.

Step 2: Solve the selected branches. For A, $\boxed{y_A = ae^{-t}}$. For B with $y > 0$, separation gives

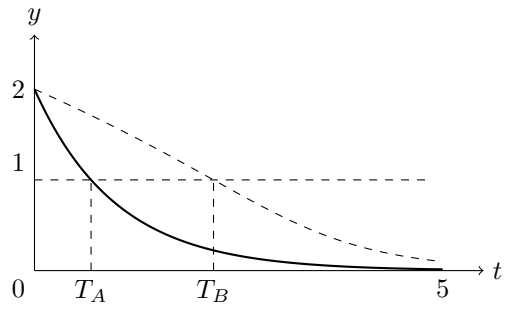
$$dt = -(1/y + y)dy, \qquad \boxed{t = \ln(a/y) + \frac{a^2 - y^2}{2}}.$$

The right side decreases strictly with y on $(0, a]$, is zero at a, and tends to infinity as $y \downarrow 0$. It therefore defines a unique positive solution for every $t \geq 0$. Implicit differentiation recovers B and confirms that zero is approached only in infinite time.

Step 3: Compare halving times. Setting $y = a/2$ yields

$$\boxed{T_A = \ln 2, \qquad T_B = \ln 2 + \frac{3a^2}{8}}.$$

The extra time in B grows quadratically with the initial amplitude. It tends to zero as $a \downarrow 0$, consistent with the identical local derivatives, but can be arbitrarily large for large a.



Solid: A; dashed curve: B; initial value $a = 2$

Step 4: Order the trajectories. For $t > 0$, the B solution satisfies $0 < y_B < a$, so its implicit equation gives $t > \ln(a/y_B)$. Hence $\boxed{y_B > ae^{-t} = y_A}$. The graph uses $a = 2$, for which $T_B = \ln 2 + 3/2$.

The factor $1/(1 + y^2)$ is positive and equals 1 at zero, preserving phase directions and the local linear rate. Away from zero it slows the motion. Equilibria and local stability data alone do not specify finite-amplitude travel times or entire solution curves.

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Question 1

Consider

$$y' = t - y, \qquad y(0) = 1.$$

For a step h, explicit Euler uses $t_{n+1} = t_n + h$ and $Y_{n+1} = Y_n + hf(t_n, Y_n)$, where Y_n approximates $y(t_n)$. Use exact arithmetic in intermediate steps unless rounding is explicitly requested.

Tasks

1. Starting at $t_0 = 0$, use $h = 1/4$ to compute every Euler value through $t = 1$. Include the slope used on each step.
2. Find the exact solution and the signed endpoint error $Y_4 - y(1)$, reporting its numerical value to six decimal places.
3. Draw the Euler polygon and the exact solution on $[0, 1]$. Explain what the slope of each straight segment represents.
4. A program obtains $Y_1 = 0.8125$ by evaluating $t - y$ at (t_1, Y_0). Identify the error and show the correct first update. Is that program implementing the stated Euler formula?

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Question 1 – Solution

Strategy. Evaluate each slope at the current numerical point, then move both time and the approximation to the next node.

Step 1: Compute the four updates. The recurrence is $Y_{n+1} = Y_n + (t_n - Y_n)/4$. Its exact values are

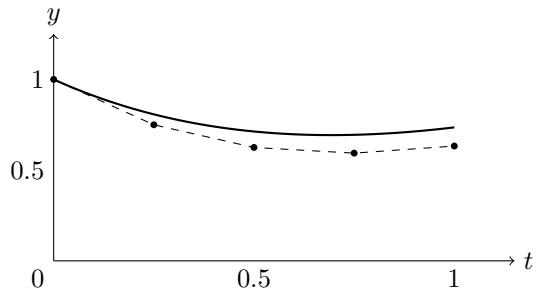
n	t_n	Y_n	$f(t_n, Y_n)$
0	0	1	-1
1	1/4	3/4	$-1/2$
2	1/2	5/8	$-1/8$
3	3/4	19/32	5/32
4	1	81/128	not needed

Thus $\boxed{Y_4 = 81/128 = 0.6328125}$.

Step 2: Check against the exact IVP. Solving $y' + y = t$ with integrating factor e^t gives

$$\boxed{y = t - 1 + 2e^{-t}}, \qquad \boxed{Y_4 - y(1) = \frac{81}{128} - \frac{2}{e} \approx -0.102946}.$$

The exact formula has $y(0) = 1$ and derivative $1 - 2e^{-t} = t - y$, so it is the correct comparison solution.



Solid: exact; dashed with nodes: Euler, $h = 1/4$

Step 3: Interpret the polygon. On $[t_n, t_{n+1}]$, the Euler segment has slope $f(t_n, Y_n)$. It is tangent to the solution through the numerical point (t_n, Y_n), which generally differs from the original exact solution after the first step. The polygon is continuous but usually has corners at its nodes.

Step 4: Diagnose the time index. The correct first update is $1 + (1/4)(0 - 1) = 0.75$. The program instead uses $1 + (1/4)(1/4 - 1) = 0.8125$, mixing the new time with the old state. It is not the stated explicit Euler method, which requires both arguments from the same current node.

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Question 2

For $y' = y$, $y(0) = 1$, apply explicit Euler on $[0, 1]$ with N equal steps and $h = 1/N$. Let Y_N denote its endpoint value. You may use $\ln(1 + h) = h - h^2/2 + O(h^3)$ and $e^z = 1 + z + O(z^2)$ as the arguments tend to zero.

Tasks

1. Derive a closed formula for Y_N and prove that it underestimates the exact endpoint for every positive integer N.
2. Compute the one-step defect when Euler starts from an exact value $y(t_n)$. Show that this defect is of order h^2.
3. Derive the leading endpoint error as $h \to 0$ and explain why its order differs from that of the one-step defect.
4. Calculate endpoint values and absolute errors for $N = 2, 4, 8$. Explain what error reduction is expected when N doubles and what the computation cost does.

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Question 2 – Solution

Strategy. Solve the discrete recurrence exactly, then compare its logarithm with the continuous exponential growth.

Step 1: Find and bound the endpoint. Euler gives $Y_{n+1} = (1+h)Y_n$, so

$$\boxed{Y_N = (1+1/N)^N}.$$

The exact solution is e^t. Since $\ln(1+h) < h$ for $h > 0$, $N\ln(1+1/N) < 1$ and hence $Y_N < e$. This is an underestimate for every grid, not merely for the three grids below.

Step 2: Define the one-step defect. Starting with the exact value e^{t_n}, the difference between the exact next value and one Euler step is

$$d_n = e^{t_n+h} - e^{t_n}(1+h) = e^{t_n}(e^h - 1 - h) = \tfrac{1}{2}e^{t_n}h^2 + O(h^3).$$

On $0 \le t_n \le 1$, this is uniformly $O(h^2)$. Here defect means an error in solution value; dividing it by h would use a different convention.

Step 3: Derive the accumulated error. Because $Nh = 1$,

$$\ln Y_N = \frac{\ln(1+h)}{h} = 1 - \frac{h}{2} + O(h^2).$$

Exponentiating gives $Y_N = e(1 - h/2 + O(h^2))$, so

$$\boxed{e - Y_N = \frac{e}{2}h + O(h^2)}.$$

There are $N = 1/h$ steps over the fixed interval, and their defects propagate through later updates. An $O(h^2)$ one-step defect therefore need not yield an $O(h^2)$ endpoint error; here the accumulated error is first order.

Step 4: Compare the grids.

N	Y_N	$e - Y_N$
2	2.250000	0.468282
4	2.441406	0.276876
8	2.565785	0.152497

The errors approach a factor-of-two reduction under each doubling of N, because their leading term is proportional to h. Exact halving is not promised on coarse grids. The number of slope evaluations doubles, exposing the first-order accuracy versus work tradeoff.

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Question 3

Consider a decaying quantity satisfying

$$y' = -4y, \qquad y(0) = 1.$$

Apply explicit Euler with a constant step $h > 0$ for arbitrarily many steps.

Tasks

1. Derive the numerical solution Y_n and determine exactly which steps give $Y_n \to 0$ as $n \to \infty$.
2. Determine the stricter step condition that keeps every numerical value nonnegative and the sequence nonincreasing. Discuss $h = 1/4$ and $h = 1/2$ separately.
3. Compute enough values for $h = 0.4$ and $h = 0.6$ to show their contrasting behavior, and compare them graphically with the exact decay.
4. Explain why decreasing the physical solution's magnitude does not by itself ensure decay of an Euler approximation, and distinguish stability from faithful positivity.

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Question 3 – Solution

Strategy. Analyze the discrete amplification factor rather than transferring the differential equation's stability directly to its numerical update.

Step 1: Solve the recurrence. The update is $Y_{n+1} = (1-4h)Y_n$, so $\boxed{Y_n = (1-4h)^n}$. It tends to zero exactly when

$$|1-4h| < 1, \qquad \boxed{0 < h < 1/2}.$$

Step 2: Impose positivity and monotonicity. All values are nonnegative and nonincreasing exactly when $0 \le 1-4h < 1$, or

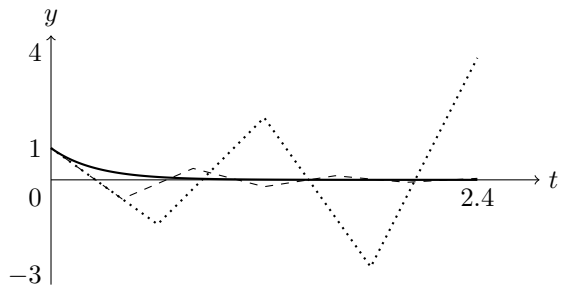
$$\boxed{0 < h \le 1/4}.$$

At $h = 1/4$, the approximation becomes zero after one step, although e^{-4t} is strictly positive at every finite time. At $h = 1/2$, $Y_n = (-1)^n$ oscillates without decay. For $h > 1/2$, its magnitude grows without bound.

Step 3: Compare two oscillating sequences.

n	0	1	2	3	4
$h = 0.4$	1	-0.6	0.36	-0.216	0.1296
$h = 0.6$	1	-1.4	1.96	-2.744	3.8416

Each row is plotted at its own times $t_n = nh$, not at common node indices interpreted as time.



Solid: exact; dashed: $h = 0.4$; dotted: $h = 0.6$

Step 4: Interpret numerical stability. The exact solution $y = e^{-4t}$ decays smoothly. Euler replaces its per-step factor e^{-4h} by $1-4h$, which can be negative or have magnitude exceeding one. At $h = 0.4$ the method is asymptotically decaying but produces unphysical sign changes; at $h = 0.6$ it also loses decay. Stability controls long-run amplification, while positivity is an additional property with a stricter step requirement.

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Question 4

A forced response obeys

$$y' = g(t) - y, \qquad y(0) = 0, \qquad g(t) = \begin{cases} 1, & 0 \le t < 0.6, \\ 3, & t \ge 0.6. \end{cases}$$

The solution is continuous at the forcing switch; the differential equation is solved classically on each side. On any grid use $Y_{n+1} = Y_n + (t_{n+1} - t_n)[g(t_n) - Y_n]$.

Tasks

1. Compute Euler values on grid A: $0, 0.4, 0.8, 1$.
2. Compute Euler values on grid B: $0, 0.4, 0.6, 1$. Use the specified value of g at the switching node.
3. Derive the exact solution by matching the two phases and compare the two absolute endpoint errors.
4. Draw both Euler polygons and the exact solution. Explain why including the switch is important, yet does not guarantee the smaller endpoint error for these particular grids.

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Question 4 – Solution

Strategy. Distinguish resolving an event from controlling the numerical error made within each phase.

Step 1: Compute grid A. The step lengths are $0.4, 0.4, 0.2$. Left-node evaluation gives

$$Y(0.4) = 0.4, \quad Y(0.8) = 0.4 + 0.4(1 - 0.4) = 0.64, \quad \boxed{Y_A(1) = 0.64 + 0.2(3 - 0.64) = 1.112}.$$

The second step continues using $g = 1$ across an interval whose forcing actually changes partway through.

Step 2: Compute grid B. The step lengths are $0.4, 0.2, 0.4$. Thus

$$Y(0.4) = 0.4, \quad Y(0.6) = 0.4 + 0.2(1 - 0.4) = 0.52, \quad \boxed{Y_B(1) = 0.52 + 0.4(3 - 0.52) = 1.512}.$$

At $t = 0.6$ the prescribed forcing is 3, so the final step uses the new phase.

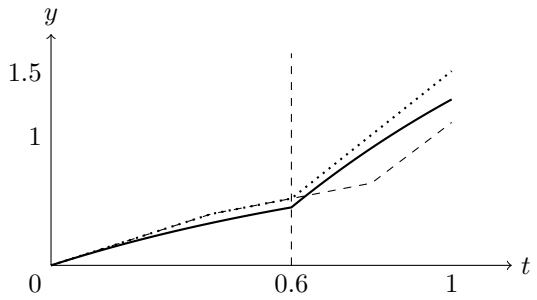
Step 3: Obtain the correct comparison value. The exact phase solutions are

$$y(t) = \begin{cases} 1 - e^{-t}, & 0 \le t \le 0.6, \\ 3 + (-2 - e^{-0.6})e^{-(t-0.6)}, & t \ge 0.6. \end{cases}$$

They agree at the switch and satisfy their respective equations. Consequently

$$y(1) = 3 - 2e^{-0.4} - e^{-1} \approx 1.291480,$$

so the absolute errors are $\boxed{0.179480 \text{ for A}, \quad 0.220520 \text{ for B}}$.



Solid: exact; dashed: grid A; dotted: grid B

Step 4: Explain the apparently better unaligned grid. Grid A undercounts the switched forcing, while grid B uses it correctly but takes a larger last step and overestimates the concave-down response. Here A happens to have the smaller endpoint error. Event alignment removes a known forcing mismatch, but does not order total errors on different coarse grids. Accurate computation also requires sufficiently small steps within the smooth phases.

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Question 5

A normalized population satisfies

$$p' = 2p(1-p), \qquad 0 \le p(0) \le 1.$$

The exact solution preserves the physical interval $[0,1]$. For explicit Euler with step $h > 0$, write $P_{n+1} = F_h(P_n)$. For the local discrete comparison, a small perturbation near $P = 1$ is multiplied to first order by $F_h'(1)$; call the linearized update contracting when $|F_h'(1)| < 1$.

Tasks

1. Derive F_h and find exactly which positive steps map all of $[0,1]$ back into $[0,1]$.
2. Prove the condition is necessary, not merely sufficient, by producing an overshooting initial value whenever it fails.
3. Compute the first two Euler updates for $h = 1$, $P_0 = 3/4$, and explain why they misrepresent the exact solution's qualitative behavior.
4. Find the step range for linearized contraction at 1. Compare it with interval preservation, and sketch the update maps for $h = 1/4$ and $h = 1$.

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Question 5 – Solution

Strategy. Check the whole physical interval, not only the behavior of tiny perturbations near its upper endpoint.

Step 1: Factor the interval constraint. Euler gives $F_h(p) = p + 2hp(1-p)$. On $[0,1]$ it is nonnegative. Also

$$1 - F_h(p) = (1-p)(1-2hp).$$

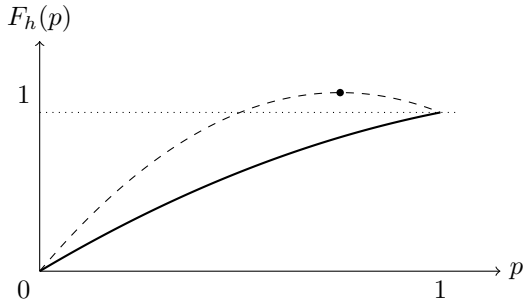
If $0 < h \le 1/2$, both factors are nonnegative for every $p \in [0,1]$, so $F_h(p) \le 1$. Therefore $\boxed{0 < h \le 1/2}$ preserves the interval at every step by induction.

Step 2: Prove necessity. If $h > 1/2$, choose p strictly between $1/(2h)$ and 1. Then $1 - p > 0$ but $1 - 2hp < 0$, so $F_h(p) > 1$. Such a value exists for every $h > 1/2$, proving the stated condition is exact.

Step 3: Exhibit overshoot. For $h = 1$, $F_1(p) = 3p - 2p^2$, giving

$$\boxed{P_1 = 9/8 = 1.125, \qquad P_2 = 27/32 = 0.84375}.$$

The numerical solution crosses above the capacity and then drops. The exact solution starting at $3/4$ increases toward 1 from below and cannot cross that equilibrium, by uniqueness.



Solid: $h = 1/4$; dashed: $h = 1$; dotted: upper bound

Step 4: Distinguish two step restrictions. Since $F_h'(p) = 1 + 2h - 4hp$, $F_h'(1) = 1 - 2h$. Linearized contraction therefore holds for $\boxed{0 < h < 1}$, a larger range than interval preservation. At $h = 1$ the linearized magnitude equals one, so the strict contraction test makes no decision about nonlinear asymptotic behavior.

A locally contracting update can still move some valid states above the physical bound. The map criterion protects the entire interval; the derivative criterion only describes first-order behavior near one fixed point.

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Question 6

Consider

$$y' = y^2, \qquad y(0) = 1.$$

Two algorithms use a grid with $h = 1/4$. Algorithm A applies explicit Euler directly to y. Algorithm B first sets $u = 1/y$, applies explicit Euler to the transformed equation, and then recovers $y = 1/u$ wherever possible.

Tasks

1. Derive both update rules and compute their approximations at $t = 1/4$ and $t = 1/2$.
2. Find the exact IVP solution and determine which algorithm is exact at these nodes. Explain why this happens here.
3. Starting from a common positive value y with $hy < 1$, compare the two one-step updates algebraically and find their exact difference.
4. Explain what happens to algorithm B at $t = 1$ and why finite direct-Euler values at or beyond that time cannot establish continuation of the original IVP.

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Question 6 – Solution

Strategy. A nonlinear change of dependent variable preserves the exact differential equation but generally does not commute with a finite Euler step.

Step 1: Derive and compute the algorithms. Direct Euler gives $Y_{n+1} = Y_n + hY_n^2$, $Y_0 = 1$. Since $u' = -y'/y^2 = -1$, transformed Euler gives $U_{n+1} = U_n - h$, $U_0 = 1$. Thus

t	A: direct Euler	B: $1/U_n$
$1/4$	$5/4$	$4/3$
$1/2$	$105/64 = 1.640625$	2

Step 2: Compare with the exact solution. Separation gives $\boxed{y = 1/(1-t)}$, whose maximal interval containing zero is $(-\infty, 1)$. Algorithm B is exact at every grid node before 1, because its transformed equation has the constant slope -1: Euler integrates that affine solution exactly. Algorithm A underestimates the two displayed exact values.

Step 3: Compare a single step. From the same positive state, the updates are

$$A(y) = y + hy^2, \qquad B(y) = \frac{1}{1/y - h} = \frac{y}{1 - hy}.$$

Their exact difference is

$$\boxed{B(y) - A(y) = \frac{h^2 y^3}{1 - hy} > 0 \quad (hy < 1)}.$$

For fixed y, the discrepancy is of order h^2 as $h \to 0$, although both are Euler approximations in their chosen coordinates. The nonlinear reciprocal map changes the finite-step rule.

Step 4: Respect the original blow-up. At $t = 1$, algorithm B has $U_4 = 0$, so its inverse is undefined, matching the exact pole. The direct recurrence uses only addition and multiplication and gives a finite real value after any fixed finite number of exact-arithmetic steps. Those values are numerical iterates, not proof of a classical solution through the pole.

The original solution tends to $+\infty$ as $t \uparrow 1$, so it cannot extend continuously across that time. Beyond 1, an inverse value from negative U_n would likewise belong to a disconnected formula branch, not a continuation of this IVP.

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Question 7

For $y' = 1 - y$, $y(0) = 0$, use explicit Euler to reach $t = 1$ with N equal steps, $h = 1/N$. Assume exact arithmetic and define the signed error $E_n = Y_n - y(t_n)$. You may use $0 \le e^{-h} - 1 + h \le h^2/2$ for $0 < h \le 1$.

Tasks

1. Find the exact solution and a closed formula for the Euler sequence. Determine the sign of E_n.
2. Derive an error recurrence using the one-step defect evaluated from the exact solution.
3. Prove $0 \le E_n \le (h/2)[1 - (1 - h)^n] \le h/2$. Use the simpler last bound to choose an integer N guaranteeing endpoint error at most 0.005.
4. For your chosen N, calculate the actual endpoint error from the closed formulas. Explain the difference between a guaranteed bound and an observed error, and state which arithmetic assumption the proof uses.

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Question 7 – Solution

Strategy. Exploit the contracting error recurrence to turn a local defect bound into a rigorous global guarantee.

Step 1: Solve both continuous and discrete problems. The exact solution is $y(t) = 1 - e^{-t}$. Euler gives $Y_{n+1} = (1-h)Y_n + h$, so

$$\boxed{Y_n = 1 - (1-h)^n}, \qquad E_n = e^{-nh} - (1-h)^n \geq 0.$$

For $0 < h < 1$, the sign follows from $1 - h < e^{-h}$; at $h = 1$, it follows directly for $n \geq 1$. At $n = 0$, both solutions are zero.

Step 2: Isolate the local defect. Starting Euler at the exact value at t_n, its overshoot of the exact next value is

$$d_n = y(t_n) + h[1 - y(t_n)] - y(t_n + h) = e^{-t_n}(e^{-h} - 1 + h).$$

Therefore $0 \leq d_n \leq h^2/2$, and subtracting the exact update from the numerical one gives

$$\boxed{E_{n+1} = (1-h)E_n + d_n, \qquad E_0 = 0}.$$

Step 3: Sum the propagated defects. Since $0 \leq 1 - h < 1$, iteration gives

$$0 \leq E_n \leq \frac{h^2}{2} \sum_{j=0}^{n-1} (1-h)^j = \frac{h}{2}[1 - (1-h)^n] \leq \frac{h}{2}.$$

Thus $1/(2N) \leq 0.005$ is sufficient. The smallest integer supplied by this simpler bound is $\boxed{N = 100}$, with $h = 0.01$. This is not a claim that no smaller N could meet the tolerance using a sharper analysis.

Step 4: Evaluate the actual error. For that grid,

$$\boxed{E_{100} = e^{-1} - (0.99)^{100} \approx 0.00184710 < 0.005}.$$

The proven bound controls the error without assuming a particular observed cancellation or leading asymptotic term. Its conservatism explains the smaller actual error. Both the recurrence and this guarantee assumed exact arithmetic; step-by-step rounding would add another error term and require a separate bound.

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Question 8

For $y' = -y$, $y(0) = 1$, let Y_h denote the Euler approximation at $t = 1$ with equal step h. Compare $h = 1/2$ and $h = 1/4$. Assume exact arithmetic. For the extrapolation argument, suppose $Y_h = y(1) + Ch + O(h^2)$ with a coefficient C independent of h as $h \to 0$.

Tasks

1. Compute $Y_{1/2}$, $Y_{1/4}$, and $R = 2Y_{1/4} - Y_{1/2}$.
2. Use the stated error expansion to explain why R cancels the leading first-order error. Relate the difference between the two Euler values to an estimate of the finer-grid error.
3. Compare the actual errors of the two Euler values and R with the exact endpoint. Is it a contradiction if R lies outside the interval between the Euler values?
4. Test whether grid agreement certifies accuracy using a different IVP, $z' = \sin^2(4\pi t)$, $z(0) = 0$. Compute both Euler endpoints on the same grids and the exact value $z(1)$, and explain the failure of the grid-difference indicator.

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Question 8 – Solution

Strategy. Separate asymptotic error cancellation from a rigorous certificate: two grids can share the same sampling blind spot.

Step 1: Compute the three approximations. For decay, $Y_h = (1-h)^{1/h}$. Hence

$$\boxed{Y_{1/2} = 1/4, \quad Y_{1/4} = 81/256, \quad R = 49/128 = 0.3828125}.$$

Step 2: Cancel the leading error term. The assumed expansions give

$$2Y_{h/2} - Y_h = y(1) + O(h^2), \qquad Y_{h/2} - Y_h = -Ch/2 + O(h^2).$$

Thus $|Y_{h/2} - Y_h|$ estimates the leading magnitude of the finer-grid error, while the extrapolated value removes that term. This argument is asymptotic; it supplies no explicit bound for the omitted terms on these particular grids.

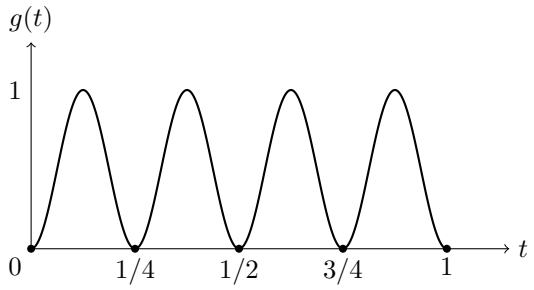
Step 3: Compare actual errors. The exact endpoint is e^{-1}. Absolute errors are

value	$Y_{1/2}$	$Y_{1/4}$	R
absolute error	0.117879	0.051473	0.014933

The finer-grid estimate is $17/256 = 0.06640625$, not its exact error. Both Euler values are below the truth, while R is above it. Extrapolation is not interpolation, so lying outside their interval is consistent with its construction and does not by itself invalidate it.

Step 4: Exhibit shared undersampling. At all left nodes of both specified grids, $\sin^2(4\pi t) = 0$. Both Euler approximations for $z(1)$ are therefore zero, and their difference is zero. But

$$\boxed{z(1) = \int_0^1 \sin^2(4\pi t)\, dt = \frac{1}{2}}.$$



$g(t) = \sin^2(4\pi t)$; all sampled forcing values are zero

The nonnegative forcing is missed between the sample nodes. Agreement of two grids does not certify accuracy; an explicit error bound or further analysis of the resolved time scales is needed.

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Question 9

Consider $y' = -0.1y$, $y(0) = 1$, with Euler step $h = 1$. Let Z_n be the exact-arithmetic Euler sequence. A second implementation rounds each new value to the nearest tenth immediately after each step, calling the result Y_n. Use decimal rounding with ties away from zero; in particular, 0.45 rounds to 0.5. Assume the stated decimal rule is applied exactly, rather than relying on binary floating-point tie behavior.

Tasks

1. Derive the exact-arithmetic and rounded recurrences. Tabulate Y_n for $n = 0, 1, \ldots, 8$ and compare its eventual behavior with Z_n and the exact solution.
2. Find every nonnegative multiple of 0.1 that is a fixed point of the rounded update.
3. Writing each rounding error as ρ_n, derive an error recurrence for $D_n = Y_n - Z_n$ and prove $|D_n| \leq 0.5(1 - 0.9^n)$.
4. Explain why a stable Euler amplification factor does not force a repeatedly rounded computation to converge to the true equilibrium. Distinguish rounding at every step from rounding only the reported final answer.

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Question 9 – Solution

Strategy. Treat rounding as a new perturbation at each step; it can create fixed points absent from the exact-arithmetic recurrence.

Step 1: Compute both updates. Exact Euler gives $Z_{n+1} = 0.9Z_n$, so $Z_n = 0.9^n$. The rounded implementation uses $Y_{n+1} = \text{round}_{0.1}(0.9Y_n)$ and yields

n	0	1	2	3	4	5	6	7	8
Y_n	1.0	0.9	0.8	0.7	0.6	0.5	0.5	0.5	0.5

Once it reaches 0.5, its unrounded next value is 0.45, which rounds back to 0.5. Both Z_n and the exact solution $e^{-0.1t}$ tend to zero, whereas this implementation stays at 0.5.

Step 2: Identify all spurious fixed points. Write a nonnegative grid value as $Y = k/10$, with integer $k \geq 0$. In units of one tenth, the unrounded update is $0.9k$. For $k > 0$, rounding back to k occurs exactly when

$$k - \tfrac{1}{2} \leq 0.9k < k + \tfrac{1}{2},$$

which reduces to $k \leq 5$. Zero is also fixed. Thus the complete set is

$$\boxed{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5}.$$

Step 3: Bound accumulated rounding. Writing $Y_{n+1} = 0.9Y_n + \rho_n$ gives $|\rho_n| \leq 0.05$. Therefore

$$D_{n+1} = 0.9D_n + \rho_n, \qquad D_0 = 0,$$

so

$$\boxed{|D_n| \leq 0.05 \sum_{j=0}^{n-1} 0.9^j = 0.5(1 - 0.9^n)}.$$

This bound approaches a nonzero limit, consistent with the observed plateau.

Step 4: Interpret the stable but biased computation. A factor 0.9 damps old perturbations, but fresh rounding occurs at every step and can balance the intended decrease. Stability does not remove continually injected errors. Rounding only the final reported answer would leave the preceding exact-arithmetic Euler trajectory unchanged and would introduce only one final reporting error, rather than altering subsequent slopes.

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Question 10

Use explicit Euler to integrate $y' = -2y$ from $t = 1$ backward to $t = 0$. Begin with the benchmark terminal value $y(1) = e^{-2}$ and use time increments $\Delta t = -1/4$. This is the explicit Euler formula on a decreasing time grid, not the implicit method sometimes called backward Euler.

Tasks

1. Derive the update on the decreasing grid and compute the approximation at $t = 0$.
2. Find the benchmark exact solution, calculate the reconstruction error at 0, and sketch the numerical polygon with its direction of computation.
3. Replace the supplied terminal value by $e^{-2} + \delta$. Determine how this perturbation affects the numerical reconstruction and the exact reconstruction.
4. Now suppose e^{-2} is a reported terminal measurement with unknown error at most 0.001. Find the interval of possible exact initial values. Can reducing the Euler step alone guarantee reconstruction within 0.005 of the unknown true initial value? Explain.

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Question 10 – Solution

Strategy. Track the sign of the time increment and distinguish discretization error from amplification of uncertain terminal data.

Step 1: March backward explicitly. Writing $t_j = 1 - j/4$, Euler gives

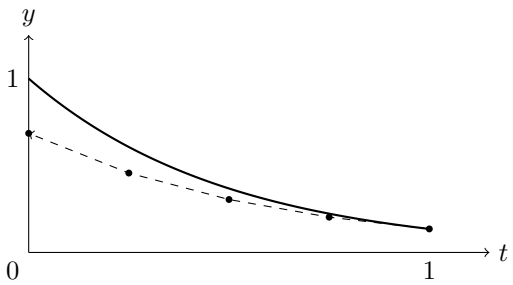
$$Y_{j+1} = Y_j - \tfrac{1}{4}(-2Y_j) = \tfrac{3}{2}Y_j.$$

After four steps,

$$\boxed{Y(0) = \left(\frac{3}{2}\right)^4 e^{-2} = \frac{81}{16}e^{-2} \approx 0.685135}.$$

Step 2: Compare with the exact benchmark. The exact solution is $y(t) = e^{-2t}$, giving $y(0) = 1$. Thus the numerical reconstruction error is

$$\boxed{Y(0) - y(0) = \frac{81}{16}e^{-2} - 1 \approx -0.314865}.$$



Solid: exact; dashed: Euler computed from $t = 1$ to $t = 0$

Each decreasing-time step amplifies the current value by 1.5. The exact amplification over a quarter-unit backward step is $e^{1/2} > 1.5$, explaining the numerical underestimate.

Step 3: Propagate a data perturbation. Linearity shows that replacing the terminal value by $e^{-2} + \delta$ changes the numerical reconstruction by

$$\boxed{(81/16)\delta},$$

whereas exact backward evolution changes it by $\boxed{e^2\delta}$. With N backward steps of size magnitude $1/N$, the numerical amplification is $(1 + 2/N)^N$, which tends to e^2, not to zero. Refinement improves the evolution calculation but does not eliminate sensitivity to data.

Step 4: Separate the uncertainty floor. The possible exact initial values form

$$\boxed{[1 - 0.001e^2,\ 1 + 0.001e^2] \approx [0.992611, 1.007389]}.$$

Its half-width is about 0.007389, exceeding 0.005. No single reconstruction from the reported value can be within 0.005 of every possible true initial value in that interval. Reducing the step alone therefore cannot supply the requested guarantee; the terminal data would also need tighter uncertainty. Forward physical decay and backward data amplification describe different directions of evolution.

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Question 1

Consider the following differential equations for a real function $y(t)$:

$$\text{A}: \ (ty')' + y = \sin t, \quad \text{B}: \ y'' + ty' + y^2 = 0,$$
$$\text{C}: \ y'' + (y')^2 = 0, \qquad \text{D}: \ y'' + t^2 y = 0.$$

An equation is linear in y if y and its derivatives occur to the first power, are not multiplied together, and have coefficients depending only on t. The terms homogeneous and nonhomogeneous here refer to linear equations with zero and nonzero forcing, respectively.

Tasks

1. Expand equation A. Classify all four equations by order and linearity; classify the linear ones by homogeneity.
2. Explain why $t^2 y$ does not destroy linearity, whereas y^2 and $(y')^2$ do. Does a zero right-hand side make B or C a homogeneous linear equation?
3. Put A in the form $y'' + p(t)y' + q(t)y = g(t)$. Find the largest open interval containing $t = 1$ on which these coefficients are continuous.
4. Suppose a C^2 solution of the undivided equation A passes through $t = 0$. Derive the necessary relation between $y(0)$ and $y'(0)$. Can $y(0) = 1$, $y'(0) = 0$ be prescribed there?

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Question 1 – Solution

Strategy. Expand derivatives before classifying, and distinguish an equation from its normalized form at a zero leading coefficient.

Step 1: Expand and classify. The product rule gives $(ty')' = ty'' + y'$. Hence A is

$$ty'' + y' + y = \sin t.$$

The classification is

equation	order	type
A	2	linear, nonhomogeneous
B	2	nonlinear
C	2	nonlinear
D	2	linear, homogeneous

A is a second-order equation whose leading coefficient vanishes at one point; its order as an equation is not reclassified solely at that point.

Step 2: Identify what is being squared. In D, t^2 is a known coefficient, while y occurs to the first power. In B and C the unknown function or its derivative is squared. A zero right-hand side does not remove that nonlinearity. Under the stated convention, neither B nor C is a homogeneous linear equation.

Step 3: Normalize on a regular interval. For $t \neq 0$, division by t yields

$$\boxed{y'' + \frac{1}{t}y' + \frac{1}{t}y = \frac{\sin t}{t}}.$$

All three normalized coefficients are continuous on $\boxed{(0, \infty)}$, the largest open interval containing 1 with this property. Although $\sin t/t$ has a removable singularity at zero, $1/t$ does not.

Step 4: Test the original equation at zero. For a C^2 solution, the undivided equation at zero requires

$$0 \cdot y''(0) + y'(0) + y(0) = 0, \qquad \boxed{y'(0) + y(0) = 0}.$$

The proposed data give $0 + 1 \neq 0$, so no such solution exists. The condition is necessary; deriving it alone is not an existence or uniqueness proof for every compatible pair. Division by the leading coefficient must not hide this compatibility requirement.

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Question 2

An unknown trajectory satisfies

$$y'' = 6t, \qquad y(1) = 2, \qquad y'(1) = -1.$$

A student writes $y = t^3 + C$ as the general solution because differentiating twice gives $6t$.

Tasks

1. Find every C^2 solution of the differential equation by integrating twice. Explain the role of each integration constant.
2. Apply both initial conditions and verify the resulting solution in the equation and the data.
3. Diagnose the student's proposed general solution. Which initial slopes at $t = 1$ can that smaller family realize?
4. Retain only $y(1) = 2$. Describe all resulting solutions and their slopes at $t = 1$. Prove directly that restoring the slope condition selects exactly one solution.

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Question 2 – Solution

Strategy. Integrate each derivative level separately, then examine which independent data the constants control.

Step 1: Recover the full family. One integration gives $y' = 3t^2 + A$; another gives

$$\boxed{y = t^3 + At + B}, \qquad A, B \in \mathbb{R}.$$

These are all solutions on any interval: if two functions have second derivative $6t$, their difference has zero second derivative and therefore is affine. The constant A changes velocity; B changes position without changing velocity.

Step 2: Fit and verify both data. At $t = 1$, the conditions become $1 + A + B = 2$ and $3 + A = -1$. Thus $A = -4$, $B = 5$, giving

$$\boxed{y = t^3 - 4t + 5}.$$

Indeed, $y' = 3t^2 - 4$, $y'' = 6t$, $y(1) = 2$, and $y'(1) = -1$. The polynomial is defined on all of $\mathbb{R}$.

Step 3: Locate the missing freedom. Every function $t^3 + C$ solves the differential equation, but every one has derivative $3t^2$. In particular, its slope at $t = 1$ is always 3. It cannot realize the required slope -1, regardless of C. Verification of a family does not by itself prove that the family is general.

Step 4: Separate position from velocity. With only $y(1) = 2$, we obtain $B = 1 - A$, so all possibilities are

$$\boxed{y = t^3 + A(t - 1) + 1}, \qquad y'(1) = 3 + A.$$

The initial slope ranges over every real number as A varies. Setting it to -1 forces $A = -4$ and hence the single solution above.

Alternatively, the difference w of any two solutions with both prescribed data satisfies $w'' = 0$, $w(1) = w'(1) = 0$. Writing $w = at + b$ gives $a = 0$ and $b = 0$. This proves uniqueness directly, rather than merely counting constants.

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Question 3

The functions e^t and e^{-t} are supplied as candidate solutions of

$$y'' - y = 0.$$

You may use this theorem: if p, q, g are continuous on an open interval I, then $y'' + py' + qy = g$ with $y(t_0) = a$, $y'(t_0) = b$, $t_0 \in I$, has exactly one solution on I.

Tasks

1. Verify that every $y = Ae^t + Be^{-t}$ solves the equation, without using a characteristic equation.
2. Determine A, B when $y(\ln 2) = 5/2$ and $y'(\ln 2) = 3/2$. Verify both data.
3. For arbitrary real data $y(\ln 2) = a$, $y'(\ln 2) = b$, solve for A, B. Show that every pair of initial data can be realized.
4. Use the theorem to prove that the displayed family contains every solution on $\mathbb{R}$. Explain why having two named constants would not be sufficient if the proposed family were $Ce^t + De^t$ instead.

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Question 3 – Solution

Strategy. Check the functions, solve the two data equations, then use uniqueness to establish completeness.

Step 1: Verify the entire family. Differentiation gives

$$y' = Ae^t - Be^{-t}, \qquad y'' = Ae^t + Be^{-t} = y.$$

Therefore $y'' - y = 0$ for every A, B. No root-solving method is required.

Step 2: Fit the stated data at their actual base point. Since $e^{\ln 2} = 2$ and $e^{-\ln 2} = 1/2$, the equations are

$$2A + B/2 = 5/2, \qquad 2A - B/2 = 3/2.$$

Adding and subtracting yields $A = 1$, $B = 1$. Thus

$$\boxed{y = e^t + e^{-t}}.$$

Its value at $\ln 2$ is $2 + 1/2 = 5/2$, and its derivative there is $2 - 1/2 = 3/2$.

Step 3: Solve the general data problem. For arbitrary a, b the same equations give

$$4A = a + b, \qquad B = a - b, \qquad \boxed{A = (a+b)/4, \quad B = a - b}.$$

There is exactly one coefficient pair for each $(a, b) \in \mathbb{R}^2$. Substitution returns $y(\ln 2) = a$ and $y'(\ln 2) = b$, so the claim includes all data, not only the numerical example.

Step 4: Establish completeness, not just verification. Here $p = 0$, $q = -1$, $g = 0$ are continuous on $\mathbb{R}$. Any solution has some data a, b at $\ln 2$. Step 3 constructs a member of the displayed family with those same data; uniqueness forces agreement on $\mathbb{R}$.

By contrast, $Ce^t + De^t = (C + D)e^t$ has only one effective parameter. Every such function satisfies $y' = y$, so at the base point it can realize only data with $b = a$. Two symbols for constants do not necessarily provide two independent freedoms.

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Question 4

For $y'' + y = 0$, you may use the complete family

$$y(t) = A\cos t + B\sin t.$$

Consider two endpoint conditions $y(0) = 0$ and $y(L) = c$, where $L > 0$ is specified. Compare these with initial data $y(0) = 0$, $y'(0) = b$.

Tasks

1. Solve the initial-data problem for an arbitrary real b. Explain why it always determines exactly one member of the family.
2. Determine all solutions of the endpoint problem for $L = \pi/2$ and arbitrary c.
3. Determine all solutions for $L = \pi$, treating $c = 0$ and $c \neq 0$ separately. Explain why two conditions need not select one solution.
4. For $L = \pi$, $c = 0$, sketch the solutions with $B = -1, 0, 1, 2$. Explain why their common endpoint values do not violate uniqueness for initial-value problems.

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Question 4 – Solution

Strategy. Translate each condition into a constraint on A, B and check whether the second constraint supplies new information.

Step 1: Use value and slope at the same point. From $y(0) = A$ and $y'(0) = B$, initial data $0, b$ give

$$\boxed{y = b\sin t}.$$

Both constants are fixed, for every real b. The coefficients of the normalized equation are continuous everywhere, consistent with initial-value uniqueness.

Step 2: Move the second datum to $\pi/2$. The first condition gives $A = 0$. At the other endpoint,

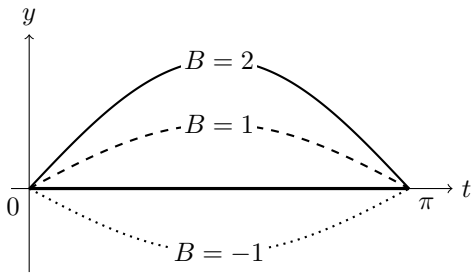
$$y(\pi/2) = B = c, \qquad \boxed{y = c\sin t}.$$

This particular pair of endpoint conditions also determines one solution.

Step 3: Detect a redundant or incompatible condition. At $L = \pi$, every function $B\sin t$ has $y(\pi) = 0$. Consequently,

$$\boxed{c = 0:\ y = B\sin t\ (B \in \mathbb{R})}, \qquad \boxed{c \neq 0:\ \text{no solution}}.$$

When $c = 0$, the second condition adds no restriction on B. When $c \neq 0$, it contradicts the first condition and the equation. Merely counting two stated conditions misses this distinction.



The horizontal solution is $B = 0$.

Step 4: Compare endpoint agreement with initial-data agreement. The four curves share their endpoint positions but have different initial slopes B. At $t = \pi$ their slopes are $-B$, also different. The initial-value theorem requires agreement of both value and first derivative at the same point. Agreement of values at two separate points is a different requirement, so the sketch creates no contradiction.

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Question 5

For a normalized linear equation $y'' + p(t)y' + q(t)y = g(t)$, continuity of p, q, g on an open interval containing the initial time guarantees a unique solution throughout that interval for any two initial data.

Consider

$$t(t-2)y'' + (t+1)y' + \ln(t+3)y = \frac{1}{t-4}.$$

All functions and solutions are real.

Tasks

1. Find the domain of the displayed coefficients before normalization. List the additional points excluded when solving for y''.
2. Find the largest theorem-guaranteed interval for initial time $t_0 = 1$, and then for $t_0 = 3$, with arbitrary finite initial value and slope.
3. Does a finite endpoint of such an interval prove that every solution blows up there? Explain precisely what the theorem does and does not say.
4. Test that distinction on the separate equation $ty'' + y' = 0$ with data $y(1) = a$, $y'(1) = b$. Solve on $(0, \infty)$ and determine exactly which resulting solutions extend as C^2 solutions of the undivided equation across zero.

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Question 5 – Solution

Strategy. Locate coefficient obstructions before applying the theorem, then compare its guaranteed interval with the behavior of actual solutions.

Step 1: Separate domain from degeneracy. The logarithm requires $t > -3$ and the forcing excludes $t = 4$. Thus the original coefficient domain is $(-3, 4) \cup (4, \infty)$. Normalization additionally excludes $t = 0, 2$, where the leading coefficient vanishes. For other allowed t,

$$p = \frac{t+1}{t(t-2)}, \qquad q = \frac{\ln(t+3)}{t(t-2)}, \qquad g = \frac{1}{(t-4)t(t-2)}.$$

Step 2: Choose the connected interval containing the data. The regular intervals are $(-3, 0)$, $(0, 2)$, $(2, 4)$ and $(4, \infty)$. Therefore the largest applicable intervals are

$$\boxed{t_0 = 1 : \ (0, 2)}, \qquad \boxed{t_0 = 3 : \ (2, 4)}.$$

The initial values do not change these coefficient-based guarantees.

Step 3: State the limitation accurately. The theorem guarantees existence and uniqueness within the selected interval. It does not classify the behavior at a singular endpoint or prove that every solution becomes unbounded there. Whether a particular solution extends requires examination of that solution and the original equation. A coefficient-domain boundary is also distinct from a pole in the solution itself.

Step 4: Solve the comparison equation completely. Since $(ty')' = ty'' + y'$, integration on $t > 0$ gives $ty' = C$, then $y = C \ln t + D$. The data at 1 imply $C = b$, $D = a$, so

$$\boxed{y = a + b \ln t \quad (t > 0)}.$$

If $b \neq 0$, y is unbounded as $t \downarrow 0$, so no continuous, hence no C^2, extension exists. If $b = 0$, the constant solution $y = a$ satisfies the undivided equation on all of $\mathbb{R}$.

Thus extension occurs exactly when $\boxed{b = 0}$. In fact it is unique among C^2 extensions: on the negative side every solution has form $C_- \ln |t| + D_-$; continuity forces $C_- = 0$, $D_- = a$. The normalized theorem alone did not provide that endpoint conclusion.

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Question 6

Let $L[y] = y'' + y$ and consider the forced equation $L[y] = t$. The functions t, $\sin t$ and $\cos t$ are supplied for direct verification.

Tasks

1. Compute $L[t]$, $L[\sin t]$ and $L[\cos t]$. Determine exactly which constants a, b, c make $at + b\sin t + c\cos t$ solve the forced equation on $\mathbb{R}$.
2. Let $u = t + \sin t$ and $v = t + \cos t$. Determine the equations satisfied by $u + v$, $u - v$ and $(u + v)/2$.
3. For arbitrary solutions u, v of $L[y] = t$, determine exactly which real weights α, β ensure $\alpha u + \beta v$ is again a solution. Explain the difference from homogeneous superposition.
4. Find a solution with $y(0) = 2$, $y'(0) = 0$. Verify it and justify uniqueness using the continuous-coefficient initial-value theorem.

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Question 6 – Solution

Strategy. Apply the linear operator to the proposed combination and track the forcing as carefully as the unknown function.

Step 1: Identify the forced and zero-forcing pieces. Direct differentiation yields

$$L[t] = t, \qquad L[\sin t] = 0, \qquad L[\cos t] = 0.$$

Linearity then gives $L[at + b\sin t + c\cos t] = at$. Equality with t for every real t requires $a = 1$; b, c are arbitrary. Hence all such candidates are

$$\boxed{y = t + b\sin t + c\cos t}.$$

The forcing fixes the coefficient of t, leaving the two homogeneous freedoms.

Step 2: Track sums, differences and averages. Because $L[u] = L[v] = t$,

$$L[u + v] = 2t, \qquad L[u - v] = 0, \qquad L[(u + v)/2] = t.$$

The sum is generally not a solution of the original forced equation. The difference solves the associated homogeneous equation, and the average preserves the original forcing.

Step 3: Determine the allowed weights. For any such u, v,

$$L[\alpha u + \beta v] = (\alpha + \beta)t.$$

This equals t on $\mathbb{R}$ exactly when $\boxed{\alpha + \beta = 1}$. The weights need not be nonnegative. For homogeneous solutions, by contrast, the right-hand side remains zero for every pair of weights. Also, the zero function is not a solution of $L[y] = t$; the forced solution set does not have all the closure properties of the homogeneous one.

Step 4: Fit and verify the initial data. For the candidate family, $y(0) = c$ and $y'(0) = 1 + b$. The data give $c = 2$, $b = -1$, so

$$\boxed{y = t - \sin t + 2\cos t}.$$

Its derivative is $1 - \cos t - 2\sin t$, its initial value and slope are 2 and 0, and $y'' + y = t$. Since $p = 0$, $q = 1$, $g = t$ are continuous on $\mathbb{R}$, the initial-value theorem proves uniqueness there. No general method for finding particular solutions was assumed; the necessary candidate was supplied.

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Question 7

Consider $y'' + 4y = 0$ with $y(0) = 1$, $y'(0) = 2$. Define

$$E(t) = (y'(t))^2 + 4(y(t))^2.$$

For direct verification you may use the candidate $y(t) = \cos(2t) + \sin(2t)$.

Tasks

1. Prove that E is constant for every solution and calculate its value for the stated data.
2. Deduce a bound on $|y(t)|$ without first substituting a solution formula. Verify the candidate and determine a time when the bound is attained.
3. In coordinates (y, v) with $v = y'$, sketch the curve forced by the energy identity, mark the initial state and indicate the direction of motion there.
4. Prove uniqueness directly: if two solutions share both initial data, apply the same identity to their difference. Explain why knowing the energy alone does not specify a unique solution.

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Question 7 – Solution

Strategy. Differentiate a sum of nonnegative terms, then use its conservation both to bound a solution and to eliminate a zero-data difference.

Step 1: Derive the conserved quantity. For any solution,

$$E' = 2y'y'' + 8yy' = 2y'(y'' + 4y) = 0.$$

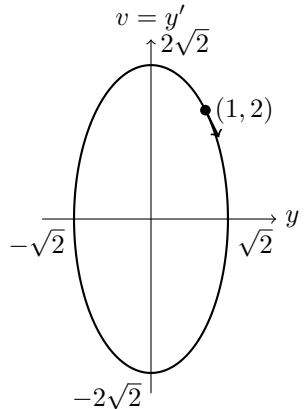
Thus $E(t) = E(0)$ throughout its interval. The prescribed data give

$$\boxed{E = 2^2 + 4(1)^2 = 8}.$$

Step 2: Obtain and attain the displacement bound. Since $(y')^2 \geq 0$, $4y^2 \leq 8$, whence $\boxed{|y| \leq \sqrt{2}}$. For the supplied candidate,

$$y' = -2\sin(2t) + 2\cos(2t), \qquad y'' = -4y.$$

It has the required data, and at $t = \pi/8$ it has $y = \sqrt{2}$, $y' = 0$. Therefore the bound is sharp, not just a convenient overestimate.



$4y^2 + v^2 = 8$; equal axis scales

Step 3: Interpret the state curve. Writing $v = y'$ gives $4y^2 + v^2 = 8$, an ellipse with intercepts $y = \pm\sqrt{2}$ and $v = \pm 2\sqrt{2}$. The initial point is $(1, 2)$. Its velocity in these coordinates is

$$(y', v') = (v, -4y) = (2, -4),$$

so it moves rightward and downward there. The axes represent state variables, not time versus displacement.

Step 4: Prove uniqueness and distinguish energy from data. The difference w of two solutions with the same initial data satisfies $w'' + 4w = 0$ and $w(0) = w'(0) = 0$. Its conserved quantity is $(w')^2 + 4w^2 = 0$. Both terms are nonnegative, so $w = 0$ everywhere on their common interval.

Energy alone only fixes an ellipse. For example, $\sqrt{2}\cos(2t)$ and $\sqrt{2}\sin(2t)$ both have energy 8 but different initial states. Initial position and velocity identify a point and its motion; the scalar energy value does not replace them.

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Question 8

Suppose both $y_1(t) = 1$ and $y_2(t) = t^2$ solve an unknown normalized homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on $(0, \infty)$.

Tasks

1. Determine p and q directly from the two supplied solutions. Check your result by substitution.
2. Can any normalized equation of this form, with continuous p, q on an interval containing zero, have both supplied functions as solutions there? Prove your answer without a determinant formula.
3. Multiply the equation from Task 1 by t. Show that the undivided equation admits $y = C_1 + C_2t^2$ on all of $\mathbb{R}$. Determine all members of this family satisfying $y(0) = a$, $y'(0) = b$.
4. Sketch $y = t^2$ and $y = 2t^2$ near zero. Explain how distinct solutions can share the same tangent there without contradicting the regular linear initial-value theorem.

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Question 8 – Solution

Strategy. Insert simple known solutions to recover coefficients, then inspect what changes when the leading coefficient vanishes.

Step 1: Recover the equation on its stated interval. Inserting $y_1 = 1$ gives $q(t) = 0$. Inserting $y_2 = t^2$ then gives $2 + 2tp(t) = 0$, so

$$\boxed{p(t) = -1/t, \quad q(t) = 0}, \qquad \boxed{y'' - \frac{1}{t}y' = 0 \quad (t > 0)}.$$

The constant solution has both derivatives zero, and $2 - (1/t)(2t) = 0$ verifies the quadratic solution.

Step 2: Rule out a regular equation through zero. At $t = 0$, the function t^2 has value 0, first derivative 0 and second derivative 2. In any normalized equation with finite coefficients at zero it would require

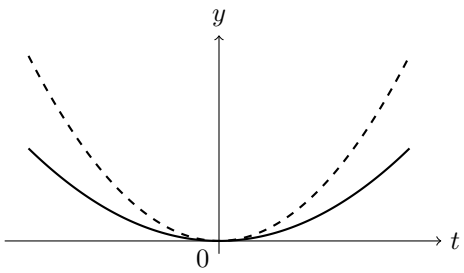
$$2 + p(0) \cdot 0 + q(0) \cdot 0 = 0,$$

which is impossible. Thus no continuous-coefficient normalized equation on an interval containing zero can have both supplied functions as solutions. The obstruction is already visible from the quadratic alone.

Step 3: Retain the degenerate leading coefficient. The undivided equation is $ty'' - y' = 0$. For $y = C_1 + C_2t^2$, its left-hand side is $t(2C_2) - 2C_2t = 0$, including at zero. The family has $y(0) = C_1$, $y'(0) = 0$, so

$$\boxed{b = 0 : \ y = a + C_2t^2 \ (C_2 \in \mathbb{R})}, \qquad \boxed{b \neq 0 : \ \text{no member}}.$$

Indeed, the undivided equation itself at zero forces $y'(0) = 0$, so nonzero b permits no C^2 solution at all.



Solid: t^2; dashed: $2t^2$; common tangent $y = 0$

Step 4: Explain the shared tangent. The two plotted functions share value 0 and slope 0 at zero but have different curvatures. The coefficient of y'' vanishes there, so the equation does not determine the second derivative from those data. Its normalized coefficient $-1/t$ is not continuous at the initial point. The regular linear theorem therefore does not apply, and the multiple zero-data solutions do not contradict it.

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Question 9

Consider the nonlinear equation

$$y'' = \sqrt{|y|}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

For a real parameter $a \geq 0$, define

$$y_a(t) = \begin{cases} 0, & t \leq a, \\ (t-a)^4/144, & t > a. \end{cases}$$

A classical solution here means a real C^2 function satisfying the equation at every point of its interval.

Tasks

1. Verify the equation on each open side of $t = a$.
2. Check the value, first derivative and second derivative at the joining point. Does the equation hold there as well?
3. Verify both initial conditions for every $a \geq 0$, and show that different finite choices of a give different solutions. Identify one additional solution.
4. Sketch y_0, y_1 and the zero solution on $[0, 3]$. Explain why continuity of the right-hand side does not justify applying the linear initial-value uniqueness theorem to this example. Does an assigned position and slope always determine a unique trajectory?

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Question 9 – Solution

Strategy. Verify the join at the derivative order required by the equation, then use the family as a counterexample to an overgeneralized uniqueness claim.

Step 1: Verify away from the join. For $t < a$, both sides of the equation are zero. For $t > a$,

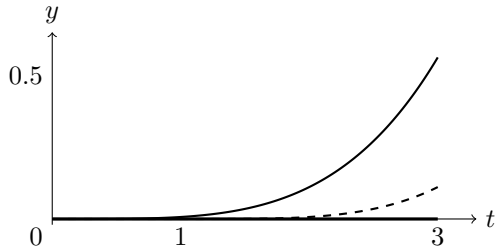
$$y_a' = \frac{(t-a)^3}{36}, \qquad y_a'' = \frac{(t-a)^2}{12}, \qquad \sqrt{|y_a|} = \sqrt{\frac{(t-a)^4}{144}} = \frac{(t-a)^2}{12}.$$

The coefficient 144 is essential to this identity.

Step 2: Verify classical regularity at the join. The right-hand limits of y_a, y_a', y_a'' as $t \downarrow a$ are all zero, matching the values on the left. The first and second derivatives at the join are also zero, as follows directly from their difference quotients. Hence y_a is C^2 on $\mathbb{R}$.

At $t = a$, the equation reads $0 = \sqrt{|0|}$, so it holds there, not merely away from the join. There is no hidden corner in position or velocity.

Step 3: Check the common data and distinctness. Since $a \geq 0$, every family member has $y_a(0) = y_a'(0) = 0$, including $a = 0$ by the join calculation. If $a_1 < a_2$, choose $a_1 < t < a_2$. Then $y_{a_1}(t) > 0$ but $y_{a_2}(t) = 0$, proving they are different solutions. The function $\boxed{y \equiv 0}$ is another solution, different from every finite-delay member.



Solid: $a = 0$; dashed: $a = 1$; horizontal: $y = 0$

Step 4: State exactly what has failed. The right-hand side $\sqrt{|y|}$ is continuous but depends nonlinearly on the unknown function. This is not an equation $y'' + p(t)y' + q(t)y = g(t)$ with prescribed continuous coefficients, so that linear theorem cannot be invoked. The verified family proves that continuity alone does not ensure uniqueness for a general nonlinear second-order equation.

Two initial data determine one solution under appropriate hypotheses, not for every differential equation. This example exhibits infinitely many solutions; it does not claim to classify all solutions of the nonlinear equation.

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Question 10

A prescribed acceleration changes abruptly at $t = 1$:

$$y'' = \begin{cases} 0, & 0 \le t < 1, \\ 2, & t > 1, \end{cases} \qquad y(0) = 0, \qquad y'(0) = 1.$$

Seek a function that is C^1 on $[0, \infty)$ and C^2 on each open side of 1, satisfying the equation there. No value for y'' at 1 is assumed. Thus position and velocity must be continuous at the switch.

Tasks

1. Integrate on the first phase and find the position and velocity arriving at $t = 1$.
2. Integrate on the second phase with two new constants, determine them using the matching conditions, and give the complete piecewise trajectory.
3. Compute the one-sided derivatives of velocity at the switch. Can any assigned value of the forcing at $t = 1$ make this trajectory a classical C^2 solution through the switch?
4. Sketch position and velocity. A student adds $k(t - 1)$ to the second-phase position while leaving the first phase unchanged. Determine which k satisfy the stated requirements, and explain why position matching alone is insufficient.

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Question 10 – Solution

Strategy. Carry both position and velocity across the switch, and distinguish a piecewise classical solution from one that is C^2 everywhere.

Step 1: Propagate the first phase. With zero acceleration and initial velocity 1, $y' = 1$ and $y = t$ for $0 \le t \le 1$. The arriving data are

$$\boxed{y(1) = 1, \qquad y'(1) = 1}.$$

Step 2: Use both matching conditions. Writing the second-phase antiderivative about the switch gives

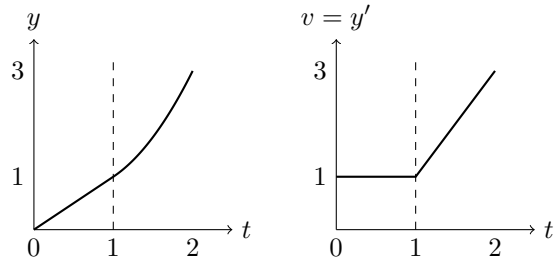
$$y = (t-1)^2 + A(t-1) + B \quad (t > 1).$$

Continuity of position sets $B = 1$; continuity of velocity sets $A = 1$. Thus

$$\boxed{y(t) = \begin{cases} t, & 0 \le t \le 1, \\ t + (t-1)^2, & t > 1, \end{cases}} \qquad y'(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 1 + 2(t-1), & t > 1. \end{cases}$$

These formulas satisfy all the stated requirements and uniquely determine both phases.

Step 3: Check the actual regularity. The derivative of velocity at 1 has left-hand value 0 and right-hand value 2. Therefore $y''(1)$ does not exist. Assigning a forcing value at that single point cannot change the unequal one-sided limits. The trajectory is C^1 and piecewise C^2, but is not a classical C^2 solution across the switch.



Position has a continuous tangent; velocity has a corner.

Step 4: Reject an unmatched velocity. Adding $k(t-1)$ on $t > 1$ keeps the position continuous and does not change the second derivative on that open phase. However, the right-hand velocity at 1 becomes $1 + k$, while the left-hand velocity stays 1. Hence the C^1 requirement forces $\boxed{k = 0}$.

A second-order problem carries two pieces of state information. Position matching alone would admit a jump in velocity that is excluded by the problem's stated regularity; checking only the differential equation away from the switch would miss it.

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Question 1

Consider

$$6y'' + y' - 2y = 0, \qquad y(1) = 7, \qquad y'(1) = 0.$$

For a constant-coefficient homogeneous second-order equation with distinct real characteristic roots r_1, r_2, the general solution is $C_1e^{r_1t} + C_2e^{r_2t}$.

Tasks

1. Substitute $y = e^{rt}$ to derive the characteristic equation. Find and verify both roots.
2. Write the general solution using exponentials based at $t = 1$, then determine its constants from the initial data.
3. Verify the resulting function in the original equation and both initial conditions. Explain why it is the unique solution on $\mathbb{R}$.
4. Classify the stationary point at $t = 1$ and determine the limits as $t \to \infty$ and $t \to -\infty$. Explain why zero initial slope does not mean a constant solution.

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Question 1 – Solution

Strategy. Keep the leading coefficient in the characteristic polynomial and choose a shifted exponential basis to simplify the data.

Step 1: Derive, factor and check the polynomial. Substitution gives $(6r^2 + r - 2)e^{rt} = 0$. Since $e^{rt} \neq 0$,

$$6r^2 + r - 2 = (2r - 1)(3r + 2) = 0, \qquad \boxed{r_1 = 1/2, \quad r_2 = -2/3}.$$

Direct evaluation gives $6(1/2)^2 + 1/2 - 2 = 0$ and $6(-2/3)^2 - 2/3 - 2 = 0$.

Step 2: Use the actual initial time. A shifted basis only rescales the arbitrary constants, so write

$$y = Ae^{(t-1)/2} + Be^{-2(t-1)/3}.$$

The data give $A + B = 7$ and $A/2 - 2B/3 = 0$. Thus $A = 4$, $B = 3$, and

$$\boxed{y = 4e^{(t-1)/2} + 3e^{-2(t-1)/3}}.$$

Step 3: Verify the equation and uniqueness. Each exponential contributes its coefficient times $6r^2 + r - 2$ to the residual, so their sum solves the original equation. At 1, its value is $4 + 3 = 7$ and its derivative is $2 - 2 = 0$. After division by 6, all coefficients are continuous on $\mathbb{R}$. The linear initial-value theorem therefore gives uniqueness there.

Step 4: Interpret the initial balance of modes. Differentiating twice gives

$$y'' = e^{(t-1)/2} + \frac{4}{3}e^{-2(t-1)/3} > 0.$$

Thus y' is strictly increasing, crosses zero at 1, and $\boxed{y(1) = 7}$ is the unique global minimum. The positive-root mode gives $y \to +\infty$ as $t \to \infty$; the negative-root mode gives $y \to +\infty$ as $t \to -\infty$.

Zero slope at one instant is a balance between two changing modes. Indeed the equation at 1 gives $y''(1) = 7/3$, not zero. A constant solution would have to satisfy $-2y = 0$ and could not have value 7.

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Question 2

Consider the family of initial-value problems

$$y'' - y' - 6y = 0, \qquad y(0) = 1, \qquad y'(0) = v.$$

Call a solution forward bounded if $|y(t)|$ is bounded for $t \geq 0$.

Tasks

1. Find the roots and express the solution coefficients in terms of v.
2. Determine exactly which initial slope produces a forward-bounded solution. State its limit.
3. Replace that slope by $v = -2 + \varepsilon$, where $0 < \varepsilon < 1$. Find when the growing and decaying terms have equal magnitude, and determine the eventual behavior.
4. For $\varepsilon = 0.01$, calculate that time to three decimal places and sketch the perturbed and unperturbed solutions on $[0, 2]$. Explain why a small initial error can change the long-time conclusion.

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Question 2 – Solution

Strategy. Express the initial data in the two modes and isolate the coefficient of the growing exponential.

Step 1: Resolve the data into modes. The characteristic polynomial is $(r-3)(r+2)$, so $y = Ae^{3t} + Be^{-2t}$. From $A + B = 1$ and $3A - 2B = v$,

$$\boxed{A = (v+2)/5, \quad B = (3-v)/5}.$$

The nonzero root separation makes this data system invertible.

Step 2: Remove the growing mode exactly. If $A \neq 0$, the term Ae^{3t} dominates and $|y| \to \infty$. Thus forward boundedness holds exactly when $A = 0$, or $\boxed{v = -2}$. The selected solution is $\boxed{y = e^{-2t}}$ and tends to zero.

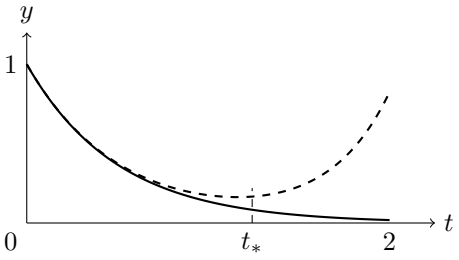
Step 3: Track a small slope perturbation. For $v = -2 + \varepsilon$,

$$y_\varepsilon = \frac{\varepsilon}{5}e^{3t} + \frac{5-\varepsilon}{5}e^{-2t}.$$

Both terms are positive. They are equal precisely when $e^{5t} = (5-\varepsilon)/\varepsilon$, giving

$$\boxed{t_* = \frac{1}{5}\ln\frac{5-\varepsilon}{\varepsilon}}.$$

This is positive for the stated range of ε. Eventually $y_\varepsilon \to +\infty$, although its initial slope is negative.



Solid: $v = -2$; dashed: $v = -1.99$

Step 4: Quantify the delayed disagreement. For $\varepsilon = 0.01$, $t_* = (\ln 499)/5 \approx \boxed{1.243}$. The exact difference is

$$y_\varepsilon - e^{-2t} = \frac{\varepsilon}{5}(e^{3t} - e^{-2t}).$$

It vanishes initially and is small with ε on any fixed finite interval, but grows without bound as $t \to \infty$ for every fixed positive ε. Forward boundedness here requires exact cancellation of a mode; it is not robust to a nonzero slope error. The time t_* compares mode magnitudes and is not the time of the trajectory's minimum.

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Question 3

A transient response satisfies

$$y'' + 5y' + 6y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Only $t \geq 0$ is considered in the qualitative questions.

Tasks

1. Solve the initial-value problem using its real distinct roots.
2. Prove that $y(t) > 0$ for $t > 0$, even though $y(t) \to 0$ as $t \to \infty$.
3. Find the unique maximum, including its time and height, and sketch the response with that point marked.
4. Calculate $\int_0^\infty y(t)\,dt$ in two ways: from the explicit formula and by integrating the differential equation. Explain why negative roots alone do not imply monotone decay from the initial value.

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Question 3 – Solution

Strategy. Separate eventual decay from the shape created by two modes with opposite coefficients.

Step 1: Solve for the two coefficients. The characteristic equation is $(r+2)(r+3)=0$, so $y = Ae^{-2t} + Be^{-3t}$. The data give $A + B = 0$ and $-2A - 3B = 1$. Hence

$$\boxed{y = e^{-2t} - e^{-3t}}.$$

Its value and derivative at zero are 0 and 1, and both exponential terms satisfy the equation.

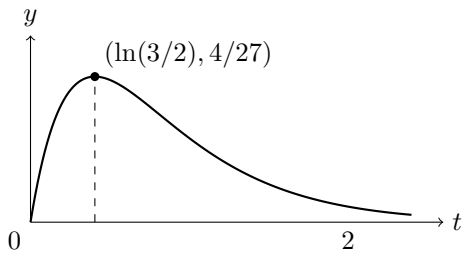
Step 2: Prove positivity and decay. For $t > 0$, $y = e^{-3t}(e^t - 1) > 0$. Both exponential terms tend to zero, so the positive response eventually decays to zero. Positivity does not require both mode coefficients to be positive.

Step 3: Locate and evaluate the only peak. The derivative factors as

$$y' = e^{-3t}(3 - 2e^t).$$

It is positive before $t_m = \ln(3/2)$ and negative afterward. At that time $e^{-t_m} = 2/3$, so

$$\boxed{t_m = \ln(3/2)}, \qquad \boxed{y(t_m) = 4/9 - 8/27 = 4/27}.$$



Positive transient with two negative roots

Step 4: Check the total area independently. Direct integration gives $\int_0^\infty y\,dt = 1/2 - 1/3 = \boxed{1/6}$. Alternatively, integrate the equation on $[0, T]$:

$$y'(T) - y'(0) + 5(y(T) - y(0)) + 6\int_0^T y\,dt = 0.$$

The explicit formula verifies $y(T), y'(T) \to 0$ and convergence of the integral. Passing to the limit gives $-1 + 6\int_0^\infty y\,dt = 0$, confirming the result.

Negative roots control the eventual limit of each mode. Their combination can initially increase: this solution starts at zero with positive slope, reaches a peak, and only then decreases.

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Question 4

For the equation

$$y'' + 3y' + 2y = 0, \qquad y(0) = a > 0, \qquad y'(0) = b,$$

classify the entire range of real initial slopes b on the time interval $t \geq 0$.

Tasks

1. Find the solution in terms of a, b.
2. Determine exactly when $y(t) \geq 0$ for every $t \geq 0$. Prove both necessity and sufficiency.
3. Determine exactly when the solution is both nonnegative and nonincreasing on $[0, \infty)$, including boundary slopes.
4. For the remaining slopes, find the unique zero when one occurs, or the unique positive-time maximum when one occurs. Summarize the cases on a diagram of the ratio b/a.

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Question 4 – Solution

Strategy. Factor out a positive exponential and reduce sign questions to affine expressions in $z = e^{-t} \in (0, 1]$.

Step 1: Fit the real-root solution. The roots are $-1, -2$. Solving $A + B = a$, $-A - 2B = b$ gives

$$\boxed{y = (2a + b)e^{-t} - (a + b)e^{-2t}}.$$

Write $A = 2a + b$, $B = -(a + b)$, so $A + B = a$.

Step 2: Classify nonnegativity. Since $y = e^{-t}(A + Bz)$, its sign is the sign of an affine function with values A at $z = 0$ and $a > 0$ at $z = 1$. If $A \geq 0$, it is positive for all $0 < z \leq 1$. If $A < 0$, it is negative for sufficiently small positive z. Therefore

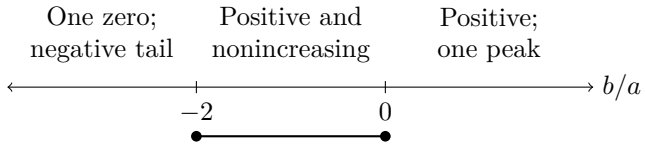
$$\boxed{y(t) \geq 0 \text{ for all } t \geq 0 \quad \Longleftrightarrow \quad b \geq -2a}.$$

At $b = -2a$, the solution is the strictly positive ae^{-2t} at every finite time.

Step 3: Impose monotonicity as well. We have $y' = -e^{-t}(A + 2Bz)$. Its bracket has endpoint values A and $A + 2B = -b$. Thus it is nonnegative on $0 < z \leq 1$ exactly when $A \geq 0$ and $b \leq 0$. Combining the conditions yields

$$\boxed{-2a \leq b \leq 0}.$$

At $b = 0$, the derivative vanishes initially but is negative for every $t > 0$. Both boundary slopes are included.



Both boundary slopes belong to the middle case.

Step 4: Resolve the two remaining regimes. If $b < -2a$, solving $A + Be^{-t} = 0$ gives the unique zero

$$\boxed{t_z = \ln \frac{a + b}{2a + b} > 0}.$$

Both numerator and denominator are negative, and their ratio exceeds 1. The solution crosses from positive to negative and tends to zero from below.

If $b > 0$, it remains positive but initially rises. Solving $y' = 0$ gives the unique positive-time maximum at

$$\boxed{t_m = \ln \frac{2(a + b)}{2a + b} > 0}.$$

The derivative changes from positive to negative there. These cases exhaust all real b; every solution still tends to zero because both roots are negative.

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Question 5

An unknown monic equation $y'' + py' + qy = 0$ has distinct real roots r_1, r_2. A solution is known to contain both modes with nonzero coefficients. Its exact samples are

$$S_n = y(n \ln 2), \qquad S_0 = 3, \quad S_1 = 1, \quad S_2 = 3/8, \quad S_3 = 5/32.$$

Set $z_j = e^{r_j \ln 2} > 0$, so $S_n = Az_1^n + Bz_2^n$.

Tasks

1. Derive a recurrence $S_{n+2} = sS_{n+1} - kS_n$ in terms of z_1, z_2.
2. Use the four samples to determine s, k. Explain why this pair is uniquely determined.
3. Recover the real roots, the coefficients p, q, the solution $y(t)$ and its initial slope. Verify all four samples.
4. Explain what would fail if the observed solution contained only one mode. Could arbitrarily many exact samples of one pure exponential identify both roots of an unknown second-order equation?

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Question 5 – Solution

Strategy. Recover the two exponential multipliers from a sample recurrence, then convert those multipliers back into differential-equation roots.

Step 1: Derive the recurrence. Each z_j satisfies $z_j^2 - (z_1 + z_2)z_j + z_1 z_2 = 0$. Multiplying by its mode coefficient and z_j^n, then summing, gives

$$\boxed{S_{n+2} = sS_{n+1} - kS_n}, \qquad s = z_1 + z_2, \quad k = z_1 z_2.$$

Step 2: Solve the sample equations exactly. For $n = 0, 1$,

$$s - 3k = 3/8, \qquad (3/8)s - k = 5/32.$$

Substitution of $s = 3/8 + 3k$ into the second equation gives $k/8 = 1/64$. Hence

$$\boxed{k = 1/8, \qquad s = 3/4}.$$

The nonzero coefficient $1/8$ in this elimination proves uniqueness. Equivalently, $S_0 S_2 - S_1^2 = 1/8 \neq 0$.

Step 3: Recover and verify the continuous model. The multipliers solve $z^2 - (3/4)z + 1/8 = (z - 1/2)(z - 1/4) = 0$. Since $r = \ln z / \ln 2$, the roots are $-1, -2$. Thus $p = 3$, $q = 2$.

Writing $y = Ae^{-t} + Be^{-2t}$, the first two samples give $A + B = 3$, $A/2 + B/4 = 1$, so $A = 1$, $B = 2$. Therefore

$$\boxed{y = e^{-t} + 2e^{-2t}}, \qquad \boxed{y'(0) = -5}.$$

The four sample values are 3, $1/2 + 2/4 = 1$, $1/4 + 2/16 = 3/8$, and $1/8 + 2/64 = 5/32$, as required. Both coefficients are nonzero.

Step 4: Identify the missing-information case. For a pure observed mode $S_n = Cz^n$, the recurrence only imposes $z^2 - sz + k = 0$. Any positive second multiplier different from z gives another pair s, k with exactly the same samples. Thus the active root can be identified, but an unexcited root remains undetermined, even with arbitrarily many noiseless samples of that one trajectory. Exact data do not reveal a mode whose coefficient is zero.

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Question 6

Consider homogeneous constant-coefficient equations with two distinct real roots $r_1 < r_2$ and endpoint data $y(0) = a$, $y(L) = b$, where $L > 0$.

Tasks

1. Prove that every pair $a, b \in \mathbb{R}$ determines exactly one solution by solving for the two mode coefficients. Identify the quantity that cannot vanish.
2. Solve $y'' - 3y' + 2y = 0$ with $y(0) = y(\ln 2) = 1$.
3. Prove that a nonzero combination $Ae^{r_1 t} + Be^{r_2 t}$ has at most one real zero. Treat vanishing coefficients explicitly, and deduce positivity between strictly positive endpoints.
4. For the solution in Task 2, find its maximum on $[0, \ln 2]$ and sketch it. Does positivity between positive endpoints mean the solution stays below its endpoint values?

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Question 6 – Solution

Strategy. Use the strict inequality of real exponentials to solve the endpoint problem and control the number of zeros.

Step 1: Invert the endpoint constraints. For $y = Ae^{r_1 t} + Be^{r_2 t}$, the conditions give $A + B = a$ and $Ae^{r_1 L} + Be^{r_2 L} = b$. Therefore

$$B = \frac{b - ae^{r_1 L}}{e^{r_2 L} - e^{r_1 L}}, \qquad A = a - B.$$

Because $r_2 > r_1$ and $L > 0$, the denominator is strictly positive. There is exactly one coefficient pair for all endpoint data, unlike some oscillatory endpoint problems.

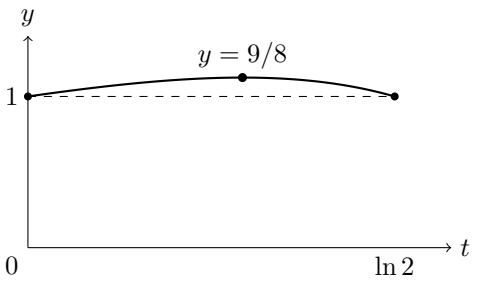
Step 2: Fit the concrete equation. Here the roots are 1 and 2. At $L = \ln 2$, the equations are $A+B = 1$ and $2A + 4B = 1$. Hence

$$\boxed{y = \frac{3}{2}e^t - \frac{1}{2}e^{2t}}.$$

Both endpoints have value 1, and each mode satisfies the original equation.

Step 3: Bound the number of zeros and deduce positivity. Factor the general solution as $e^{r_1 t}(A + Be^{(r_2 - r_1)t})$. If $B \neq 0$, the bracket is strictly monotone, so it has at most one zero. If $B = 0$, a nonzero solution is a nonvanishing pure exponential; the case $A = 0$ is likewise nonvanishing.

If a solution with positive endpoints became negative inside, continuity would force two distinct zeros. If it merely touched zero inside while staying nonnegative, that interior minimum would have $y = y' = 0$; initial-value uniqueness would force the identically zero solution, contradicting the endpoints. Thus it is strictly positive throughout the interval.



Positive endpoints do not bound the interior maximum.

Step 4: Distinguish positivity from an upper bound. For the concrete solution, $y' = e^t(3/2 - e^t)$ changes from positive to negative at $t = \ln(3/2)$, which lies between 0 and $\ln 2$. Its value there is

$$\boxed{y_{\max} = \frac{3}{2}\frac{3}{2} - \frac{1}{2}\frac{9}{4} = \frac{9}{8} > 1}.$$

The curve stays positive but exceeds both endpoint values. A restriction on zeros is not a maximum principle bounding the response by its endpoints.

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Question 7

For a real parameter α, consider

$$y'' + 2\alpha y' + (\alpha^2 - 4)y = 0.$$

A solution is forward bounded if it is bounded on $[0, \infty)$. Distinguish statements about every solution from statements about selected nonzero solutions.

Tasks

1. Find both roots and show that they are real and distinct for every real α.
2. Determine exactly when every solution tends to zero as $t \to \infty$.
3. Determine exactly when every solution is forward bounded. Treat $\alpha = 2$ explicitly.
4. For all $\alpha \leq 2$, classify the individual forward-bounded solutions, including $\alpha = -2$. Give a complete parameter table and explain why a zero root must not be treated as a negative root.

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Question 7 – Solution

Strategy. Track the two root signs and retain the distinction between a mode being available and its coefficient actually being present.

Step 1: Keep the root separation visible. The characteristic polynomial is $(r+\alpha)^2 - 4$, so

$$\boxed{r_+ = 2-\alpha, \quad r_- = -2-\alpha}, \qquad r_+ - r_- = 4.$$

Consequently $y = Ae^{(2-\alpha)t} + Be^{(-2-\alpha)t}$ for every real α; no repeated or nonreal roots occur.

Step 2: Require both modes to decay. Every solution tends to zero exactly when both roots are negative. Since r_+ is the larger root, this is equivalent to

$$\boxed{\alpha > 2}.$$

If $\alpha \leq 2$, choosing $A = 1$, $B = 0$ gives a constant or growing mode, disproving universal decay.

Step 3: Allow a constant mode for boundedness. Every solution is forward bounded exactly when both roots are nonpositive, or $\boxed{\alpha \geq 2}$. At $\alpha = 2$,

$$y = A + Be^{-4t} \longrightarrow A.$$

All solutions are bounded, but only those with $A = 0$ tend to zero. A zero root supplies a constant mode rather than a decaying one.

Step 4: Classify the selected bounded solutions. Using the coefficients A, B from Step 1, the full classification is

α	forward-bounded solutions	their limits
$\alpha < -2$	$y = 0$	0
$\alpha = -2$	$y = B \quad (A = 0)$	B
$-2 < \alpha < 2$	$y = Be^{(-2-\alpha)t} \quad (A = 0)$	0
$\alpha = 2$	$y = A + Be^{-4t}$	A
$\alpha > 2$	all solutions	0

For $\alpha < -2$, both roots are positive. If $A \neq 0$, the faster mode forces unbounded magnitude; if $A = 0$ and $B \neq 0$, the remaining positive mode still grows. Distinct growth rates cannot cancel for all large times.

At $\alpha = -2$ the roots are 4 and 0, so only constant solutions are bounded. For $-2 < \alpha < 2$, one positive mode must be removed, leaving a decaying mode. These boundary cases show why a classification based only on whether roots are real would be incomplete.

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Question 8

The positive response

$$y(t) = e^{-t} + 9e^{-3t} \qquad (t \geq 0)$$

solves a homogeneous second-order equation with real distinct roots. Define its instantaneous decay rate by $R(t) = -y'(t)/y(t)$.

Tasks

1. Recover the monic differential equation and verify the initial value and slope.
2. Express R as a weighted average of the two decay rates. Find $R(0)$, its limit, and the unique time when $R = 2$.
3. Differentiate R using the differential equation to prove that it is strictly decreasing. Sketch R with its limiting level marked.
4. A constant-rate extrapolation from the initial data predicts $\widehat{y}(t) = 10e^{-(14/5)t}$. Prove whether this estimate lies above or below the actual response for every $t > 0$, using the curvature of $\ln y$.

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Question 8 – Solution

Strategy. A mixture of two exponential modes need not have a constant effective decay rate; use its logarithmic derivative to quantify the change.

Step 1: Recover and check the equation. The roots are $-1, -3$, so the monic characteristic polynomial is $r^2 + 4r + 3$. Hence $\boxed{y'' + 4y' + 3y = 0}$. Direct differentiation gives $y' = -e^{-t} - 27e^{-3t}$, so

$$\boxed{y(0) = 10, \qquad y'(0) = -28}.$$

The response is positive because both mode terms are positive.

Step 2: Read the rate as a changing weighted average. Let $z = 9e^{-2t} > 0$. Then

$$R = \frac{e^{-t} + 27e^{-3t}}{e^{-t} + 9e^{-3t}} = \frac{1 + 3z}{1 + z}.$$

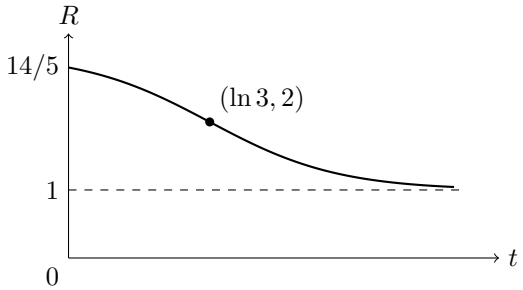
This averages 1 and 3 with positive weights $1, z$, so $1 < R < 3$. In particular,

$$\boxed{R(0) = 14/5}, \qquad \boxed{R \to 1}, \qquad R = 2 \iff z = 1 \iff \boxed{t = \ln 3}.$$

Step 3: Derive the rate equation. Using $y'' = -4y' - 3y$ and $y'/y = -R$,

$$R' = -\frac{y''}{y} + (y'/y)^2 = R^2 - 4R + 3 = (R - 1)(R - 3) < 0.$$

Thus R is strictly decreasing; its long-time limit is the slower mode's rate, rather than an unchanging average of the roots.



The effective decay rate decreases toward 1.

Step 4: Compare with a constant-rate prediction. Set $f = \ln y$. Then $f' = -R$ and $f'' = -R' > 0$. Since f' is strictly increasing, integration on $[0, t]$ for $t > 0$ gives

$$f(t) > f(0) + tf'(0) = \ln 10 - \frac{14}{5}t.$$

Exponentiation preserves this inequality, so $\boxed{y(t) > 10e^{-(14/5)t}}$ for every $t > 0$. The extrapolation matches the initial value and slope but decays too quickly afterward. The changing mode weights, not an error in the initial data, cause the discrepancy.

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Question 9

Let D denote differentiation with respect to t. Consider

$$y'' - 4y' + 3y = 0, \qquad y(0) = 2, \qquad y'(0) = 4.$$

A student reasons that $(D-3)(D-1)y = 0$ means every solution must satisfy either $(D-1)y = 0$ or $(D-3)y = 0$.

Tasks

1. Verify the operator factorization by expanding it on a twice-differentiable function.
2. Set $z = y' - y$. Derive and solve its first-order initial-value problem.
3. Solve $y' - y = z$ using an integrating factor and the original value $y(0) = 2$. Verify the resulting solution and both initial data.
4. Test the student's either/or claim on your solution. Explain why a product of differential operators must not be interpreted as a product of two numbers acting separately on y.

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Question 9 – Solution

Strategy. Factorization creates a sequence of two first-order equations, not a rule requiring one of two first-order residuals to vanish.

Step 1: Expand the operator composition. Applying $(D-3)$ to $(D-1)y = y' - y$ gives

$$(y' - y)' - 3(y' - y) = y'' - 4y' + 3y.$$

Thus the factorization is valid and its characteristic roots are 1 and 3.

Step 2: Solve for the intermediate quantity. With $z = y' - y$, the equation is $z' - 3z = 0$. The data imply $z(0) = 4 - 2 = 2$. Separating or using an integrating factor yields

$$\boxed{z = 2e^{3t}}.$$

The intermediate residual is nonzero, yet it is annihilated by $D - 3$.

Step 3: Recover the original solution. We must solve $y' - y = 2e^{3t}$. Multiplication by e^{-t} gives

$$(e^{-t}y)' = 2e^{2t}.$$

Integrating from 0 to t and using $y(0) = 2$ gives $e^{-t}y - 2 = e^{2t} - 1$. Hence

$$\boxed{y = e^t + e^{3t}}.$$

Indeed $y(0) = 2$, $y'(0) = 1 + 3 = 4$, and

$$y'' - 4y' + 3y = (1 - 4 + 3)e^t + (9 - 12 + 3)e^{3t} = 0.$$

The construction recovers both modes by first-order methods and gives the unique solution with the prescribed data.

Step 4: Refute the incorrect either/or inference. For this solution,

$$(D-1)y = 2e^{3t} \neq 0, \qquad (D-3)y = -2e^t \neq 0.$$

Nevertheless, $(D-3)((D-1)y) = (D-3)(2e^{3t}) = 0$. An operator composition first produces a new function and then applies another operator to it. A nonzero function can be sent to zero by differentiation combined with multiplication. The scalar zero-product rule therefore does not imply the proposed either/or claim. Sums of the two modes must be retained, not just the two separate pure-mode families.

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Question 10

For $0 < \varepsilon \leq 1$, consider

$$y'' - 2y' + (1 - \varepsilon^2)y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

All parameter values in the problem have real distinct roots. You may use, for $z \geq 0$, the Taylor bound

$$0 \leq \frac{e^z - e^{-z}}{2} - z \leq \frac{z^3 e^z}{6}.$$

Tasks

1. Find the roots and the exact solution y_ε, including both mode coefficients.
2. For fixed $t \geq 0$, find the limit of $y_\varepsilon(t)$ as $\varepsilon \downarrow 0$. Explain how diverging mode coefficients can still produce a finite limit.
3. Prove an explicit bound on $|y_\varepsilon(t) - te^t|$ for $0 \leq t \leq T$, where $T > 0$ is fixed. Deduce uniform convergence on that interval.
4. Explain the numerical danger in subtracting the two close exponentials when εt is small. Derive an equivalent expression using $H(z) = (e^z - e^{-z})/(2z)$, with $H(0) = 1$, and a small-z expansion that avoids that subtraction. Do not solve the limiting differential equation as a separate root case.

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Question 10 – Solution

Strategy. Keep the roots distinct while studying cancellation in the data-to-coefficient representation and a controlled limit of its solutions.

Step 1: Solve the distinct-root IVP. The characteristic polynomial is $(r-1)^2 - \varepsilon^2$, with roots $1+\varepsilon$, $1-\varepsilon$. Writing their coefficients as A, B gives $A + B = 0$, $(1+\varepsilon)A + (1-\varepsilon)B = 1$. Thus

$$\boxed{A = 1/(2\varepsilon), \quad B = -1/(2\varepsilon)}, \qquad \boxed{y_\varepsilon = \frac{e^{(1+\varepsilon)t} - e^{(1-\varepsilon)t}}{2\varepsilon}}.$$

Both modes satisfy the equation; the value and derivative at zero are 0 and 1.

Step 2: Take a function limit with cancellation intact. Factoring out e^t and using the derivative of the exponential at zero gives

$$y_\varepsilon = e^t \frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2\varepsilon} \longrightarrow \boxed{te^t}.$$

At $t = 0$ the identity and limit are both zero. The coefficients grow without bound, but the two modes approach each other and their leading contributions cancel. Large coefficients in this basis do not alone imply large solution values on a fixed bounded interval.

Step 3: Control the limit on the whole finite interval. Apply the supplied bound with $z = \varepsilon t$ and multiply by e^t/ε:

$$0 \le y_\varepsilon - te^t \le \frac{\varepsilon^2 t^3 e^{(1+\varepsilon)t}}{6} \le \boxed{\frac{\varepsilon^2 T^3 e^{2T}}{6}} \qquad (0 \le t \le T).$$

The last expression tends to zero independently of t, proving uniform convergence on each fixed $[0, T]$. It is not a uniform statement over all $t \ge 0$.

Step 4: Separate algebraic cancellation from its numerical evaluation. The direct numerator subtracts two nearly equal numbers, potentially losing relative accuracy in rounded arithmetic. An equivalent expression is

$$\boxed{y_\varepsilon = te^t H(\varepsilon t)}, \qquad H(z) = 1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6) \quad (z \to 0).$$

This expansion follows by subtracting the Taylor series of e^z and e^{-z} before numerical evaluation. Evaluating H by this series for small z avoids the close-number subtraction; merely renaming the original quotient would not. The limit concerns this distinct-root family and does not require a general solution method for a repeated root.

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Question 1

Consider

$$2y'' + 4y' + 10y = 0, \qquad y(1) = 2, \qquad y'(1) = 2.$$

You may use Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ and the continuous-coefficient linear initial-value uniqueness theorem.

Tasks

1. Find the characteristic roots. Use their complex exponential solutions to construct two real solutions.
2. Write a real solution family based at $t = 1$ and determine the coefficients from the initial data.
3. Verify the solution in the original equation and both data. Show that the real family can realize arbitrary initial value and slope at 1, and hence is complete.
4. In the complex form $c_+e^{(-1+2i)(t-1)} + c_-e^{(-1-2i)(t-1)}$, determine c_+, c_- for this IVP. Explain why taking both coefficients real and equal would lose some real solutions.

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Question 1 – Solution

Strategy. Pair conjugate exponentials to obtain real functions, retaining two independent real data freedoms.

Step 1: Find the roots and real building blocks. The characteristic equation is $2r^2 + 4r + 10 = 0$, or $(r+1)^2 + 4 = 0$, giving

$$\boxed{r = -1 \pm 2i}.$$

The half-sum and the difference divided by $2i$ of $e^{(-1\pm 2i)t}$ give, respectively, $e^{-t}\cos 2t$ and $e^{-t}\sin 2t$. These solve the real equation because it is linear with real coefficients.

Step 2: Fit the shifted family. Set $x = t - 1$ and write $y = e^{-x}(A\cos 2x + B\sin 2x)$. At $x = 0$, $y = A$ and $y' = -A + 2B$. Thus $A = 2$ and $B = 2$, so

$$\boxed{y = 2e^{-(t-1)}(\cos(2(t-1)) + \sin(2(t-1)))}.$$

Step 3: Verify the equation and completeness. Writing $y = e^{-x}u$ gives $y' = e^{-x}(u' - u)$ and $y'' = e^{-x}(u'' - 2u' + u)$. Hence

$$2y'' + 4y' + 10y = 2e^{-x}(u'' + 4u) = 0.$$

The value and slope at 1 are 2 and $-2 + 4 = 2$. More generally, data $y(1) = a$, $y'(1) = b$ give $A = a$, $B = (a+b)/2$, uniquely. Any solution shares its data with one family member; the linear theorem forces equality on $\mathbb{R}$.

Step 4: Interpret the complex constants correctly. The real combination satisfies

$$A\cos 2x + B\sin 2x = \frac{A - iB}{2}e^{2ix} + \frac{A + iB}{2}e^{-2ix}.$$

Thus $\boxed{c_+ = 1 - i, \quad c_- = 1 + i}$ for this IVP. The constants are conjugates, so the imaginary parts cancel. Equal real constants would force $B = 0$, permitting only initial slopes $b = -a$ in the shifted basis. Real solutions require conjugate coefficients, not necessarily real coefficients in the complex basis.

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Question 2

Consider

$$y'' + 2y' + 5y = 0, \qquad y(0) = -1, \qquad y'(0) = 3.$$

Use the phase convention $y = Re^{-t}\cos(2t - \delta)$ with $R > 0$ and $0 \le \delta < 2\pi$.

Tasks

1. Solve the IVP in sine/cosine form.
2. Determine R, δ in the stated convention and verify the initial value and derivative in phase form.
3. Find every zero for $t \ge 0$, the sign immediately after the first zero, and the spacing between successive zeros.
4. Explain why solving only $\tan\delta = B/A$ can give the wrong phase quadrant. Diagnose the choice $\delta = -\pi/4$ with positive amplitude for this IVP.

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Question 2 – Solution

Strategy. Determine the sine and cosine coefficients before choosing the phase quadrant; then use the phase to locate zeros.

Step 1: Solve the real IVP. The roots are $-1 \pm 2i$. For $y = e^{-t}(A\cos 2t + B\sin 2t)$, the data give $A = -1$ and $-A + 2B = 3$. Thus $B = 1$ and

$$\boxed{y = e^{-t}(-\cos 2t + \sin 2t)}.$$

Step 2: Choose the correct quadrant. Expansion of $R\cos(2t - \delta)$ gives $A = R\cos\delta$, $B = R\sin\delta$. Therefore

$$R = \sqrt{A^2 + B^2} = \sqrt{2}, \qquad \cos\delta = -1/\sqrt{2}, \quad \sin\delta = 1/\sqrt{2}.$$

The prescribed convention selects $\boxed{\delta = 3\pi/4}$, yielding

$$\boxed{y = \sqrt{2}e^{-t}\cos(2t - 3\pi/4)}.$$

At zero the value is -1, while the derivative is $\sqrt{2}[-\cos(-3\pi/4) - 2\sin(-3\pi/4)] = 3$, as required.

Step 3: Enumerate the nonnegative zeros. The exponential is never zero. The first zero occurs when $2t - 3\pi/4 = -\pi/2$, so all nonnegative zeros are

$$\boxed{t_n = \frac{\pi}{8} + \frac{n\pi}{2}, \qquad n = 0, 1, 2, \ldots}.$$

At t_0, the derivative is $2\sqrt{2}e^{-\pi/8} > 0$, so the solution crosses from negative to positive. Successive zeros are separated by $\pi/2$, half the trigonometric period π. The decaying solution itself is not periodic.

Step 4: Resolve the tangent ambiguity. The equation $\tan\delta = -1$ does not distinguish quadrants II and IV. With positive $R = \sqrt{2}$, the phase $-\pi/4$ gives $A = 1$, $B = -1$, which is the negative of the required solution. It has value 1 and slope -3 at zero. The signs of both sine and cosine, together with the amplitude convention, determine the phase; a tangent value alone does not.

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Question 3

For the response $y(t) = e^{-t}\cos 2t$ on $t \geq 0$, the curves $y = \pm e^{-t}$ are called its exponential envelopes because $|y(t)| \leq e^{-t}$.

Tasks

1. Find a monic homogeneous equation with this solution and verify its initial value and slope.
2. Find all positive-time stationary points, classify them, and identify the first minimum and the first positive-time maximum.
3. Determine where the response touches an envelope. Are these times stationary points? Find the ratio of each stationary-point magnitude to the envelope height there.
4. Sketch the response and envelopes on $[0, 2\pi]$, marking the first minimum. Determine the ratio of consecutive positive local maxima and explain why envelope contacts and extrema must be distinguished.

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Question 3 – Solution

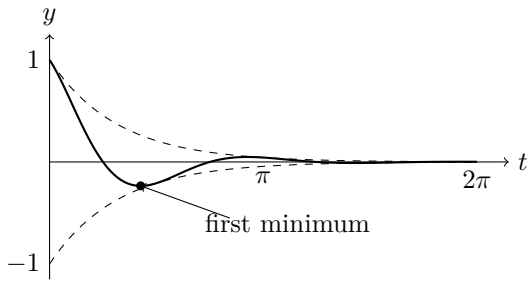
Strategy. Differentiate the entire product, including the exponential factor, before locating peaks.

Step 1: Verify the complex-root model. The roots $-1 \pm 2i$ give $\boxed{y'' + 2y' + 5y = 0}$. Differentiation yields $y' = e^{-t}(-\cos 2t - 2\sin 2t)$, so $y(0) = 1$ and $y'(0) = -1$. Substituting a second derivative verifies the equation.

Step 2: Locate and classify the extrema. Let $\theta = \arctan(1/2) \in (0, \pi/2)$. The stationary condition is $\cos 2t + 2\sin 2t = 0$, giving

$$\boxed{t_k = (k\pi - \theta)/2, \qquad k = 1, 2, \dots}.$$

At these times $\cos 2t_k = (-1)^k \cos\theta = (-1)^k 2/\sqrt{5}$. Since $y' = 0$, the ODE gives $y'' = -5y$. Thus odd k give negative minima and even k give positive maxima. The first minimum is at $(\pi - \theta)/2$; the first positive-time maximum is at $\pi - \theta/2$.



Solid: response; dashed: envelopes

Step 3: Compare extrema with envelope contacts. Equality $|y| = e^{-t}$ requires $|\cos 2t| = 1$, or $t = n\pi/2$ for $n \geq 0$. At those times $y' = -y \neq 0$, so they are not stationary points. The response does share the tangent of the corresponding envelope there.

At every stationary point,

$$\boxed{|y(t_k)| = \frac{2}{\sqrt{5}} e^{-t_k} < e^{-t_k}}.$$

The decreasing envelope shifts the extrema away from cosine peaks.

Step 4: Compare like-signed peaks. Consecutive positive local maxima have indices k and $k + 2$, so their time separation is π. Their common trigonometric factor cancels, giving the height ratio $\boxed{e^{-\pi}}$. This is the envelope decay over a full cycle, even though the maxima occur strictly below the envelopes. At $t = 0$ the solution has a one-sided endpoint maximum; it is not a positive-time stationary maximum.

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Question 4

A nonzero solution of an unknown monic equation $y'' + py' + qy = 0$ has characteristic roots $\alpha \pm i\beta$, where $\beta > 0$. Two consecutive positive local maxima occur at $t = 1$ and $t = 3$, with respective heights 4 and 1. The equation has real constant coefficients, and there is no forcing.

Tasks

1. Determine β and α from the peak times and heights. Justify why positive maxima are separated by a full cycle even when $\alpha \neq 0$.
2. Recover p, q and write the equation exactly.
3. Reconstruct the unique solution from the peak data at $t = 1$. Verify that the point at $t = 3$ is also a positive local maximum of height 1.
4. If the word consecutive were removed, would the two peak observations still uniquely determine the equation? Describe the resulting possibilities for β and explain the ambiguity.

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Question 4 – Solution

Strategy. Use the interval between like-signed extrema for frequency and their height ratio for the real part of the roots.

Step 1: Extract frequency and decay. Write $y = Re^{\alpha t}\cos(\beta t - \delta)$. Stationarity requires $\alpha\cos(\beta t - \delta) - \beta\sin(\beta t - \delta) = 0$. Successive extrema have phase separation π and alternate sign, so consecutive positive maxima are separated by $2\pi/\beta$.

The observed separation is 2, giving $\boxed{\beta = \pi}$. The trigonometric factors at those maxima agree, so $1/4 = e^{2\alpha}$ and

$$\boxed{\alpha = -\ln 2}.$$

Step 2: Recover the real coefficients. Let $k = \ln 2 > 0$. The characteristic polynomial is $(r+k)^2 + \pi^2$, hence

$$\boxed{y'' + 2ky' + (k^2 + \pi^2)y = 0}.$$

Step 3: Reconstruct from value and zero slope. Using $x = t - 1$, write $y = e^{-kx}(A\cos\pi x + B\sin\pi x)$. The peak gives $A = 4$ and $-kA + \pi B = 0$, so

$$\boxed{y = e^{-k(t-1)}[4\cos(\pi(t-1)) + \frac{4k}{\pi}\sin(\pi(t-1))]}.$$

At $t = 3$, $x = 2$, and $y = 4e^{-2k} = 1$, $y' = 0$. At either point the equation gives $y'' = -(k^2 + \pi^2)y < 0$, verifying strict positive local maxima. The solution is unique because its value and slope at 1 are specified for a regular linear equation.

Step 4: Identify what consecutiveness contributes. Without that condition, the time interval 2 could contain m full cycles, where m is any positive integer. Then $\boxed{\beta = m\pi}$ while the height ratio still fixes $\alpha = -\ln 2$. Each choice gives a different equation and a solution with the stated peaks, but for $m > 1$ there are additional positive maxima between them. The observed heights alone do not resolve the missing cycle count.

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Question 5

For a real parameter a, consider

$$y'' + 2ay' + (a^2 + 9)y = 0.$$

All qualitative claims concern $t \geq 0$. A solution is nontrivial if it is not identically zero.

Tasks

1. Find the roots and the real general solution for every a.
2. Classify when every solution tends to zero and when every solution is bounded. Describe the nontrivial behavior at $a = 0$.
3. Prove that every nontrivial solution has infinitely many simple zeros, with the same spacing for all a. Can any such solution be eventually of one strict sign?
4. For $a < 0$, distinguish unbounded magnitude from the assertion $|y(t)| \to \infty$. Give explicit sequences of times that prove the correct conclusions.

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Question 5 – Solution

Strategy. Separate the exponential amplitude from the trigonometric zero pattern; growth of the former does not erase zeros of the latter.

Step 1: Find the universal complex-root form. The polynomial is $(r+a)^2+9$, so the roots are $\boxed{-a \pm 3i}$. Hence

$$y = e^{-at}(A\cos 3t + B\sin 3t).$$

For a nontrivial solution, $R = \sqrt{A^2+B^2} > 0$ and some phase δ give $y = Re^{-at}\cos(3t-\delta)$.

Step 2: Classify decay and boundedness. If $a > 0$, $|y| \le Re^{-at} \to 0$. At $a = 0$, all solutions are bounded, and every nontrivial one is periodic with least positive period $2\pi/3$ and does not tend to zero. If $a < 0$, the magnitudes at trigonometric peaks grow without bound. Therefore

$$\boxed{\text{all solutions decay} \iff a > 0}, \qquad \boxed{\text{all solutions are bounded} \iff a \ge 0}.$$

The zero solution is bounded and decays for every parameter value.

Step 3: Track the zeros and their slopes. Zeros occur when $3t - \delta = \pi/2 + n\pi$, so

$$t_n = \frac{\delta + \pi/2 + n\pi}{3},$$

with all integers n giving nonnegative times retained. There are infinitely many, spaced by $\boxed{\pi/3}$. At each zero,

$$y'(t_n) = -3Re^{-at_n}\sin(\pi/2 + n\pi) \neq 0.$$

Thus every zero is simple and the sign changes there. No nontrivial solution is eventually strictly positive or strictly negative, even when its amplitude decays.

Step 4: State growth without an incorrect limit. For $a < 0$, choose large integers n and set $s_n = (\delta + 2n\pi)/3$. Then $y(s_n) = Re^{-as_n} \to +\infty$. At $u_n = (\delta + (2n+1)\pi)/3$, $y(u_n) = -Re^{-au_n} \to -\infty$. But along the zero sequence, $|y(t_n)| = 0$.

Consequently the solution has unbounded magnitude, $\limsup y = +\infty$ and $\liminf y = -\infty$, while $\liminf |y| = 0$. The limit $|y(t)| \to \infty$ is false. A growing envelope describes arbitrarily large excursions, not a lower bound on the response at every late time.

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Question 6

For constants $a \in \mathbb{R}$, $b > 0$, consider

$$y'' + 2ay' + (a^2 + b^2)y = 0.$$

Define transformed coordinates

$$u = e^{at}y, \qquad v = \frac{e^{at}(y' + ay)}{b}.$$

The axes in a plot of (u, v) are these transformed quantities, not time and displacement.

Tasks

1. Differentiate u, v and prove $u' = bv$, $v' = -bu$.
2. Deduce a conserved quantity and express its value using $y(0) = Y$, $y'(0) = V$. Relate it to the amplitude of the trigonometric factor in y.
3. For $a = 1$, $b = 2$, $Y = 1$, $V = -1$, find y, u, v. Sketch the transformed curve with its initial point, direction and quarter-cycle points.
4. Explain why the circle in these transformed coordinates does not imply constant amplitude of the original response. Determine exactly which original initial data make the conserved quantity zero and what solution follows.

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Question 6 – Solution

Strategy. Remove the exponential factor and scale the remaining derivative so the two trigonometric coordinates have equal amplitudes.

Step 1: Derive the transformed equations. Directly, $u' = e^{at}(y' + ay) = bv$. Also,

$$v' = \frac{e^{at}}{b}(y'' + 2ay' + a^2 y) = \frac{e^{at}}{b}(-b^2 y) = -bu.$$

The original equation is essential in the second equality.

Step 2: Identify the invariant and amplitude. Differentiating gives $(u^2+v^2)' = 2u(bv)+2v(-bu) = 0$. At zero,

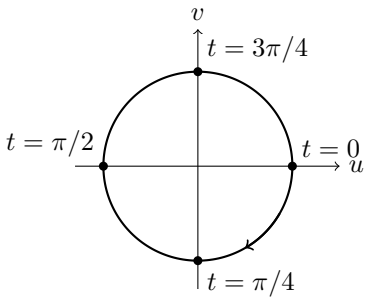
$$\boxed{u^2 + v^2 = Y^2 + \frac{(V + aY)^2}{b^2} = R^2}.$$

The real solution has trigonometric coefficients $A = Y$, $B = (V + aY)/b$, so $R = \sqrt{A^2 + B^2}$ is exactly its trigonometric amplitude.

Step 3: Trace the concrete circle. For the stated data, $A = 1$, $B = 0$. Thus

$$\boxed{y = e^{-t}\cos 2t}, \qquad u = \cos 2t, \quad v = -\sin 2t.$$

The initial point is $(1, 0)$ and its velocity is $(0, -2)$, so motion is clockwise. At times $0, \pi/4, \pi/2, 3\pi/4$ the points are $(1, 0), (0, -1), (-1, 0), (0, 1)$, respectively; the cycle closes at $t = \pi$.



$u^2 + v^2 = 1$; equal axis scales

Step 4: Translate back to the original quantity. The transformation removed e^{-at}: the original response satisfies $|y| \le Re^{-at}$. Its amplitude decays for $a > 0$, is constant for $a = 0$, and grows for $a < 0$. A circle in (u, v) describes a normalized oscillation, not constant original displacement amplitude or a claim about physical energy.

Because $b > 0$, the sum of squares R^2 is zero exactly when $Y = 0$ and $V + aY = 0$, hence exactly when $Y = V = 0$. These data give $y \equiv 0$, consistent with initial-value uniqueness.

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Question 7

An unknown frequency $\beta > 0$ appears in

$$y'' + \beta^2 y = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Exact observations at all nonnegative integer times report $y(n) = 0$.

Tasks

1. Solve the IVP in terms of β and determine every frequency compatible with the integer observations.
2. Explain why the observations, even with the known initial slope, do not imply the zero solution and do not identify a unique frequency.
3. An additional exact observation gives $y(1/2) = 1/\pi$. Determine which of the previously possible frequencies remain. Prove that no larger candidate frequency can fit it.
4. Sketch the candidates with $\beta = \pi$ and $\beta = 3\pi$ on $[0, 2]$, marking the integer sampling times. Explain what feature of the observation schedule hides their different oscillations.

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Question 7 – Solution

Strategy. Solve before interpreting the data; a sampling grid can repeatedly land on zeros of a nonzero oscillation.

Step 1: Determine all admissible frequencies. The roots are $\pm i\beta$, and the data give

$$\boxed{y(t) = \frac{\sin \beta t}{\beta}}.$$

In particular, $y(1) = 0$ implies $\sin \beta = 0$, so $\beta = m\pi$ for a positive integer m. Conversely, each such frequency makes every integer observation zero. The complete candidate set is therefore

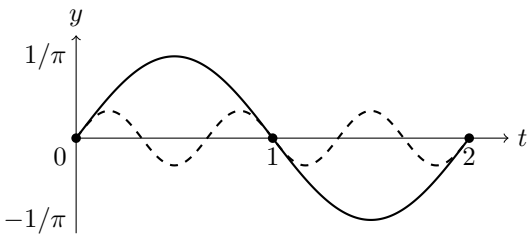
$$\boxed{\beta = m\pi, \qquad m = 1, 2, 3, \ldots}.$$

Step 2: Retain the slope information correctly. Each candidate has $y'(t) = \cos(m\pi t)$ and hence $y'(0) = 1$, so each is nonzero. The zero function would violate that slope. The slope sets the sine coefficient to $1/(m\pi)$, but still allows infinitely many frequencies. A known initial slope does not rescue a schedule that misses all intermediate oscillations.

Step 3: Use the extra observation to select a frequency. For the candidate indexed by m,

$$y(1/2) = \frac{\sin(m\pi/2)}{m\pi}.$$

For $m = 1$ this equals $1/\pi$. For every $m > 1$, its absolute value is at most $1/(m\pi) < 1/\pi$, making agreement impossible. Thus the new datum selects exactly $\boxed{\beta = \pi}$ and $y = \sin(\pi t)/\pi$.



Solid: $\beta = \pi$; dashed: $\beta = 3\pi$; dots: samples

Step 4: Explain the indistinguishable samples. The integer grid samples an integer number of half-cycles for every candidate, so it always observes a zero. The two plotted curves differ between those times, including in amplitude, yet share all grid values and the initial slope. This is an explicit sampling ambiguity, not a violation of uniqueness: each fixed equation has one solution for its data, while different frequencies define different equations.

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Question 8

Consider the endpoint problem

$$y'' + 2y' + 5y = 0, \qquad y(0) = A, \qquad y(L) = B, \qquad L > 0.$$

The numbers A, B are endpoint values, not mode coefficients.

Tasks

1. Derive the endpoint equation for the remaining sine coefficient. Determine all lengths L for which every pair of endpoint values gives a unique solution.
2. At each excluded length, give the exact compatibility condition for existence and classify the number of solutions.
3. For $A = 0$, $B = 1$, solve when $L = \pi/4$ and determine what happens at $L = \pi/2$.
4. Keep $A = 0$, $B = 1$ but take $L = \pi/2 + \varepsilon$, where $0 < \varepsilon < \pi/4$. Find $y(\pi/4)$ and its leading behavior as $\varepsilon \downarrow 0$. Explain how unique solvability for every such ε can coexist with arbitrarily large interior values.

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Question 8 – Solution

Strategy. An exponential prefactor never vanishes, so solvability and sensitivity are governed by the sine factor in the endpoint data map.

Step 1: Reduce to one coefficient. The roots are $-1 \pm 2i$. The first endpoint fixes the cosine coefficient, giving $y = e^{-t}(A\cos 2t + C\sin 2t)$. The second requires

$$C\sin 2L = Be^{L} - A\cos 2L.$$

If $\sin 2L \neq 0$, there is exactly one C, so unique solvability for arbitrary data holds exactly when

$$\boxed{L \neq n\pi/2 \quad (n = 1, 2, 3, \ldots)}.$$

Step 2: Classify the exceptional lengths. At $L = n\pi/2$, the condition becomes $0 = Be^{L} - (-1)^{n}A$. Thus

$$\boxed{B = (-1)^{n}e^{-L}A}$$

is necessary and sufficient for existence. If it holds, every real C works, giving infinitely many solutions. If it fails, there is no solution. The initial-value uniqueness theorem does not promise endpoint uniqueness at these lengths.

Step 3: Apply the two specified lengths. For $A = 0$, $B = 1$, $L = \pi/4$, we have $C = e^{\pi/4}$ and

$$\boxed{y = e^{\pi/4 - t}\sin 2t}.$$

At $L = \pi/2$, compatibility would require $B = 0$, contradicting $B = 1$, so no solution exists.

Step 4: Quantify sensitivity near an exceptional length. For $L = \pi/2 + \varepsilon$, $\sin 2L = -\sin 2\varepsilon \neq 0$. Hence

$$y(t) = -\frac{e^{L-t}\sin 2t}{\sin 2\varepsilon}, \qquad \boxed{y(\pi/4) = -\frac{e^{\pi/4+\varepsilon}}{\sin 2\varepsilon}}.$$

Using $\sin 2\varepsilon \sim 2\varepsilon$ and $e^{\varepsilon} \to 1$ gives

$$\boxed{y(\pi/4) \sim -\frac{e^{\pi/4}}{2\varepsilon} \longrightarrow -\infty}.$$

For each positive ε there is one finite coefficient, but the denominator approaches zero as the length approaches the incompatible endpoint problem. Uniqueness at each parameter value is not a uniform bound on the dependence on that parameter.

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Question 9

For the decaying oscillation $y(t) = e^{-t}\sin t$, define its signed total area and its total absolute area by

$$I = \int_0^\infty y(t)\,dt, \qquad J = \int_0^\infty |y(t)|\,dt.$$

Tasks

1. Find the corresponding monic homogeneous equation and initial data. State the signs on consecutive half-cycles.
2. Find an antiderivative of $e^{-t}\sin t$ and calculate I.
3. Calculate J exactly by comparing the absolute areas of successive half-cycles. Explain why $J > I$ even though both are finite.
4. Sketch the first two half-cycles with their signed areas indicated. Bound the absolute tail after an arbitrary $T \geq 0$, and give its exact value when $T = n\pi$ for a nonnegative integer n.

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Question 9 – Solution

Strategy. Separate cancellation between lobes from decay of each lobe, then sum the absolute areas as a geometric series.

Step 1: Identify the equation and signs. The roots are $-1 \pm i$, so $\boxed{y'' + 2y' + 2y = 0}$. The derivative is $e^{-t}(\cos t - \sin t)$, giving $y(0) = 0$, $y'(0) = 1$. On $(n\pi, (n+1)\pi)$, the sign is $(-1)^n$: positive first, negative next, and alternating thereafter.

Step 2: Integrate the signed response. Integration by parts twice, or differentiation of the result, gives

$$F(t) = -\frac{1}{2}e^{-t}(\sin t + \cos t), \qquad F' = e^{-t}\sin t.$$

Since $F(t) \to 0$ as $t \to \infty$ and $F(0) = -1/2$,

$$\boxed{I = 1/2}.$$

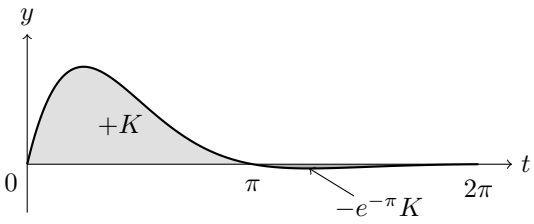
Step 3: Sum the magnitudes without cancellation. The first positive lobe has area

$$K = F(\pi) - F(0) = \frac{1 + e^{-\pi}}{2}.$$

The substitution $t = n\pi + s$ shows that the absolute area of the nth lobe is $e^{-n\pi}K$. Thus the convergent geometric sum gives

$$\boxed{J = \frac{K}{1 - e^{-\pi}} = \frac{1 + e^{-\pi}}{2(1 - e^{-\pi})} > \frac{1}{2}}.$$

The negative lobes reduce I but contribute positively to J. Infinitely many oscillations do not prevent absolute convergence because the lobe magnitudes decrease geometrically.



Signed lobe areas; both magnitudes contribute to J.

Step 4: Control the unobserved tail. Since $|\sin t| \le 1$,

$$\boxed{\int_T^\infty |y(t)|\,dt \le \int_T^\infty e^{-t}\,dt = e^{-T}}.$$

At a half-cycle boundary, substitution by $n\pi$ preserves $|\sin|$, so the exact absolute tail is $\boxed{e^{-n\pi}J}$. The signed area of the second plotted lobe is $-e^{-\pi}K$, while its contribution to J is $+e^{-\pi}K$.

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Question 10

Consider

$$y'' + 4y' + 13y = 0, \qquad y(0) = 3, \qquad y'(0) = 6.$$

A settling certificate at time $T \geq 0$ means a proof that $|y(t)| \leq 0.1$ for every $t \geq T$. Distinguish a certificate for this specific solution from one valid for every phase with the same trigonometric amplitude.

Tasks

1. Find the solution, its amplitude R, and its phase $\delta \in [0, 2\pi)$ in the form $Re^{-2t}\cos(3t - \delta)$.
2. Use the envelope to find the smallest certificate time valid for every phase with amplitude R. Prove the claimed optimality over phases.
3. For the stated IVP, show that $T = 1.9$ is already a certificate. Use the value at 1.9 and the magnitudes of all later stationary points, justifying their ordering. Numerical evaluations should be supported by exact expressions.
4. Sketch the response for $t \geq 1.5$ with the tolerance levels and envelope certificate marked. Explain why the envelope certificate need not be the earliest settling time for one fixed phase, and whether Task 3 identifies that exact earliest time.

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Question 10 – Solution

Strategy. First obtain a phase-independent bound, then sharpen it for one solution by controlling every later extremum.

Step 1: Resolve the data and phase. The roots are $-2 \pm 3i$. Writing $y = e^{-2t}(A\cos 3t + B\sin 3t)$ gives $A = 3$ and $-2A + 3B = 6$, hence $B = 4$. Thus

$$\boxed{y = e^{-2t}(3\cos 3t + 4\sin 3t) = 5e^{-2t}\cos(3t - \delta)}, \qquad \delta = \arctan(4/3) \in (0, \pi/2).$$

Step 2: Obtain the best certificate for every phase. The envelope condition is $5e^{-2t} \le 0.1$, so

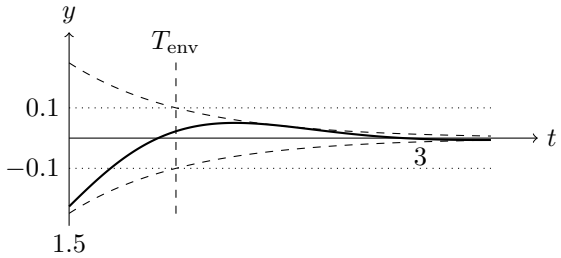
$$\boxed{T_{\text{env}} = \tfrac{1}{2}\ln 50 \approx 1.956012}.$$

It guarantees the tolerance at every later time. For any proposed $0 \le T < T_{\text{env}}$, choose a phase with $3T - \delta$ an integer multiple of 2π. That solution has $|y(T)| = 5e^{-2T} > 0.1$. Thus no earlier time works for every phase with amplitude 5.

Step 3: Check the fixed phase more sharply. Let $\theta = \arctan(2/3)$. Stationary times are $t_k = (\delta - \theta + k\pi)/3$, and their magnitudes are $(15/\sqrt{13})e^{-2t_k}$. In particular,

$$t_1 \approx 1.160295 < 1.9 < t_2 \approx 2.207493, \qquad |y(t_2)| \approx 0.050317 < 0.1.$$

Also $y(1.9) = e^{-3.8}(3\cos 5.7 + 4\sin 5.7) \approx 0.006742$. Between 1.9 and t_2 there is no stationary point, so the response is monotone and its absolute value is bounded by the larger endpoint magnitude. Each subsequent stationary magnitude is multiplied by $e^{-2\pi/3} < 1$. Monotonicity between those extrema then proves $\boxed{|y(t)| \le 0.1 \text{ for all } t \ge 1.9}$.



Solid: response; dashed: envelopes; dotted: tolerance

Step 4: Distinguish a certificate from the exact last crossing. The envelope treats all phases, whereas the specified phase can have small values while the envelope is still above 0.1. The preceding negative minimum has magnitude greater than 0.1, so a final tolerance crossing occurs between t_1 and 1.9. Task 3 proves a sufficient time, not that 1.9 is the earliest one; locating the exact last crossing requires solving the corresponding trigonometric-exponential level equation.

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Question 1

Consider

$$4y'' - 12y' + 9y = 0, \qquad y(2) = -1, \qquad y'(2) = 1/2.$$

A student proposes $y = C_1e^{3t/2} + C_2e^{3t/2}$ as the general solution because the characteristic root occurs twice.

Tasks

1. Find the characteristic root and explain precisely which initial-data freedom the proposed family lacks.
2. Put $x = t - 2$ and $y = e^{3x/2}u(x)$. Differentiate to reduce this particular equation to an equation for u, and obtain the full solution family.
3. Fit the initial data and verify the resulting solution in the original equation and both data.
4. Show that the derived family can represent arbitrary value and slope at $t = 2$. Justify completeness and explain why multiplying one exponential by $t - 2$ creates a new freedom while merely naming a second coefficient does not.

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Question 1 – Solution

Strategy. Remove the common exponential in this constant-coefficient equation and determine what equation remains for its multiplier.

Step 1: Identify the repeated root and lost freedom. The characteristic polynomial is $(2r-3)^2$, so $\boxed{r=3/2}$ is repeated. The student's family equals $(C_1+C_2)e^{3t/2}$ and has only one effective constant. It always satisfies $y'=(3/2)y$, so at $t=2$ a value of -1 would force slope $-3/2$, not $1/2$.

Step 2: Derive the missing multiplier. With $x=t-2$ and $y=e^{3x/2}u$,

$$y'=e^{3x/2}(u'+3u/2), \qquad y''=e^{3x/2}(u''+3u'+9u/4).$$

Substitution gives $4y''-12y'+9y=4e^{3x/2}u''$. Thus $u''=0$ and $u=A+Bx$, yielding

$$\boxed{y=(A+B(t-2))e^{3(t-2)/2}}.$$

Step 3: Apply and verify the data. At $t=2$, $y=A$ and $y'=B+3A/2$. Hence $A=-1$, $B=2$, so

$$\boxed{y=(-1+2(t-2))e^{3(t-2)/2}}.$$

The multiplier has zero second derivative, making the original residual zero by Step 2. At 2, the value is -1 and the derivative is $2-3/2=1/2$.

Step 4: Establish the two independent data freedoms. For arbitrary data $y(2)=a$, $y'(2)=b$, we obtain $A=a$, $B=b-3a/2$, uniquely. The transformation is invertible because the exponential never vanishes, and integrating $u''=0$ gives all multipliers. Alternatively, continuous-coefficient IVP uniqueness proves completeness after the data map is verified.

The function $(t-2)e^{3(t-2)/2}$ has zero value but nonzero derivative at the base point, allowing slope to change without changing the prescribed value. A second copy of the same exponential cannot do that.

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Question 2

Consider

$$y'' + 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = -3.$$

All qualitative conclusions below concern $t \geq 0$.

Tasks

1. Solve the IVP using the repeated-root family and verify both data.
2. Find every nonnegative zero and determine the sign on either side. Does a repeated negative root prevent the response from changing sign?
3. Find every positive-time stationary point, classify it and calculate its value. Determine the limit and the side from which it is approached.
4. Sketch the response, marking the zero and minimum. Explain why the extra polynomial factor can change the shape without defeating eventual exponential decay.

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Question 2 – Solution

Strategy. Use the nonvanishing exponential to separate zeros of the affine multiplier from zeros of the derivative.

Step 1: Fit the repeated-root solution. The characteristic equation is $(r+2)^2 = 0$, so $y = (A+Bt)e^{-2t}$. The data give $A = 1$, $B - 2A = -3$, hence $B = -1$. Therefore

$$\boxed{y = (1-t)e^{-2t}}.$$

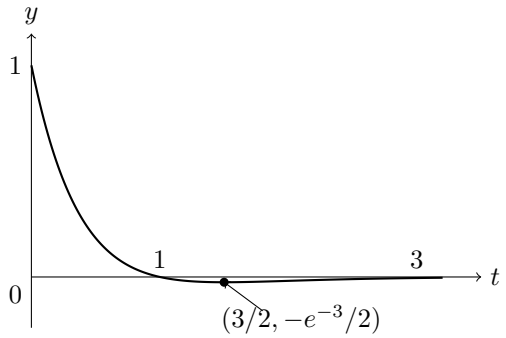
Its derivative is $(2t-3)e^{-2t}$, with value -3 at zero, and substitution verifies $y'' + 4y' + 4y = 0$.

Step 2: Locate the sign change. Since $e^{-2t} > 0$, the sign is that of $1 - t$. The unique nonnegative zero is $\boxed{t = 1}$, with positive values before and negative values after. At the zero, $y'(1) = -e^{-2} \neq 0$, so this is a crossing rather than a tangency.

Step 3: Locate the minimum and the limiting behavior. The derivative vanishes only at $t = 3/2$. It is negative before and positive afterward, so the point is the unique global minimum on $[0, \infty)$:

$$\boxed{y(3/2) = -\tfrac{1}{2}e^{-3}}.$$

The factor t grows more slowly than e^{2t}, so $y \to 0$ as $t \to \infty$, approaching from below.



One crossing, then a minimum and approach from below.

Step 4: Interpret the roles of the two factors. The negative repeated root ensures decay of both e^{-2t} and te^{-2t}. It does not determine the sign of their combination or prevent a finite-time crossing. The affine multiplier supplies that sign change, while the derivative's affine multiplier determines a later turning point. The response crosses zero at 1, reaches its minimum at 3/2, and then rises toward zero without crossing again.

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Question 3

For $\lambda > 0$ and $a > 0$, consider

$$y'' + 2\lambda y' + \lambda^2 y = 0, \qquad y(0) = a, \qquad y'(0) = b.$$

Classify all real slopes b on $t \geq 0$.

Tasks

1. Find the solution in terms of a, b, λ.
2. Prove exactly when the response remains nonnegative for every $t \geq 0$.
3. Prove exactly when it is both nonnegative and nonincreasing, treating the boundary slopes separately.
4. For the remaining slopes, find the unique zero or positive-time maximum, as appropriate. Summarize the cases on a diagram of $b/(\lambda a)$ and explain how the repeated root enters the classification.

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Question 3 – Solution

Strategy. Reduce both sign and monotonicity to affine functions of time after factoring out a positive exponential.

Step 1: Resolve the data. The root is $-\lambda$ twice. Writing $y = (A + Bt)e^{-\lambda t}$ gives $A = a$, $B = b + \lambda a$. Thus

$$\boxed{y = (a + (b + \lambda a)t)e^{-\lambda t}}.$$

For brevity let $B = b + \lambda a$.

Step 2: Classify nonnegativity. The multiplier starts at $a > 0$. It stays positive at every finite nonnegative time exactly when $B \geq 0$; if $B < 0$, it crosses zero and becomes negative. Therefore

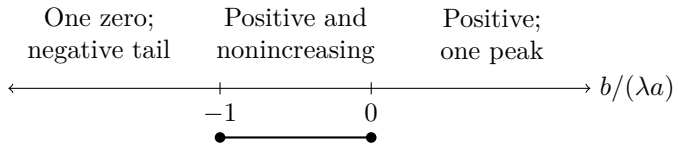
$$\boxed{y(t) \geq 0 \text{ for all } t \geq 0 \quad \Longleftrightarrow \quad b \geq -\lambda a}.$$

At equality, the polynomial factor is constant and $y = ae^{-\lambda t}$.

Step 3: Impose nonincrease as well. Differentiation gives $y' = e^{-\lambda t}(b - \lambda Bt)$. With $B \geq 0$, this derivative is nonpositive for all $t \geq 0$ exactly when $b \leq 0$. Hence

$$\boxed{-\lambda a \leq b \leq 0}.$$

At $b = -\lambda a$, the response is a decreasing pure exponential. At $b = 0$, it has zero initial slope but negative derivative for every $t > 0$ because $B = \lambda a > 0$.



Both boundary slopes are included in the middle case.

Step 4: Classify the exterior ranges. If $b < -\lambda a$, then $B < 0$, and the unique zero is

$$\boxed{t_z = -\frac{a}{b + \lambda a} > 0}.$$

The response crosses from positive to negative and tends to zero from below. If $b > 0$, then $B > 0$ and the unique maximum occurs at

$$\boxed{t_m = \frac{b}{\lambda(b + \lambda a)} > 0}.$$

Here y' changes from positive to negative, and the response remains positive. The repeated root produces an affine multiplier; the zeros of that multiplier and its differentiated counterpart govern these thresholds. All cases still decay to zero.

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Question 4

An unknown monic homogeneous second-order equation has a repeated real root r. A solution has exact samples

$$S_n = y(n), \qquad S_0 = 1, \quad S_1 = 4/e, \quad S_2 = 7/e^2.$$

Write $y(t) = (A + Bt)e^{rt}$ and set $z = e^r > 0$.

Tasks

1. Derive the recurrence $S_{n+2} = 2zS_{n+1} - z^2S_n$.
2. Determine every positive z compatible with the three observations, then find the corresponding r, A, B. Explain why the observations do not yet identify a unique equation.
3. An additional sample gives $S_3 = 10/e^3$. Determine which candidate survives and recover the monic equation and initial slope.
4. Verify all four observations for the selected solution and calculate the rejected candidate's prediction at time 3. Contrast their long-time behavior and explain why three exact observations were insufficient here.

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Question 4 – Solution

Strategy. Use the repeated-root sample recurrence, but retain every admissible root of the nonlinear identification equation.

Step 1: Derive the recurrence. Since $S_n = (A + Bn)z^n$,

$$2zS_{n+1} - z^2S_n = \big(2(A + B(n + 1)) - (A + Bn)\big)z^{n+2} = S_{n+2}.$$

Step 2: Find both admissible models. The observed values imply $7/e^2 = 8z/e - z^2$, or

$$(z - 1/e)(z - 7/e) = 0.$$

Both roots are positive. Since $A = S_0 = 1$ and $B = S_1/z - A$, the candidates are

z	r	A	B
$1/e$	-1	1	3
$7/e$	$\ln 7 - 1$	1	$-3/7$

Each gives a different repeated-root equation $y'' - 2ry' + r^2y = 0$, and each fits the original three observations. Selecting only one positive root of the identification quadratic would be unjustified.

Step 3: Use the fourth observation. For the first candidate, $S_3 = (1 + 9)e^{-3} = 10/e^3$, matching the new datum. The other gives

$$S_3 = (1 - 9/7)(7/e)^3 = \boxed{-98/e^3},$$

so it is rejected. The identified equation and data are

$$\boxed{y'' + 2y' + y = 0}, \qquad \boxed{y = (1 + 3t)e^{-t}}, \qquad \boxed{y'(0) = 2}.$$

Step 4: Verify and interpret the ambiguity. At $n = 0, 1, 2, 3$, the selected multiplier $1 + 3n$ gives $1, 4, 7, 10$, with the required exponential factors. This solution tends to zero. The rejected candidate has $r = \ln 7 - 1 > 0$ and negative leading multiplier coefficient, so it eventually tends to $-\infty$.

The initial three sample constraints lead to a quadratic equation with two admissible positive multipliers, despite being exact. The fourth observation distinguishes them. Counting unknown parameters and observations alone does not prove a nonlinear inverse problem has a unique solution.

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Question 5

For a real parameter r, consider

$$y'' - 2ry' + r^2 y = 0.$$

A solution is forward bounded if bounded on $[0, \infty)$ and two-sided bounded if bounded on all of $\mathbb{R}$.

Tasks

1. Give the complete solution family for every real r, including $r = 0$.
2. Classify exactly which solutions are forward bounded and which tend to zero as $t \to \infty$, for each sign of r.
3. Classify all two-sided bounded solutions. Justify why a polynomial factor cannot cancel exponential growth in the growing time direction.
4. Explain why a repeated zero root behaves differently from a single zero root paired with a negative root. Give a counterexample to the statement that nonpositive characteristic roots always make every solution bounded.

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Question 5 – Solution

Strategy. Treat the zero root separately and distinguish decay in one time direction from boundedness in both directions.

Step 1: Retain both repeated-root freedoms. For every r, the family is $\boxed{y = (A + Bt)e^{rt}}$. At $r = 0$ this becomes the affine family $y = A + Bt$ solving $y'' = 0$, not merely the constant family.

Step 2: Classify forward behavior. For $r < 0$, both e^{rt} and te^{rt} tend to zero, so every solution decays and is forward bounded. For $r = 0$, boundedness requires $B = 0$, while decay to zero requires $A = B = 0$. For $r > 0$, only the zero solution is forward bounded or tends to zero.

r	forward-bounded solutions	solutions tending to zero
$r < 0$	all solutions	all solutions
$r = 0$	$y = A$	$y = 0$
$r > 0$	$y = 0$	$y = 0$

For negative r, boundedness follows from the finite limit and continuity on every finite initial interval.

Step 3: Inspect the other time direction. If $B \neq 0$, the magnitude of $A + Bt$ is at least $|B||t|/2$ for sufficiently large $|t|$. If $B = 0$ but $A \neq 0$, it is a nonzero constant. Thus neither nonzero case can cancel the exponential in its growing direction.

When $r > 0$, that direction is $t \to \infty$; when $r < 0$, it is $t \to -\infty$. Hence the complete two-sided classification is

$$\boxed{r \neq 0 : \ y = 0 \text{ only}}, \qquad \boxed{r = 0 : \ y = A \text{ for any constant } A}.$$

Step 4: Explain the exceptional repeated zero. A single zero root paired with a negative root gives a constant mode plus a decaying exponential, so every solution is forward bounded. A repeated zero root additionally gives the unbounded mode t.

For example, $\boxed{y(t) = t}$ solves $y'' = 0$, whose roots are both zero and therefore nonpositive, but is unbounded forward. Root signs alone are insufficient at a repeated zero; the polynomial multiplier must be included. For a strictly negative repeated root, the polynomial affects the rate and shape but not eventual decay.

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Question 6

The repeated-root response $y(t) = te^{-t}$ solves $y'' + 2y' + y = 0$ with $y(0) = 0$, $y'(0) = 1$. Consider exponential bounds on $t \geq 0$.

Tasks

1. Find the maximum of y and verify that it tends to zero.
2. Determine whether any finite constant C can satisfy $y(t) \leq Ce^{-t}$ for every $t \geq 0$.
3. For a fixed $0 < \varepsilon < 1$, find the smallest C_ε such that $y(t) \leq C_\varepsilon e^{-(1-\varepsilon)t}$ for every $t \geq 0$. Prove optimality and sketch the bound for $\varepsilon = 1/2$ with its contact point.
4. Extend the argument to $w(t) = (A + Bt)e^{-t}$ by giving an explicit finite bound with rate $1 - \varepsilon$. Explain why eventual decay does not automatically give a bound with the exact exponent 1.

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Question 6 – Solution

Strategy. Divide by the proposed exponential bound and maximize the resulting ratio, rather than judging only the root's sign.

Step 1: Find the transient peak. We have $y' = (1-t)e^{-t}$, so the response rises until $t = 1$ and falls thereafter. Hence

$$\boxed{\max_{t\geq 0} y(t) = 1/e}.$$

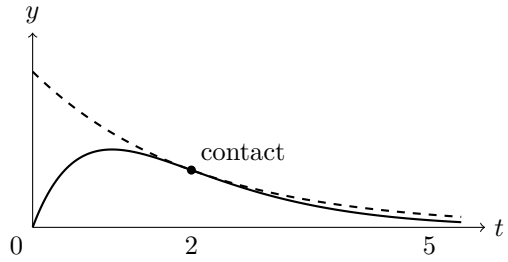
Also $t/e^t \to 0$, for example by l'Hôpital's rule, proving eventual decay.

Step 2: Test the exact root exponent. The proposed bound is equivalent to $t \leq C$ for all $t \geq 0$, which no finite C can satisfy. The exponential decay is modified by an unbounded polynomial prefactor.

Step 3: Find the sharp slightly slower bound. Dividing by $e^{-(1-\varepsilon)t}$ gives the ratio $te^{-\varepsilon t}$. Its derivative is $e^{-\varepsilon t}(1-\varepsilon t)$, so its unique maximum occurs at $t = 1/\varepsilon$ and equals $1/(e\varepsilon)$. Thus

$$\boxed{C_\varepsilon = \frac{1}{e\varepsilon}}.$$

This constant is sufficient by the maximum calculation and necessary because equality holds at that time.



Solid: te^{-t}; dashed: $(2/e)e^{-t/2}$

For $\varepsilon = 1/2$, the bound is $(2/e)e^{-t/2}$ and contact occurs at $t = 2$, where both values are $2e^{-2}$. Both derivatives equal $-e^{-2}$, confirming tangency rather than a crossing.

Step 4: Bound every fixed repeated-root solution. The triangle inequality gives

$$|w(t)|e^{(1-\varepsilon)t} \leq |A|e^{-\varepsilon t} + |B|te^{-\varepsilon t} \leq |A| + \frac{|B|}{e\varepsilon}.$$

Therefore $\boxed{|w(t)| \leq (|A| + |B|/(e\varepsilon))e^{-(1-\varepsilon)t}}$. This displayed constant need not be optimal for arbitrary A, B. A bound with the exact exponent 1 exists precisely when $B = 0$; for $B \neq 0$, the ratio $|A + Bt|$ is unbounded even though w itself decays.

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Question 7

Consider the parameter family

$$y'' + 2y' + (1 - \mu)y = 0, \qquad y(0) = 0, \qquad y'(0) = 1, \qquad -1 < \mu < 1.$$

For convenience, $\sinh z = (e^z - e^{-z})/2$. You may use $\sin z/z \to 1$ and $\sinh z/z \to 1$ as $z \to 0$.

Tasks

1. Solve the IVP separately for $\mu > 0$, $\mu = 0$ and $\mu < 0$, identifying the root type in each case.
2. Prove that the formulas from both sides tend to the repeated-root solution for each fixed $t \geq 0$ as $\mu \to 0$.
3. Determine which of these solutions have positive-time zeros. For $\mu < 0$, find the first positive zero and its limit as $\mu \uparrow 0$.
4. Prove that every solution of every equation in the stated parameter range tends to zero as $t \to \infty$. Explain why convergence of the displayed IVP solutions on any fixed time window does not preserve their number of positive-time zeros.

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Question 7 – Solution

Strategy. Center the characteristic equation at its repeated root and compare the resulting real, repeated and complex formulas without conflating their global zero patterns.

Step 1: Solve each root case. The characteristic polynomial is $(r+1)^2-\mu$. If $\mu > 0$, set $s = \sqrt{\mu} \in (0,1)$. The roots are $-1 \pm s$ and the data give

$$y_\mu = e^{-t}\frac{\sinh(st)}{s}.$$

At $\mu = 0$, the root is -1 twice, and the solution is $\boxed{y_0 = te^{-t}}$. If $\mu < 0$, set $b = \sqrt{-\mu} > 0$; the roots are $-1 \pm ib$, giving

$$y_\mu = e^{-t}\frac{\sin(bt)}{b}.$$

Each formula has value 0 and derivative 1 at zero and solves its stated equation.

Step 2: Recover the repeated-root limit from either side. For $t > 0$, factor the first formula as $te^{-t}\sinh(st)/(st)$ and the third as $te^{-t}\sin(bt)/(bt)$. The supplied limits yield $y_\mu(t) \to te^{-t}$ from either side. At $t = 0$, all values are zero directly. Thus the repeated-root solution is the common fixed-time limit, rather than a missing special value.

Step 3: Locate zeros before taking their limit. For $\mu \geq 0$, the displayed solution is strictly positive for every $t > 0$, since $t > 0$ and $\sinh(st) > 0$. For $\mu < 0$, its positive zeros are

$$\boxed{t_n = \frac{n\pi}{\sqrt{-\mu}}, \qquad n = 1, 2, \ldots}.$$

In particular, $t_1 \to \infty$ as $\mu \uparrow 0$. The infinitely many oscillations move beyond every fixed finite observation window.

Step 4: Separate decay from zero counts. For $0 < \mu < 1$, both roots $-1 \pm \sqrt{\mu}$ are negative; at zero, both repeated-root modes decay; for $\mu < 0$, the exponential factor e^{-t} bounds every fixed sine/cosine combination. Hence every solution decays throughout the stated parameter range.

On any fixed $[0, T]$, the ratios in Step 2 converge uniformly because their arguments lie in intervals shrinking to zero and both ratios extend continuously with value 1 there. Yet for every negative μ there are infinitely many later zeros, while y_0 has none for $t > 0$. Convergence on fixed windows does not control zeros that escape to infinity.

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Question 8

For the dimensionless equation

$$y'' + 2y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 1,$$

define the nonnegative quantity $E(t) = ((y'(t))^2 + (y(t))^2)/2$.

Tasks

1. Solve the IVP and find the time and height of the displacement maximum.
2. Differentiate E using the equation. Prove that E is strictly decreasing between any two distinct nonnegative times, even though its derivative can vanish at an isolated time.
3. Compute $E(0)$ and its value at the displacement maximum. Sketch y and E, and explain why increasing displacement is consistent with decreasing E.
4. Use the energy identity to calculate $\int_0^\infty (y')^2\,dt$, and confirm the answer by integrating the explicit derivative.

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Question 8 – Solution

Strategy. Distinguish a coordinate's amplitude from a quantity combining position and derivative, then integrate the exact dissipation identity.

Step 1: Solve and locate the transient maximum. The repeated root is -1. The data give $y = (1+2t)e^{-t}$ and $y' = (1-2t)e^{-t}$. Hence the unique displacement maximum is

$$\boxed{t_m = 1/2, \qquad y(t_m) = 2e^{-1/2} > 1}.$$

Step 2: Derive and interpret dissipation. Differentiating E and substituting $y'' = -2y' - y$ gives

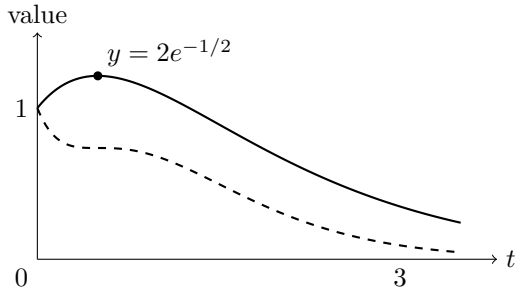
$$E' = y'y'' + yy' = y'(y'' + y) = \boxed{-2(y')^2 \le 0}.$$

The derivative vanishes only at $t = 1/2$. On any interval $[s,t]$ with $s < t$, $(y')^2$ is positive except possibly at that one point, so its integral is strictly positive. Thus $E(t) - E(s) = -2\int_s^t (y')^2\,d\tau < 0$, proving strict decrease between distinct times.

Step 3: Compare the two quantities. Initially $\boxed{E(0) = 1}$. At the displacement peak, $y' = 0$, so

$$\boxed{E(1/2) = \tfrac{1}{2}(2e^{-1/2})^2 = 2/e < 1}.$$

The explicit expression is $E = (1+4t^2)e^{-2t}$.



Solid: y; dashed: E (dimensionless quantities)

Displacement initially increases, but the derivative contribution to E decreases enough for their sum to fall. The two curves represent the defined dimensionless quantities, not two competing claims about the same amplitude. Negative repeated roots ensure eventual decay without prohibiting an initial increase in y.

Step 4: Check the total integral in two ways. Since $y, y' \to 0$, also $E \to 0$. Integrating the dissipation identity gives $0 - 1 = -2\int_0^\infty (y')^2\,dt$, so the integral is $\boxed{1/2}$.

Directly, $(y')^2 = (1 - 4t + 4t^2)e^{-2t}$. Integration by parts gives the moments $\int_0^\infty e^{-2t}dt = 1/2$, $\int_0^\infty te^{-2t}dt = 1/4$ and $\int_0^\infty t^2e^{-2t}dt = 1/4$. The resulting value $1/2 - 4(1/4) + 4(1/4) = 1/2$ agrees.

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Question 9

For $L > 0$ and strictly positive endpoint values a, b, consider

$$y'' + 2y' + y = 0, \qquad y(0) = a, \qquad y(L) = b.$$

The same solution is defined beyond the observed interval.

Tasks

1. Find the unique solution and justify uniqueness for every $L > 0$.
2. Prove that it is strictly positive throughout $[0, L]$ using the transformed quantity $u = e^t y$.
3. Determine exactly which endpoint pairs make the response nonnegative for every future time $t \geq 0$, not just between observations.
4. For $L = 1$, $a = 1$, $b = e^{-2}$, find the first zero after the observed interval and determine the sign for all later times. Explain why positive endpoint observations did not guarantee permanent positivity.

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Question 9 – Solution

Strategy. The exponential transformation turns the repeated-root endpoint problem into linear interpolation, but its continuation beyond the interval is extrapolation.

Step 1: Solve the endpoint equations. Write $y = (A + Bt)e^{-t}$. The first datum sets $A = a$, and the second gives $a + BL = be^L$. Thus

$$\boxed{y(t) = [a + \frac{be^L - a}{L}t]e^{-t}}.$$

The two constants are uniquely determined because $L \neq 0$. This also proves existence by substitution into the repeated-root family.

Step 2: Prove positivity between the data points. For $0 \leq t \leq L$,

$$u(t) = e^t y(t) = (1 - t/L)a + (t/L)be^L.$$

The weights are nonnegative and sum to 1, while both endpoint values are positive. Hence $u > 0$ and therefore $y > 0$ throughout the closed interval.

Step 3: Determine when positivity persists. The affine function $u = a+Bt$ starts positive. It remains nonnegative for all $t \geq 0$ exactly when $B \geq 0$. Equivalently,

$$\boxed{b \geq ae^{-L}}.$$

At equality, u is constant and $y = ae^{-t}$. If $b < ae^{-L}$, the slope of u is negative and its eventual zero is unavoidable despite both observed values being positive.

Step 4: Check the concrete continuation. Here $B = e^{-1} - 1 < 0$, so

$$y(t) = [1 - (1 - e^{-1})t]e^{-t}.$$

Its unique zero is

$$\boxed{t_z = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} > 1}.$$

The response is negative for every $t > t_z$ and tends to zero from below. The given endpoint at 1 is indeed $[e^{-1}]e^{-1} = e^{-2}$. Linear interpolation of positive transformed data stays positive between them; extending that same negative-slope line beyond the interval eventually crosses zero. The endpoint data determine the continuation uniquely, but do not force it to preserve the observed sign forever.

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Question 10

A dimensionless family of repeated-root responses satisfies

$$y'' + 2\lambda y' + \lambda^2 y = 0, \qquad y(0) = 1, \qquad y'(0) = 0, \qquad \lambda > 0.$$

The design requirement is $y(t) \le 0.05$ for every $t \ge 2$, with the additional constraint $|y''(0)| \le 4$.

Tasks

1. Solve the IVP and prove that the response is positive and strictly decreasing for $t > 0$.
2. Show that the first time at level 0.05 is z_*/λ, where $z_* > 0$ is the unique root of $(1 + z)e^{-z} = 0.05$. Locate z_* between two consecutive four-decimal-place numbers by a justified numerical calculation.
3. Translate both the settling requirement and the initial-acceleration constraint into restrictions on λ. Determine whether they can be satisfied simultaneously.
4. Find the earliest achievable settling time under the acceleration constraint alone. Sketch the responses for $\lambda = 1, 2$ with the tolerance and required time marked, and explain why simply increasing λ is not an admissible unlimited improvement.

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Question 10 – Solution

Strategy. Scale time by the repeated decay rate, then compare the performance requirement with the independent constraint on the initial acceleration.

Step 1: Solve and establish monotonicity. The repeated root is $-\lambda$. The data give

$$\boxed{y = (1+\lambda t)e^{-\lambda t}}, \qquad y' = -\lambda^2 t e^{-\lambda t}.$$

The response is positive, strictly decreasing for $t > 0$, starts at 1 and tends to zero. Therefore crossing the tolerance once is enough to guarantee it thereafter.

Step 2: Reduce the crossing calculation to one dimensionless number. Let $f(z) = (1+z)e^{-z}$ for $z \geq 0$. Then $f(0) = 1$, $f(z) \to 0$ and $f'(z) = -ze^{-z} < 0$ for $z > 0$. The intermediate value theorem and strict decrease give exactly one positive root $f(z_*) = 0.05$.

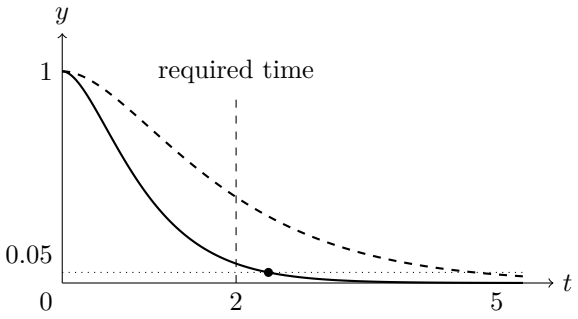
Direct evaluation gives $f(4.7438) > 0.05$ and $f(4.7439) < 0.05$. Thus

$$\boxed{4.7438 < z_* < 4.7439}, \qquad t_{\text{settle}} = z_*/\lambda.$$

Numerically $z_* \approx 4.743865$; the bracket, rather than a rounded equality, supports the feasibility comparisons below.

Step 3: Test simultaneous feasibility. Settling by time 2 requires $z_*/\lambda \leq 2$, or $\lambda \geq z_*/2 > 2.3719$. But the equation at zero gives $y''(0) = -\lambda^2$, so the acceleration constraint requires $\lambda \leq 2$. These ranges do not intersect: $\boxed{\text{the two requirements are incompatible}}$.

At the largest admissible value $\lambda = 2$, the response at the deadline is $y(2) = 5e^{-4} \approx 0.091578 > 0.05$, confirming the failure directly.



Solid: $\lambda = 2$; dashed: $\lambda = 1$; dotted: tolerance

Step 4: Optimize within the allowed family. The settling time z_*/λ strictly decreases as λ increases. Under $0 < \lambda \leq 2$, it is minimized at $\lambda = 2$, giving

$$\boxed{t_{\min} = z_*/2 \approx 2.371932}.$$

This is later than the required time 2. Larger λ would improve the settling time but violate the initial-acceleration bound. The conclusion optimizes this explicitly specified family and constraint, not every possible differential-equation model or control law.

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Question 1

Let p, q be continuous on an open interval I, let $t_0 \in I$, and let y_1 be a known solution of

$$y'' + p(t)y' + q(t)y = 0$$

that never vanishes on I. Write $P(t) = \int_{t_0}^{t} p(s)\,ds$.

Tasks

1. Derive the equation for $w = v\prime$ after substituting $y = y_1 v$.
2. Solve the first-order equation for w and give an integral formula for y.
3. Construct a solution z with $z(t_0) = 0$ and $z\prime(t_0) = 1$. Prove that z is not a constant multiple of y_1.
4. Express the solution with $y(t_0) = a$, $y\prime(t_0) = b$ in terms of y_1, z. Explain why the nonvanishing hypothesis matters.

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Question 1 – Solution

Strategy. Factor out the known solution, then normalize the new solution by its initial slope.

Step 1: Reduce the order. Product differentiation gives

$$y' = y_1'v + y_1v', \qquad y'' = y_1''v + 2y_1'v' + y_1v''.$$

The coefficient of v is zero because y_1 solves the equation. Hence

$$y_1v'' + (2y_1' + py_1)v' = 0, \qquad w' + (2y_1'/y_1 + p)w = 0.$$

Step 2: Integrate twice. An integrating factor is $y_1^2e^P$, so

$$w = C\frac{e^{-P}}{y_1^2}, \qquad y = y_1(t)[A + C\int_{t_0}^{t} \frac{e^{-P(s)}}{y_1(s)^2}\,ds].$$

Both integrations contribute a constant. The formula includes $w = 0$.

Step 3: Normalize the companion. Set

$$\boxed{z(t) = y_1(t_0)y_1(t)\int_{t_0}^{t} \frac{e^{-P(s)}}{y_1(s)^2}\,ds.}$$

The integral vanishes at t_0. Since $P(t_0) = 0$, differentiation gives $z'(t_0) = 1$. If $z = ky_1$, its zero value forces $k = 0$, contradicting this slope.

Step 4: Match arbitrary data. The two constants are fixed by

$$\boxed{y = \frac{a}{y_1(t_0)}y_1 + [b - \frac{ay_1'(t_0)}{y_1(t_0)}]z.}$$

This is the unique solution on I by the regular linear initial-value theorem. Division by y_1 and the integral kernel require $y_1 \neq 0$. A zero of the seed may obstruct this substitution without obstructing the original equation.

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Question 2

On $t > 0$, consider

$$t^2y'' - 3ty' + 4y = 0, \qquad y(1) = 1, \quad y'(1) = 3.$$

A proposed known solution is $y_1 = t^2$.

Tasks

1. Verify y_1 and use reduction of order to find the general solution.
2. Solve the initial-value problem and verify both data.
3. Find every zero and stationary point of this solution on $t > 0$, and classify the stationary point.
4. Determine whether this solution has a C^1 extension and whether it has a C^2 extension through $t = 0$. Justify using one-sided limits.

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Question 2 – Solution

Strategy. Normalize the equation before reduction, then use exact derivatives to study the singular endpoint.

Step 1: Find the missing solution. For t^2, the residual is $(2-6+4)t^2=0$. With $y=t^2v$, the divided equation reduces to

$$v''+\frac{1}{t}v'=0, \qquad v'=D/t, \qquad \boxed{y=t^2(C+D\ln t).}$$

Step 2: Impose the data. At 1, $y=C$ and $y'=2C+D$. Thus $C=D=1$ and

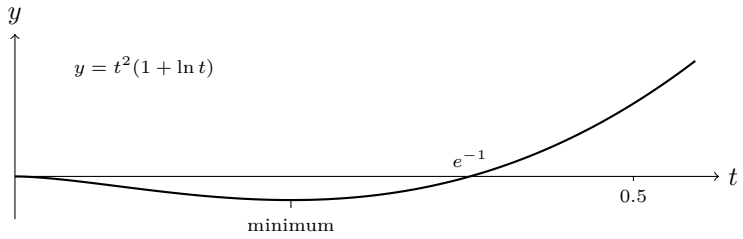
$$\boxed{y=t^2(1+\ln t)}, \qquad y'=t(3+2\ln t), \quad y''=5+2\ln t.$$

These give $y(1)=1$ and $y'(1)=3$.

Step 3: Locate the features. The sole zero is $t=e^{-1}$. The derivative changes from negative to positive at $t=e^{-3/2}$, giving the unique global minimum

$$(t,y)=(e^{-3/2},-\tfrac{1}{2}e^{-3}).$$

Step 4: Test endpoint regularity. As $t\downarrow 0$, both y and y' tend to zero, but $y''\to-\infty$. Setting $y=0$ for $t\le 0$ gives a C^1 extension (the derivative at zero is also $\lim y(t)/t=0$). No C^2 extension is possible. The leading coefficient vanishes at zero, so the regular normalized theorem does not apply there.



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Question 3

Consider the equation on $\mathbb{R}$

$$y'' - 2ty' - 2y = 0, \qquad y(0) = 0, \quad y'(0) = 1,$$

with known solution $y_1 = e^{t^2}$. An integral is an acceptable exact answer.

Tasks

1. Verify the seed and derive the reduced first-order equation.
2. Find the initial-value solution as a definite integral and verify its residual without evaluating the integral.
3. Prove that the solution is odd and strictly increasing on $\mathbb{R}$.
4. Prove $t \le y(t) \le te^{t^2}$ for $t \ge 0$ and compute $y(1)$ to six decimal places. Explain why an unevaluated integral still defines a valid solution.

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Question 3 – Solution

Strategy. Keep the definite integral and use the fundamental theorem of calculus for verification and bounds.

Step 1: Reduce. The seed has derivatives $2te^{t^2}$ and $(2+4t^2)e^{t^2}$, so its residual vanishes. With $y = e^{t^2}v$,

$$w' + 2tw = 0, \qquad w = v' = Ce^{-t^2}.$$

Step 2: Fit and verify. The zero initial value sets the constant in v to zero; the initial slope sets $C = 1$. Therefore

$$\boxed{y(t) = e^{t^2}\int_0^t e^{-s^2}\,ds.}$$

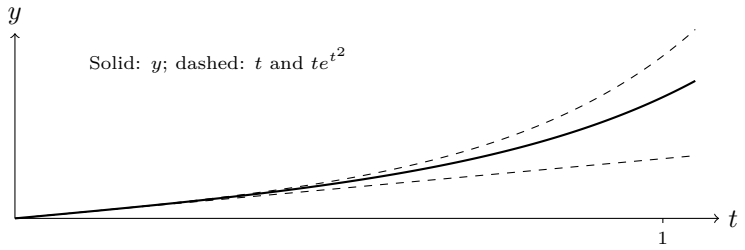
Differentiation gives $y' = 2ty + 1$, then $y'' = 2y + 2ty'$. The residual is zero, and at zero the value and slope are $0, 1$.

Step 3: Use symmetry and sign. The integrand is positive and even, so its integral from zero is odd and has the sign of t. Multiplication by e^{t^2} preserves both properties. Consequently $ty \geq 0$ and $y' = 2ty + 1 \geq 1$ everywhere: y is strictly increasing.

Step 4: Bound and evaluate. For $0 \leq s \leq t$,

$$e^{-t^2} \leq e^{-s^2} \leq 1 \quad \Longrightarrow \quad te^{-t^2} \leq \int_0^t e^{-s^2}\,ds \leq t.$$

Multiplying by e^{t^2} proves the bounds. Numerical integration gives $y(1) \approx 2.030078$. The integrand is smooth, so the definite integral defines a smooth function on all of $\mathbb{R}$; the exact residual check proves it solves the equation.



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Question 4

For $y'' + y = 0$, suppose the known solution is $y_1 = \sin t$. First work on $(0, \pi)$, where the seed does not vanish.

Tasks

1. Use $y = (\sin t)v$ to construct a second solution on $(0, \pi)$.
2. Explain whether the poles of the quotient v prevent the second solution from extending to $\mathbb{R}$.
3. Use the extended solutions to solve $y(0) = 1$, $y\prime(0) = 0$, and justify uniqueness.
4. A proposed continuation is $\cos t + a \sin t$ for $t < 0$ and $\cos t + b \sin t$ for $t > 0$, with $y(0) = 1$. Find the conditions for continuity, for C^1 regularity, and for being a classical solution through zero.

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Question 4 – Solution

Strategy. Separate a failure of division by the seed from a singularity of the original equation.

Step 1: Reduce on the valid interval. Substitution gives

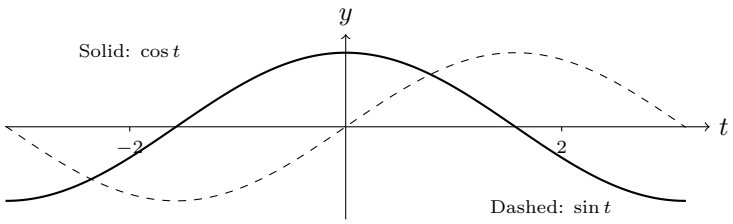
$$v'' + 2\cot t\, v' = 0, \qquad v' = C\csc^2 t, \qquad v = -C\cot t + D.$$

Thus $y = -C\cos t + D\sin t$; a second solution is $\cos t$.

Step 2: Extend the product. Although $\cos t/\sin t = \cot t$ has poles at multiples of π, its product with $\sin t$ is $\cos t$ on each valid interval. The latter is smooth and directly satisfies $y'' + y = 0$ on all of $\mathbb{R}$. The poles belong to the chosen quotient.

Step 3: Solve at the seed's zero. In $y = A\cos t + B\sin t$, the data give $A = 1$, $B = 0$. Thus $\boxed{y = \cos t}$. Continuous coefficients and nonzero leading coefficient ensure global uniqueness for the data at zero.

Step 4: Match derivatives. Both one-sided values are 1, so continuity holds for all real a, b. The one-sided slopes are a, b, so C^1 regularity requires $a = b$. In that case one smooth formula holds on both sides, and it is a classical solution. If $a \neq b$, a derivative at zero does not exist; matching only position is insufficient.



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Question 5

On $t > 0$, consider $t^2y'' - 2ty' + 2y = 0$ with seed $y_1 = t$. Use the following *unnormalized* companion convention:

$$Y_a(t) = y_1(t) \int_a^t \frac{s^2}{y_1(s)^2}\, ds, \qquad a > 0.$$

The numerator s^2 is held fixed when changing a or rescaling the seed.

Tasks

1. Verify the seed and derive the kernel in this convention from the normalized equation.
2. Compute Y_1, Y_2, and the companion obtained with seed $3t$ and lower limit 1.
3. Explain why these choices give the same solution space. Why does an arbitrary additive integration constant not create a third solution freedom?
4. Use $y = At + BY_1(t)$ to solve $y(1) = 2$, $y\prime(1) = 1$. Verify the equation and data.

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Question 5 – Solution

Strategy. Compare the actual functions produced by changes in normalization, rather than counting written constants.

Step 1: Identify the kernel. The seed residual is $-2t + 2t = 0$. After division by t^2, $p = -2/t$ and

$$P(t) = \int_1^t -2/s\, ds = -2\ln t, \qquad e^{-P(t)} = t^2.$$

Reduction gives $v' = Ct^2/y_1^2$, which yields the specified convention.

Step 2: Compute three companions. For $y_1 = t$, the integrand equals 1, so

$$Y_1 = t(t-1) = t^2 - t, \qquad Y_2 = t(t-2) = t^2 - 2t.$$

For seed $3t$, the integrand is $1/9$, hence

$$\widetilde{Y}_1 = 3t(t-1)/9 = (t^2 - t)/3.$$

Step 3: Compare freedoms. We have $Y_2 = Y_1 - t$ and $\widetilde{Y}_1 = Y_1/3$. Adding a seed multiple or multiplying a companion by a nonzero constant leaves the family unchanged: every choice spans $\{At + Bt^2 : A, B \in \mathbb{R}\}$. An additive constant inside the antiderivative produces only another multiple of the seed, already present in the family.

Step 4: Fit and verify. Since $Y_1(1) = 0$, $Y_1'(1) = 1$, the data give $A = 2$ and $A + B = 1$. Therefore

$$\boxed{y = 2t - (t^2 - t) = 3t - t^2.}$$

Its derivatives are $3 - 2t$ and -2, and

$$t^2(-2) - 2t(3 - 2t) + 2(3t - t^2) = 0.$$

At $t = 1$ the value is 2 and the slope is 1. Two effective constants are sufficient and necessary for arbitrary regular initial data.

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Question 6

On $t > 0$, consider

$$ty'' - (t+1)y' + y = 0, \qquad y_1 = e^t.$$

A student inserts the undivided coefficient $-(t+1)$ into the usual normalized reduction formula and obtains $v' = e^{t^2/2-t}$.

Tasks

1. Verify the seed, normalize the equation, and derive the correct equation for $w = v\prime$.
2. Find the full solution family and solve $y(1) = 0$, $y\prime(1) = 1$.
3. Compute the residual of the student's candidate $y = e^t v$, with $v\prime = e^{t^2/2-t}$. Explain why testing only $t = 1$ would miss the error.
4. For the original undivided equation at $t = 0$, find the necessary relation between value and slope. Exhibit two distinct smooth solutions with zero value and slope there.

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Question 6 – Solution

Strategy. The coefficient in the standard formula is the coefficient after division by the leading term.

Step 1: Normalize. For e^t, the residual is $[t-(t+1)+1]e^t = 0$. Division gives $p = -1 - 1/t$, so

$$w' + (1 - 1/t)w = 0, \qquad w = Cte^{-t}.$$

Step 2: Integrate and fit. Since $\int te^{-t}\,dt = -(t+1)e^{-t}$,

$$y = Ae^t + B(t+1).$$

At 1 the data give $Ae + 2B = 0$ and $Ae + B = 1$. Thus $B = -1$, $A = 2/e$, and

$$\boxed{y = 2e^{t-1} - (t+1).}$$

Both basis functions have zero residual, and substitution at 1 gives $0, 1$.

Step 3: Diagnose the error. For any v, the undivided residual of $e^t v$ is

$$e^t[tw' + (t-1)w].$$

The student's w satisfies $w' = (t-1)w$, giving residual

$$e^t(t^2-1)e^{t^2/2-t}.$$

This is zero at $t = 1$ but equals $3e^2$ at $t = 2$. A single vanishing residual does not establish a differential identity.

Step 4: Examine the singular initial point. At zero the original equation requires $y'(0) = y(0)$. Both $y = 0$ and $y = e^t - (t+1)$ are smooth solutions through zero with value and slope zero; the second is nonzero, for example because its second derivative at zero is 1. The regular initial-value theorem fails here because the leading coefficient vanishes. Compatibility does not imply uniqueness.

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Question 7

On $\mathbb{R}$, consider $(1+t^2)y''-2y=0$ with known solution $y_1=1+t^2$. Let y_2 denote the solution with $y_2(0)=0$, $y_2'(0)=1$, and define $I=y_2/(1+t^2)$.

Tasks

1. Use reduction of order to find y_2 explicitly. Evaluate the required integral.
2. Verify the equation and the two normalization data directly.
3. Find the exact range and monotonicity of $I(t)$ on $\mathbb{R}$, including the limiting values at infinity.
4. For every real C, classify the finite zeros and the sign of $y_C=C(1+t^2)-y_2$. Include the two boundary parameter values.

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Question 7 – Solution

Strategy. Divide the solution family by its positive seed so that zeros become level crossings of a monotone function.

Step 1: Construct the companion. The seed residual is $2(1+t^2) - 2(1+t^2) = 0$. Since $p = 0$,

$$I(t) = \int_0^t \frac{ds}{(1+s^2)^2} = \frac{t}{2(1+t^2)} + \frac{1}{2}\arctan t.$$

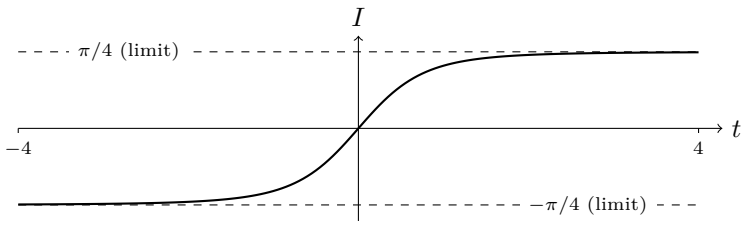
The integral follows by setting $s = \tan u$ and integrating $\cos^2 u$. Hence

$$\boxed{y_2 = \tfrac{1}{2}t + \tfrac{1}{2}(1+t^2)\arctan t.}$$

Step 2: Verify. We obtain $y_2' = 1 + t\arctan t$ and $y_2'' = \arctan t + t/(1+t^2) = 2y_2/(1+t^2)$. The equation holds and the data are $0, 1$.

Step 3: Determine the ratio range. Since $I' = 1/(1+t^2)^2 > 0$, the ratio is strictly increasing. Its limits are $-\pi/4$ and $\pi/4$; neither is attained. Its range is exactly $(-\pi/4, \pi/4)$ by continuity.

Step 4: Classify all parameters. The sign of y_C is the sign of $C - I(t)$. If $|C| < \pi/4$, there is exactly one finite zero, with positive values before it and negative values after it. If $C \geq \pi/4$, the solution is strictly positive everywhere; if $C \leq -\pi/4$, it is strictly negative everywhere. Equality in these last two cases does not create a finite zero because the limiting ratio levels are excluded.



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Question 8

On $\mathbb{R}$, solve the boundary-at-infinity problem

$$y'' + \frac{2t}{1+t^2}y' = 0, \qquad \lim_{t\to-\infty} y(t) = -1, \quad \lim_{t\to\infty} y(t) = 3.$$

The constant function $y_1 = 1$ is a known homogeneous solution.

Tasks

1. Use reduction of order to obtain the full solution family.
2. Find the unique solution satisfying the two limiting conditions and verify its equation.
3. Find its value and slope at zero, monotonicity and exact range. Is every solution of this equation bounded?
4. Compute $\lim_{t\to\infty} t[3 - y(t)]$. Decide whether constants $M, c > 0$ can satisfy $3 - y(t) \le Me^{-ct}$ for all sufficiently large t.

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Question 8 – Solution

Strategy. A constant seed reduces the equation directly to a first-order equation for the slope.

Step 1: Integrate the slope. Set $w = y'$. An integrating factor is $1 + t^2$, so

$$((1 + t^2)w)' = 0, \qquad w = B/(1 + t^2), \qquad y = A + B \arctan t.$$

Step 2: Fit the two limits. The equations $A - B\pi/2 = -1$ and $A + B\pi/2 = 3$ give

$$\boxed{y = 1 + \frac{4}{\pi} \arctan t.}$$

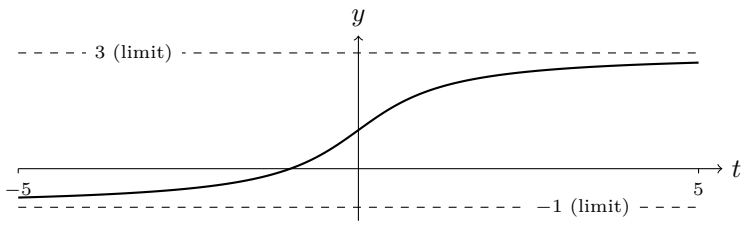
This is the unique choice of the two constants. Its derivatives $4/[\pi(1 + t^2)]$ and $-8t/[\pi(1 + t^2)^2]$ cancel in the original equation.

Step 3: Describe the trajectory. The value and slope at zero are 1 and $4/\pi$. The slope is positive everywhere; continuity and the limiting values give range $(-1, 3)$. Every member $A + B \arctan t$ is bounded, since $|\arctan t| < \pi/2$.

Step 4: Determine the tail rate. For $t > 0$,

$$3 - y(t) = \frac{4}{\pi} \arctan(1/t), \qquad \boxed{\lim_{t\to\infty} t[3 - y(t)] = \frac{4}{\pi}.}$$

Here $\arctan u/u \to 1$ as $u \to 0$. An exponential bound would force this positive limit to be at most $\lim Mte^{-ct} = 0$, a contradiction. The approach to the limit has an algebraic $1/t$ tail.



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Question 9

Consider the original undivided equation $t^2y'' - 6y = 0$. First work on $t > 0$ with known solution $y_1 = t^3$. A classical solution through zero means a C^2 function satisfying the undivided equation on an open interval containing zero.

Tasks

1. Use reduction of order to find every solution on $t > 0$.
2. Which right-hand solutions admit a C^2 extension through zero? Find the forced value, slope and second derivative at zero.
3. For each admissible right-hand solution, classify all classical continuations to $t < 0$. Explain whether the zero initial data determine a unique continuation.
4. Which of those continuations are C^3 at zero? Explain why the usual uniqueness theorem does not settle this problem.

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Question 9 – Solution

Strategy. Solve on each side of the singular coefficient, then compare one-sided derivatives at the join.

Step 1: Reduce on the positive side. The seed has residual $t^2(6t) - 6t^3 = 0$. In normalized form $p = 0$, so

$$v' = Ct^{-6}, \quad v = D - \tfrac{1}{5}Ct^{-5}, \quad \boxed{y = At^3 + Bt^{-2} \quad (t > 0).}$$

These two powers each satisfy the original equation by direct substitution.

Step 2: Exclude the divergent mode. If $B \neq 0$, then y diverges at zero, so even a continuous extension is impossible. If $B = 0$, the limits of y, y', y'' are $0, 0, 0$. These are the required values for a C^2 extension.

Step 3: Classify continuations. On $t < 0$ the same calculation gives $at^3 + bt^{-2}$; continuity forces $b = 0$. Thus all classical continuations of the right branch At^3 are

$$\boxed{y(t) = \begin{cases} at^3, & t < 0, \\ 0, & t = 0, \\ At^3, & t > 0, \end{cases} \qquad a \in \mathbb{R}.}$$

The first and second derivatives match at zero for every a. At zero the undivided residual is $0 - 6(0) = 0$. Therefore these are sufficient as well as necessary, and infinitely many continuations share $y(0) = y'(0) = 0$ even with the right branch fixed.

Step 4: Require one more derivative. The one-sided third derivatives are $6a$ and $6A$; a C^3 join occurs exactly when $a = A$. In that case the single polynomial At^3 holds on both sides. The leading coefficient is zero at the joining point, and the normalized coefficient $-6/t^2$ is undefined there, so the regular uniqueness theorem has no applicable interval containing zero.

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Question 10

On $t > 0$, consider the forced equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t, \qquad y(1) = 0, \quad y'(1) = 0.$$

The function $y_1 = t$ is supplied as a solution of the *associated homogeneous equation.*

Tasks

1. Verify the stated role of y_1 and explain why it is not a solution of the forced equation.
2. Substitute $y = tv$ and derive a reduced equation. Find the full forced solution family, retaining both integration constants.
3. Solve the given initial-value problem.
4. Check the original residual and data directly. Explain the error in applying the homogeneous reduction formula without carrying along the forcing term.

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Question 10 – Solution

Strategy. The known homogeneous solution still cancels the term containing v, but the nonzero right-hand side remains.

Step 1: Verify the seed's role. For $y_1 = t$, the left-hand side is $0 - 2/t + 2/t = 0$. This satisfies the homogeneous equation, but cannot equal the positive forcing t on the specified domain.

Step 2: Reduce with the forcing present. With $y = tv$,

$$y' = v + tv', \qquad y'' = 2v' + tv''.$$

Substitution cancels the terms in v and v':

$$2v' + tv'' - \frac{2}{t}(v + tv') + \frac{2}{t}v = tv'' = t.$$

Since $t > 0$, $v'' = 1$. Equivalently the first-order reduced equation is $w' = 1$, with $w = v'$. Integrating twice gives

$$v = \tfrac{1}{2}t^2 + At + B, \qquad \boxed{y = \tfrac{1}{2}t^3 + At^2 + Bt.}$$

Step 3: Impose the data. At 1 the equations are $1/2 + A + B = 0$ and $3/2 + 2A + B = 0$. Thus $A = -1$, $B = 1/2$, yielding

$$\boxed{y = \tfrac{1}{2}t(t-1)^2.}$$

Step 4: Verify and diagnose. In expanded form the solution is $t^3/2 - t^2 + t/2$, with $y' = 3t^2/2 - 2t + 1/2$ and $y'' = 3t - 2$. Its left-hand side is

$$(3t - 2) + (-3t + 4 - 1/t) + (t - 2 + 1/t) = t.$$

At 1 its value and slope are zero. Discarding the forcing would instead give $w' = 0$, producing only $At^2 + Bt$, whose left-hand side is zero. Those two terms provide the homogeneous freedom, but the cubic term is essential to match the forcing.

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Question 1

A fundamental set for a homogeneous linear equation on an interval is a pair of solutions whose constant linear combinations represent every solution uniquely. Let p, q be continuous on an open interval I. For two solutions of $y'' + py' + qy = 0$, define

$$D(t_0) = u(t_0)v'(t_0) - u'(t_0)v(t_0), \qquad t_0 \in I.$$

Tasks

1. Using existence and uniqueness for initial data, prove that the pair is fundamental on I if and only if $D(t_0) \neq 0$.
2. For $y'' + 2y' + 5y = 0$ on $\mathbb{R}$, verify $u = e^{-t}\cos 2t$, $v = e^{-t}\sin 2t$ and apply the criterion at zero.
3. Construct solutions ϕ, ψ with initial data $(\phi(0), \phi'(0)) = (1, 0)$ and $(\psi(0), \psi'(0)) = (0, 1)$.
4. Use this normalized pair to solve $y(0) = 2$, $y'(0) = -3$, and verify the data.

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Question 1 – Solution

Strategy. A fundamental pair must supply every value-and-slope pair, so test the two-by-two initial-data system.

Step 1: Prove the criterion. If $D(t_0) \neq 0$, the equations

$$Au(t_0) + Bv(t_0) = a, \qquad Au'(t_0) + Bv'(t_0) = b$$

have unique constants A, B for every a, b. The resulting combination solves the equation; uniqueness makes it equal to every solution with those data. Conversely, if $D(t_0) = 0$, some nonzero constant pair (A, B) gives zero value and slope. By uniqueness, $Au + Bv$ is identically zero, contradicting unique representation. Thus the criterion is necessary and sufficient.

Step 2: Verify the candidates. Their derivatives are

$$u' = e^{-t}(-\cos 2t - 2\sin 2t), \quad u'' = e^{-t}(-3\cos 2t + 4\sin 2t),$$

$$v' = e^{-t}(-\sin 2t + 2\cos 2t), \quad v'' = e^{-t}(-3\sin 2t - 4\cos 2t).$$

Both residuals vanish. At zero their data are $(1, -1)$ and $(0, 2)$, so $D(0) = 2 \neq 0$ and the pair is fundamental on $\mathbb{R}$.

Step 3: Normalize the data. The required choices are

$$\boxed{\phi = u + \tfrac{1}{2}v, \qquad \psi = \tfrac{1}{2}v.}$$

Their data are $(1, 0)$ and $(0, 1)$ respectively, so they too form a fundamental set.

Step 4: Fit the solution. The normalized coefficients equal the data:

$$\boxed{y = 2\phi - 3\psi = e^{-t}(2\cos 2t - \tfrac{1}{2}\sin 2t).}$$

At zero its value is 2; its slope is $-2 - 1 = -3$. It satisfies the equation by linearity and is unique on $\mathbb{R}$.

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Question 2

For the equation $y'' - y = 0$ on $\mathbb{R}$, consider the parameter-dependent pair

$$u = e^t + e^{-t}, \qquad v_\alpha = e^t + \alpha e^{-t}, \quad \alpha \in \mathbb{R}.$$

Coefficients in a fundamental set must be constant in t.

Tasks

1. Verify both solutions and find all α for which the pair is fundamental.
2. For those parameters, find the coefficients of the solution with $y(0) = 0$, $y\prime(0) = 1$ in this pair.
3. As $\alpha \to 1$, compare the behavior of these coefficients with the behavior of the represented solution. Can large coefficients alone establish large solution values?
4. Construct from u, v_α a companion with data $(0, 1)$ that extends to a well-defined function at $\alpha = 1$. Explain why simply setting $\alpha = 1$ in the original pair fails.

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Question 2 – Solution

Strategy. Track the initial-data vectors as the two proposed basis elements approach dependence.

Step 1: Test independence. Each exponential solves $y'' - y = 0$, so both combinations do. The data vectors at zero are $(2, 0)$ and $(1 + \alpha, 1 - \alpha)$, with determinant $2(1 - \alpha)$. The pair is fundamental exactly when $\alpha \neq 1$. At 1, both functions equal u.

Step 2: Fit the data. In $y = Au + Bv_\alpha$, the data require

$$2A + (1 + \alpha)B = 0, \qquad (1 - \alpha)B = 1.$$

Thus

$$\boxed{A = -\frac{1 + \alpha}{2(1 - \alpha)}, \qquad B = \frac{1}{1 - \alpha}.}$$

Step 3: Simplify before taking a limit. Substitution cancels the large terms exactly:

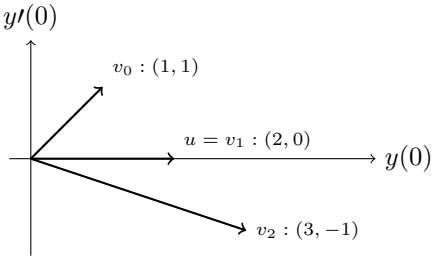
$$Au + Bv_\alpha = \tfrac{1}{2}(e^t - e^{-t}) = \sinh t.$$

Both coefficients grow without bound in magnitude near $\alpha = 1$, while the function is independent of α. Large coordinates in a nearly dependent pair do not imply large values of the function on a fixed interval.

Step 4: Replace the collapsing direction. Define, for $\alpha \neq 1$,

$$z_\alpha = \frac{v_\alpha - (1 + \alpha)u/2}{1 - \alpha} = \sinh t.$$

This extends by $z_1 = \sinh t$ and has data $(0, 1)$. The pair u, z_1 remains fundamental. The original pair at $\alpha = 1$ contains only one effective direction and cannot supply the requested nonzero slope.



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Question 3

For $y'' - 3y' + 2y = 0$ on $\mathbb{R}$, let

$$f = e^t, \quad g = e^{2t}, \quad h = f + g, \quad k = 2f - g.$$

A student writes $y = Af + Bg + Ch + Dk$ and claims this gives four independent solution freedoms.

Tasks

1. Verify that all four functions solve the equation and that f, g form a fundamental set.
2. Determine which of the six unordered pairs chosen from f, g, h, k are fundamental. Justify every pair.
3. Find every quadruple (A, B, C, D) that represents the zero function, and explain the error in the student's claim.
4. Find all quadruples representing the solution with $y(0) = 1$, $y\prime(0) = 3$. Give its unique representation using only h, k.

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Question 3 – Solution

Strategy. Express every candidate in the already verified pair f, g, then count effective coefficients.

Step 1: Verify the underlying pair. Substitution of e^{rt} gives residual $(r^2 - 3r + 2)e^{rt}$, which vanishes for $r = 1, 2$. Linearity verifies h, k. The data determinant for f, g at zero is $2 - 1 = 1$, so this pair is fundamental.

Step 2: Check all six pairs. The coefficient columns relative to f, g are $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(2, -1)$. Their determinants, in the listed order, are

pair	(f,g)	(f,h)	(f,k)	(g,h)	(g,k)	(h,k)
determinant	1	1	-1	-1	-2	-3

Every determinant is nonzero, so every pair is an invertible change of the fundamental pair f, g and is itself fundamental.

Step 3: Identify redundancy. The proposed expression is

$$y = (A + C + 2D)f + (B + C - D)g.$$

It is zero exactly when $A = -C - 2D$ and $B = -C + D$, where C, D are arbitrary. These nontrivial constant relations show that the four functions are not independent. Only two effective coefficients are present.

Step 4: Fit the specified data. Writing $y = af + bg$ gives $a + b = 1$, $a + 2b = 3$, hence $a = -1, b = 2$. All four-coefficient representations are

$$\boxed{(A, B, C, D) = (-1 - C - 2D,\ 2 - C + D,\ C,\ D).}$$

Since $h - k = -f + 2g$, the unique representation in the pair h, k is $\boxed{y = h - k}$. Its data are $-1 + 2 = 1$ and $-1 + 4 = 3$.

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Question 4

The smooth functions $u = t^2$ and $v = t^3$ are proposed as a fundamental set for an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients on $\mathbb{R}$.

Tasks

1. Prove that u, v are linearly independent as functions on $\mathbb{R}$.
2. Prove that no such regular normalized equation on $\mathbb{R}$ can have both as solutions. Explain why independence alone is insufficient.
3. Find the normalized equation having both as solutions on $(0, \infty)$ and verify that they form a fundamental set there.
4. On $(0, \infty)$, solve $y(1) = 0$, $y\prime(1) = 1$ using the pair. Explain why its zero value and slope at zero do not contradict your conclusion about regular equations.

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Question 4 – Solution

Strategy. Check both requirements: independence and membership in the solution space of one regular equation on the stated interval.

Step 1: Prove independence. If $At^2 + Bt^3 = 0$ for every real t, then $A + Bt = 0$ for every nonzero t. Evaluating at two distinct nonzero points forces $A = B = 0$.

Step 2: Reject the proposed regular equation. The nonzero function $u = t^2$ has $u(0) = u'(0) = 0$. Uniqueness for continuous normalized coefficients would force it to equal the zero solution on $\mathbb{R}$. Equivalently, substitution at zero would require $u''(0) = 2 = 0$. Thus these independent functions cannot be a fundamental set for any equation of the stated kind.

Step 3: Work on a valid interval. Substituting the two functions for $t > 0$ gives

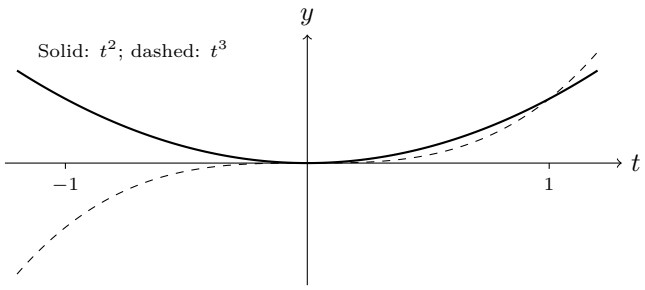
$$2 + 2tp + t^2q = 0, \qquad 6t + 3t^2p + t^3q = 0.$$

Subtracting t times the first equation from the second yields $p = -4/t$, then $q = 6/t^2$. Hence

$$\boxed{y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0.}$$

Both residuals vanish, the coefficients are continuous on $(0, \infty)$, and $uv' - u'v = t^4 > 0$ there, so the pair is fundamental.

Step 4: Fit and interpret. The data give $A + B = 0$, $2A + 3B = 1$, so $\boxed{y = t^3 - t^2}$. Its polynomial extension has zero value and slope at zero, but the normalized equation is undefined there. A theorem requiring continuous coefficients on an interval containing zero cannot be applied to that extension.



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Question 5

The pair $\cos t, \sin t$ is fundamental for $y'' + y = 0$ on $\mathbb{R}$. Fix $L > 0$ and prescribe two position measurements

$$y(0) = a, \qquad y(L) = b, \qquad a, b \in \mathbb{R}.$$

Tasks

1. Find the coefficient equations and determine when the measurements select a unique solution.
2. For every exceptional length, give necessary and sufficient conditions for existence and describe all solutions when they exist.
3. At $L = \pi$, describe all solutions for $(a, b) = (1, -1)$ and decide whether any solution exists for $(a, b) = (1, 1)$.
4. Explain why the exceptional lengths do not mean that the fundamental pair has become dependent. Identify which additional measurement at zero would uniquely determine a solution.

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Question 5 – Solution

Strategy. Distinguish a fundamental set of functions from the invertibility of a particular measurement map.

Step 1: Solve at ordinary lengths. Every solution is $A\cos t + B\sin t$. The position measurements give

$$A = a, \qquad a\cos L + B\sin L = b.$$

If $\sin L \neq 0$, the unique solution is

$$\boxed{y = a\cos t + \frac{b - a\cos L}{\sin L}\sin t.}$$

Thus uniqueness holds exactly at lengths that are not positive integer multiples of π.

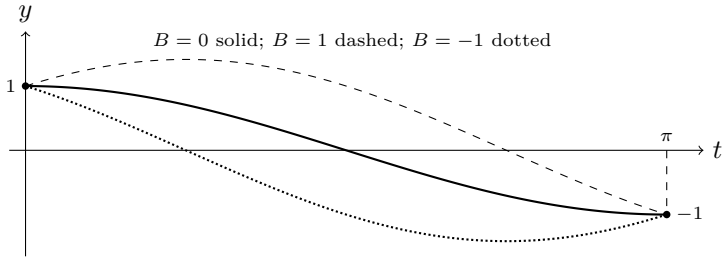
Step 2: Classify exceptional data. If $L = n\pi$, $n = 1, 2, \ldots$, the second equation reduces to $b = (-1)^n a$. If this fails, there is no solution. If it holds, B is arbitrary and all solutions are $a\cos t + B\sin t$; there are infinitely many.

Step 3: Apply the classification. At $L = \pi$, the data $(1, -1)$ give

$$\boxed{y = \cos t + B\sin t, \qquad B \in \mathbb{R}.}$$

The data $(1, 1)$ fail the necessary condition $b = -a$ and give no solution. The figure shows three members of the compatible family.

Step 4: Explain what failed. The data vectors of $\cos t, \sin t$ using value and slope at zero are $(1, 0)$ and $(0, 1)$, so the pair remains fundamental. At $L = n\pi$, both endpoint positions are insensitive to the coefficient B. Measuring $y'(0) = c$ fixes $B = c$ uniquely; the endpoint data must still be compatible with that solution.



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Question 6

Let $D = \mathbb{R} \setminus \{0\}$. On this disconnected domain, consider $y'' = 0$ and the two functions

$$u(t) = 1, \qquad v(t) = \begin{cases} -1, & t < 0, \\ 1, & t > 0. \end{cases}$$

A solution means a C^2 function on D; no matching condition at zero is imposed.

Tasks

1. Verify that both functions solve the equation and are linearly independent on D.
2. Do their constant linear combinations represent every solution on D? Give a decisive counterexample and compare their independence on each connected component.
3. Find the full solution family on D and an explicit independent spanning set. Prove why at least four functions are needed.
4. Describe every solution satisfying $y(1) = 0$, $y\prime(1) = 1$. Explain why those data do not determine a unique solution on all of D.

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Question 6 – Solution

Strategy. Each connected component has its own integration constants because zero is excluded and no matching rule is imposed.

Step 1: Verify and test independence. Both functions are constant on each component, so both have zero second derivative on D. If $Au + Bv = 0$ on D, the positive side gives $A + B = 0$ and the negative side gives $A - B = 0$. Thus $A = B = 0$.

Step 2: Test spanning. Every combination $Au + Bv$ is constant on each side, so it cannot represent the solution $y = t$. The pair is independent on D but does not span its solution space. On either individual component, $v = u$ or $v = -u$, so the restricted pair is dependent.

Step 3: Exhibit all four freedoms. Integrating separately gives

$$\boxed{y(t) = \begin{cases} A_- + B_- t, & t < 0, \\ A_+ + B_+ t, & t > 0. \end{cases}}$$

Let χ_+ equal 1 for $t > 0$ and 0 for $t < 0$, and let $\chi_- = 1 - \chi_+$. An independent spanning set is

$$\{\chi_-,\ t\chi_-,\ \chi_+,\ t\chi_+\}.$$

Restriction to either side proves independence by the independence of $1, t$. Each of the four displayed constants can be chosen freely, so the space has dimension four and no set with fewer than four functions spans it. All four functions are smooth on D.

Step 4: Impose data on one component. On the positive side, $B_+ = 1$ and $A_+ = -1$. Thus $y = t - 1$ for $t > 0$, while A_-, B_- remain arbitrary for $t < 0$. Initial-value uniqueness controls the connected interval containing the initial point; it supplies no information across the excluded point.

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Question 7

Consider $y'' + 4y = 0$ on $\mathbb{R}$. Allow complex-valued solutions initially and define

$$z_+ = e^{2it}, \qquad z_- = e^{-2it}, \qquad i^2 = -1.$$

Then restrict attention to real-valued solutions.

Tasks

1. Verify the two complex solutions and show that they form a fundamental set over the complex numbers.
2. Find the necessary and sufficient relation between complex constants A, B for $Az_+ + Bz_-$ to be real for every real t.
3. Derive the correspondence with real coefficients in $a\cos 2t + b\sin 2t$. Explain why z_+, z_- themselves are not a fundamental set of the real-valued solution space.
4. Solve $y(0) = 1$, $y\prime(0) = -2$ in both representations and verify the coefficients.

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Question 7 – Solution

Strategy. Specify the scalar field and impose real-valuedness on the entire function, not just its value at one point.

Step 1: Verify complex completeness. Each candidate has second derivative -4 times itself. Their data columns at zero are $(1, 2i)$ and $(1, -2i)$, with determinant $-4i \neq 0$. The initial-data criterion applies over $\mathbb{C}$ as well: real and imaginary parts each satisfy the real equation and its uniqueness theorem.

Step 2: Impose reality. Since $\overline{z_+} = z_-$, the conjugate of $Az_+ + Bz_-$ is $\overline{B}z_+ + \overline{A}z_-$. Independence implies equality to its conjugate exactly when

$$\boxed{B = \overline{A}.}$$

This relation is sufficient too, since the two terms then are conjugates.

Step 3: Convert coordinates. Euler's formula gives

$$Az_+ + Bz_- = (A + B)\cos 2t + i(A - B)\sin 2t.$$

For real a, b, the inverse relations are

$$\boxed{A = (a - ib)/2, \qquad B = (a + ib)/2.}$$

They automatically satisfy the reality condition. The exponentials are not themselves real-valued functions on $\mathbb{R}$, so they are not elements of the real-valued solution space. A real fundamental set is $\cos 2t, \sin 2t$.

Step 4: Match the data. The real representation has $a = 1$ and $2b = -2$, hence

$$\boxed{y = \cos 2t - \sin 2t = \tfrac{1}{2}(1 + i)e^{2it} + \tfrac{1}{2}(1 - i)e^{-2it}.}$$

Here $A + B = 1$ and $2i(A - B) = -2$, verifying the value and slope. The conjugate coefficients ensure that the resulting solution is real everywhere.

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Question 8

Two measured solution profiles are $u(t) = t$ and $v(t) = e^{-t}$. Seek a normalized equation

$$y'' + p(t)y' + q(t)y = 0$$

with both as solutions on a connected open interval containing $t = 0$.

Tasks

1. Use the two profiles to determine p, q wherever the coefficient equations are solvable.
2. Find the largest interval containing zero on which the reconstructed coefficients are continuous, and prove that the profiles form a fundamental set there.
3. Prove that no choice of continuous normalized coefficients on an interval containing -1 can have both profiles as solutions. Does their smoothness resolve this obstruction?
4. Use the fundamental pair to solve $y(0) = 2$, $y\prime(0) = -1$. Verify the data and explain the distinction between extending this formula and extending the normalized equation.

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Question 8 – Solution

Strategy. Treat the two known solutions as simultaneous equations for the unknown coefficients, then check the interval where the result is regular.

Step 1: Reconstruct the equation. Substituting $u = t$ and $v = e^{-t}$ gives

$$p + tq = 0, \qquad 1 - p + q = 0.$$

For $t \neq -1$, these have the unique solution

$$\boxed{p = \frac{t}{t+1}, \qquad q = -\frac{1}{t+1}.}$$

Thus both residuals vanish in $y'' + [t/(t+1)]y' - y/(t+1) = 0$ wherever these coefficients are defined.

Step 2: Find a fundamental interval. The largest connected open interval containing zero and avoiding the pole is $\boxed{(-1, \infty)}$. The initial-data determinant, or its value at any point of this interval, is

$$uv' - u'v = -(t+1)e^{-t} \neq 0.$$

In particular it equals -1 at zero, so the two solutions form a fundamental set throughout the interval.

Step 3: Locate the obstruction. At $t = -1$, the coefficient equations would require $p - q = 0$ and $1 - p + q = 0$, an immediate contradiction. Even finite pointwise coefficient values cannot satisfy both conditions there. Smoothness of the two functions does not guarantee a common regular normalized equation across that point.

Step 4: Solve and distinguish extensions. In $y = At + Be^{-t}$, the data give $B = 2$ and $A - B = -1$, hence

$$\boxed{y = t + 2e^{-t} \quad (-1 < t < \infty).}$$

The value and slope at zero are $2, -1$, and linearity verifies its equation. The formula is smooth for all real t, but the reconstructed normalized equation is undefined at -1. Extending a function does not remove that coefficient singularity. Multiplying by $t + 1$ instead gives a different, undivided formulation with a vanishing leading coefficient there.

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Question 9

For $y'' + y = 0$ on $\mathbb{R}$, let the state at a time s mean the ordered pair $(y(s), y'(s))$. Fix a real reference time s.

Tasks

1. Construct a fundamental pair ϕ_s, ψ_s with state data $(1, 0)$ and $(0, 1)$ at time s. Express any solution in that pair.
2. Derive explicit formulas for the state at $s + h$ in terms of the state (a, b) at s.
3. Show that moving the reference time by h and then k gives the same result as moving it by $h + k$, and that the move by $-h$ reverses the move by h. Prove that $y^2 + (y\prime)^2$ is preserved.
4. For initial state $(3, 4)$ at zero, find the first positive time when the slope is zero and the value then. Describe the direction of travel in the state plane.

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Question 9 – Solution

Strategy. Changing the reference time changes the coordinates in a normalized fundamental pair, while the represented solution remains the same.

Step 1: Normalize at the reference time. Set

$$\phi_s(t) = \cos(t - s), \qquad \psi_s(t) = \sin(t - s).$$

Both solve the equation and have the required data. Their data determinant is 1, so any solution is $y(t) = a\cos(t - s) + b\sin(t - s)$, where $a = y(s)$ and $b = y'(s)$.

Step 2: Move the state. Evaluation and differentiation give

$$\boxed{y(s + h) = a\cos h + b\sin h, \qquad y'(s + h) = -a\sin h + b\cos h.}$$

Step 3: Compose and reverse. Applying the same formulas with increment k yields first component

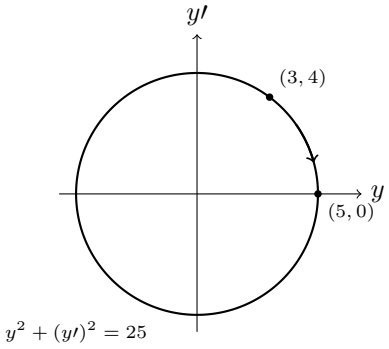
$$a(\cos h\cos k - \sin h\sin k) + b(\sin h\cos k + \cos h\sin k).$$

The addition formulas make this $a\cos(h+k)+b\sin(h+k)$; the second component is $-a\sin(h+k)+b\cos(h+k)$. Taking $k = -h$ gives (a, b). Squaring the two components and adding cancels the cross terms and gives a^2+b^2.

Step 4: Locate the first turning point. Here $y = 3\cos t + 4\sin t$ and $y' = -3\sin t + 4\cos t$. The first positive zero of y' is

$$\boxed{t_* = \arctan(4/3), \qquad y(t_*) = 5.}$$

Indeed $\cos t_* = 3/5$, $\sin t_* = 4/5$, and the slope is positive before this time. In coordinates (y, y'), the state travels clockwise on the circle of radius 5: at $(5, 0)$ its derivative is $(0, -5)$.



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Question 10

For $y'' - y = 0$ on $\mathbb{R}$, fix $L > 0$. Seek solutions ℓ, r with endpoint values

$$\ell(0) = 1, \quad \ell(L) = 0, \qquad r(0) = 0, \quad r(L) = 1.$$

Use $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

Tasks

1. Construct ℓ, r from the exponential fundamental pair and prove that they also form a fundamental set.
2. Use this pair to solve $y(0) = a$, $y(L) = b$ for arbitrary real a, b. Prove uniqueness.
3. For $a, b \geq 0$ and not both zero, prove positivity on $0 < t < L$ and $y(t) \leq \max(a, b)$ on $[0, L]$. Determine whether $\ell(t) + r(t) = 1$ in the interior.
4. For $a = b = 1$, locate the unique minimum on $[0, L]$ and give its value. Explain why positive interpolation weights do not imply straight-line interpolation.

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Question 10 – Solution

Strategy. Choose basis functions adapted to the two measurements, then inspect their positive weights.

Step 1: Build an endpoint basis. Write $r = Ae^{t} + Be^{-t}$. The data give $A + B = 0$ and $A(e^{L} - e^{-L}) = 1$, so $A = 1/(2\sinh L)$ and $B = -A$. Reflecting r about the midpoint gives $\ell(t) = r(L - t)$, hence

$$\boxed{\ell(t) = \frac{\sinh(L-t)}{\sinh L}, \qquad r(t) = \frac{\sinh t}{\sinh L}.}$$

Both solve $y'' - y = 0$ and have the stated endpoint values. Their determinant at zero is $\ell(0)r'(0) - \ell'(0)r(0) = 1/\sinh L \neq 0$, so they form a fundamental set on $\mathbb{R}$.

Step 2: Fit the measurements. In this pair the endpoint coefficients are exactly the prescribed values:

$$\boxed{y = a\ell + br.}$$

Every solution has a unique representation, and evaluating at $0, L$ forces its coefficients to be a, b. Thus the endpoint problem has exactly one solution.

Step 3: Bound the positive weights. On $0 < t < L$, both weights are positive. The hyperbolic addition identity gives

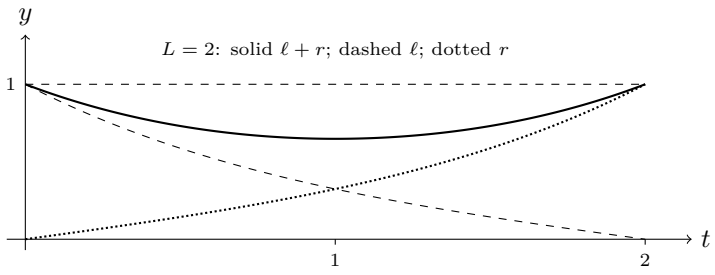
$$\ell(t) + r(t) = \frac{\cosh(t - L/2)}{\cosh(L/2)} \leq 1 \qquad (0 \leq t \leq L),$$

with strict inequality in the interior, since $|t - L/2| < L/2$. If $M = \max(a, b)$, then $0 < y$ in the interior and $y \leq M(\ell + r) \leq M$ on the full closed interval. At an endpoint the solution can equal zero if the corresponding datum is zero.

Step 4: Examine equal endpoint data. For $a = b = 1$,

$$y = \frac{\cosh(t - L/2)}{\cosh(L/2)}, \qquad \boxed{y_{\min} = \frac{1}{\cosh(L/2)} \text{ at } t = L/2.}$$

The minimum is unique because $\cosh x$ has its unique minimum at zero. The weights are positive but sum to less than one inside; this is not the straight-line interpolation of equal endpoint values.



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Question 1

For two real solutions u, v of $y'' + p(t)y' + q(t)y = 0$ on a connected open interval I, define the ordered Wronskian

$$W[u, v](t) = u(t)v'(t) - u'(t)v(t).$$

Assume p, q are continuous and $t_0 \in I$.

Tasks

1. Derive a first-order equation for W and solve it in terms of $W(t_0)$. Your derivation must include the case $W(t_0) = 0$.
2. Prove that W either vanishes everywhere on I or never vanishes there. Explain the consequences for independence of the two solutions.
3. For $(1 + t^2)y'' + 2ty' + (1 + t^2)y = 0$ on $\mathbb{R}$, find $W(t)$ if $W(0) = -3$, without solving the second-order equation.
4. Find the sign and limiting values of this Wronskian. Does its approach to zero at infinity imply that the solutions become dependent at a finite time?

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Question 1 – Solution

Strategy. Differentiate the determinant and substitute the common equation before integrating.

Step 1: Derive Abel's identity. Differentiation cancels the mixed terms:

$$W' = uv'' - u''v = u(-pv' - qv) - (-pu' - qu)v = -pW.$$

Multiplication by $\exp(\int_{t_0}^{t} p(s)\,ds)$ gives a function with zero derivative. Thus, without dividing by W,

$$\boxed{W(t) = W(t_0)\exp[-\int_{t_0}^{t} p(s)\,ds].}$$

This includes $W \equiv 0$ when $W(t_0) = 0$.

Step 2: Use the nonzero exponential. The exponential factor is positive and finite at every point of I. A nonzero initial Wronskian stays nonzero with the same sign. If it is zero, the initial-data columns are dependent; a nontrivial constant combination has zero value and slope at t_0. Regular uniqueness forces that combination to vanish on I. Therefore, for solutions of this common regular equation, $W \equiv 0$ is equivalent to dependence.

Step 3: Normalize the example. Divide by $1 + t^2$, giving $p = 2t/(1 + t^2)$. Integration gives

$$\int_0^t \frac{2s}{1+s^2}\,ds = \ln(1+t^2), \qquad \boxed{W(t) = -\frac{3}{1+t^2}.}$$

Thus the Wronskian is determined without finding either solution. The undivided coefficient $2t$ is not the coefficient to use in Abel's identity.

Step 4: Interpret the limits. We have $W < 0$ everywhere, with $W \to 0$ from below as $t \to \pm\infty$. The pair is independent on all of $\mathbb{R}$, and its data columns are independent at every finite time. A limiting zero outside the interval is not a zero at a point where the initial-data criterion applies.

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Question 2

Let u, v solve $y'' + 2y' + q(t)y = 0$ on $\mathbb{R}$, where q is continuous, with data

$$(u(0), u'(0)) = (1, 0), \qquad (v(0), v'(0)) = (0, 1).$$

Define $f = 2u - v$ and $g = 3u + 4v$. At each time, regard (f, f') and (g, g') as vectors in the value–slope plane.

Tasks

1. For arbitrary constants a, b, c, d, derive the formula relating $W[au + bv, cu + dv]$ to $W[u, v]$.
2. Find $W[f, g](t)$ and decide whether f, g form a fundamental set. Explain the effect of reversing their order.
3. Interpret the Wronskian as oriented parallelogram area. Find the ordinary area at time zero and the first nonnegative time at which it is half that size.
4. Can the two state vectors become parallel at a finite time? Explain what their area tending to zero does and does not imply.

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Question 2 – Solution

Strategy. Separate the constant determinant of a basis change from the time evolution supplied by Abel's identity.

Step 1: Expand the determinant. Terms containing uu' or vv' cancel, leaving

$$\boxed{W[au+bv, cu+dv] = (ad-bc)W[u,v].}$$

The constants may be factored through differentiation; this step would require extra terms for time-dependent coefficients.

Step 2: Apply the two determinants. Here $W[u,v](0) = 1$ and $p = 2$, so $W[u,v] = e^{-2t}$. The coefficient determinant is $2 \cdot 4 - (-1) \cdot 3 = 11$, hence

$$\boxed{W[f,g] = 11e^{-2t} > 0.}$$

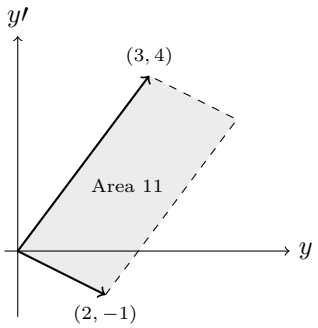
Both combinations solve the equation and are independent, so they are fundamental. Reversing their order negates the Wronskian but leaves independence unchanged.

Step 3: Compute the area. The columns (f, f'), (g, g') have signed area $W[f,g]$ and ordinary area $|W[f,g]|$. At zero they are $(2,-1)$ and $(3,4)$, with area 11. The half-area equation gives

$$11e^{-2t} = 11/2, \qquad \boxed{t = \tfrac{1}{2}\ln 2.}$$

The area is strictly decreasing for $t \geq 0$, so this is the first such time.

Step 4: Distinguish collapse from a limit. The determinant never vanishes at finite t, so the two state vectors never become parallel there. The area tends to zero as $t \to \infty$; this does not destroy finite-time independence. Area alone also does not specify the lengths of the two vectors or their angle separately.



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Question 3

Consider the functions $u(t) = t^3$ and $v(t) = |t|^3$ on $\mathbb{R}$. A student claims that a Wronskian which is identically zero always proves linear dependence of two C^2 functions.

Tasks

1. Prove that both functions are C^2 on $\mathbb{R}$, explicitly checking the derivatives of v at zero.
2. Compute $W[u, v]$ everywhere and determine whether the two functions are linearly independent on $\mathbb{R}$.
3. Explain precisely why the common regular-equation theorem does not contradict your result. Could both functions solve a normalized equation with continuous coefficients on $\mathbb{R}$?
4. Verify that both solve the undivided equation $t^2y'' - 2ty' = 0$ on $\mathbb{R}$. Compare the ratio v/u on the two sides of zero.

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Question 3 – Solution

Strategy. Check the counterexample directly, then identify the hypothesis missing from the student's assertion.

Step 1: Check regularity. For $t \neq 0$, $v' = 3t|t|$ and $v'' = 6|t|$. At zero,

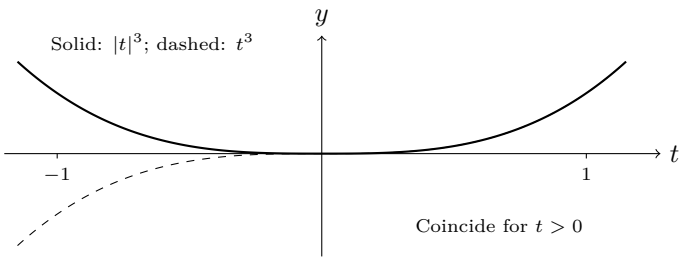
$$v'(0) = \lim_{h\to 0} |h|^3/h = 0, \qquad v''(0) = \lim_{h\to 0} 3h|h|/h = 0.$$

These agree continuously with the formulas away from zero, so v is C^2; u is a polynomial and is smooth.

Step 2: Compare the Wronskian and independence. On $t > 0$, $v = u$; on $t < 0$, $v = -u$. Thus $W = 0$ on each side, and at zero both data columns are zero, so $\boxed{W \equiv 0}$. Yet $Au + Bv = 0$ on all of $\mathbb{R}$ forces $A + B = 0$ on the positive side and $A - B = 0$ on the negative side. Hence $A = B = 0$: the functions are independent on $\mathbb{R}$.

Step 3: Restore the missing hypothesis. The zero-Wronskian implication requires solutions of one common regular homogeneous linear equation on a connected interval, not merely two C^2 functions. The nonzero solution u would have zero value and slope at zero, contradicting regular uniqueness. Thus no normalized equation with continuous coefficients on $\mathbb{R}$ can have both as solutions.

Step 4: Verify a singular equation. For u, the residual is $t^2(6t) - 2t(3t^2) = 0$. On each side $v = \pm u$, so its residual also vanishes; at zero both terms vanish. The leading coefficient is zero there. The ratio v/u is 1 on $t > 0$ and -1 on $t < 0$, and is undefined at the shared zero. Local constant ratios need not join into one global constant.



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Question 4

A regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ has a known solution $u = e^t$. A desired companion v must satisfy

$$W[u, v](t) = 2e^{-t^2}, \qquad v(0) = 0.$$

Assume p, q are continuous; an unevaluated definite integral is an acceptable exact answer.

Tasks

1. Determine the only possible coefficients p, q.
2. Construct v using the prescribed Wronskian and initial value, and find $v\prime(0)$.
3. Verify both the Wronskian and the differential equation without evaluating the integral.
4. If the known solution u were not supplied, would the prescribed Wronskian alone determine q? Justify your answer using initial-data normalization for arbitrary continuous q.

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Question 4 – Solution

Strategy. Abel's identity determines the first-derivative coefficient; the known nonzero solution then determines the remaining coefficient.

Step 1: Recover the equation. Since W never vanishes,

$$p = -W'/W = 2t.$$

Substituting $u = e^t$ gives $1 + p + q = 0$, hence

$$\boxed{y'' + 2ty' - (1 + 2t)y = 0.}$$

Step 2: Recover the companion. The identity $W = e^t(v' - v)$ gives $v' - v = 2e^{-t^2-t}$. Multiplying by e^{-t} and integrating from zero yields

$$\boxed{v(t) = e^t \int_0^t 2e^{-s^2-2s}\, ds.}$$

The integral is smooth on all of $\mathbb{R}$. It gives $v(0) = 0$ and $v'(0) = 2$.

Step 3: Verify by differentiation. Write J for the integral, so $J' = 2e^{-t^2-2t}$. Then $v' = v + 2e^{-t^2-t}$ and $e^t(v' - v) = 2e^{-t^2}$, as prescribed. If $L[y] = y'' + py' + qy$, direct expansion gives

$$uL[v] - vL[u] = W' + pW.$$

Here $L[u] = 0$ and $W' + 2tW = 0$. Since $u = e^t \neq 0$, this proves $L[v] = 0$ everywhere.

Step 4: Identify what the Wronskian omits. With $p = 2t$ and any continuous choice of q, regular existence supplies solutions with data $(1, 0)$ and $(0, 2)$ at zero. Their initial Wronskian is 2, so Abel's identity gives the same $2e^{-t^2}$ for every such q. Thus the Wronskian alone fixes p but not q; the supplied nonzero seed was essential for the latter.

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Question 5

Two independent solutions of a regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ have measured ordered Wronskians

$$W(0) = 6, \qquad W(2) = 3, \qquad W(5) = 12.$$

Assume p, q are continuous and the measurements are exact.

Tasks

1. Find $\int_0^2 p(t)\,dt$ and $\int_2^5 p(t)\,dt$, and the average value of p on each interval.
2. Prove that p is positive somewhere in $(0, 2)$, negative somewhere in $(2, 5)$, and zero somewhere in $(0, 5)$.
3. Under the additional assumption that $p(t) = a + bt$, determine a, b and the full Wronskian function. Check all three measurements.
4. For this affine model, find the unique minimum of W on $[0, 5]$. Explain why three Wronskian samples alone do not prove that the true coefficient is affine.

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Question 5 – Solution

Strategy. Ratios of Wronskian measurements give integrals of the coefficient, rather than its pointwise values.

Step 1: Extract the averages. Abel's identity gives

$$\int_0^2 p = \ln 2, \qquad \int_2^5 p = -\ln 4.$$

The average values are $\boxed{(\ln 2)/2}$ and $\boxed{-(\ln 4)/3}$ respectively.

Step 2: Force a sign change. A continuous function with positive integral must be positive at an interior point; likewise the negative integral requires an interior negative value. Choosing one point from each interval, the intermediate value theorem supplies a zero between them. This conclusion does not specify the value of p at time 2.

Step 3: Fit the affine model. The coefficient equations are $2a + 2b = \ln 2$ and $3a + (21/2)b = -2\ln 2$. Solving gives

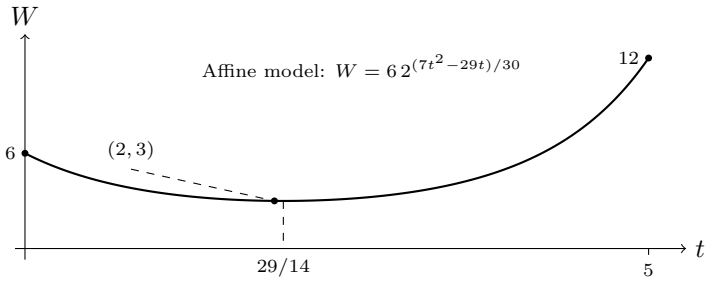
$$p(t) = \frac{29 - 14t}{30}\ln 2, \qquad \boxed{W(t) = 6\,2^{(7t^2-29t)/30}.}$$

At $t = 0, 2, 5$, the exponents are $0, -1, 1$, giving $6, 3, 12$ as required.

Step 4: Locate and qualify the minimum. Since $W' = -pW$ and $W > 0$, the Wronskian decreases until $t = 29/14$ and increases afterward. Its unique minimum on $[0, 5]$ is

$$\boxed{W(29/14) = 6\,2^{-841/840}.}$$

Affineness is an extra modeling assumption. For example, adding any multiple of $\cos(2\pi t)$ to this p leaves both measured integrals unchanged, but generally gives a nonaffine coefficient and a different Wronskian between the measurement times.



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Question 6

Let u, v be independent real solutions of $y'' + p(t)y' + q(t)y = 0$ on an open interval I, with continuous coefficients. Suppose $a < b$ lie in I and are consecutive zeros of u: $u(a) = u(b) = 0$ and $u(t) \neq 0$ for $a < t < b$.

Tasks

1. Prove that the zeros at a, b are simple and that $v(a), v(b)$ are nonzero.
2. Use the sign of the Wronskian at the endpoints to prove that $v(a)$ and $v(b)$ have opposite signs.
3. Differentiate v/u on (a, b) and use it to prove that v has exactly one zero there.
4. Apply the result to $u = \sin t$, $v = \cos t + 2\sin t$ between 0 and π, verifying independence and finding the unique zero explicitly.

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Question 6 – Solution

Strategy. Use the constant sign of the Wronskian first at the endpoints and then in the derivative of a ratio.

Step 1: Exclude repeated and shared zeros. If u and u' both vanished at an endpoint, regular uniqueness would make u identically zero, contrary to independence. Thus both zeros are simple. Since W never vanishes, $W(a) = -u'(a)v(a)$ and $W(b) = -u'(b)v(b)$ force $v(a), v(b) \neq 0$.

Step 2: Compare endpoint signs. Multiply u by -1 if necessary so that $u > 0$ on (a, b). Simplicity then implies $u'(a) > 0$ and $u'(b) < 0$. Abel's identity says $W(a), W(b)$ have the same sign. The two endpoint formulas therefore force $v(a), v(b)$ to have opposite signs. Continuity gives at least one zero of v inside.

Step 3: Prove uniqueness of that zero. On the zero-free interval for u,

$$\boxed{(v/u)' = \frac{uv' - u'v}{u^2} = \frac{W}{u^2}.}$$

The derivative has a fixed nonzero sign, so the ratio is strictly monotone. Its zeros are precisely the zeros of v there. It can vanish at most once; combined with the preceding existence argument, this proves exactly one zero.

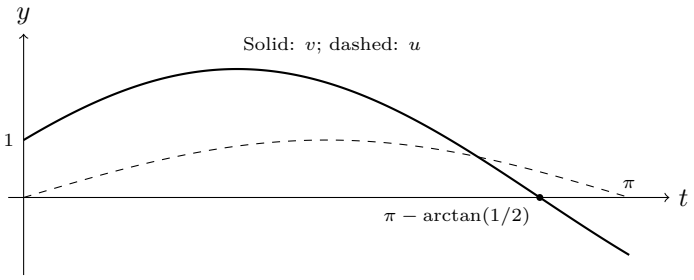
Step 4: Compute the example. Both candidates solve $y'' + y = 0$, and

$$W[\sin t, \cos t + 2\sin t] = -1.$$

The endpoint values of v are $1, -1$. In $(0, \pi)$, the equation $\cos t + 2\sin t = 0$ has its root in the second quadrant, with

$$\boxed{t = \pi - \arctan(1/2).}$$

The separation argument proves there are no other roots between these consecutive zeros of $\sin t$.



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Question 7

Let p be C^1 and q continuous on an open interval I containing zero, and set $P(t) = \int_0^t p(s)\,ds$. Consider the substitution $y = e^{-P/2}z$ in $y'' + py' + qy = 0$.

Tasks

1. Derive the differential equation satisfied by z and show that its first-derivative term vanishes.
2. For two transformed solutions, derive the relation between their Wronskian and the original pair's Wronskian. Explain why the transformed Wronskian is constant.
3. Apply the method to $y'' + 2ty' + (t^2 + 3)y = 0$ on $\mathbb{R}$. Construct a fundamental pair normalized to data $(1, 0)$ and $(0, 1)$ at zero and compute its Wronskian.
4. Solve $y(0) = 1$, $y'(0) = 2$ and explain why both members of your fundamental pair tending to zero as $t \to \infty$ does not contradict independence.

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Question 7 – Solution

Strategy. A common nonzero multiplier changes the Wronskian by its square and can remove the first-derivative coefficient.

Step 1: Differentiate the substitution. Put $h = e^{-P/2}$, so $h'/h = -p/2$ and $h''/h = -p'/2 + p^2/4$. Substitution of $y = hz$ gives

$$z'' + (2h'/h + p)z' + (h''/h + ph'/h + q)z = 0,$$

and therefore

$$\boxed{z'' + [q - p'/2 - p^2/4]z = 0.}$$

Step 2: Transform the determinant. Expanding derivatives cancels the terms containing hh':

$$\boxed{W[hz_1, hz_2] = h^2 W[z_1, z_2] = e^{-P} W[z_1, z_2].}$$

The transformed equation has no first-derivative term, so its Wronskian has derivative zero. The nonzero multiplier preserves independence.

Step 3: Solve the transformed example. Here $P = t^2$ and $q - p'/2 - p^2/4 = t^2 + 3 - 1 - t^2 = 2$. Thus $z'' + 2z = 0$, and the normalized pair is

$$\boxed{\phi = e^{-t^2/2}\cos(\sqrt{2}t), \qquad \psi = e^{-t^2/2}\frac{\sin(\sqrt{2}t)}{\sqrt{2}}.}$$

At zero, $h = 1$ and $h' = 0$, so the data remain $(1, 0)$ and $(0, 1)$. The transformed Wronskian is 1, giving $W[\phi, \psi] = e^{-t^2} > 0$ everywhere.

Step 4: Match and interpret. The normalized data give

$$\boxed{y = \phi + 2\psi = e^{-t^2/2}[\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t)].}$$

Its value and slope at zero are $1, 2$. Both basis functions decay to zero because their trigonometric factors are bounded and the Gaussian factor decays. Dependence would require an exact constant relation on the interval; the strictly positive finite-time Wronskian rules that out.

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Question 8

Suppose p, q are real continuous functions on $\mathbb{R}$, each periodic with period $T > 0$. Consider $y'' + p(t)y' + q(t)y = 0$. A T-periodic solution satisfies $y(t + T) = y(t)$ for every real t.

Tasks

1. Prove that the existence of two independent T-periodic solutions requires $\int_0^T p(t)\,dt = 0$.
2. If $p \geq 0$ and p is not identically zero, decide whether two such periodic solutions can exist.
3. Is the zero-integral condition sufficient? Analyze $y'' - y = 0$ as a counterexample and justify the absence of nonzero periodic solutions.
4. Does the existence of just one nonzero T-periodic solution force the same integral condition? Analyze $y'' + y' = 0$ and identify a nonperiodic companion.

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Question 8 – Solution

Strategy. Periodicity forces the Wronskian to return to its starting value, while Abel's identity specifies its multiplicative change.

Step 1: Compare one full period. Differentiating $y(t+T) = y(t)$ shows that the slope is periodic too. Thus two periodic solutions have $W(T) = W(0)$. Independence gives $W(0) \neq 0$, so Abel's identity implies

$$\exp[-\int_0^T p(t)\,dt] = 1.$$

The integral is real; consequently

$$\boxed{\int_0^T p(t)\,dt = 0.}$$

Step 2: Exclude nontrivial nonnegative damping. By continuity and periodicity, if $p \geq 0$ and is not identically zero, it is positive on some interval within a period. Hence its integral over a period is strictly positive. The necessary condition fails, so two independent T-periodic solutions cannot exist.

Step 3: Disprove sufficiency. For $y'' - y = 0$, $p = 0$ satisfies the integral condition for every T. Every solution is $Ae^t + Be^{-t}$. A continuous periodic function is bounded on $\mathbb{R}$; boundedness as $t \to \infty$ forces $A = 0$, and boundedness as $t \to -\infty$ forces $B = 0$. Thus there are no nonzero periodic solutions, much less two independent ones.

Step 4: Test a single periodic solution. In $y'' + y' = 0$, the constant solution $u = 1$ is nonzero and periodic for every $T > 0$, while $p = 1$ has integral $T \neq 0$. A companion is $v = e^{-t}$, with $W[u, v] = -e^{-t} \neq 0$; it is not periodic. One periodic solution does not make a nonzero pair Wronskian periodic and imposes no such zero-integral requirement.

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Question 9

On $\mathbb{R}$, consider the original undivided equation

$$t^2y'' - 2ty' + 2y = 0$$

and the smooth candidates $u = t$, $v = t^2$. A classical global solution here means a C^2 function on $\mathbb{R}$ satisfying this undivided equation at every point.

Tasks

1. Verify both candidates and compute their Wronskian. Determine their independence on $\mathbb{R}$ and identify every zero of the Wronskian.
2. Apply Abel's identity separately on the two largest regular intervals. Explain why its zero-or-never-zero conclusion cannot be applied across zero.
3. Find every classical global solution by matching the general solutions from the negative and positive sides through zero.
4. Find all global solutions with $y(0) = 0$, $y\prime(0) = 1$. Explain both the failure of uniqueness and why no regular normalized equation through zero can have this candidate pair as solutions.

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Question 9 – Solution

Strategy. The singular leading coefficient changes which interval hypotheses are available; explicitly match derivatives to determine global solutions.

Step 1: Verify and compute. The residuals are $-2t + 2t = 0$ and $2t^2 - 4t^2 + 2t^2 = 0$. Also

$$\boxed{W[t, t^2] = t(2t) - 1(t^2) = t^2.}$$

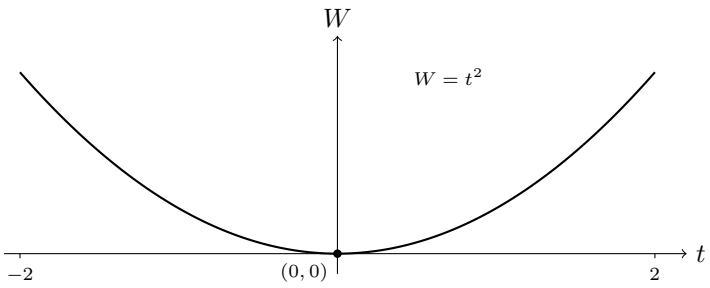
It is zero only at $t = 0$. The functions are independent on $\mathbb{R}$, since $At + Bt^2 \equiv 0$ forces $A = B = 0$.

Step 2: Respect the regular intervals. Division by t^2 gives $p = -2/t$, $q = 2/t^2$, continuous on $(-\infty, 0)$ and $(0, \infty)$. On either interval Abel's identity gives $W = Ct^2$, with the constant fixed within that interval. For this pair it is $C = 1$ on both sides. There is no regular interval containing zero. Abel's conclusion applies separately on the two intervals and makes no assertion across zero.

Step 3: Match all global solutions. The pair is fundamental on either side, so write $A_-t + B_-t^2$ for $t < 0$ and $A_+t + B_+t^2$ for $t > 0$. Continuity forces $y(0) = 0$. Matching first derivatives forces $A_- = A_+$; matching second derivatives forces $B_- = B_+$. Conversely every resulting polynomial solves the equation globally. Thus

$$\boxed{y = At + Bt^2 \quad \text{on } \mathbb{R}.}$$

Step 4: Inspect the singular data. The data give $A = 1$ but leave B arbitrary, so all solutions are $\boxed{y = t + Bt^2}$. Uniqueness fails despite this two-dimensional global family. No regular normalized equation through zero could include $v = t^2$: at zero its value and slope vanish while its second derivative is 2, making the normalized equation impossible there.



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Question 10

Let u, v be C^2 real functions on an interval and let θ be a real C^2 function. Define time-dependent combinations

$$f = u\cos\theta + v\sin\theta, \qquad g = -u\sin\theta + v\cos\theta.$$

At every time the two-by-two coefficient matrix has determinant 1.

Tasks

1. Derive the actual expression for $W[f, g]$ in terms of u, v, θ and their derivatives. Explain why the constant change-of-basis formula cannot simply be used.
2. Take $u = \cos t$, $v = \sin t$, $\theta = t$ on $\mathbb{R}$. Compute f, g and their Wronskian. Does the pointwise invertibility preserve a fundamental set for $y'' + y = 0$?
3. Keep the same u, v but take $\theta = t/2$. Show that a nonzero resulting Wronskian still does not make the new pair solutions of the original equation.
4. For these trigonometric u, v, classify every real C^2 function θ on $\mathbb{R}$ for which both f, g solve $y'' + y = 0$.

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Question 10 – Solution

Strategy. Differentiate the moving coefficients too; pointwise invertibility is not a constant linear transformation of solution functions.

Step 1: Include the extra derivative terms. In the derivatives of f, g, the terms involving u', v' contribute $W[u, v]$, as in a constant rotation. The remaining terms are $\theta' g$ in f' and $-\theta' f$ in g'. Hence

$$\boxed{W[f, g] = W[u, v] - \theta'(f^2 + g^2) = W[u, v] - \theta'(u^2 + v^2).}$$

The additional term is exactly what the constant-coefficient formula omits.

Step 2: Observe a complete collapse. For $u = \cos t$, $v = \sin t$, the addition formulas give $f = \cos(t-\theta)$, $g = \sin(t - \theta)$. At $\theta = t$, $f = 1$ and $g = 0$, so $W = 0$. They are dependent, and $f'' + f = 1 \neq 0$. Pointwise invertibility of the moving coefficient matrix does not preserve the solution space.

Step 3: Test a nonzero determinant. At $\theta = t/2$, the functions are $\cos(t/2), \sin(t/2)$ and $W = 1/2$. Each has $y''+y = (3/4)y$, so neither solves the original equation identically. They instead form a fundamental set for $y'' + y/4 = 0$.

Step 4: Classify all valid rotations. Put $\varphi = t - \theta$. The residuals for $f = \cos\varphi$, $g = \sin\varphi$ are

$$(1 - \varphi'^2)\cos\varphi - \varphi''\sin\varphi, \qquad (1 - \varphi'^2)\sin\varphi + \varphi''\cos\varphi.$$

Both vanish exactly when $\varphi'^2 = 1$ and $\varphi'' = 0$, since the displayed rotation is invertible. On the connected real line this gives $\varphi = \pm t + C$, hence

$$\boxed{\theta(t) = C \quad \text{or} \quad \theta(t) = 2t + C, \qquad C \in \mathbb{R}.}$$

In the second case the final functions are still constant combinations of $\cos t, \sin t$, despite their time-dependent defining coefficients.

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Question 1

Let $L[y] = y'' + p(t)y' + q(t)y$ on a connected open interval I, where p, q, g are continuous. Suppose y_p is one solution of $L[y] = g$, and u, v form a fundamental set for $L[y] = 0$.

Tasks

1. Prove that every solution of $L[y] = g$ has the form $y_p + Au + Bv$, with unique constants A, B. Prove both directions.
2. If initial data $y(t_0) = a$, $y\prime(t_0) = b$ are prescribed, state the initial data that the homogeneous correction must satisfy and explain uniqueness.
3. For $y\prime\prime - 3y\prime + 2y = 2t^2 - 6t + 2$ on $\mathbb{R}$, verify the supplied particular solution $y_p = t^2$ and find the general solution.
4. Solve the example with $y(0) = 1$, $y\prime(0) = 0$, verifying the equation and initial data.

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Question 1 – Solution

Strategy. Subtract one known forced solution so that the remaining unknown satisfies a homogeneous equation.

Step 1: Prove the representation. If $L[y] = L[y_p] = g$, linearity gives $L[y - y_p] = 0$. Completeness of u, v then gives $y - y_p = Au + Bv$. Conversely, $L[y_p + Au + Bv] = g + A(0) + B(0) = g$. Independence makes the two constants unique for each solution. Thus

$$\boxed{y = y_p + Au + Bv.}$$

Step 2: Transfer the initial data. The correction $h = y - y_p$ must satisfy

$$h(t_0) = a - y_p(t_0), \qquad h'(t_0) = b - y_p'(t_0).$$

Regular homogeneous existence and uniqueness determine h on I. Therefore the forced initial-value problem is unique too. The coefficients in the correction must fit these adjusted data, not automatically the original data.

Step 3: Verify and complete the example. For $y_p = t^2$, the left-hand side is $2 - 6t + 2t^2$, as required. The characteristic polynomial of the associated homogeneous equation is $(r - 1)(r - 2)$, so

$$\boxed{y = t^2 + Ae^t + Be^{2t}.}$$

The exponentials are independent homogeneous solutions, and the representation theorem proves that no further solution freedom is missing.

Step 4: Fit and check. At zero, the equations are $A + B = 1$ and $A + 2B = 0$. They give $A = 2$, $B = -1$, hence

$$\boxed{y = t^2 + 2e^t - e^{2t}.}$$

Its initial value is $2 - 1 = 1$ and its slope is $0 + 2 - 2 = 0$. The polynomial contributes the entire forcing, while each exponential has zero residual, verifying the differential equation everywhere.

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Question 2

For $y'' + y = 1$ on $\mathbb{R}$, define

$$y_1 = 1, \qquad y_2 = 1 + \cos t, \qquad y_3 = 1 + \sin t.$$

Consider constant combinations $y = ay_1 + by_2 + cy_3$, where $a, b, c \in \mathbb{R}$.

Tasks

1. Verify all three solutions and find the necessary and sufficient condition on a, b, c for their combination to solve the same forced equation.
2. Prove that every solution of $y'' + y = 1$ has exactly one representation satisfying that condition.
3. Explain whether the solution set is closed under sums, differences and midpoints. Identify the equation satisfied by the difference of two forced solutions.
4. Find the unique coefficients a, b, c producing $y(0) = 0$, $y'(0) = 2$, and verify the resulting solution.

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Question 2 – Solution

Strategy. A fixed nonzero forcing is preserved by affine combinations, whose coefficients sum to one.

Step 1: Track the forcing. Each listed function has $y_j'' + y_j = 1$. Therefore

$$L[ay_1 + by_2 + cy_3] = a + b + c.$$

The combination solves the same equation exactly when $\boxed{a + b + c = 1}$.

Step 2: Describe the full set. Subtracting the particular solution 1 leaves $h'' + h = 0$, so all solutions are $1 + B\cos t + C\sin t$. The proposed combination expands to

$$(a + b + c) + b\cos t + c\sin t.$$

It represents the specified solution exactly with $b = B$, $c = C$, $a = 1 - B - C$. Uniqueness follows from the independence of $1, \cos t, \sin t$: if their combination vanishes, adding its second derivative to itself first forces the constant coefficient to zero, then the two harmonic coefficients vanish.

Step 3: Compare operations. The sum of two solutions has forcing 2, and their difference has forcing 0, so neither operation stays in the set with forcing 1. Their midpoint has forcing $(1 + 1)/2 = 1$ and does stay in the set. The zero function is absent. Thus this is an affine translate of a homogeneous solution space, rather than a vector space itself.

Step 4: Fit the data. In $1 + B\cos t + C\sin t$, the data give $B = -1$, $C = 2$. Hence

$$\boxed{(a, b, c) = (0, -1, 2), \qquad y = 1 - \cos t + 2\sin t.}$$

The coefficients sum to one. Directly, $y'' = \cos t - 2\sin t$, so $y'' + y = 1$; the initial value and slope are $0, 2$.

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Question 3

For $y'' + 2y' + y = e^{-t}$ on $\mathbb{R}$, two proposed particular solutions are

$$p(t) = \tfrac{1}{2}t^2e^{-t}, \qquad \widetilde{p}(t) = e^{-t}(\tfrac{1}{2}t^2 + 3 - 2t).$$

Two solvers write $y = p + (A + Bt)e^{-t}$ and $y = \widetilde{p} + (\alpha + \beta t)e^{-t}$.

Tasks

1. Verify both particular solutions and explain why their difference must be homogeneous.
2. Derive the exact relation between the two pairs of homogeneous constants. Do the two formulas describe different solution sets?
3. Solve $y(0) = 2$, $y\prime(0) = -1$ in both representations and verify the data.
4. For the selected solution on $t \geq 0$, determine positivity, monotonicity and its limit. Explain why the particular and homogeneous pieces are not uniquely determined by the trajectory.

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Question 3 – Solution

Strategy. Changing the particular solution transfers a homogeneous term between the two pieces without changing the full solution.

Step 1: Verify efficiently. For any twice differentiable F, differentiation gives

$$(Fe^{-t})'' + 2(Fe^{-t})' + Fe^{-t} = e^{-t}F''.$$

Both supplied polynomials have second derivative 1, so both are particular solutions. Their difference $(3 - 2t)e^{-t}$ has zero residual, either by this identity or by subtracting the two forced equations.

Step 2: Match coefficients. Equality of the two representations is equivalent to

$$\boxed{A = \alpha + 3, \qquad B = \beta - 2.}$$

This is an invertible reassignment of constants, so the two formulas describe the same complete solution family.

Step 3: Impose the data. In the first representation, $y(0) = A$ and $y'(0) = B - A$, because $p(0) = p'(0) = 0$. Thus $A = 2$, $B = 1$, while the second representation has $\alpha = -1$, $\beta = 3$. Both give

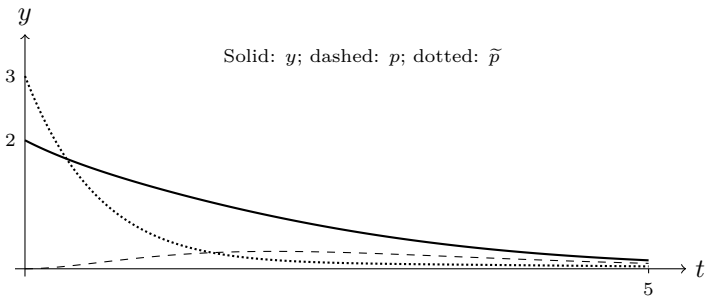
$$\boxed{y = e^{-t}(2 + t + \tfrac{1}{2}t^2).}$$

Its initial value is 2. Differentiation gives $y' = -e^{-t}(1 + \frac{1}{2}t^2)$, whose value at zero is -1, as required.

Step 4: Interpret the complete trajectory. The derivative follows from subtracting the full polynomial when differentiating its exponential factor:

$$y' = e^{-t}[(1 + t) - (2 + t + \tfrac{1}{2}t^2)] = -e^{-t}(1 + \tfrac{1}{2}t^2).$$

For $t \geq 0$, $y > 0$ and $y' < 0$; the polynomial times e^{-t} tends to zero. The same trajectory admits both decompositions above, so neither piece is unique without a chosen particular solution.



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Question 4

For the operator $L[y] = y'' + y$, consider the supplied responses

$$u(t) = \frac{\cos t - \cos 2t}{3}, \qquad v(t) = 2(t - \sin t)$$

on $\mathbb{R}$.

Tasks

1. Verify that $L[u] = \cos 2t$, $L[v] = 2t$, and that both responses have zero value and slope at zero.
2. Construct the solution of $y'' + y = \cos 2t + 2t$ with $y(0) = 1$, $y'(0) = -2$. Verify the data.
3. State and prove how both forcing and initial data combine when a constant linear combination of two responses is formed.
4. Let $r = u + \cos t$ and $s = v + \cos t$. Each now has data $(1, 0)$. Correct the naive sum $r + s$ so that it has the combined forcing but only the single prescribed data pair $(1, 0)$.

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Question 4 – Solution

Strategy. Superposition combines the initial data as well as the forcing; use a homogeneous correction to achieve the desired data.

Step 1: Verify the supplied responses. We have $u'' = (-\cos t + 4\cos 2t)/3$, so $u'' + u = \cos 2t$. Also $v' = 2(1 - \cos t)$ and $v'' = 2\sin t$, so $v'' + v = 2t$. At zero both values and both slopes vanish.

Step 2: Add the forcing and fit the state. The sum $u + v$ handles the combined forcing with zero initial data. Adding the homogeneous solution $\cos t - 2\sin t$ gives

$$\boxed{y = u + v + \cos t - 2\sin t = \tfrac{4}{3}\cos t - \tfrac{1}{3}\cos 2t + 2t - 4\sin t.}$$

At zero the value is $4/3 - 1/3 = 1$ and the slope is $2 - 4 = -2$. Linearity verifies the combined forcing, and regular uniqueness proves this is the initial-value solution.

Step 3: Track the full linear map. If $L[y_j] = g_j$ and the initial data are (a_j, b_j), then $Ay_1 + By_2$ has forcing $Ag_1 + Bg_2$ and data

$$(Aa_1 + Ba_2,\ Ab_1 + Bb_2).$$

This follows directly from linearity of L, evaluation and differentiation. Ignoring the data part can produce the right equation with the wrong initial state.

Step 4: Remove duplicated initial data. The naive sum $r + s$ has data $(2, 0)$. Subtract one copy of the homogeneous solution $\cos t$:

$$\boxed{r + s - \cos t = u + v + \cos t.}$$

The forcing is unchanged, while the data become $(1, 0)$. This correction differs from Step 2 because the required initial slope here is zero, not -2.

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Question 5

For $y'' - y = 2\cos t$ on $\mathbb{R}$, a proposed particular solution is $p = -\cos t$. Prescribe real initial data $y(0) = a$, $y'(0) = b$.

Tasks

1. Verify the particular solution and find the full initial-value solution in terms of a, b.
2. Find a necessary and sufficient relation between a, b for boundedness on $[0, \infty)$. For those data, determine the limiting difference $y(t) - p(t)$.
3. Find the unique solution bounded on all of $\mathbb{R}$. Does the selected bounded solution necessarily have a limit as $t \to \infty$?
4. Compare the trajectories from $(a, b) = (0, -1)$ and $(0, 0)$. Explain why bounded forcing alone does not make every solution bounded.

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Question 5 – Solution

Strategy. The difference from a particular solution follows the homogeneous equation, whose growing mode determines boundedness.

Step 1: Verify and fit. Since $p'' = \cos t$, $p'' - p = 2\cos t$. The general solution is $-\cos t + Ae^t + Be^{-t}$. The data give $A + B = a + 1$, $A - B = b$, hence

$$\boxed{y = -\cos t + \frac{a+1+b}{2}e^t + \frac{a+1-b}{2}e^{-t}.}$$

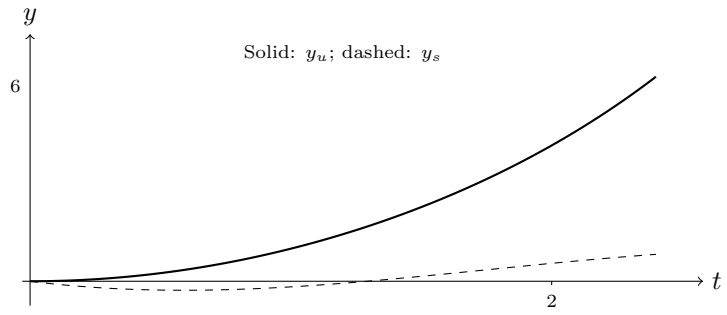
Step 2: Eliminate the growing mode. The bounded oscillatory and decaying terms cannot cancel a nonzero Ae^t as $t \to \infty$. Thus forward boundedness holds exactly when

$$\boxed{b = -a - 1.}$$

In that case $y = -\cos t + (a+1)e^{-t}$, so $y - p \to 0$. This convergence is to a time-varying particular solution, not necessarily to a constant.

Step 3: Require both time directions. Boundedness as $t \to -\infty$ also forces $B = 0$. Thus the unique globally bounded solution is $\boxed{y = -\cos t}$, with data $(-1, 0)$. It has no limit at positive infinity: its values at $2\pi n$ and $(2n+1)\pi$ are -1 and 1. The forward-bounded family also has these two limiting subsequences because its decaying correction tends to zero.

Step 4: Compare the selected data. Data $(0, -1)$ give $y_s = -\cos t + e^{-t}$, which is bounded forward. Data $(0, 0)$ give $y_u = -\cos t + \cosh t$, which is unbounded forward. Both have the same bounded forcing; the second retains a growing homogeneous component. The finite plot illustrates the distinction; the mode argument proves the global claim.



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Question 6

For $y'' + y = \cos t$ on $\mathbb{R}$, a proposed particular solution is $p(t) = \frac{1}{2} t \sin t$. The forcing is bounded in absolute value by one.

Tasks

1. Verify the particular solution and find the full solution family.
2. Solve the initial-value problem with arbitrary data $y(0) = a$, $y\prime(0) = b$. Identify the zero-data response.
3. Prove that every solution is unbounded above and below on $[0, \infty)$, using two explicit sequences of times. Can a homogeneous correction remove this behavior?
4. For the zero-data response, give exact upper and lower envelope bounds on $t \geq 0$ and the times when they are attained. Explain why envelope contacts need not be stationary points.

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Question 6 – Solution

Strategy. Verify the supplied growing response and use exact phase samples to rule out cancellation by bounded homogeneous terms.

Step 1: Verify and complete. Differentiation gives

$$p' = \tfrac{1}{2}\sin t + \tfrac{1}{2}t\cos t, \qquad p'' = \cos t - \tfrac{1}{2}t\sin t.$$

Thus $p'' + p = \cos t$ and the general solution is

$$\boxed{y = A\cos t + B\sin t + \tfrac{1}{2}t\sin t.}$$

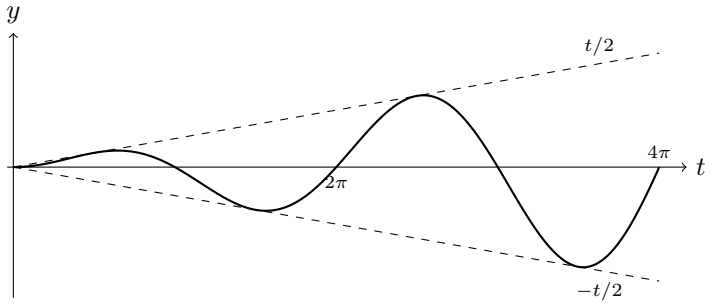
Step 2: Fit the data. Since $p(0) = p'(0) = 0$, the data give $A = a$, $B = b$. In particular the zero-data response is exactly $\boxed{y_0 = \tfrac{1}{2}t\sin t}$.

Step 3: Prove both unbounded excursions. At $t_n = \pi/2 + 2\pi n$, $n = 0, 1, \ldots$, we have $y(t_n) = b + t_n/2 \to \infty$. At $s_n = 3\pi/2 + 2\pi n$, we have $y(s_n) = -b - s_n/2 \to -\infty$. These values do not depend on a. Thus no constant homogeneous correction removes the unbounded behavior, even though the forcing is bounded.

Step 4: Distinguish bounds from extrema. On $t \geq 0$,

$$\boxed{-t/2 \leq y_0(t) \leq t/2.}$$

For $t > 0$, the upper envelope is attained exactly at t_n and the lower at s_n; both bounds also coincide with y_0 at zero. At a positive upper contact $y_0' = 1/2$, and at a lower contact $y_0' = -1/2$. These slopes match those of the respective envelopes and are not zero, so the contacts are not stationary points of y_0.



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Question 7

On $t > 0$, consider

$$y'' = 1/t, \qquad y(1) = 0, \quad y'(1) = 0,$$

with supplied particular solution $p(t) = t \ln t - t$. The associated homogeneous equation has the globally smooth solutions $1, t$.

Tasks

1. Verify the particular solution and find every solution on $(0, \infty)$.
2. Solve the initial-value problem and determine its monotonicity, convexity and minimum.
3. Find the right-hand limit at zero and decide whether the selected solution admits continuous, C^1, or C^2 extension through zero.
4. Can any homogeneous correction to p produce a C^1 extension through zero? Explain why smooth homogeneous solutions do not guarantee continuation of the forced problem.

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Question 7 – Solution

Strategy. The forcing can impose endpoint behavior that no homogeneous correction can cancel.

Step 1: Verify and complete. Since $p' = \ln t$ and $p'' = 1/t$, it is a particular solution. Adding all affine homogeneous functions gives

$$\boxed{y = t\ln t - t + At + B \qquad (t > 0).}$$

Step 2: Fit and analyze. The slope at 1 gives $A = 0$, and the value then gives $B = 1$. Thus

$$\boxed{y = t\ln t - t + 1, \qquad y' = \ln t, \quad y'' = 1/t > 0.}$$

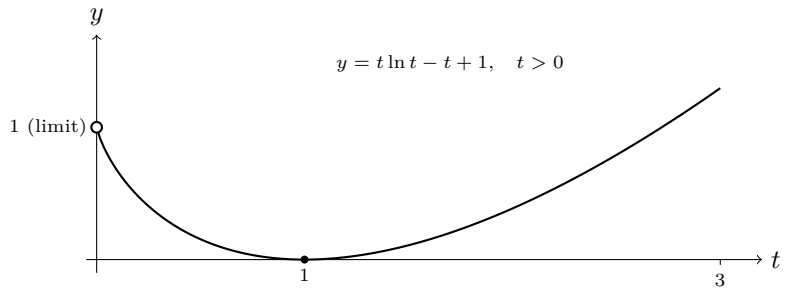
The function decreases on $(0, 1)$, increases on $(1, \infty)$, and is strictly convex. Its unique global minimum is $y(1) = 0$.

Step 3: Test extension of the selected function. Since $t\ln t \to 0$ as $t \downarrow 0$, we have $y(t) \to 1$. Setting $y(0) = 1$ and, for example, $y(t) = 1$ for $t < 0$ gives a continuous extension. A finite derivative at zero is impossible because

$$\frac{y(t) - 1}{t} = \ln t - 1 \longrightarrow -\infty.$$

Hence there is no C^1 extension and therefore no C^2 extension. The limiting point in the graph is excluded from the original domain.

Step 4: Test every correction. For general A, B, continuity would require $y(0) = B$, but the difference quotient is $\ln t - 1 + A$, again tending to $-\infty$. No finite homogeneous constants cure this. Although $1, t$ extend smoothly, the forcing $1/t$ is undefined at zero. The maximal regular interval containing the initial point is $(0, \infty)$, not all of $\mathbb{R}$.



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Question 8

Let u, v satisfy the same forced equation $y'' + p(t)y' + q(t)y = g(t)$ on a regular interval, and write $W = uv' - u'v$. For a concrete example on $\mathbb{R}$, take

$$u = 1 - \cos t, \qquad v = 1 - \cos t + \sin t.$$

Tasks

1. Derive the differential identity for W when the common forcing is nonzero.
2. Verify that the example functions solve $y'' + y = 1$. Compute their Wronskian and explain why its zero at zero does not force it to vanish everywhere.
3. Prove that the two example functions are linearly independent as functions, and identify the homogeneous solution obtained from their difference.
4. Classify the constant affine combinations of u, v that solve the same forced equation. Which initial values and slopes at zero can they attain, and do they give the entire forced solution set?

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Question 8 – Solution

Strategy. Recompute the Wronskian identity with the forcing included, then distinguish linear independence from a homogeneous fundamental set.

Step 1: Keep the forcing terms. Differentiating and substituting gives

$$W' = uv'' - u''v = u(g - pv' - qv) - v(g - pu' - qu),$$

so

$$\boxed{W' = -pW + g(u - v).}$$

The extra term is absent from Abel's homogeneous identity and generally does not vanish.

Step 2: Verify the example. We have $u'' + u = \cos t + 1 - \cos t = 1$, and adding $\sin t$ preserves this forcing. Direct computation gives

$$\boxed{W = (1 - \cos t)\cos t - \sin^2 t = \cos t - 1.}$$

It is zero at 0 but equals -2 at π. Its derivative is $-\sin t$, matching $g(u - v) = -\sin t$. The homogeneous zero-or-never-zero conclusion is inapplicable to this forced pair.

Step 3: Test independence. If $Au + Bv = 0$, evaluation at π gives $2A + 2B = 0$ and at $\pi/2$ gives $A + 2B = 0$. Hence $A = B = 0$. The difference $v - u = \sin t$ solves the associated homogeneous equation. Independence of the forced functions does not make them a fundamental set of homogeneous solutions.

Step 4: Describe their affine family. The forcing of $Au + Bv$ is $A + B$, so retaining forcing 1 requires $A + B = 1$. Such combinations are

$$\boxed{u + B\sin t = 1 - \cos t + B\sin t.}$$

They have initial value zero and arbitrary slope B. They give exactly the forced solutions with initial value zero, but miss every solution with a different initial value. The full family is $1 + C\cos t + D\sin t$, with two independent initial-data freedoms.

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Question 9

Let y solve $y'' + y = 1$ with $y(0) = y'(0) = 0$. On $0 \le t \le 1$, consider the approximation $Y(t) = t^2/2$ and the error $E = Y - y$. You may use $0 \le \sin r \le r$ for $0 \le r \le 1$.

Tasks

1. Compute the residual $R = Y'' + Y - 1$ and derive the differential equation and initial data satisfied by E.
2. Verify by differentiating twice that $\int_0^t \sin(t-s)s^2/2\,ds$ solves this error initial-value problem. Justify that it equals E.
3. Prove the rigorous error bound $0 \le E(t) \le t^4/24$ on $[0,1]$, including the direction of the approximation error.
4. Find the exact solution y by verification, compute the actual error at 1 to six decimal places, and compare it with the bound. Why does a residual vanishing only at zero not prove exactness?

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Question 9 – Solution

Strategy. Treat the approximation residual as a forcing for the error, then bound the resulting integral with a positive kernel.

Step 1: Form the error equation. Since $Y'' = 1$, the residual is $R = t^2/2$. Subtraction gives

$$\boxed{E'' + E = t^2/2, \qquad E(0) = E'(0) = 0.}$$

Step 2: Verify the integral. Set $F(t) = \int_0^t \sin(t-s)s^2/2\,ds$. Leibniz differentiation gives

$$F' = \int_0^t \cos(t-s)s^2/2\,ds, \qquad F'' = t^2/2 - \int_0^t \sin(t-s)s^2/2\,ds.$$

Thus $F'' + F = t^2/2$ and $F(0) = F'(0) = 0$. The continuous coefficients and forcing give uniqueness, so $F = E$.

Step 3: Bound the error. For $0 \le s \le t \le 1$, the supplied sine inequality gives

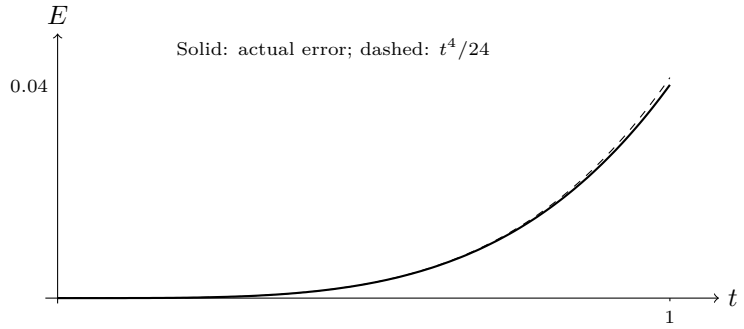
$$0 \le E(t) \le \frac{1}{2}\int_0^t (t-s)s^2\,ds = \frac{1}{2}(t^4/3 - t^4/4) = \boxed{t^4/24}.$$

Hence Y overestimates the solution. The bound vanishes at zero and remains valid across the entire stated interval.

Step 4: Check against the exact response. The function $y = 1 - \cos t$ has the required equation and zero data. Therefore

$$\boxed{E(1) = \cos 1 - \tfrac{1}{2} \approx 0.040302 < 1/24 \approx 0.041667.}$$

The residual is zero at the initial point but is positive elsewhere on $(0, 1]$. A solution must satisfy its equation at every point, so agreement at one point and in the initial data does not make Y exact.



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Question 10

On $0 \le t \le 1$, design a forcing g for $y'' + 2y' + y = g(t)$ so that the trajectory has

$$y(0) = 0, \quad y'(0) = 0, \qquad y(1) = 1, \quad y'(1) = 0.$$

Restrict the desired trajectory to polynomials of degree at most three.

Tasks

1. Find the unique polynomial meeting the four endpoint requirements.
2. Derive the forcing g that makes this polynomial an exact solution and verify all four requirements.
3. Prove that this forcing changes sign exactly once on $[0, 1]$, locating the zero between $3/4$ and $4/5$. Can a nonnegative forcing realize this same cubic trajectory?
4. Keep the designed forcing but perturb the initial data to $y(0) = \varepsilon$, $y\prime(0) = 0$. Find the exact position and slope errors at $t = 1$, and determine whether the target state is still reached when $\varepsilon \neq 0$.

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Question 10 – Solution

Strategy. Prescribe a trajectory and compute its forcing directly; then use a homogeneous difference equation to analyze initial-data error.

Step 1: Fit the cubic trajectory. Write $y = c_0 + c_1 t + c_2 t^2 + c_3 t^3$. The initial conditions give $c_0 = c_1 = 0$. The final conditions give $c_2 + c_3 = 1$ and $2c_2 + 3c_3 = 0$, so

$$\boxed{y_*(t) = 3t^2 - 2t^3.}$$

The coefficient equations have a unique solution, including among all polynomials of lower degree.

Step 2: Compute the forcing. The derivatives are $y_*' = 6t - 6t^2$ and $y_*'' = 6 - 12t$. Thus

$$\boxed{g = y_*'' + 2y_*' + y_* = 6 - 9t^2 - 2t^3.}$$

The derivative formulas give zero initial value and slope, and final value $3 - 2 = 1$ with slope $6 - 6 = 0$. The computed forcing verifies the equation identically.

Step 3: Locate the sign change. We have $g(0) = 6$, $g(1) = -5$, and $g' = -18t - 6t^2 < 0$ for $t > 0$. Therefore there is exactly one zero. More precisely,

$$g(3/4) = 3/32 > 0, \qquad g(4/5) = -98/125 < 0,$$

so the zero lies strictly between $3/4$ and $4/5$. This same cubic fixes g pointwise, so a nonnegative forcing cannot realize it on the whole interval. This conclusion concerns the specified cubic, not every possible trajectory.

Step 4: Propagate the initial error. The difference $h = y - y_*$ satisfies $h'' + 2h' + h = 0$, with $h(0) = \varepsilon$, $h'(0) = 0$. Hence $h = \varepsilon(1 + t)e^{-t}$ and $h' = -\varepsilon t e^{-t}$. At the endpoint,

$$\boxed{y(1) - 1 = 2\varepsilon/e, \qquad y'(1) = -\varepsilon/e.}$$

For nonzero ε the target state is missed. The forcing is unchanged; the error evolves entirely through the homogeneous correction.

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Question 1

Consider the initial-value problem on $\mathbb{R}$

$$y'' - 3y' + 2y = t^2, \qquad y(0) = 0, \quad y'(0) = 0.$$

A student proposes the particular trial $y_p = At^2$ because the forcing contains only a quadratic term.

Tasks

1. Explain why the proposed trial is insufficient. Choose the smallest complete polynomial trial and justify its degree.
2. Determine all coefficients of that trial and verify its residual.
3. Find the full initial-value solution, checking both data.
4. Prove that the polynomial particular solution is unique among polynomials, even though particular solutions in general are not unique.

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Question 1 – Solution

Strategy. Differentiation creates lower polynomial powers, so include them before matching every coefficient.

Step 1: Complete the trial. For At^2, the residual is $2At^2 - 6At + 2A$, which cannot equal t^2. Use $y_p = at^2 + bt + c$. The nonzero coefficient of y preserves the highest polynomial degree, so degree two is necessary and sufficient here.

Step 2: Match coefficients. Substitution yields

$$2at^2 + (2b - 6a)t + (2a - 3b + 2c) = t^2.$$

Thus $a = 1/2$, $b = 3/2$, $c = 7/4$, giving

$$\boxed{y_p = \tfrac{1}{2}t^2 + \tfrac{3}{2}t + \tfrac{7}{4}.}$$

The displayed coefficient equations verify the residual identically.

Step 3: Fit the homogeneous correction. The characteristic roots are $1, 2$, so $y = y_p + Ce^t + De^{2t}$. At zero,

$$C + D = -7/4, \qquad C + 2D = -3/2.$$

Hence $C = -2$, $D = 1/4$, and

$$\boxed{y = \tfrac{1}{2}t^2 + \tfrac{3}{2}t + \tfrac{7}{4} - 2e^t + \tfrac{1}{4}e^{2t}.}$$

Its value is $7/4 - 2 + 1/4 = 0$ and its slope is $3/2 - 2 + 1/2 = 0$. Adding homogeneous modes does not change the verified forcing.

Step 4: Prove polynomial uniqueness. The difference of two polynomial particular solutions would be a polynomial h with $h'' - 3h' + 2h = 0$. If h were nonzero with degree m and leading coefficient a_m, the leading term of this residual would be $2a_m t^m$, which cannot vanish. Thus $h = 0$. Nonpolynomial homogeneous additions still give infinitely many other particular solutions.

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Question 2

On $\mathbb{R}$, solve

$$y'' - 4y' + 4y = e^{2t}(6t + 2), \qquad y(0) = 1, \quad y'(0) = 0.$$

Three candidate trial forms are $e^{2t}(At + B)$, $te^{2t}(At + B)$ and $t^2e^{2t}(At + B)$.

Tasks

1. Find the homogeneous roots and explain the resonance multiplicity. Derive the identity obtained by writing $y = e^{2t}F(t)$.
2. Test the first two trial forms and explain exactly why neither can produce the forcing.
3. Use the third trial to find a particular solution and then solve the initial-value problem.
4. Explain why the trial omits the constant and linear powers of F, and why their omission loses no solution freedom. Verify the final equation and data.

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Question 2 – Solution

Strategy. Shift the exponential factor out of the operator; the repeated characteristic root reveals the necessary power of t.

Step 1: Identify the repeated root. The characteristic polynomial is $(r-2)^2$. Product differentiation gives

$$(e^{2t}F)'' - 4(e^{2t}F)' + 4e^{2t}F = e^{2t}F''.$$

Thus the transformed equation is $F'' = 6t + 2$.

Step 2: Reject incomplete shifts. The first trial has $F = At + B$, so $F'' = 0$. The second has $F = At^2 + Bt$, so $F'' = 2A$, which cannot supply the linear term $6t$. Multiplying the usual degree-one trial by only one power of t is still insufficient at a double root.

Step 3: Use the correct trial. With $F = At^3 + Bt^2$, coefficient matching gives $6At + 2B = 6t + 2$, hence $A = B = 1$. The complete solution is

$$y = e^{2t}(C + Dt + t^2 + t^3).$$

The initial value sets $C = 1$, and the slope at zero is $2C + D$, so $D = -2$. Therefore

$$\boxed{y = e^{2t}(1 - 2t + t^2 + t^3).}$$

Step 4: Account for every constant. Constant and linear terms in F have zero second derivative and correspond exactly to e^{2t} and te^{2t}, the homogeneous modes. Leaving them out of the particular trial avoids redundant coefficients; the constants C, D restore them in the general solution. For the final polynomial, $F'' = 2 + 6t$, so the equation holds. Also $F(0) = 1$ and $2F(0) + F'(0) = 2 - 2 = 0$, verifying both data.

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Question 3

Consider $y'' + y = t\cos 2t$ on $\mathbb{R}$, with $y(0) = y'(0) = 0$. A proposed trial includes only the cosine term $(at + b)\cos 2t$.

Tasks

1. Explain why a polynomial sine component must also be included, and write the smallest complete trigonometric-polynomial trial.
2. Determine its coefficients by matching both sine and cosine terms.
3. Solve the zero-data initial-value problem and verify the data.
4. Explain why nonresonance with frequency 2 does not guarantee a bounded response to this forcing. Prove that the selected response is unbounded on $[0, \infty)$.

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Question 3 – Solution

Strategy. Polynomial differentiation couples the sine and cosine components even when only one appears in the forcing.

Step 1: Complete the trial space. Use

$$y_p = (at + b)\cos 2t + (ct + d)\sin 2t.$$

Frequency 2 differs from the homogeneous frequency 1, so no extra power of t is needed for resonance. Nevertheless, differentiating the proposed cosine-only term twice produces $-4a\sin 2t$, which cannot be ignored.

Step 2: Match all terms. Applying $D^2 + 1$ to the complete trial gives

$$(-3at - 3b + 4c)\cos 2t + (-3ct - 3d - 4a)\sin 2t.$$

Thus $a = -1/3$, $c = 0$, $b = 0$, $d = 4/9$, and

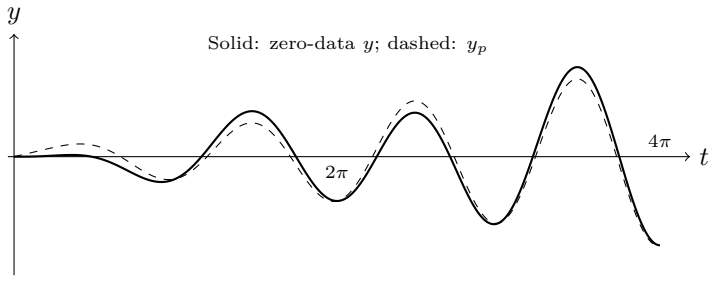
$$\boxed{y_p = -\tfrac{1}{3}t\cos 2t + \tfrac{4}{9}\sin 2t.}$$

Step 3: Fit the initial state. The particular data are $y_p(0) = 0$ and $y_p'(0) = -1/3 + 8/9 = 5/9$. The homogeneous correction is therefore $-(5/9)\sin t$, giving

$$\boxed{y = -\tfrac{1}{3}t\cos 2t + \tfrac{4}{9}\sin 2t - \tfrac{5}{9}\sin t.}$$

Its value and slope at zero vanish, and the correction leaves the verified forcing unchanged.

Step 4: Separate resonance from forcing growth. The forcing itself has a factor t and is unbounded. At $t_n = 2\pi n$, the solution is $y(t_n) = -t_n/3 \to -\infty$. At $s_n = \pi/2 + 2\pi n$, it is $y(s_n) = s_n/3 - 5/9 \to \infty$. Thus it is unbounded in both directions despite the absence of frequency resonance. Nonresonance alone is not a boundedness guarantee when the forcing amplitude grows.



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Question 4

On $\mathbb{R}$, solve

$$y'' - y = (t+1)e^t + 2\cosh t, \qquad y(0) = y'(0) = 0,$$

where $\cosh t = (e^t + e^{-t})/2$.

Tasks

1. Rewrite the forcing into distinct exponential families and identify every resonant component.
2. Choose one combined trial with no duplicate unknown coefficients. Explain why matching separate overlapping trials would introduce redundancy.
3. Determine a particular solution by coefficient matching.
4. Add the homogeneous correction, solve the initial-value problem, and verify both the equation and initial data.

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Question 4 – Solution

Strategy. Combine overlapping forcing families before choosing the trial, then apply the resonance correction separately to each distinct exponential.

Step 1: Simplify the forcing. The right side becomes

$$(t+2)e^t + e^{-t}.$$

The characteristic roots of $y'' - y = 0$ are $1, -1$, both simple. Therefore each exponential family is resonant once.

Step 2: Choose a nonredundant trial. Use

$$y_p = e^t(at^2 + bt) + cte^{-t}.$$

A separate trial for the e^t inside $2\cosh t$ would repeat the te^t term already in the first family's trial. Combining the forcing first avoids two coefficients multiplying the same function.

Step 3: Match coefficients. Product differentiation gives

$$(D^2-1)(e^tF) = e^t(F'' + 2F'), \qquad (D^2-1)(e^{-t}G) = e^{-t}(G'' - 2G').$$

Thus the trial produces $e^t(4at + 2a + 2b) - 2ce^{-t}$. Matching gives $a = 1/4$, $b = 3/4$, $c = -1/2$, so

$$\boxed{y_p = e^t(t^2/4 + 3t/4) - \tfrac{1}{2}te^{-t}.}$$

Step 4: Fit and verify. The particular data are $(0, 1/4)$. With $y = y_p + Ce^t + De^{-t}$, the zero data require $C + D = 0$, $C - D = -1/4$, hence $C = -1/8$, $D = 1/8$. Therefore

$$\boxed{y = e^t(t^2/4 + 3t/4 - 1/8) + e^{-t}(1/8 - t/2).}$$

Its value is $-1/8+1/8 = 0$ and its slope is $1/4-1/8-1/8 = 0$. The two added exponentials are homogeneous, so the coefficient-matched forcing is preserved.

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Question 5

For $t \in \mathbb{R}$, consider

$$y'' + 2y' + 5y = e^{-t}\cos 2t, \qquad y(0) = y'(0) = 0.$$

Tasks

1. Find the homogeneous roots and choose a correct resonant trial. Derive a useful operator identity with $y = e^{-t}F$.
2. Determine the particular solution and the complete zero-data response.
3. Despite the resonance, prove that every solution tends to zero as $t \to \infty$.
4. Find a simple envelope for the zero-data response and prove the bound $|y(t)| \leq 1/(4e)$ for $t \geq 0$. Is equality actually attained by the response?

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Question 5 – Solution

Strategy. Resonance adds a polynomial factor, but the sign of the exponential rate still controls long-time decay.

Step 1: Identify the resonance. The roots are $-1 \pm 2i$. Thus the exponential-trigonometric forcing matches a simple complex root pair, and a suitable trial is $te^{-t}(A\cos 2t + B\sin 2t)$. Differentiation gives

$$(e^{-t}F)'' + 2(e^{-t}F)' + 5e^{-t}F = e^{-t}(F'' + 4F).$$

Step 2: Match and fit. For $F = t(A\cos 2t + B\sin 2t)$, the transformed residual is $-4A\sin 2t + 4B\cos 2t$. Hence $A = 0$, $B = 1/4$. The particular solution and its first derivative vanish at zero, so

$$\boxed{y_0 = \tfrac{1}{4}te^{-t}\sin 2t.}$$

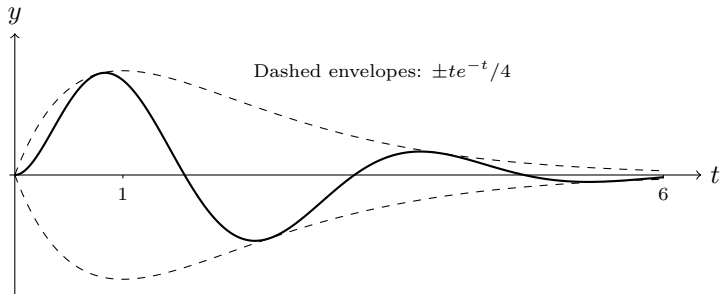
The full family is $e^{-t}[C\cos 2t + D\sin 2t + (t/4)\sin 2t]$. Zero initial data force $C = D = 0$.

Step 3: Prove decay for every solution. For fixed C, D and $t \geq 0$, its absolute value is at most $e^{-t}(|C| + |D| + t/4)$, which tends to zero. The resonant factor t does not overcome the negative exponential rate.

Step 4: Distinguish an envelope bound from an attained maximum. We have $|y_0(t)| \leq te^{-t}/4$. The envelope attains its unique maximum at $t = 1$, since its derivative is $e^{-t}(1-t)/4$. Thus

$$\boxed{|y_0(t)| \leq 1/(4e).}$$

Equality would require both $t = 1$ and $|\sin 2t| = 1$. But $|\sin 2| < 1$, so the response never attains this bound. It is a valid envelope estimate, not the exact largest displacement.



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Question 6

Let $F > 0$ and $\omega \geq 0$. Consider $y'' + 2y' + 5y = F\cos(\omega t)$ on $\mathbb{R}$. For $\omega > 0$, seek a particular response $y_p = A\cos\omega t + B\sin\omega t$ and define its amplitude $R = \sqrt{A^2 + B^2}$. At $\omega = 0$, define R as the absolute value of the constant particular response.

Tasks

1. Derive A, B for $\omega > 0$ and find the constant response at zero frequency.
2. Find the frequency maximizing R on $\omega \geq 0$ and the maximum amplitude.
3. At $\omega = \sqrt{5}$, describe the phase of the response relative to the forcing and compare its amplitude with the maximum.
4. For $\omega > 0$, prove that this is the unique $2\pi/\omega$-periodic solution and that every other solution approaches it as $t \to \infty$.

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Question 6 – Solution

Strategy. Solve the two coefficient equations, then minimize the amplitude denominator instead of differentiating a complicated square root.

Step 1: Match the harmonic components. The equations are

$$(5-\omega^2)A + 2\omega B = F, \qquad -2\omega A + (5-\omega^2)B = 0.$$

Writing $\Delta = (5-\omega^2)^2 + 4\omega^2 > 0$, we obtain

$$\boxed{A = F(5-\omega^2)/\Delta, \qquad B = 2F\omega/\Delta.}$$

At $\omega = 0$, use the constant particular solution $F/5$.

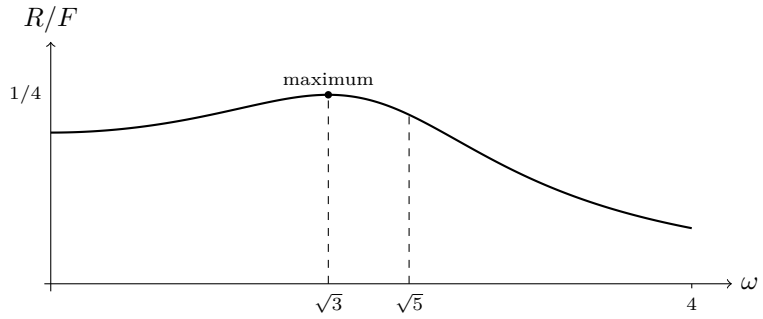
Step 2: Maximize the amplitude. For all $\omega \geq 0$, including this constant-response convention,

$$R = \frac{F}{\sqrt{\omega^4 - 6\omega^2 + 25}} = \frac{F}{\sqrt{(\omega^2-3)^2 + 16}}.$$

The denominator is minimized exactly at $\omega = \sqrt{3}$, giving $\boxed{R_{\max} = F/4}$.

Step 3: Locate the quarter-cycle phase lag. At $\omega = \sqrt{5}$, $A = 0$ and $B = F/(2\sqrt{5})$, so $y_p = [F/(2\sqrt{5})]\sin(\sqrt{5}t)$. This is a cosine delayed by phase $\pi/2$. Its amplitude $F/(2\sqrt{5})$ is smaller than $F/4$; the quarter-cycle phase frequency is not the amplitude-maximizing frequency.

Step 4: Prove the steady response is unique. Every difference from y_p is $e^{-t}(C\cos 2t + D\sin 2t)$ and tends to zero forward. If a second solution had the same period $T = 2\pi/\omega$, this difference would be periodic too. For any fixed t, its value equals its value at $t + nT$, whose limit is zero. Hence the difference is identically zero, proving uniqueness of the periodic response.



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Question 7

For $\omega > 0$, let y_ω solve

$$y'' + y = \cos(\omega t), \qquad y(0) = y'(0) = 0.$$

Tasks

1. Use undetermined coefficients to find y_ω for $\omega \neq 1$, including the homogeneous correction.
2. At $\omega = 1$, choose a resonant trial and solve the same initial-value problem directly.
3. For every fixed t, compute $\lim_{\omega \to 1} y_\omega(t)$. Explain why the divergent particular coefficient for $\omega \neq 1$ does not imply a divergent fixed-time limit of the full response.
4. Compare boundedness on $[0, \infty)$ in the resonant and nonresonant cases. Can the convergence as $\omega \to 1$ be uniform on that entire half-line?

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Question 7 – Solution

Strategy. Fit the initial data before taking the frequency limit; the homogeneous correction cancels the divergent constant part.

Step 1: Solve off resonance. The trial $A\cos\omega t$ gives $A(1-\omega^2)=1$. Fitting zero data adds $-\cos t/(1-\omega^2)$, so

$$\boxed{y_\omega(t)=\frac{\cos(\omega t)-\cos t}{1-\omega^2}, \qquad \omega\neq 1.}$$

Both the value and slope at zero vanish.

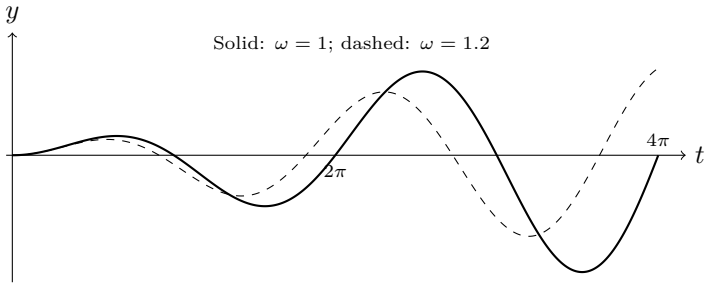
Step 2: Solve at resonance. Use $t(A\cos t+B\sin t)$. Applying D^2+1 gives $-2A\sin t+2B\cos t$, hence $A=0$, $B=1/2$. Its data are already zero, so $\boxed{y_1=(t/2)\sin t}$.

Step 3: Take the fixed-time limit. Differentiating numerator and denominator with respect to ω gives

$$\boxed{\lim_{\omega\to 1} y_\omega(t)=\frac{-t\sin t}{-2}=\tfrac{1}{2}t\sin t.}$$

This includes $t=0$. The particular coefficient and the cosine correction separately diverge, but their full numerator vanishes at the same rate as the denominator. They must be combined before taking the limit.

Step 4: Compare long-time behavior. For fixed $\omega\neq 1$, $|y_\omega|\le 2/|1-\omega^2|$ on the whole half-line. The resonant response is unbounded, for example at $t=\pi/2+2\pi n$. Consequently, for each fixed nonresonant ω, the difference $y_\omega-y_1$ is unbounded on that half-line. Uniform convergence there is impossible, although the fixed-time limit exists. A finite-window plot cannot replace this distinction.



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Question 8

Consider the standard finite polynomial–exponential–trigonometric version of undetermined coefficients for constant-coefficient equations. Then examine $y'' + y = 1/(1+t)$ on $I = (-1, \infty)$.

Tasks

1. Which forcings fit the standard finite trial families: $e^{-t}(t^2+1)$, $t\sin 3t$, $\cos^2 t$, e^{t^2}, and $1/(1+t)$? Explain any algebraic rewriting needed.
2. Test the rational trial $y_p = A/(1+t)$ for the stated equation and prove that no constant A works.
3. Prove that no finite sum $\sum_{k=1}^{N} a_k(1+t)^{-k}$ with real coefficients can be a particular solution.
4. Does the failure of these trials imply that an initial-value solution does not exist? State the existence and uniqueness conclusion for any initial point in I.

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Question 8 – Solution

Strategy. The standard method depends on closure in a finite differentiation family; a plausible-looking trial outside that family need not close.

Step 1: Classify the forcings. The first two belong to the standard families. The third does too after $\cos^2 t = (1 + \cos 2t)/2$. The functions e^{t^2} and $1/(1 + t)$ do not: successive derivatives introduce polynomial factors of ever-increasing degree in the first case and poles of ever-increasing order in the second. The usual finite constant-coefficient matching prescription does not apply to them.

Step 2: Test one inverse power. Put $x = 1 + t > 0$. For $y_p = A/x$,

$$y_p'' + y_p = A/x + 2A/x^3.$$

Equality to $1/x$ would require $A(x^2 + 2) = x^2$ for all $x > 0$. Matching x^2 forces $A = 1$, while matching the constant term forces $2A = 0$, a contradiction.

Step 3: Exclude any finite inverse-power sum. If such a sum were nonzero, choose its largest index N with $a_N \neq 0$. Its second derivative contains

$$a_N N(N + 1)x^{-N-2}.$$

No other differentiated term, undifferentiated term or forcing term has that most singular power. Equivalently, multiply the proposed equation by x^{N+2} and let $x \downarrow 0$: the left side tends to $a_N N(N + 1) \neq 0$, while the right side tends to zero. The zero sum also fails to give the nonzero forcing.

Step 4: Separate method failure from nonexistence. On I, the normalized coefficients and the forcing $1/(1 + t)$ are continuous. For every $t_0 \in I$ and every finite value and slope at t_0, the regular linear theorem gives exactly one solution on I. A failed finite ansatz is a limitation of the trial class, not a proof that the differential equation lacks solutions.

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Question 9

An unknown forcing in $y'' - 3y' + 2y = Ae^{3t} + Bt$ has two real constants A, B. An exact response on $\mathbb{R}$ has measured data

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y^{(3)}(0) = 0.$$

Tasks

1. Determine A, B using the equation and its derivative at zero before solving for y.
2. Choose a complete nonresonant trial and compute a particular solution for the identified forcing.
3. Fit the homogeneous correction and verify all four measured quantities.
4. Explain the distinct roles of the four measurements. Once A, B are fixed, are four freely prescribed initial quantities available for this second-order equation?

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Question 9 – Solution

Strategy. Use acceleration and its derivative to identify the forcing, then use value and slope to select the trajectory.

Step 1: Identify the forcing parameters. At zero, the equation gives $1 = A$. Differentiating once gives $y^{(3)} - 3y'' + 2y' = 3Ae^{3t} + B$, so at zero $-3 = 3A + B$. Therefore

$$\boxed{A = 1, \qquad B = -6.}$$

Step 2: Match a particular solution. Neither 3 nor 0 is a root of $(r-1)(r-2)$, so use $y_p = ce^{3t} + dt + k$. The residual is $2ce^{3t} + 2dt + (2k - 3d)$. Matching $e^{3t} - 6t$ gives

$$\boxed{y_p = \tfrac{1}{2}e^{3t} - 3t - \tfrac{9}{2}.}$$

Step 3: Fit and verify. Write $y = y_p + Ce^t + De^{2t}$. Since $y_p(0) = -4$ and $y_p'(0) = -3/2$, the zero data give $C + D = 4$, $C + 2D = 3/2$. Hence

$$\boxed{y = \tfrac{1}{2}e^{3t} + \tfrac{13}{2}e^t - \tfrac{5}{2}e^{2t} - 3t - \tfrac{9}{2}.}$$

At zero its value and slope are $1/2 + 13/2 - 5/2 - 9/2 = 0$ and $3/2 + 13/2 - 5 - 3 = 0$. Its second and third derivatives there are $9/2 + 13/2 - 10 = 1$ and $27/2 + 13/2 - 20 = 0$. The coefficient matching already verifies the forcing everywhere.

Step 4: Interpret the information count. Before calibration there are two unknown forcing parameters and two trajectory constants. The four measurements determine those four quantities here. Once the forcing is fixed, only the value and slope are freely assignable: the equation and its derivative determine the higher derivatives. Arbitrary additional acceleration measurements would be compatibility tests, not extra solution freedoms.

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Question 10

For a real parameter c, consider on $\mathbb{R}$

$$y'' - y = e^{-t} + ce^{-2t}, \qquad y(0) = y'(0) = 0.$$

Both forcing components tend to zero as $t \to \infty$.

Tasks

1. Choose resonant and nonresonant trials as appropriate and find the full zero-data solution in terms of c.
2. Find the unique value of c making this solution bounded on $[0, \infty)$.
3. For that tuned value, prove the sign of the solution for every $t > 0$ and find its limiting value. You may use the strict inequality $e^{-t} > 1 - t$ for $t > 0$.
4. If the tuned parameter is replaced by $c + \delta$ with $\delta \neq 0$, determine the leading long-time term and its sign. Explain why decaying forcing alone does not guarantee a decaying response.

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Question 10 – Solution

Strategy. Determine which forcing combination cancels the growing homogeneous mode after the initial data are fitted.

Step 1: Match and fit. The root -1 is simple, while -2 is not a root. Use $y_p = ate^{-t} + be^{-2t}$. Since $(D^2-1)(te^{-t}) = -2e^{-t}$ and $(D^2-1)e^{-2t} = 3e^{-2t}$, we obtain $a = -1/2$, $b = c/3$. Fitting the zero data gives

$$\boxed{y = (\tfrac{1}{4} + \tfrac{c}{6})e^{t} + (-\tfrac{1}{4} - \tfrac{c}{2})e^{-t} - \tfrac{1}{2}te^{-t} + \tfrac{c}{3}e^{-2t}.}$$

The value is zero because the constant coefficients sum to zero; the slope is zero because the derivative contributions sum to zero.

Step 2: Cancel the unstable mode. Every term except the e^t term tends to zero. Thus boundedness holds exactly when $1/4 + c/6 = 0$, giving $\boxed{c = -3/2}$. The tuned response is

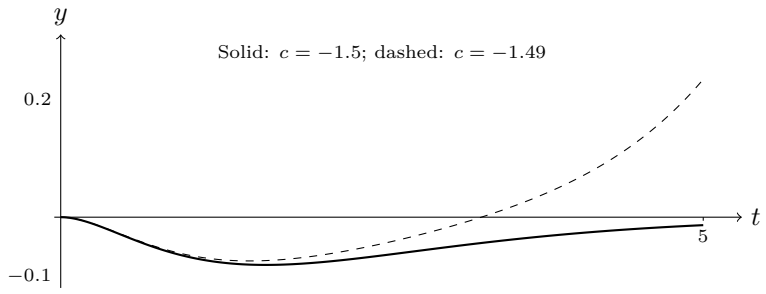
$$\boxed{y_* = \tfrac{1}{2}e^{-t}(1 - t - e^{-t}).}$$

Step 3: Prove sign and decay. For $t > 0$, the supplied strict exponential inequality gives $1 - t - e^{-t} < 0$, so $y_* < 0$. All its terms decay, hence $y_* \to 0$ from below. It has zero initial value and slope despite remaining strictly negative afterward.

Step 4: Examine imperfect cancellation. Replacing the tuned value by $-3/2 + \delta$ makes the growing-mode coefficient $\delta/6$. More explicitly, the change in solution is

$$\delta[e^{t}/6 - e^{-t}/2 + e^{-2t}/3].$$

Consequently $y(t)/e^t \to \delta/6$: the response tends to positive infinity for $\delta > 0$ and negative infinity for $\delta < 0$. The forcing still decays; the initial-data adjustment excites a growing homogeneous component unless the cancellation is exact.



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Question 1

On $t > 0$, consider

$$t^2y'' - 2ty' + 2y = t^3, \qquad y(1) = y'(1) = 0.$$

Two homogeneous solutions are $y_1 = t$ and $y_2 = t^2$.

Tasks

1. Normalize the equation and compute the Wronskian. Derive the two equations for the parameter derivatives using the auxiliary condition $u_1'y_1 + u_2'y_2 = 0$.
2. Use variation of parameters to find a particular solution and then fit the initial data.
3. Check the final solution in the original, unnormalized equation and at the initial point.
4. A student inserts t^3, rather than the normalized forcing, into the parameter formulas. Find the resulting particular expression and its actual forcing in the original equation. Explain the error.

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Question 1 – Solution

Strategy. Divide by the leading coefficient before solving for the parameter derivatives; check the answer in the original equation.

Step 1: Normalize and derive. The normalized forcing is $g(t) = t$, and $W = t(2t) - t^2 = t^2 \neq 0$. With $y = u_1 t + u_2 t^2$, impose $tu_1' + t^2 u_2' = 0$. Differentiating after this cancellation gives

$$tu_1' + t^2 u_2' = 0, \qquad u_1' + 2tu_2' = t.$$

Thus $u_1' = -t$ and $u_2' = 1$.

Step 2: Integrate and fit. Choose $u_1 = -t^2/2$, $u_2 = t$; constants of integration contribute only homogeneous terms. Hence $y_p = t^3/2$ and

$$y = Ct + Dt^2 + t^3/2.$$

The data give $C + D = -1/2$ and $C + 2D = -3/2$, so $C = 1/2$, $D = -1$. Therefore

$$\boxed{y = \tfrac{1}{2}t(t-1)^2.}$$

Step 3: Verify the original equation. For the operator $L[y] = t^2 y'' - 2ty' + 2y$, one has $L[t] = L[t^2] = 0$ and $L[t^3] = 2t^3$. Thus $L[y] = t^3$. Also $y(1) = 0$ and $y' = (3t^2 - 4t + 1)/2$, so $y'(1) = 0$. The normalized coefficients are continuous on $(0, \infty)$, giving uniqueness there.

Step 4: Diagnose the missing division. Using $g = t^3$ instead gives $u_1' = -t^3$, $u_2' = t^2$ and

$$\widetilde{y}_p = -t^5/4 + t^5/3 = t^5/12.$$

Since $L[t^5] = (20 - 10 + 2)t^5 = 12t^5$, this expression satisfies $L[\widetilde{y}_p] = t^5$, not t^3. Adding homogeneous terms cannot repair its forcing. The parameter formulas apply to the normalized equation; omitting the division multiplies the intended original forcing by t^2.

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Question 2

Solve on $I = (-\pi/2, \pi/2)$:

$$y'' + y = \sec t, \qquad y(0) = y'(0) = 0.$$

Use the homogeneous pair $\cos t, \sin t$.

Tasks

1. Derive the parameter derivatives and evaluate the integrals with lower endpoint zero.
2. Simplify the initial-value solution and verify its equation and both data.
3. Determine the limits of y and y' as $t \to (\pi/2)^-$. Can the solution extend as a C^1 function through that endpoint?
4. Explain why a finite limit of y does not enlarge the interval on which this initial-value problem has a classical solution.

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Question 2 – Solution

Strategy. The forcing is singular at the interval endpoints. Keep its domain visible even when the solution itself has a finite limit.

Step 1: Integrate the parameter system. Here $W = 1$. The equations $u_1' \cos t + u_2' \sin t = 0$ and $-u_1' \sin t + u_2' \cos t = \sec t$ give

$$u_1' = -\tan t, \qquad u_2' = 1.$$

Since $\cos t > 0$ on I, integration from zero yields $u_1 = \ln(\cos t)$, $u_2 = t$.

Step 2: Verify the response. The zero lower endpoints fit both data, and

$$\boxed{y = \cos t \ln(\cos t) + t \sin t.}$$

Differentiation simplifies to

$$y' = -\sin t \ln(\cos t) + t \cos t, \quad y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t \sin t.$$

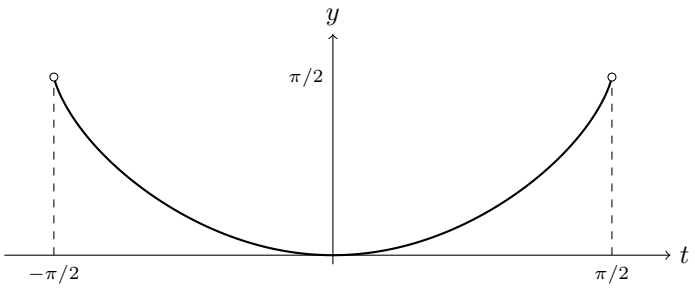
Therefore $y'' + y = (\sin^2 t + \cos^2 t)/\cos t = \sec t$, and both data vanish.

Step 3: Examine the endpoint. Using $x \ln x \to 0$ as $x \to 0^+$, we obtain

$$\lim_{t \to (\pi/2)^-} y(t) = \pi/2, \qquad \lim_{t \to (\pi/2)^-} y'(t) = +\infty.$$

Thus even a C^1 extension is impossible. The graph approaches a finite height with an unbounded positive slope; the endpoint is excluded.

Step 4: Keep the equation's domain. The right side $\sec t$ is undefined at $\pi/2$ and $-\pi/2$. A continuous assignment to y at an endpoint would not define the differential equation there. The maximal open interval containing zero on which this IVP is classical is exactly I.



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Question 3

On $\mathbb{R}$, consider

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, \qquad y(0) = y'(0) = 0.$$

Use $y_1 = e^t$ and $y_2 = te^t$.

Tasks

1. Compute the Wronskian and use variation of parameters to solve the initial-value problem.
2. Independently verify the result by writing $y = e^t F(t)$ and simplifying the differential equation for F.
3. Prove that the selected response is strictly positive for every $t > 0$.
4. Find $\lim_{t\to\infty} y(t)/(te^t)$ and interpret the effect of the factor $1/(1+t^2)$ in the forcing.

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Question 3 – Solution

Strategy. The repeated homogeneous root is handled by the Wronskian; an exponential substitution provides an independent residual check.

Step 1: Compute and integrate. The Wronskian is $e^t(1+t)e^t - te^{2t} = e^{2t}$. Hence

$$u_1' = -\frac{t}{1+t^2}, \qquad u_2' = \frac{1}{1+t^2}.$$

Integrating from zero gives $u_1 = -\frac{1}{2}\ln(1+t^2)$ and $u_2 = \arctan t$. Therefore

$$\boxed{y = e^t[t\arctan t - \tfrac{1}{2}\ln(1+t^2)].}$$

Zero integration endpoints ensure $y(0) = y'(0) = 0$.

Step 2: Check by a different calculation. Writing $y = e^t F$ gives $y'' - 2y' + y = e^t F''$. For the displayed bracket,

$$F' = \arctan t, \qquad F'' = \frac{1}{1+t^2}, \qquad F(0) = F'(0) = 0.$$

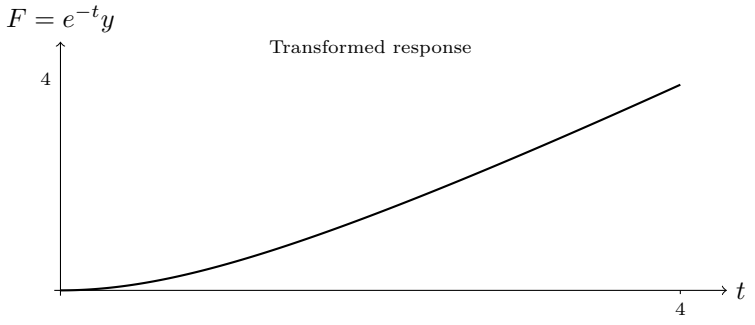
These identities verify the equation and both data independently. All coefficients and the forcing are continuous on $\mathbb{R}$, so this is the unique global IVP solution.

Step 3: Prove the sign. For $t > 0$, $F(t) = \int_0^t \arctan s\, ds > 0$. Equivalently, $F(t) = \int_0^t (t-s)/(1+s^2)\, ds$ has a positive integrand in the interior. Since $e^t > 0$, the response is strictly positive.

Step 4: Compare the growth rate. For $t > 0$,

$$\frac{y}{te^t} = \arctan t - \frac{\ln(1+t^2)}{2t} \longrightarrow \boxed{\pi/2}.$$

Although the forcing is smaller than e^t by a factor tending to zero, the zero-data response is asymptotic to $(\pi/2)te^t$. The graph shows the transformed response $F = e^{-t}y$, whose slope tends to $\pi/2$; it is not a plot of y itself.



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Question 4

Let g be continuous on $\mathbb{R}$. For $y'' + y = g(t)$, compare the homogeneous bases

$$(y_1, y_2) = (\cos t, \sin t), \qquad (z_1, z_2) = (\cos t + \sin t, \cos t - \sin t).$$

In both parameter constructions, set the two parameter values to zero at $t = 0$.

Tasks

1. Derive the parameter derivatives for both bases, including the sign of each Wronskian.
2. Prove directly that both constructions give the same integral formula for the zero-data response.
3. If the lower endpoint is changed from zero to a, express the difference between the two particular solutions as a homogeneous combination.
4. Apply the formula to $g(t) = e^{t^2}$ with $y(0) = 2$, $y'(0) = -1$. Verify the solution without assuming that its integrals have elementary antiderivatives.

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Question 4 – Solution

Strategy. A basis changes the parameter functions, but the integral response with fixed initial data must remain the same.

Step 1: Compute both parameter systems. For the first basis $W = 1$, so $u_1' = -g\sin t$, $u_2' = g\cos t$. The constant change-of-basis determinant is -2, hence $W_z = -2$. Thus

$$v_1' = \tfrac{1}{2}g(\cos t - \sin t), \qquad v_2' = -\tfrac{1}{2}g(\cos t + \sin t).$$

The negative Wronskian is essential to these signs.

Step 2: Compare the reconstructed functions. For the first basis, the integrand multiplying $g(s)$ is $\sin t\cos s - \cos t\sin s = \sin(t-s)$. For the second, it is

$$\tfrac{1}{2}[(\cos t + \sin t)(\cos s - \sin s) - (\cos t - \sin t)(\cos s + \sin s)],$$

which expands to the same expression. Therefore both give

$$\boxed{Y_0(t) = \int_0^t \sin(t-s)g(s)\,ds.}$$

This also holds for negative t using oriented integrals.

Step 3: Change the lower endpoint. Set $Y_a(t) = \int_a^t \sin(t-s)g(s)\,ds$. Then

$$Y_a - Y_0 = \cos t\int_0^a \sin s\, g(s)\,ds - \sin t\int_0^a \cos s\, g(s)\,ds.$$

Both coefficients are constants in t. Changing the lower endpoint changes a particular solution only by a homogeneous term; the data must be fitted again.

Step 4: Use a nonelementary forcing. The requested solution is

$$\boxed{y = 2\cos t - \sin t + \int_0^t \sin(t-s)e^{s^2}\,ds.}$$

Leibniz differentiation gives $Y_0' = \int_0^t \cos(t-s)e^{s^2}\,ds$ and $Y_0'' = e^{t^2} - Y_0$. Also $Y_0(0) = Y_0'(0) = 0$. Thus the equation and data hold. A definite-integral formula is a complete exact solution even without an elementary antiderivative.

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Question 5

On $t > 0$, solve

$$ty'' - (t+1)y' + y = t^2, \qquad y(1) = y'(1) = 0.$$

The functions $t + 1$ and e^t solve the homogeneous equation. Also consider the original undivided equation at $t = 0$.

Tasks

1. Normalize the equation, compute the Wronskian, and obtain a particular solution by variation of parameters.
2. Fit the initial data and check the result directly in the undivided equation.
3. Does this selected function extend smoothly through zero and still satisfy the undivided equation? Find its value and slope there.
4. Prove that these same value and slope at zero do not select a unique smooth solution of the undivided equation near zero.

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Question 5 – Solution

Strategy. Use variation of parameters where the leading coefficient is nonzero, then examine the original equation separately at its singular point.

Step 1: Normalize and integrate. The normalized forcing is $g = t$, and $W = (t+1)e^t - e^t = te^t$. Hence

$$u_1' = -1, \qquad u_2' = (t+1)e^{-t}.$$

Choose $u_1 = -t$, $u_2 = -(t+2)e^{-t}$, giving

$$y_p = -t(t+1) - (t+2) = -t^2 - 2t - 2.$$

Thus $y = C(t+1) + De^t - t^2 - 2t - 2$ on $(0, \infty)$.

Step 2: Fit and verify. The conditions are $2C + De = 5$ and $C + De = 4$, so $C = 1$, $D = 3/e$ and

$$\boxed{y = 3e^{t-1} - t^2 - t - 1.}$$

Its derivatives are $y' = 3e^{t-1} - 2t - 1$, $y'' = 3e^{t-1} - 2$. Substitution cancels the exponential terms; the polynomial terms yield

$$-2t + (t+1)(2t+1) - t^2 - t - 1 = t^2.$$

Both data at one vanish. The normalized problem is regular on $(0, \infty)$, where its solution is unique.

Step 3: Continue the selected function. The formula is analytic on $\mathbb{R}$, and the same residual identity holds there. At zero,

$$y(0) = y'(0) = 3/e - 1.$$

The undivided equation reduces to $-y'(0) + y(0) = 0$, which these values satisfy. This extension does not require dividing by t at zero.

Step 4: Demonstrate nonuniqueness at zero. The nonzero homogeneous function $h = e^t - (t+1)$ is analytic, with $h(0) = h'(0) = 0$. For every real k, $y + kh$ solves the undivided equation and has the same two data at zero. These are distinct functions since $h''(0) = 1$. The Wronskian vanishes at zero precisely where the leading coefficient vanishes; the regular normalized uniqueness theorem does not apply there.

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Question 6

Let $T > 0$ and let g be continuous on $[0, T]$. Consider

$$y'' - y = g(t), \qquad y(0) = y'(0) = 0.$$

Use the homogeneous basis e^t, e^{-t}.

Tasks

1. Derive a single definite-integral formula for the response by variation of parameters, and verify it by differentiation.
2. If $|g(t)| \le M$ with $M > 0$, find a sharp bound for $|y(t)|$ on $[0, T]$ and exhibit forcing that attains it.
3. If $g \ge 0$, prove $y \ge 0$ and give a precise condition guaranteeing $y(t) > 0$ at a specified time $t > 0$.
4. For $g(t) = (1 + \cos 2t)/2$, find an elementary response and compare it with the bound for $M = 1$.

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Question 6 – Solution

Strategy. The sign of the integral kernel gives comparison and error bounds without evaluating each forcing integral.

Step 1: Derive the response kernel. Here $W = -2$, so $u_1' = e^{-t}g/2$ and $u_2' = -e^{t}g/2$. Integrating from zero and combining gives

$$\boxed{y(t) = \int_0^t \sinh(t-s)g(s)\,ds.}$$

Differentiation gives $y' = \int_0^t \cosh(t-s)g(s)\,ds$ and $y'' = g(t) + y(t)$. Both initial data vanish.

Step 2: Prove sharpness. For $0 \le s \le t$, $\sinh(t-s) \ge 0$. Consequently

$$|y(t)| \le M\int_0^t \sinh(t-s)\,ds = \boxed{M(\cosh t - 1)}.$$

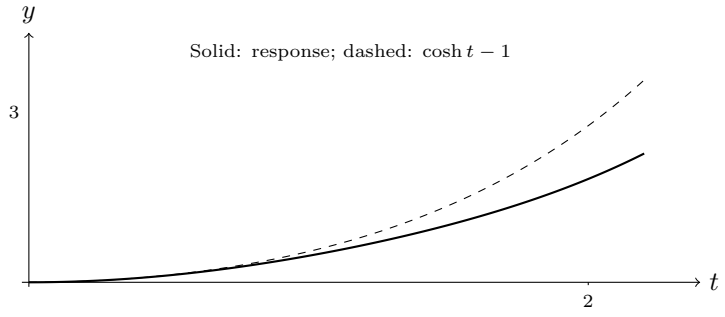
Constant forcing $g = M$ gives equality for every t, and $g = -M$ gives equality in magnitude. The bound is therefore sharp for this forcing class; it also bounds response error when g denotes a forcing error.

Step 3: Prove positivity carefully. If $g \ge 0$, the integral is nonnegative. At fixed $t > 0$, it is strictly positive exactly when g is not identically zero on $[0, t]$. Indeed, continuity then gives a positive subinterval lying before t, where the kernel is strictly positive. Forcing that is positive only at later times cannot make the response positive at this time.

Step 4: Evaluate a comparison example. For $g = (1 + \cos 2t)/2$, an elementary particular solution is $-1/2 - \cos(2t)/10$. Fitting zero data gives

$$\boxed{y = \tfrac{3}{5}\cosh t - \tfrac{1}{2} - \tfrac{1}{10}\cos 2t.}$$

Direct differentiation yields $y'' - y = (1 + \cos 2t)/2$ and zero data. Since $0 \le g \le 1$, the kernel proves $0 < y(t) < \cosh t - 1$ for $t > 0$: both g and $1 - g$ are nonzero on every such interval.



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Question 7

On $t \geq 0$, consider the family

$$y'' - y = \frac{1}{1+t^2}, \qquad y(0) = 0, \quad y'(0) = v.$$

Write $f(t) = 1/(1+t^2)$ and $I = \int_0^\infty e^{-s} f(s)\, ds$. An answer containing a convergent definite integral is exact.

Tasks

1. Use variation of parameters to express y_v as an integral. Prove that I converges.
2. Find the only possible value of v for which y_v is bounded on $[0, \infty)$.
3. For that value, rewrite the response using the positive kernel $K(t,s) = \sinh(\min\{t,s\})e^{-\max\{t,s\}}$. Prove boundedness and determine its sign for $t > 0$.
4. Prove that the bounded response tends to zero, and determine the long-time behavior when the initial slope differs from the selected value.

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Question 7 – Solution

Strategy. Cancel the growing mode using a convergent integral, then prove the resulting tail expression is bounded.

Step 1: Construct the family. Variation of parameters with e^t, e^{-t} gives

$$y_v = v \sinh t + \int_0^t \sinh(t-s) f(s)\, ds.$$

Since $0 < f \le 1$, $0 < I \le \int_0^\infty e^{-s}\, ds = 1$, with strict inequality $I < 1$.

Step 2: Identify the necessary slope. Expanding the hyperbolic sine yields

$$e^{-t} y_v = \frac{v}{2}(1 - e^{-2t}) + \frac{1}{2}\int_0^t e^{-s} f(s)\, ds - \frac{1}{2} e^{-2t} \int_0^t e^{s} f(s)\, ds.$$

The last term has magnitude at most $(e^{-t} - e^{-2t})/2$. Thus $e^{-t} y_v \to (v + I)/2$. Boundedness requires $\boxed{v_* = -I}$.

Step 3: Prove sufficiency and sign. Splitting the integral at $s = t$ and substituting $v = -I$ gives

$$y_*(t) = -\int_0^\infty K(t,s) f(s)\, ds, \quad K(t,s) = \begin{cases} e^{-t} \sinh s, & s \le t, \\ e^{-s} \sinh t, & s \ge t. \end{cases}$$

This identity follows by grouping the e^t and e^{-t} terms above. Moreover,

$$\int_0^\infty K(t,s)\, ds = e^{-t}(\cosh t - 1) + e^{-t} \sinh t = 1 - e^{-t}.$$

Therefore $|y_*| \le 1 - e^{-t} < 1$. For $t > 0$, the kernel and forcing are positive for $s > 0$, so $y_*(t) < 0$.

Step 4: Prove decay and sensitivity. Given $\varepsilon > 0$, choose A with $f(s) \le \varepsilon$ for $s \ge A$. For $t > A$, the integral over $[0, A]$ has magnitude at most $e^{-t} \int_0^A \sinh s\, ds \to 0$; the remaining magnitude is at most $\varepsilon \int_A^\infty K \le \varepsilon$. Hence $y_* \to 0$. If $v = v_* + \delta$, then $y_v = y_* + \delta \sinh t$, so $y_v / e^t \to \delta/2$. It tends to $+\infty$ for $\delta > 0$ and $-\infty$ for $\delta < 0$.

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Question 8

For $t \geq 0$, solve $y'' + y = g(t)$ with zero initial value and slope. First use the continuous finite-duration forcing

$$g_2(t) = \begin{cases} \sin 2t, & 0 \leq t \leq \pi, \\ 0, & t > \pi. \end{cases}$$

Later compare it with g_3, defined in the same way using $\sin 3t$.

Tasks

1. Derive the integral response and solve explicitly for g_2 on $[0, \pi]$.
2. Find the response after π, checking the matching value and slope. Does turning off the forcing return the solution to rest?
3. For any continuous forcing supported in $[0, \pi]$, derive the two integral conditions equivalent to $y(t) = 0$ for every $t \geq \pi$.
4. Test g_3 against these conditions and find its response. Explain how a nonzero forcing can leave no later motion.

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Question 8 – Solution

Strategy. After the forcing ends, its two weighted integrals become the coefficients of the remaining homogeneous oscillation.

Step 1: Solve while the first forcing acts. Variation of parameters gives $y(t) = \int_0^t \sin(t-s)g(s)\,ds$. For g_2 on $[0,\pi]$, evaluation gives

$$y_2(t) = \frac{2\sin t - \sin 2t}{3}.$$

Its value and slope at zero vanish, and $y_2'' + y_2 = \sin 2t$, independently verifying the integral evaluation.

Step 2: Match at the shutoff time. At π, $y_2 = 0$ and $y_2' = -4/3$. The subsequent homogeneous response is

$$\boxed{y_2(t) = -\tfrac{4}{3}\sin(t-\pi) = \tfrac{4}{3}\sin t, \quad t \geq \pi.}$$

It matches both data. Since $g_2(\pi) = 0$, $y'' = g_2 - y$ also matches. The resulting function is a classical C^2 solution, but it does not return to rest.

Step 3: Derive the cancellation conditions. For any such forcing and $t \geq \pi$,

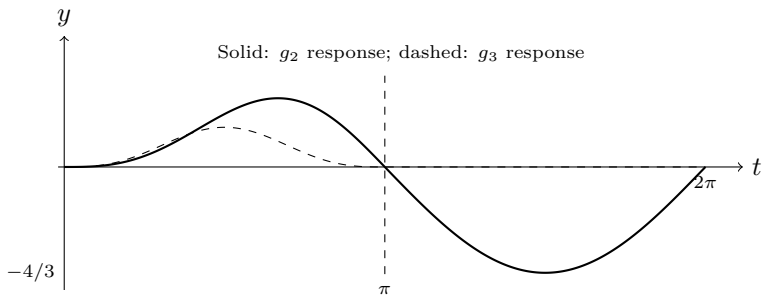
$$y(t) = \sin t \int_0^\pi \cos s\, g(s)\,ds - \cos t \int_0^\pi \sin s\, g(s)\,ds.$$

Independence of sine and cosine shows that the response vanishes identically after π exactly when both displayed integrals are zero. Cancellation of just one moment is insufficient.

Step 4: Construct a return to rest. For g_3, product-to-sum identities give $\int_0^\pi \cos s \sin 3s\,ds = 0$ and $\int_0^\pi \sin s \sin 3s\,ds = 0$. Explicitly,

$$\boxed{y_3(t) = \begin{cases} (3\sin t - \sin 3t)/8, & 0 \leq t \leq \pi, \\ 0, & t \geq \pi. \end{cases}}$$

On the first interval it has residual $\sin 3t$ and zero initial data; at π its value and slope are zero. The forcing's signed contributions cancel both final-state components, despite a nonzero response during the forcing interval.



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Question 9

Let f be continuous on $[0, \pi]$. Seek C^2 solutions of

$$y'' + y = f(t), \qquad y(0) = y(\pi) = 0.$$

Use variation of parameters; no general boundary-value theory is required.

Tasks

1. Write every solution satisfying the first endpoint condition, using its free initial slope A.
2. Derive a necessary and sufficient condition on f for the second endpoint condition. Explain uniqueness or nonuniqueness when the condition holds.
3. Apply the criterion to $f = 1$ and $f = \cos t$, and find all solutions in the latter case.
4. For $f = \cos t$, impose the additional condition $\int_0^\pi y(t)\,dt = 0$. Determine whether it selects a unique solution and find it.

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Question 9 – Solution

Strategy. Variation of parameters supplies all initial-value responses; evaluating the integral at the second endpoint reveals the obstruction.

Step 1: Keep the free slope. The homogeneous pair is $\cos t, \sin t$, with $W = 1$. Integrating the parameter derivatives from zero and imposing $y(0) = 0$ gives every candidate:

$$y(t) = A\sin t + \int_0^t \sin(t-s)f(s)\,ds, \qquad y'(0) = A.$$

Differentiating this formula verifies the equation and shows no candidate has been lost.

Step 2: Evaluate the second endpoint. Since $\sin\pi = 0$ and $\sin(\pi - s) = \sin s$, the remaining condition is

$$\boxed{\int_0^\pi \sin s\, f(s)\,ds = 0.}$$

If it fails, no choice of A helps. If it holds, every real A works: there are infinitely many solutions, differing by $A\sin t$. Thus this endpoint pair never selects exactly one solution for a compatible forcing.

Step 3: Compare the two forcings. For $f = 1$, the integral is 2, so there is no solution. For $f = \cos t$, it is $\int_0^\pi \sin s\cos s\,ds = 0$. Evaluation of the response integral, or a direct residual check, gives

$$\boxed{y(t) = (A + t/2)\sin t, \quad A \in \mathbb{R}.}$$

Indeed, $(t\sin t/2)'' + t\sin t/2 = \cos t$, and both endpoints vanish.

Step 4: Apply the extra measurement. The needed integrals are $\int_0^\pi \sin t\,dt = 2$ and, by integration by parts, $\int_0^\pi t\sin t\,dt = \pi$. Thus

$$0 = 2A + \pi/2, \qquad A = -\pi/4, \quad \boxed{y(t) = (t/2 - \pi/4)\sin t.}$$

The coefficient of A in this measurement is nonzero, so it selects exactly one member of the compatible family. This does not contradict the earlier nonuniqueness: the added integral condition supplies new information.

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Question 10

For $t \geq 0$, solve

$$y'' + y = e^{-t^2}, \qquad y(0) = y'(0) = 0.$$

Define the convergent integrals $C = \int_0^\infty e^{-s^2} \cos s \, ds$ and $S = \int_0^\infty e^{-s^2} \sin s \, ds$. No special-function evaluations of C or S are required.

Tasks

1. Use variation of parameters to express the exact solution through two finite definite integrals, and verify its initial data and residual.
2. Show that the solution approaches the periodic function $P(t) = C \sin t - S \cos t$ in the sense that $y(t) - P(t) \to 0$.
3. Prove the quantitative estimate $|y(t) - P(t)| \leq e^{-t^2}/(2t)$ for every $t > 0$.
4. Prove $S > 0$ by pairing successive positive and negative half-waves of sine. Use this to show that y does not tend to zero, even though the forcing does.

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Question 10 – Solution

Strategy. Keep the variation-of-parameters integrals exact, then bound their tails to identify the surviving oscillation.

Step 1: Write and check the exact response. With $W = 1$, the parameter derivatives are $u_1' = -e^{-t^2}\sin t$ and $u_2' = e^{-t^2}\cos t$. Hence

$$\boxed{y(t) = \sin t \int_0^t e^{-s^2}\cos s\, ds - \cos t \int_0^t e^{-s^2}\sin s\, ds.}$$

Equivalently, $y = \int_0^t \sin(t-s)e^{-s^2}\, ds$. Differentiating gives $y' = \int_0^t \cos(t-s)e^{-s^2}\, ds$ and $y'' = e^{-t^2} - y$; both initial data vanish.

Step 2: Isolate the tails. Absolute convergence follows from the integrability of e^{-s^2} (for $s \geq 1$, it is at most e^{-s}). Subtracting the full integrals gives

$$y(t) - P(t) = -\int_t^\infty \sin(t-s)e^{-s^2}\, ds.$$

Thus $|y - P| \leq \int_t^\infty e^{-s^2}\, ds \to 0$.

Step 3: Obtain an explicit error bound. For $s \geq t > 0$, $1 \leq s/t$. Consequently

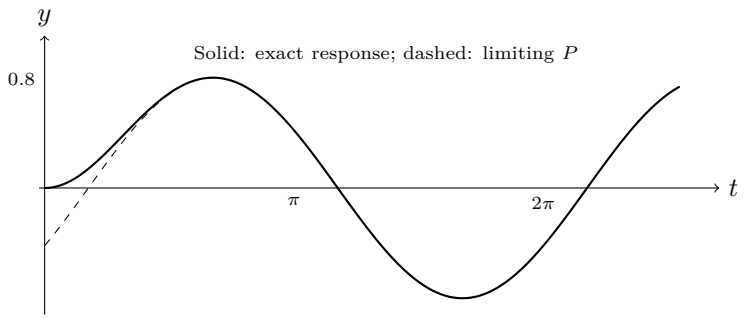
$$|y - P| \leq \frac{1}{t}\int_t^\infty s e^{-s^2}\, ds = \boxed{\frac{e^{-t^2}}{2t}}.$$

Combining the tails before estimating avoids an unnecessary sum of two bounds.

Step 4: Prove a nonzero surviving oscillation. Absolute convergence permits pairing intervals. Their kth pair contributes

$$\int_0^\pi \sin u[e^{-(2k\pi+u)^2} - e^{-((2k+1)\pi+u)^2}], du > 0, \qquad k \geq 0.$$

Therefore $S > 0$. Along $t = 2n\pi$, $y(t) \to -S$; along $t = (2n+1)\pi$, $y(t) \to S$. The solution has no limit and does not decay to zero. A decaying forcing can leave a persistent homogeneous oscillation.



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Question 1

A 2 kg mass stretches an ideal vertical spring by 0.196 m at static equilibrium. Use $g = 9.8\,\mathrm{m/s^2}$ and neglect damping. Let x be downward displacement from equilibrium. At $t = 0$, the mass is 0.05 m below equilibrium and moving upward at 0.20 m/s.

Tasks

1. Determine the spring constant and derive the equation for x from Newton's law, starting with extension measured from the spring's natural length.
2. Solve the initial-value problem and find its oscillation amplitude and angular frequency.
3. Find the first positive time the mass crosses equilibrium and its signed velocity then.
4. Use mechanical energy to verify the crossing speed. Find the maximum spring extension from natural length and explain why gravity is absent from the final equation for x.

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Question 1 – Solution

Strategy. Distinguish extension from natural length from displacement about equilibrium, and keep the upward initial velocity negative.

Step 1: Establish the coordinate and model. Write $z = \ell + x$ for downward extension, where $\ell = 0.196\,\text{m}$. The static balance is $k\ell = mg$, so $k = 19.6/0.196 = 100\,\text{N/m}$. Newton's law gives

$$2x'' = mg - k(\ell + x) = -100x, \qquad x(0) = 0.05, \quad x'(0) = -0.20.$$

Thus $\omega_0 = \sqrt{k/m} = 5\sqrt{2}\,\text{s}^{-1}$.

Step 2: Solve and find the amplitude. Fitting both data gives

$$\boxed{x(t) = \frac{1}{20}\cos(5\sqrt{2}t) - \frac{1}{25\sqrt{2}}\sin(5\sqrt{2}t).}$$

The amplitude satisfies $R^2 = x(0)^2 + [x'(0)/\omega_0]^2 = 1/400 + 1/1250 = 33/10000$. Hence $\boxed{R = \sqrt{33}/100\,\text{m}}$. Differentiation verifies $x'' + 50x = 0$ and the two data.

Step 3: Locate the first crossing. The first zero lies in $0 < \omega_0 t < \pi/2$, where the motion is upward. Solving $x = 0$ gives

$$\boxed{t_* = \frac{1}{5\sqrt{2}}\arctan\frac{5\sqrt{2}}{4}.}$$

At that crossing, $x' = -\omega_0 R = -\sqrt{66}/20\,\text{m/s}$. The negative sign identifies upward travel; using an arbitrary branch of arctangent could miss the first crossing.

Step 4: Check energy and physical extension. Relative to equilibrium, the conserved energy is $E = mx'^2/2 + kx^2/2$. Initially,

$$E = \tfrac{1}{2}(2)(0.20)^2 + \tfrac{1}{2}(100)(0.05)^2 = 0.165\,\text{J}.$$

At $x = 0$, $E = x'^2$, giving $|x'| = \sqrt{0.165} = \sqrt{66}/20\,\text{m/s}$. The maximum natural-length extension is $\ell + R = 0.196 + \sqrt{33}/100\,\text{m}$, about $0.25345\,\text{m}$. Gravity has not been neglected: its constant force cancels the spring's equilibrium force after shifting the coordinate.

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Question 2

A mass of 1.5 kg vibrates freely about equilibrium with viscous damping:

$$mx'' + cx' + kx = 0, \qquad c > 0, \quad k > 0.$$

It is released from rest at $x(0) = 0.08\,\text{m}$. Measured successive positive displacement maxima are separated by $\pi\,\text{s}$, and each is $e^{-\pi/2}$ times the preceding one. Treat these measurements as exact.

Tasks

1. Infer the decay rate and damped angular frequency, explaining why same-sign peaks must be used. Determine c and k with units.
2. Find the response and verify that its positive maxima really have the measured spacing and ratio.
3. Determine the ratio of mechanical energies at successive positive maxima. Why is it not the displacement ratio?
4. If the mass had not been supplied, identify exactly which combinations of m, c, k these measurements determine, and exhibit the remaining ambiguity.

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Question 2 – Solution

Strategy. Peak spacing determines the oscillation frequency; peak ratios determine exponential decay. A known mass converts these rates into physical coefficients.

Step 1: Infer the coefficients. An underdamped response has form $e^{-\alpha t}(A\cos\omega_d t + B\sin\omega_d t)$, with $\alpha = c/(2m)$ and $\omega_d^2 = k/m - \alpha^2$. Same-sign peaks are one damped period apart, so

$$\omega_d = \frac{2\pi}{\pi} = 2\,\mathrm{s}^{-1}, \qquad e^{-\alpha\pi} = e^{-\pi/2} \implies \alpha = \tfrac{1}{2}\,\mathrm{s}^{-1}.$$

Thus $\boxed{c = 3/2\,\mathrm{N\,s/m}}$ and $\boxed{k = 51/8\,\mathrm{N/m}}$. Adjacent positive and negative extrema would be only half a period apart.

Step 2: Reconstruct and verify the peaks. The initial value gives $A = 0.08$, and $x'(0) = 0$ gives $B = 0.02$. Hence

$$\boxed{x = 0.08e^{-t/2}[\cos 2t + \tfrac{1}{4}\sin 2t],} \qquad x' = -0.17e^{-t/2}\sin 2t.$$

The characteristic roots are $-1/2 \pm 2i$, verifying the equation. Extrema occur at $t = n\pi/2$; positive maxima occur at $t = n\pi$, where $x = 0.08e^{-n\pi/2}$. The derivative changes from positive to negative at each interior positive maximum, and the release at zero starts a decrease.

Step 3: Compare energies. At an extremum, velocity is zero, so $E = kx^2/2$. Successive positive-peak energies therefore have ratio

$$\boxed{e^{-\pi}},$$

not $e^{-\pi/2}$. Energy depends quadratically on displacement at a turning point.

Step 4: State the identification limit. Without a known mass, only $c/m = 2\alpha = 1\,\mathrm{s}^{-1}$ and $k/m = \alpha^2 + \omega_d^2 = 17/4\,\mathrm{s}^{-2}$ are determined. Multiplying m, c, k by the same positive constant leaves the normalized equation, the complete free response, and all peak measurements unchanged. The three coefficients cannot then be identified separately.

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Question 3

A mass–spring–damper system has $m = 1\,\text{kg}$, $c = 2\,\text{N s/m}$ and $k = 5\,\text{N/m}$. It starts at equilibrium with velocity $1\,\text{m/s}$ in the positive direction, with no external forcing. Define

$$E(t) = \tfrac{1}{2}mx'(t)^2 + \tfrac{1}{2}kx(t)^2.$$

Tasks

1. Find the displacement and velocity, including the damping regime.
2. Derive the energy balance directly from the equation. Does decreasing energy imply that $|x|$ must decrease from the start?
3. Find the first positive turning time, its displacement, and the energy dissipated up to that time.
4. Explain why $E' = 0$ at a turning point does not mean damping has ceased permanently. Find the total energy dissipated as $t \to \infty$.

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Question 3 – Solution

Strategy. Track energy in both position and velocity. Displacement can grow while total mechanical energy falls.

Step 1: Solve the free motion. The equation is $x'' + 2x' + 5x = 0$, with roots $-1 \pm 2i$, so the motion is underdamped. The data give

$$\boxed{x = \tfrac{1}{2}e^{-t}\sin 2t, \qquad x' = e^{-t}[\cos 2t - \tfrac{1}{2}\sin 2t].}$$

These expressions have $x(0) = 0$, $x'(0) = 1$ and residual zero.

Step 2: Derive the dissipation law. Multiplying the equation by x' gives

$$E' = x'(mx'' + kx) = -cx'^2 = \boxed{-2x'^2 \le 0}.$$

Nevertheless, $x'(0) = 1$, so x and $|x|$ initially increase from zero. Kinetic energy is partly converted into spring energy while the damper removes energy. Monotonic total energy is not monotonic displacement.

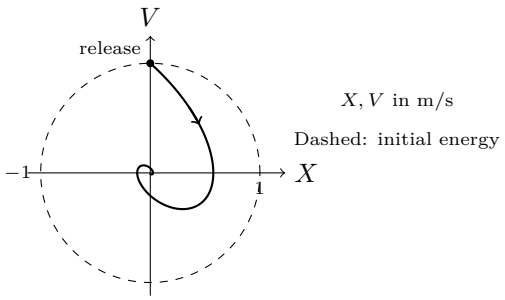
Step 3: Find the first turning point. The first root of $x' = 0$ satisfies $\tan 2t = 2$, in $0 < 2t < \pi/2$. Set $a = \frac{1}{2}\arctan 2$. Then

$$\boxed{t_* = a, \qquad x(t_*) = e^{-a}/\sqrt{5}\,\text{m}.}$$

The velocity changes from positive to negative, so this is a positive maximum. Initially $E(0) = 1/2\,\text{J}$. At the turning point,

$$E(a) = \tfrac{5}{2}x(a)^2 = \tfrac{1}{2}e^{-2a}\,\text{J}, \quad \boxed{E(0) - E(a) = \tfrac{1}{2}(1 - e^{-2a})\,\text{J}.}$$

Step 4: Interpret zero instantaneous loss. At a turning point, $x' = 0$, so damper power is instantaneously zero. But $x \neq 0$ there and $x'' = -5x \neq 0$, so motion resumes immediately. The displayed response and velocity both tend to zero; hence $E \to 0$ and total dissipated energy is $1/2\,\text{J}$. In the figure, $X = \sqrt{k/m}\,x$ and $V = x'$ both have velocity units, and $E = m(X^2 + V^2)/2$.



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Question 4

A critically damped system has $m = 1\,\text{kg}$, $c = 6\,\text{N s/m}$ and $k = 9\,\text{N/m}$. Its initial data are $x(0) = a > 0$ and $x'(0) = v$, with a in meters and v in meters per second. There is no forcing.

Tasks

1. Derive and verify the response for arbitrary a, v.
2. Classify exactly which initial velocities cause a finite equilibrium crossing. Find the crossing time when one exists.
3. Classify exactly which initial velocities make the displacement stay positive and decrease monotonically toward zero.
4. For $a = 0.10\,\text{m}$ and $v = -0.60\,\text{m/s}$, find the crossing and the subsequent minimum. Use the result to assess the claim that critical damping always prevents overshoot.

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Question 4 – Solution

Strategy. Critical damping fixes the repeated root, but the sign of the linear factor still depends on the initial velocity.

Step 1: Fit the repeated-root solution. The characteristic polynomial is $(r+3)^2$. Thus

$$\boxed{x(t) = e^{-3t}[a + (v+3a)t], \qquad x'(t) = e^{-3t}[v - 3(v+3a)t].}$$

Here the numerical rate 3 has units s^{-1}. The two data follow immediately, and substitution gives $x'' + 6x' + 9x = 0$.

Step 2: Classify crossings. Since $e^{-3t} > 0$, a positive-time zero occurs exactly when $v + 3a < 0$. In that case it is unique and

$$\boxed{t_c = \frac{a}{-v - 3a}.}$$

The linear factor changes sign there, so this is a crossing, not a touch. If $v \geq -3a$, the factor remains positive for all finite $t \geq 0$, including the equality case $x = ae^{-3t}$.

Step 3: Classify monotone positive return. For $v \geq -3a$, the displacement is positive. Its derivative is nonpositive for all $t \geq 0$ exactly when $v \leq 0$, because $v - 3(v+3a)t$ then begins nonpositive and decreases. Thus the required range is

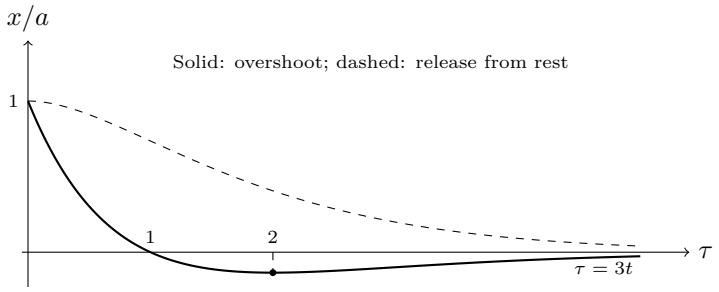
$$\boxed{-3a \leq v \leq 0.}$$

For $v > 0$, the mass first moves farther from equilibrium. For $v < -3a$, it crosses equilibrium and cannot be a positive monotone return.

Step 4: Exhibit overshoot. For the stated data, $x = 0.10e^{-3t}(1 - 3t)$ and $x' = e^{-3t}(-0.60 + 0.90t)$. Therefore

$$\boxed{t_c = 1/3\,\mathrm{s}, \quad t_{\min} = 2/3\,\mathrm{s}, \quad x_{\min} = -0.10e^{-2}\,\mathrm{m}.}$$

The derivative changes from negative to positive at the minimum. Critical damping eliminates sustained oscillations, but sufficiently negative initial velocity still causes one overshoot. The comparison uses $\tau = 3t$ and normalized displacement x/a.



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Question 5

A 1 kg mass on a spring with $k = 4\,\mathrm{N/m}$ is released from $x(0) = a = 0.01\,\mathrm{m}$, with $x'(0) = 0$. Compare viscous damping coefficients $c = 4$ and $c = 5\,\mathrm{N\,s/m}$; write their responses as x_4, x_5.

Tasks

1. Determine the damping regimes and find both exact responses.
2. Prove that each response stays positive and strictly decreases for $t > 0$.
3. Prove $x_5(t) > x_4(t)$ for every $t > 0$. You may form the equation for $w = x_5 - x_4$ and derive its zero-data integral response.
4. Explain the implication for the first time the displacement reaches any fixed level ε with $0 < \varepsilon < a$. For general $c > 4$, examine the slow characteristic root as $c \to \infty$.

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Question 5 – Solution

Strategy. Compare complete release responses, rather than assuming a larger damping force always gives a faster return.

Step 1: Solve both release problems. For $c = 4$, the repeated root is -2, giving critical damping. For $c = 5$, roots $-1, -4$ give overdamping. Fitting the data yields

$$\boxed{x_4 = a(1 + 2t)e^{-2t}, \qquad x_5 = \frac{a}{3}(4e^{-t} - e^{-4t}).}$$

Their characteristic forms verify the equations; both have value a and slope zero at the release.

Step 2: Check positivity and decrease. The first expression is positive, and the second is positive because $4e^{-t} > e^{-4t}$. Their derivatives are

$$x_4' = -4ate^{-2t} < 0, \qquad x_5' = \tfrac{4a}{3}(e^{-4t} - e^{-t}) < 0 \quad (t > 0).$$

Both tend to zero.

Step 3: Prove the comparison for every time. Applying $D^2 + 5D + 4$ to $w = x_5 - x_4$ gives

$$w'' + 5w' + 4w = -x_4', \qquad w(0) = w'(0) = 0.$$

Variation of parameters with e^{-t}, e^{-4t} yields

$$w(t) = \int_0^t \frac{e^{-(t-s)} - e^{-4(t-s)}}{3}[-x_4'(s)]\, ds.$$

For $0 < s < t$, both factors are strictly positive. Hence $\boxed{x_5(t) > x_4(t)}$ for every $t > 0$. This integral proof avoids relying on a finite plot or only on large-time rates.

Step 4: Interpret settling and heavy damping. Each response reaches any $\varepsilon \in (0, a)$ exactly once, by continuity and strict decrease. At the time $x_4 = \varepsilon$, one still has $x_5 > \varepsilon$, so the more heavily damped response reaches the level later. For general $c > 4$ in this SI model,

$$r_{\text{slow}} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = \frac{-2k}{c + \sqrt{c^2 - 4mk}} \longrightarrow 0^-.$$

The release excites this mode with coefficient $-ar_{\text{fast}}/(r_{\text{slow}} - r_{\text{fast}}) > 0$. Excessive damping therefore introduces a slow return, with $r_{\text{slow}} \sim -k/c$; damping strength alone is not a measure of settling speed.

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Question 6

An undamped 1 kg mass on a 25 N/m spring is initially at rest at equilibrium. For $t \geq 0$, apply the force $2\cos 4t$ N.

Tasks

1. Derive and solve the forced initial-value problem.
2. Express the response as a product that displays beats. Identify the carrier angular frequency and the period of the nonnegative amplitude envelope.
3. At $t = 2\pi$ s, find both displacement and velocity. Does this mean the mass stays at rest afterward? Justify your answer using the equation.
4. Replace the forcing by $2\cos 5t$ N and find the new zero-data response. Explain the difference between beats and exact resonance.

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Question 6 – Solution

Strategy. The homogeneous correction needed to fit rest data creates the nearby-frequency interference; exact resonance requires a different particular form.

Step 1: Solve the nonresonant problem. Newton's law gives $x'' + 25x = 2\cos 4t$, $x(0) = x'(0) = 0$. Since $25 - 16 = 9$, a particular response is $(2/9)\cos 4t$. Fitting zero data gives

$$\boxed{x = \tfrac{2}{9}(\cos 4t - \cos 5t).}$$

Its second derivative plus $25x$ equals $2\cos 4t$, and both initial data vanish.

Step 2: Separate the two time scales. The cosine-difference identity gives

$$\boxed{x = \tfrac{4}{9}\sin(9t/2)\sin(t/2).}$$

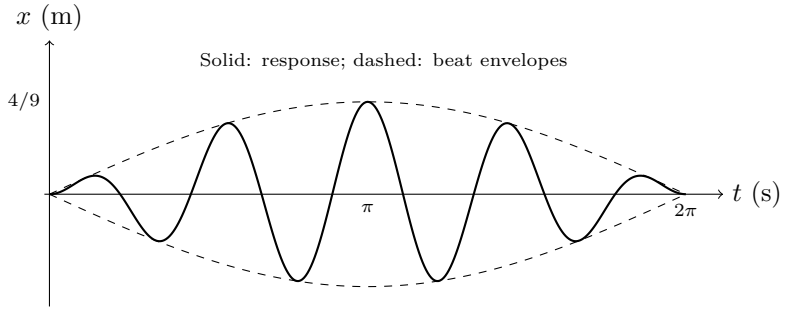
The carrier angular frequency is $9/2\,\mathrm{s}^{-1}$. The nonnegative envelope is $A(t) = (4/9)|\sin(t/2)|\,\mathrm{m}$, with period $2\pi\,\mathrm{s}$. Its zeros occur at $t = 2n\pi$, and its largest value is $4/9\,\mathrm{m}$, attained by x at $t = \pi$. The envelope period uses the absolute value; the signed slow sine has period 4π.

Step 3: Distinguish a rest state from remaining at rest. Here $x' = \frac{2}{9}(-4\sin 4t + 5\sin 5t)$, so $x(2\pi) = x'(2\pi) = 0$. Yet the applied force then equals $2\,\mathrm{N}$; the equation gives $x''(2\pi) = 2\,\mathrm{m/s}^2$. The mass immediately accelerates again. Reaching the equilibrium rest state while a nonzero force persists does not keep it there.

Step 4: Solve at exact resonance. For $x'' + 25x = 2\cos 5t$, use $x_p = At\sin 5t$. Its residual is $10A\cos 5t$, so $A = 1/5$ and the data are already zero:

$$\boxed{x_{\mathrm{res}} = \tfrac{1}{5}t\sin 5t.}$$

At $t_n = (\pi/2 + 2n\pi)/5$, it equals $t_n/5 \to \infty$, whereas the beating response is bounded by $4/9$. Beats arise from interference of distinct frequencies; resonance produces a growing factor when the forcing matches the natural frequency.



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Question 7

A mass–spring–damper system has $m = 2\,\mathrm{kg}$, $c = 4\,\mathrm{N\,s/m}$ and $k = 20\,\mathrm{N/m}$. Forcing is $F_0 \cos \omega t$, where $F_0 > 0$ is fixed and $\omega > 0$. Consider its steady periodic response. Let X be displacement amplitude and V be velocity amplitude.

Tasks

1. Derive the cosine and sine coefficients of the steady response and express X and V in terms of ω.
2. Find the frequency that maximizes displacement amplitude and its maximum value.
3. Independently find the frequency that maximizes velocity amplitude and its maximum value.
4. At the velocity-maximizing frequency, determine the displacement phase relative to the applied force. Explain why a request for the 'resonant frequency' needs a specified response quantity.

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Question 7 – Solution

Strategy. Different measured quantities introduce different frequency factors; optimize their amplitudes separately.

Step 1: Match harmonic coefficients. Set $x = A\cos\omega t + B\sin\omega t$. The equation $2x'' + 4x' + 20x = F_0\cos\omega t$ gives

$$(20 - 2\omega^2)A + 4\omega B = F_0, \qquad -4\omega A + (20 - 2\omega^2)B = 0.$$

Writing $\Delta = (20 - 2\omega^2)^2 + 16\omega^2 > 0$, we obtain

$$A = \frac{F_0(20 - 2\omega^2)}{\Delta}, \quad B = \frac{4F_0\omega}{\Delta}, \quad \boxed{X = \frac{F_0}{\sqrt{\Delta}}, \quad V = \omega X.}$$

The homogeneous roots are $-1 \pm 3i$, so all initial-data transients decay and this periodic response is the long-time motion.

Step 2: Maximize displacement. Complete the square:

$$\Delta = 4[(\omega^2 - 8)^2 + 36].$$

It has its unique minimum for $\omega > 0$ at $\omega_X = 2\sqrt{2}\,\mathrm{s}^{-1}$. Therefore $\boxed{X_{\max} = F_0/12}$ in meters when F_0 is expressed in newtons.

Step 3: Maximize velocity. Divide the amplitude denominator by ω:

$$V = \frac{F_0}{\sqrt{(20/\omega - 2\omega)^2 + 16}}.$$

This is largest exactly when $20/\omega - 2\omega = 0$, giving

$$\boxed{\omega_V = \sqrt{10}\,\mathrm{s}^{-1}, \qquad V_{\max} = F_0/4}$$

in meters per second when F_0 is in newtons. The spring and inertial contributions to this denominator cancel there.

Step 4: Interpret the phase and the two maxima. At $\omega = \sqrt{10}$, $A = 0$ and $B = F_0/(4\sqrt{10}) > 0$. Thus $x = [F_0/(4\sqrt{10})]\sin(\sqrt{10}t)$, a displacement lag of $\pi/2$ behind the cosine force. Its displacement amplitude is smaller than $F_0/12$, even though its velocity amplitude is largest. The displacement peak, velocity peak and undamped natural frequency need not all coincide; the response quantity must be named.

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Question 8

A $1\,\mathrm{kg}$ mass is connected to a moving support by a spring with $k = 9\,\mathrm{N/m}$ and a parallel viscous damper with $c = 2\,\mathrm{N\,s/m}$. Let $b(t) = B\cos\omega t$ be support displacement and $x(t)$ the mass's absolute displacement, measured in the same positive direction about equilibrium. Here $B > 0$ and $\omega > 0$. Spring and damping forces depend on relative displacement and relative velocity.

Tasks

1. Derive the equations for absolute displacement x and relative displacement $z = x - b$.
2. Find the steady amplitude ratios X/B and Z/B, where X and Z are the amplitudes of x and z.
3. Determine exactly when the absolute amplitude is smaller than the support amplitude, including the equality frequency.
4. Find the low- and high-frequency limits of both ratios. Explain why good isolation of the mass does not imply small relative spring deformation.

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Question 8 – Solution

Strategy. The damper responds to velocity relative to the support. Distinguish motion of the mass from deformation of the suspension.

Step 1: Derive the two models. Newton's law gives $x'' = -9(x-b) - 2(x'-b')$. Therefore

$$x'' + 2x' + 9x = 2b' + 9b, \qquad \boxed{z'' + 2z' + 9z = -b'' = B\omega^2 \cos \omega t.}$$

The inertial term in the relative equation comes from $x'' = z'' + b''$; it is not an extra spring force.

Step 2: Compute both amplitude ratios. Let $\Delta = (9-\omega^2)^2 + 4\omega^2$. Matching sine and cosine in the relative equation gives

$$z = A\cos \omega t + D \sin \omega t, \quad A = \frac{B\omega^2(9-\omega^2)}{\Delta}, \quad D = \frac{2B\omega^3}{\Delta}.$$

Thus $Z/B = \omega^2/\sqrt{\Delta}$. Since $x = z + B\cos \omega t$, its cosine coefficient is $B + A$, not its amplitude $B + Z$. Simplifying $(B+A)^2 + D^2$ gives

$$\boxed{\frac{X}{B} = \frac{\sqrt{81+4\omega^2}}{\sqrt{\Delta}}, \qquad \frac{Z}{B} = \frac{\omega^2}{\sqrt{\Delta}}.}$$

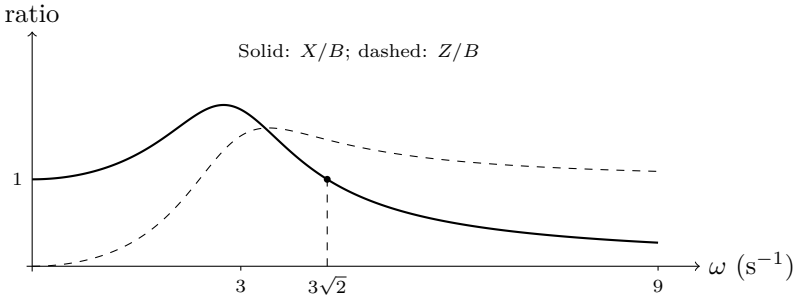
The homogeneous roots $-1 \pm 2\sqrt{2}i$ ensure the initial-data transients decay.

Step 3: Find the isolation threshold. Since $\Delta > 0$, $X/B < 1$ exactly when

$$\Delta - (81 + 4\omega^2) = \omega^2(\omega^2 - 18) > 0.$$

For $\omega > 0$, this means $\boxed{\omega > 3\sqrt{2}\,\mathrm{s}^{-1}}$. Equality holds at $3\sqrt{2}$; below that positive frequency the absolute motion is amplified.

Step 4: Interpret the limiting motions. As $\omega \to 0^+$, $X/B \to 1$ and $Z/B \to 0$: the mass follows the slowly moving support. As $\omega \to \infty$, $X/B \to 0$ but $Z/B \to 1$. In fact $A/B \to -1$ and $D/B \to 0$, so $z \approx -b$ while $x \approx 0$. An almost stationary mass can still have relative deformation comparable to the base motion.



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Question 9

A $1\,\mathrm{kg}$ mass with $c = 2\,\mathrm{N\,s/m}$ and $k = 5\,\mathrm{N/m}$ obeys $x'' + 2x' + 5x = F(t)$ in SI units. An actuator must move it from $x(0) = 0.10\,\mathrm{m}$, $x'(0) = 0$ to $x(1) = x'(1) = 0$, then turn off for $t > 1$. Before $t = 0$, it is held motionless at $x = 0.10\,\mathrm{m}$ by a constant force.

Tasks

1. Find the unique cubic trajectory satisfying the four prescribed endpoint data on $[0, 1]$.
2. Compute its required force. Examine the force and acceleration when it joins the held state at zero and the rest state at one.
3. Instead require zero acceleration at both endpoints as well. Find the unique polynomial trajectory of degree at most five satisfying all six conditions.
4. Compute the new force and verify that it joins the pre-motion holding force and the zero post-motion force continuously. Show that the mass remains at rest after one second.

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Question 9 – Solution

Strategy. Position and velocity matching ensure a continuous state, but continuous actuator force also requires compatible acceleration.

Step 1: Fit the cubic. Write $x = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. The initial conditions give $a_0 = 0.10$, $a_1 = 0$; the final conditions give $a_2 + a_3 = -0.10$, $2a_2 + 3a_3 = 0$. Hence

$$\boxed{x_c = 0.10(1 - 3t^2 + 2t^3).}$$

The nonsingular two-equation system makes this cubic unique.

Step 2: Check the actuator joins. Differentiating gives $x_c' = 0.60(t^2 - t)$ and $x_c'' = 1.20t - 0.60$. Thus

$$\boxed{F_c = x_c'' + 2x_c' + 5x_c = t^3 - 0.30t^2 - 0.10 \quad \text{N.}}$$

The holding force before zero is $5(0.10) = 0.50\,\text{N}$. Yet $F_c(0) = -0.10\,\text{N}$ and $F_c(1) = 0.60\,\text{N}$, so both joins require force jumps. Acceleration jumps from zero to -0.60 initially and from 0.60 to zero finally. The joined trajectory is C^1 and piecewise C^2, not globally C^2.

Step 3: Add acceleration matching. For a polynomial of degree at most five, the initial conditions fix $a_0 = 0.10$, $a_1 = a_2 = 0$. The endpoint conditions become

$$a_3 + a_4 + a_5 = -0.10, \quad 3a_3 + 4a_4 + 5a_5 = 0, \quad 6a_3 + 12a_4 + 20a_5 = 0.$$

Solving gives $a_3 = -1$, $a_4 = 1.5$, $a_5 = -0.6$, uniquely. Therefore

$$\boxed{x_q = 0.10(1 - 10t^3 + 15t^4 - 6t^5).}$$

Step 4: Verify force continuity and rest. Substitution into the model yields

$$\boxed{F_q = 0.50 - 6t + 12t^2 - 5t^3 + 1.5t^4 - 3t^5 \quad \text{N.}}$$

It has $F_q(0) = 0.50$ and $F_q(1) = 0$, matching the constant forces outside the motion interval. All six trajectory conditions hold, so the joined motion is C^2. For $t > 1$, zero forcing and zero value and slope at one have the unique solution $x = 0$. The improvement is continuous force and acceleration; no claim of a force-minimizing trajectory is needed.

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Question 10

A 1 kg mass with damping $c = 2\,\mathrm{N\,s/m}$ and spring constant $k = 9\,\mathrm{N/m}$ is driven by $F(t) = 6\cos\omega t\,\mathrm{N}$, with $\omega > 0$. Consider only the steady periodic response. For a periodic quantity h, define its cycle average by

$$\langle h \rangle = \frac{1}{T}\int_0^T h(t)\,dt, \qquad T = 2\pi/\omega.$$

Tasks

1. Derive the instantaneous energy balance and the relation between average input power and average damper dissipation.
2. Find the steady response coefficients and derive an explicit formula for average input power as a function of ω.
3. Find the unique positive frequency maximizing average input power and the maximum power.
4. At that frequency, find the instantaneous response, stored mechanical energy, and work supplied in one period. Explain how positive work per cycle can coexist with constant stored energy.

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Question 10 – Solution

Strategy. The work done by the actuator splits into stored energy and damping loss. Periodicity removes the net storage change over a cycle.

Step 1: Derive the power balance. For $E = x'^2/2 + 9x^2/2$, multiplying $x'' + 2x' + 9x = F$ by x' gives

$$\boxed{E' = Fx' - 2x'^2.}$$

The steady energy is periodic, so integrating over one period gives $\langle Fx' \rangle = 2\langle x'^2 \rangle$. This is an average statement for general ω.

Step 2: Compute the mean input. Write $x = A\cos\omega t + B\sin\omega t$. Coefficient matching yields

$$\Delta = (9-\omega^2)^2 + 4\omega^2, \quad A = \frac{6(9-\omega^2)}{\Delta}, \quad B = \frac{12\omega}{\Delta}.$$

Since $x' = -\omega A\sin\omega t + \omega B\cos\omega t$, averaging the products gives

$$\boxed{\overline{P} = \langle 6\cos\omega t\, x' \rangle = 3\omega B = \frac{36\omega^2}{\Delta} \quad \text{W}.}$$

Also $2\langle x'^2 \rangle = \omega^2(A^2+B^2) = 36\omega^2/\Delta$, independently checking the energy balance.

Step 3: Maximize average power. For $\omega > 0$,

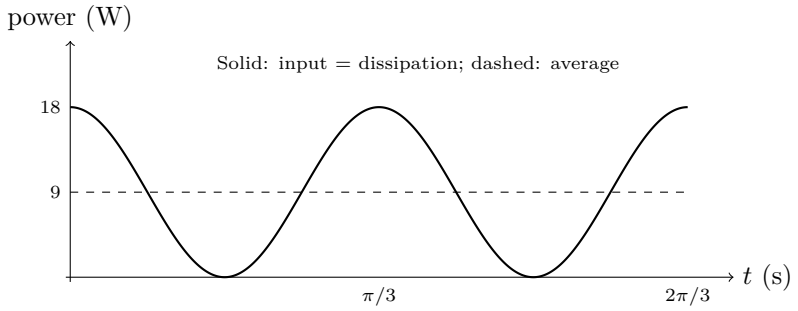
$$\overline{P} = \frac{36}{(9/\omega - \omega)^2 + 4} \le 9\,\text{W}.$$

Equality holds uniquely when $\omega^2 = 9$. Thus $\boxed{\omega = 3\,\text{s}^{-1}, \quad \overline{P}_{\max} = 9\,\text{W}}$.

Step 4: Account for the work at the maximum. At this frequency $A = 0$, $B = 1$, so $x = \sin 3t\,\text{m}$ and $x' = 3\cos 3t\,\text{m/s}$. Therefore

$$E = \tfrac{9}{2}(\cos^2 3t + \sin^2 3t) = \boxed{4.5\,\text{J}}.$$

The instantaneous input and damping powers are both $18\cos^2 3t\,\text{W}$: here $E' = 0$ at every instant, not just on average. Over $T = 2\pi/3\,\text{s}$, the supplied work is $9T = \boxed{6\pi\,\text{J}}$, exactly the energy dissipated. Constant stored energy does not mean zero energy throughput.



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Question 1

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Let $f(t) = e^{2t}(1 + t)$ for $t \geq 0$. A student obtains the expression $1/(s - 2) + 1/(s - 2)^2$ and claims that it gives a transform value at every real $s \neq 2$.

Tasks

1. Evaluate the truncated defining integral from zero to R, treating $s = 2$ separately.
2. Determine exactly which real s give convergence, and find $F(s)$ on that set.
3. Evaluate the student's expression at $s = 1$ and explain why it cannot be the Laplace integral there.
4. Prove directly from the integral, without differentiating the final formula, that F is positive and strictly decreasing throughout its convergence interval.

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Question 1 – Solution

Strategy. Keep the upper endpoint finite until the parameter-dependent boundary terms have been examined.

Step 1: Evaluate the finite integral. Write $a = s - 2$. Integration by parts gives, for $a \neq 0$,

$$I_R = \int_0^R (1+t)e^{-at}\,dt = \frac{1}{a} + \frac{1}{a^2} - e^{-aR}[\frac{1+R}{a} + \frac{1}{a^2}].$$

At $a = 0$, direct integration gives $I_R = R + R^2/2$. The formula with denominators cannot be used at that parameter.

Step 2: Determine the convergence set. For $a > 0$, both e^{-aR} and Re^{-aR} tend to zero. Therefore

$$\boxed{F(s) = \frac{1}{s-2} + \frac{1}{(s-2)^2}, \qquad s > 2.}$$

For $a = 0$, the displayed polynomial diverges. For $a < 0$, the integrand $(1+t)e^{|a|t}$ is at least one, so $I_R \geq R \to \infty$. Thus $s > 2$ is the entire real convergence set, and convergence there is absolute.

Step 3: Reject an algebraic continuation as an integral value. At $s = 1$, the rational expression equals $-1 + 1 = 0$. But the actual integral is

$$I_R = \int_0^R (1+t)e^{t}\,dt = Re^R \longrightarrow \infty.$$

The discarded boundary term grows instead of vanishing. A finite value of the algebraic expression outside the convergence interval does not assign a value to the defining improper integral.

Step 4: Prove order properties from the kernel. For $s > 2$, the integrand is positive, so $F(s) > 0$. If $s_2 > s_1 > 2$, then for every $t > 0$,

$$0 < e^{-s_2 t} f(t) < e^{-s_1 t} f(t).$$

Both integrals converge, and their difference is strictly positive on any fixed interval of positive length inside $(0, \infty)$. Hence $F(s_2) < F(s_1)$. This argument shows why positivity and monotonicity hold, independently of the rational simplification.

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Question 2

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Consider the finite-duration triangular signal

$$f(t) = \begin{cases} t, & 0 \le t \le 1, \\ 2 - t, & 1 < t \le 2, \\ 0, & t > 2. \end{cases}$$

Tasks

1. Compute $F(s)$ directly by integrating over the two nonzero pieces, first for $s \neq 0$.
2. Determine its full real convergence set and compute $F(0)$ directly. Is the apparent singularity at zero a true divergence?
3. Prove $F(-s) = e^{2s} F(s)$ by a change of variable, using the symmetry of the triangle.
4. If the value at $t = 1$ alone is changed to 100, does the transform change? Explain what this example says about recovering individual point values from an integral transform.

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Question 2 – Solution

Strategy. Finite support guarantees convergence for every real parameter. Apparent poles in a simplified expression must be checked against the original integral.

Step 1: Integrate both pieces. For $s \neq 0$, integration by parts gives

$$\int_0^1 te^{-st}\,dt = \frac{1-(1+s)e^{-s}}{s^2}, \qquad \int_1^2 (2-t)e^{-st}\,dt = \frac{(s-1)e^{-s}+e^{-2s}}{s^2}.$$

Adding yields

$$\boxed{F(s) = \frac{(1-e^{-s})^2}{s^2} \quad (s \neq 0).}$$

Step 2: Restore the missing parameter value. For any fixed real s, the integrand is bounded and piecewise continuous on $[0,2]$, and vanishes afterward. Thus the real convergence set is all of $\mathbb{R}$. At zero, the integral is the triangle's area:

$$\boxed{F(0) = 1.}$$

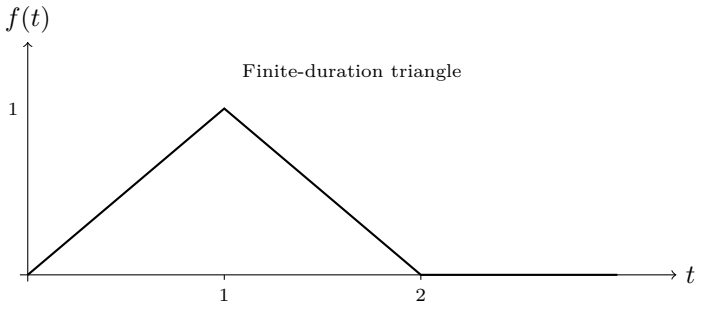
Also $(1-e^{-s})/s \to 1$ as $s \to 0$, so the simplified formula has a removable singularity, not divergent integral behavior.

Step 3: Use reflection symmetry. On $[0,2]$, $f(2-u) = f(u)$. Substituting $u = 2-t$ gives

$$F(-s) = \int_0^2 e^{st} f(t)\,dt = e^{2s}\int_0^2 e^{-su} f(2-u)\,du = e^{2s}F(s).$$

This holds at zero as well. Equivalently, $e^s F(s)$ is an even function of s, expressing symmetry about the triangle's midpoint.

Step 4: Distinguish a point from an interval. Changing one finite point value does not change any of these Riemann integrals, so the transform is unchanged. The modified function is different at one point but agrees elsewhere. Therefore the transform alone cannot recover arbitrary isolated point assignments; this observation does not contradict uniqueness results that impose continuity.



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Question 3

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin 3t$ for $t \geq 0$. Its truncated transform is $F_R(s) = \int_0^R e^{-st} \sin 3t\,dt$.

Tasks

1. Evaluate $F_R(s)$ for real s, retaining its upper-endpoint contribution.
2. Find the exact real convergence interval for $F(s)$ and determine whether convergence there is absolute.
3. At $s = 0$, exhibit two sequences of upper endpoints giving different values of $F_R(0)$.
4. Compare $\lim_{s\to 0^+} F(s)$ with the existence of $F(0)$. Why does a finite parameter limit not justify interchanging the two limits $R \to \infty$ and $s \to 0^+$?

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Question 3 – Solution

Strategy. Damping by e^{-st} can create convergence that disappears at the boundary parameter.

Step 1: Keep the endpoint term. Two integrations by parts, or differentiation of the resulting antiderivative, give

$$\boxed{F_R(s) = \frac{3 - e^{-sR}[s\sin 3R + 3\cos 3R]}{s^2+9}.}$$

There is no real zero of the denominator, but that alone says nothing about the limit as $R \to \infty$.

Step 2: Classify convergence. If $s > 0$, the endpoint term vanishes and

$$\boxed{F(s) = \frac{3}{s^2+9}, \qquad s > 0.}$$

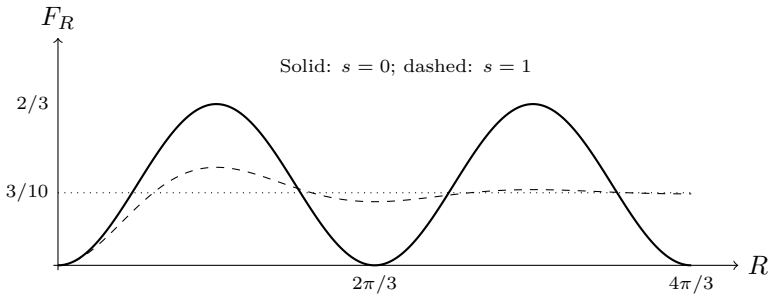
Absolute convergence follows from $|e^{-st}\sin 3t| \le e^{-st}$. If $s < 0$, take $R_n = 2n\pi/3$. Then $F_{R_n}(s) = 3(1 - e^{|s|R_n})/(s^2+9) \to -\infty$, so the improper integral cannot converge.

Step 3: Test the boundary directly. At $s = 0$, $F_R(0) = (1 - \cos 3R)/3$. For

$$R_n = 2n\pi/3, \quad F_{R_n}(0) = 0; \qquad Q_n = (2n+1)\pi/3, \quad F_{Q_n}(0) = 2/3.$$

The two sequences tend to infinity but have different integral values. Thus $F(0)$ does not exist, and the full real convergence interval is $(0, \infty)$.

Step 4: Separate the two limiting procedures. The damped transforms satisfy $\lim_{s\to 0^+} F(s) = 1/3$. In contrast, setting $s = 0$ first produces an oscillating truncated integral with no limit. For fixed R, the limit in s exists, but convergence in R is not uniform as damping disappears. The value $1/3$ is a limit of damped integrals, not the value of the undamped improper integral. The graph compares accumulated integrals, not the original sine signal.



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Question 4

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

For $t > 0$, let $f(t) = t^{-1/2}e^{-t}$; assign any finite value to $f(0)$. For the defining integral, treat zero as an improper endpoint as well as infinity. You may use $\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}/2$.

Tasks

1. Analyze convergence near zero and near infinity separately, and find the exact real convergence set.
2. Evaluate $F(s)$ on that set by a substitution.
3. Repeat the endpoint analysis for $h(t) = t^{-1}e^{-t}$, $t > 0$. Does exponential decay guarantee a transform for this function?
4. Explain why the first example does not contradict a theorem giving piecewise continuity on each finite interval and exponential order as sufficient conditions for existence.

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Question 4 – Solution

Strategy. An improper transform can fail at either endpoint. Rapid decay at infinity cannot repair a nonintegrable singularity at zero.

Step 1: Separate the endpoints. Write $a = s + 1$. Near zero, e^{-at} is bounded above and below by positive constants, and $\int_0^1 t^{-1/2}\,dt = 2$, so this endpoint is integrable for every fixed real s. At infinity, $a > 0$ gives exponential decay. If $a = 0$, the integral of $t^{-1/2}$ diverges; if $a < 0$, the integrand eventually exceeds one. Hence convergence occurs exactly for $s > -1$, and is absolute.

Step 2: Evaluate the convergent integral. For $a > 0$, use $u = \sqrt{at}$, so $t = u^2/a$ and $dt = 2u\,du/a$. Then

$$F(s) = \frac{2}{\sqrt{a}} \int_0^\infty e^{-u^2}\,du = \boxed{\frac{\sqrt{\pi}}{\sqrt{s+1}}, \qquad s > -1.}$$

Both the substitution and the square root require the already established condition $a > 0$. An assigned value at the single point zero does not change the improper integral.

Step 3: Test the stronger singularity. For h, the integrand is $t^{-1}e^{-(s+1)t}$. On a sufficiently small interval $(0, \delta)$, its exponential factor is at least $1/2$. Thus

$$\int_0^\delta t^{-1}e^{-(s+1)t}\,dt \geq \tfrac{1}{2} \int_0^\delta \frac{dt}{t} = \infty.$$

This holds for every real s. Consequently h has no transform under this definition, even for parameters that give fast decay at infinity.

Step 4: Interpret the sufficient theorem. The first function is unbounded as $t \to 0^+$, so it does not satisfy the usual finite-limit requirement for piecewise continuity on $[0, A]$. Yet its endpoint singularity is integrable, and the direct calculation establishes existence for $s > -1$. A sufficient theorem guarantees existence when its hypotheses hold; failure of a hypothesis does not prove nonexistence. Here direct endpoint analysis gives the sharper conclusion.

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Question 5

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin t/t$ for $t > 0$, with $f(0) = 1$. In this problem examine parameters $s \geq 0$. You may derive the positive-parameter transform using $\sin t/t = \int_0^1 \cos(ut)\,du$.

Tasks

1. For $s > 0$, justify interchanging the two integrations and evaluate $F(s)$.
2. For $s \geq 0$ and $R > 0$, prove the tail bound $|\int_R^\infty e^{-st} \sin t/t\,dt| \leq 2/R$, including existence of the tail at $s = 0$.
3. Use that uniform tail bound and finite-interval convergence as $s \to 0^+$ to determine $F(0)$.
4. Determine whether the integral at $s = 0$ converges absolutely, and contrast it with the positive-parameter integrals.

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Question 5 – Solution

Strategy. A uniform tail estimate can justify passage to the boundary parameter; absolute and conditional convergence must still be distinguished.

Step 1: Evaluate for positive damping. For $s > 0$, the double integral of the absolute integrand is at most $\int_0^\infty e^{-st}\,dt = 1/s$, so interchange is justified. Integration by parts gives $\int_0^\infty e^{-st}\cos(ut)\,dt = s/(s^2+u^2)$. Therefore

$$F(s) = \int_0^1 \frac{s}{s^2+u^2}\,du = \boxed{\arctan(1/s), \qquad s > 0.}$$

Step 2: Control the tail uniformly. Set $w(t) = e^{-st}/t$ for $t \geq R$. For every $s \geq 0$, w decreases to zero and $w' \leq 0$. Integration by parts on $[R, L]$ gives

$$\int_R^L w(t)\sin t\,dt = [-w(t)\cos t]_R^L + \int_R^L w'(t)\cos t\,dt.$$

The integral of $|w'|$ is $w(R) - w(L)$, so the right side has magnitude at most $2w(R) \leq 2/R$. Applying the same bound with any larger lower endpoint proves the Cauchy criterion as $L \to \infty$, including at $s = 0$. The bound persists for the improper tail.

Step 3: Pass to the boundary parameter. On each fixed $[0, R]$, $e^{-st}f(t) \to f(t)$ uniformly as $s \to 0^+$, since f is continuous and bounded there. The tails at both s and zero have magnitude at most $2/R$. Thus

$$|F(s) - F(0)| \leq \left|\int_0^R (e^{-st} - 1)f(t)\,dt\right| + 4/R.$$

First choose large R, then let $s \to 0^+$. This proves continuity at the boundary and hence $\boxed{F(0) = \pi/2}$.

Step 4: Test absolute convergence. On $[n\pi + \pi/6, n\pi + 5\pi/6]$, $|\sin t| \geq 1/2$ and $t \leq (n+1)\pi$. Each such interval contributes at least $1/[3(n+1)]$ to $\int |\sin t|/t\,dt$. The harmonic series diverges, so convergence at zero is conditional. For $s > 0$, $|\sin t/t| \leq 1$ gives absolute convergence by comparison with e^{-st}. The boundary value here exists as an ordinary improper integral, not merely as a limit of damped values.

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Question 6

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Compare $f_+(t) = e^{t^2}$ and $f_-(t) = e^{-t^2}$ for $t \geq 0$, and write their transforms as F_+ and F_-.

Tasks

1. Determine the real convergence set of each transform directly from the defining integral.
2. Complete the square to express $F_-(s)$ through a Gaussian tail for every real s. No special-function notation is required.
3. For a truncation at T with $T + s/2 > 0$, prove an explicit upper bound for the omitted tail of $F_-(s)$.
4. At $s = -2$, find the smallest integer $T > 1$ for which your bound certifies an error at most 10^{-4}. Explain why negative s does not destroy convergence for f_-.

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Question 6 – Solution

Strategy. Compare the quadratic growth or decay with the linear term supplied by the Laplace weight, and use the completed square to bound numerical truncation.

Step 1: Classify both integrals. For any real s, $t^2 - st \geq 0$ when $t \geq \max\{s, 0\}$, so $e^{t^2-st} \geq 1$ on a half-line. Hence F_+ diverges for every real s. For F_-,

$$-t^2 - st = -(t + s/2)^2 + s^2/4.$$

A shifted Gaussian has an integrable tail and no finite-endpoint singularity, so F_- converges absolutely for every real s.

Step 2: Express the exact transform. Substitution $u = t + s/2$ gives

$$\boxed{F_-(s) = e^{s^2/4} \int_{s/2}^{\infty} e^{-u^2}\, du, \qquad s \in \mathbb{R}.}$$

The lower endpoint is allowed to be negative; the finite portion of the integral then remains harmless. This definite integral is an exact answer.

Step 3: Bound the omitted tail. Let $a = T + s/2 > 0$. Since $1 \leq u/a$ for $u \geq a$,

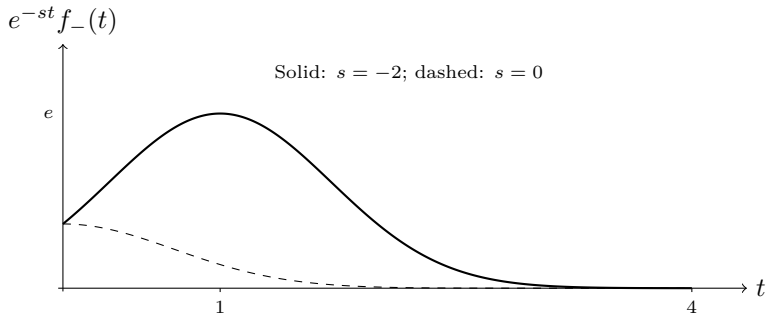
$$\int_a^{\infty} e^{-u^2}\, du \leq \frac{1}{a} \int_a^{\infty} u e^{-u^2}\, du = \frac{e^{-a^2}}{2a}.$$

Consequently the positive truncation error satisfies

$$\boxed{\int_T^{\infty} e^{-t^2-st}\, dt \leq \frac{e^{-T^2-sT}}{2T + s}.}$$

The condition $2T + s > 0$ is part of the bound and cannot be dropped.

Step 4: Certify a cutoff. For $s = -2$, the bound is $e^{-T^2+2T}/(2T - 2)$, decreasing for $T > 1$. At $T = 3$ it is $e^{-3}/4 \approx 0.01245$, while at $T = 4$ it is $e^{-8}/6 \approx 5.59 \times 10^{-5}$. Thus the smallest integer cutoff certified by this bound is $\boxed{T = 4}$. Negative s makes the weight grow exponentially, but the quadratic decay still dominates. The graph shows the weighted integrands, whose areas give the transforms.



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Question 7

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

For each integer $n \geq 1$, let $w_n = e^{-2n^2}$. Define a nonnegative signal by

$$f(t) = \begin{cases} e^{n^2}, & n \leq t \leq n + w_n \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

A function is of exponential order if $|f(t)| \leq Me^{at}$ for all sufficiently large t, for some constants $M > 0$ and real a.

Tasks

1. Explain why f is piecewise continuous on every finite interval but is not of exponential order.
2. Prove that its Laplace integral nevertheless converges absolutely for every real s.
3. Express $F(s)$ as a convergent series, treating $s = 0$ separately.
4. Explain how the spike widths resolve the apparent conflict between arbitrarily large heights and convergence, and why the usual exponential-order theorem is not contradicted.

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Question 7 – Solution

Strategy. The integral depends on weighted areas, not just pointwise heights. Quantify the widths before invoking a growth criterion.

Step 1: Examine local regularity and height. Since $w_n < 1$, the spike intervals are disjoint. Any bounded time interval meets only finitely many spikes, each with finite height and finite one-sided limits. Thus f is piecewise continuous locally. If an exponential-order bound held, evaluating at $t = n$ for large integers would give $e^{n^2} \le Me^{an}$, or $n^2 - an \le \ln M$. The left side tends to infinity, a contradiction.

Step 2: Estimate each weighted area. Fix real s. On the nth spike, $t \le n + 1$ and $e^{-st} \le e^{|s|(n+1)}$. Its contribution is therefore at most

$$e^{n^2} w_n e^{|s|(n+1)} = e^{-n^2+|s|(n+1)}.$$

For sufficiently large n, $|s|(n+1) \le n^2/2$, so these terms are at most $e^{-n^2/2} \le e^{-n/2}$. The comparison series converges. Since the integrand is nonnegative, the bound proves absolute convergence for every real s.

Step 3: Sum the exact contributions. Split finite integrals into their finitely many spike intervals, then pass to the limit using the convergent positive series. For $s \neq 0$,

$$\boxed{F(s) = \sum_{n=1}^{\infty} e^{n^2-sn} \frac{1 - e^{-sw_n}}{s}.}$$

The quotient is positive even for negative s. At $s = 0$, each contribution is simply height times width, giving

$$\boxed{F(0) = \sum_{n=1}^{\infty} e^{-n^2}.}$$

No closed form for this convergent series is required.

Step 4: Interpret the theorem correctly. The heights grow faster than every exponential, but the widths shrink fast enough that their areas are e^{-n^2}. Any fixed exponential weight still leaves a summable sequence of areas. The common existence theorem provides sufficient conditions; exponential order is not necessary for an individual function to have a Laplace transform. This example establishes that distinction by direct estimates, rather than by trying to apply a theorem whose hypothesis fails.

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Question 8

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

An unknown signal belongs to the family $f(t) = 0$ for $0 \le t < a$ and $f(t) = A$ for $t \ge a$, where $A > 0$ and $a \ge 0$. Exact measurements are

$$F(1) = 2/e, \qquad F(2) = e^{-2}.$$

Tasks

1. Derive the transform and its real convergence set directly, then recover A and a from the measurements.
2. For arbitrary positive measurements $M_1 = F(1)$ and $M_2 = F(2)$, give a necessary and sufficient consistency condition for this family and explicit recovery formulas.
3. For $h(t) = e^{-t}(t^2 - 5t/3 + 1/3)$, compute its transform at $s > 0$ directly by integration by parts and show that it vanishes at both measured parameters.
4. Use h to explain precisely why the two measurements identify the signal within the stated family but do not identify an arbitrary real-valued signal.

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Question 8 – Solution

Strategy. Distinguish uniqueness of a few model parameters from uniqueness among all possible functions.

Step 1: Integrate and calibrate. For $s > 0$,

$$F(s) = A\int_a^\infty e^{-st}\,dt = \frac{Ae^{-as}}{s}.$$

For $s \le 0$, the positive tail integral diverges. Dividing the two measured equations gives $2F(2)/F(1) = e^{-a} = e^{-1}$, so $\boxed{a = 1,\ A = 2}$. These values reproduce both measurements.

Step 2: Characterize all consistent data. For positive M_1, M_2, the family requires $2M_2/M_1 = e^{-a} \in (0, 1]$. Thus consistency is exactly $\boxed{2M_2 \le M_1}$. In that case the unique parameters are

$$\boxed{a = \ln\frac{M_1}{2M_2}, \qquad A = \frac{M_1^2}{2M_2}.}$$

They satisfy $a \ge 0$, $A > 0$ and both equations, proving sufficiency as well as necessity. Equality corresponds to onset at zero.

Step 3: Construct an invisible perturbation. Set $b = s + 1 > 0$. Integration by parts gives the three integrals of $1, t, t^2$ against e^{-bt} as $1/b, 1/b^2, 2/b^3$. Therefore

$$H(s) = \frac{2}{(s+1)^3} - \frac{5}{3(s+1)^2} + \frac{1}{3(s+1)} = \boxed{\frac{(s-1)(s-2)}{3(s+1)^3}}.$$

The exponential boundary terms vanish, justifying those integrations. In particular, $H(1) = H(2) = 0$, although h is not the zero function.

Step 4: State the identification limit. For every real λ, the signal $f + \lambda h$ has a convergent transform for $s > 0$. Splitting its defining integral gives $F(s) + \lambda H(s)$, so both recorded measurements are unchanged. Distinct λ give distinct real-valued signals; for nonzero λ they are not members of the original two-parameter family. The recovery formulas establish uniqueness only under that family assumption, not among arbitrary signals.

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Question 9

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

Let f have period two, with $f(t) = 1$ for $0 \le t < 1$ and $f(t) = 0$ for $1 \le t < 2$, repeated for all $t \ge 0$. Use the definition rather than a precomputed periodic-transform formula.

Tasks

1. Split the integral into complete periods, sum the resulting series, and determine the exact real convergence set.
2. Simplify the transform on its convergence interval.
3. Compute $\lim_{s\to 0^+} sF(s)$ and $\lim_{s\to\infty} sF(s)$. Relate these to the period average and to the signal just after zero.
4. If the defining integral is truncated after N complete periods, find its exact relative error for $s > 0$. At $s = 0.2$, find the smallest integer N making that error at most 10^{-3}.

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Question 9 – Solution

Strategy. A time translation by one period multiplies the weighted area by a fixed factor, producing a geometric series.

Step 1: Build the series and its domain. For $s > 0$, the nth nonzero interval contributes

$$\int_{2n}^{2n+1} e^{-st}\,dt = e^{-2ns}\frac{1-e^{-s}}{s}.$$

The ratio e^{-2s} lies in $(0,1)$, so the sum converges. At $s = 0$, each interval contributes one and the sum diverges. For $s < 0$, $e^{-st} \geq 1$ on each such interval, so it also diverges. The real convergence set is exactly $(0,\infty)$, with absolute convergence since $f \geq 0$.

Step 2: Simplify the geometric sum. Summing gives

$$F(s) = \frac{1-e^{-s}}{s(1-e^{-2s})} = \boxed{\frac{1}{s(1+e^{-s})}, \qquad s > 0.}$$

The integral over only one period is the first term of this sum, not the transform of the endlessly repeated signal.

Step 3: Interpret the two normalized limits. The formula gives

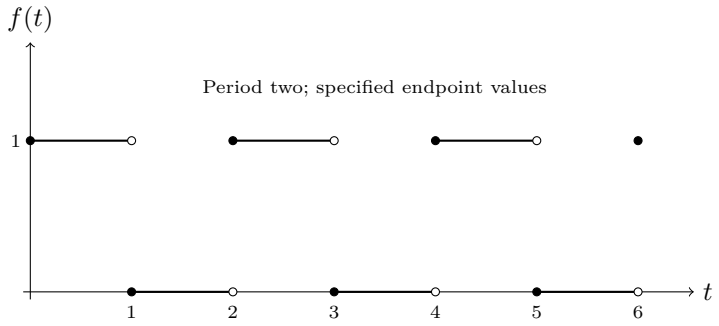
$$\boxed{\lim_{s\to 0^+} sF(s) = \tfrac{1}{2}, \qquad \lim_{s\to\infty} sF(s) = 1.}$$

The first value is the period average $(1/2)\int_0^2 f(t)\,dt = 1/2$. The second agrees with $f(0^+) = 1$: large s weights short times most strongly. These normalized limits do not assert that $F(0)$ exists; it does not.

Step 4: Quantify finite-window error. Writing $F_N = \int_0^{2N} e^{-st}f(t)\,dt$, the finite geometric sum gives

$$F_N = F(s)(1-e^{-2Ns}), \qquad \boxed{\frac{F(s)-F_N}{F(s)} = e^{-2Ns}.}$$

At $s = 0.2$, the requirement is $e^{-0.4N} \leq 10^{-3}$, so $N \geq \ln(1000)/0.4 \approx 17.27$. The least integer is $\boxed{N = 18}$, corresponding to a cutoff time of 36. Endpoint assignments do not affect the integrals; the graph uses the values specified in the question.



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Question 10

For real s, use $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ whenever this ordinary improper integral converges.

A continuous signal on $[0, \infty)$ satisfies $0 \le f(t) \le 1$, and its observed part is $f(t) = t$ on $[0, 1]$. Its future after one is unknown. Consider $s > 0$.

Tasks

1. Derive explicit lower and upper bounds for $F(s)$ using only this information.
2. Decide which bound is attained within the continuous signal class. Construct admissible signals approaching any unattained bound.
3. State the exact resulting range of possible values of $F(1)$, justifying that every value in that range is possible.
4. Find the width of the uncertainty interval for general $s > 0$, prove it decreases with s, and find the smallest positive integer s making the width at most 0.01.

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Question 10 – Solution

Strategy. Positivity of the Laplace weight gives bounds; continuity at the last observed point decides whether an extremum is actually attained.

Step 1: Bound the unknown tail. The observed contribution is

$$L(s) = \int_0^1 te^{-st}\,dt = \frac{1-(1+s)e^{-s}}{s^2}.$$

The remaining integral lies between zero and $\int_1^\infty e^{-st}\,dt = e^{-s}/s$. Therefore

$$L(s) \le F(s) \le U(s), \qquad \boxed{U(s) = \frac{1-e^{-s}}{s^2}.}$$

All integrals converge absolutely because $0 \le f \le 1$ and $s > 0$.

Step 2: Resolve attainment. The upper bound is attained by $f(t) = \min\{t, 1\}$. The lower bound is not attained: continuity and $f(1) = 1$ force a strictly positive contribution on some interval immediately after one. For $\varepsilon > 0$, instead continue linearly from $(1, 1)$ to $(1+\varepsilon, 0)$ and then stay zero. This continuous f_ε satisfies

$$0 < F_\varepsilon(s) - L(s) \le \varepsilon e^{-s} \longrightarrow 0.$$

Thus $L(s)$ is the infimum, while $U(s)$ is a maximum.

Step 3: Give the full range at one. Substitution gives the interval

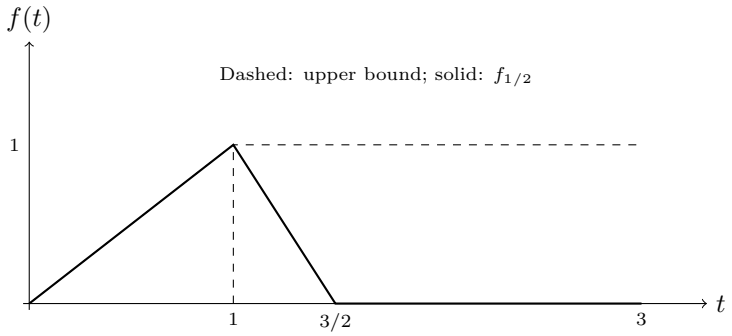
$$\boxed{1 - 2/e < F(1) \le 1 - 1/e.}$$

For any target strictly above the lower endpoint, choose ε small enough that $F_\varepsilon(1)$ is below it. A convex combination of f_ε and the upper-bound signal remains continuous, in $[0, 1]$, and agrees with the observed ramp. Splitting the defining integral shows its transform is the same convex combination. Choosing the weight appropriately reaches the target, proving no intermediate values are missing.

Step 4: Measure the uncertainty. The interval width is $W(s) = U(s) - L(s) = e^{-s}/s$, with

$$W'(s) = -e^{-s}(s+1)/s^2 < 0.$$

Since $W(3) = e^{-3}/3 \approx 0.01660$ and $W(4) = e^{-4}/4 \approx 0.00458$, the least positive integer meeting the tolerance is $\boxed{s = 4}$. Increasing s reduces sensitivity to the unknown late-time portion; it does not supply missing pointwise information about that future.



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Question 1

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Let $f(t) = \sin^2(3t)$. A student proposes

$$\mathcal{L}\{\sin^2(3t)\}(s) = [\mathcal{L}\{\sin 3t\}(s)]^2.$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$ and $\mathcal{L}\{\sin bt\} = b/(s^2 + b^2)$ for $s > 0$.

Tasks

1. Find the correct transform using a trigonometric identity and linearity.
2. Determine the full real convergence set and compare the correct result with the student's proposed expression.
3. Use $|\sin x| \le |x|$ to prove a direct upper bound for the transform when $s > 0$.
4. Use $u = st$ in the defining integral to derive $\lim_{s\to\infty} s^3 F(s)$ independently of the rational formula. Explain how this limit diagnoses the proposed product rule.

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Question 1 – Solution

Strategy. Linearity applies to sums and scalar multiples, not to pointwise products. A large-parameter check can expose the distinction.

Step 1: Rewrite the function before transforming. Since $\sin^2(3t) = (1 - \cos 6t)/2$, linearity gives

$$F(s) = \frac{1}{2s} - \frac{s}{2(s^2+36)} = \boxed{\frac{18}{s(s^2+36)}, \qquad s > 0.}$$

Step 2: Check convergence and the proposed rule. For $s > 0$, boundedness of f gives absolute convergence. For $s \le 0$, the weight is at least one and $\sin^2(3t)$ has positive integral on each period, so the nonnegative improper integral diverges. Thus the real convergence set is exactly $(0, \infty)$. The proposed expression is

$$\frac{9}{(s^2+9)^2},$$

which is not the correct transform. For example, at $s = 3$ the correct value is $2/15$, while the proposed value is $1/36$. Squaring a time function is not transformed by squaring its transform.

Step 3: Obtain a bound from the integral. The inequality $0 \le \sin^2(3t) \le 9t^2$ yields

$$0 < F(s) \le 9\int_0^\infty t^2 e^{-st}\,dt = \boxed{18/s^3}.$$

The last integral is $2/s^3$, by two integrations by parts. Its boundary terms vanish for $s > 0$.

Step 4: Recover the leading scale independently. With $u = st$,

$$s^3 F(s) = \int_0^\infty e^{-u} s^2 \sin^2(3u/s)\,du.$$

The integrand tends to $9u^2e^{-u}$ and is bounded by that integrable function. Dominated convergence therefore gives $\boxed{s^3F(s) \to 18}$. This reflects the local behavior $\sin^2(3t) \sim 9t^2$. In contrast, multiplying the proposed expression by s^3 gives a limit of zero. The proposed rule even predicts the wrong leading power of $1/s$.

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Question 2

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Find the transform of $f(t) = t^2 e^{-2t}\cos 3t$ by combining exponential shifting with differentiation in the transform parameter. You may use $C(p) = \mathcal{L}\{\cos 3t\}(p) = p/(p^2+9)$ for $p > 0$.

Tasks

1. Justify the identity $\mathcal{L}\{t^2 g(t)\}(s) = G''(s)$ in this example, including the convergence condition needed for differentiation.
2. Compute and simplify the transform explicitly.
3. Find all its zeros and determine its sign within the convergence interval. Explain why a zero transform value need not mean that the original function is zero.
4. The simplified expression also has value zero at $s = -2$. Does that value represent an ordinary Laplace integral? Check the boundary directly and determine whether any smaller real parameter can work.

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Question 2 – Solution

Strategy. Account for the exponential shift first, then differentiate the correct parameter expression and retain its convergence domain.

Step 1: Justify parameter differentiation. The factor e^{-2t} changes the effective parameter to $p = s+2$. For any $p_0 > 0$, a sufficiently small neighborhood has $p \geq p_0/2$. Differentiating the kernel twice produces $t^2 e^{-pt}\cos 3t$, dominated by $t^2 e^{-p_0 t/2}$, an integrable function. The first derivative is justified similarly. Thus, for $s > -2$, the desired transform is $C''(s+2)$.

Step 2: Differentiate and simplify. Direct calculation gives

$$C'(p) = \frac{9-p^2}{(p^2+9)^2}, \qquad C''(p) = \frac{2p(p^2-27)}{(p^2+9)^3}.$$

Therefore

$$\boxed{F(s) = \frac{2(s+2)[(s+2)^2-27]}{[(s+2)^2+9]^3}, \qquad s > -2.}$$

The exponential times a polynomial gives absolute convergence on this interval.

Step 3: Interpret its zero and sign. For $p > 0$, the denominator and $2p$ are positive. Hence the only zero in the convergence interval is $\boxed{s = 3\sqrt{3} - 2}$. The transform is negative below that value and positive above it. At the zero, weighted positive and negative portions of $t^2 e^{-2t}\cos 3t$ cancel; the function itself is certainly not identically zero.

Step 4: Test the boundary rather than the rational expression. At $s = -2$, the integral becomes $\int_0^R t^2 \cos 3t\, dt$. Integration by parts gives

$$\frac{R^2 \sin 3R}{3} + \frac{2R\cos 3R}{9} - \frac{2\sin 3R}{27}.$$

At $R_n = 2n\pi/3$, this equals $2R_n/9 \to \infty$. For $p = s+2 < 0$, consider $[R_n, R_n + \pi/9]$, where $\cos 3t \geq 1/2$. The integral on that interval is at least $(\pi/18)R_n^2 e^{-pR_n}$ and does not tend to zero, violating the Cauchy criterion. Thus no real $s \leq -2$ works; the algebraic boundary value is not a transform value.

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Question 3

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Let $g(t) = e^{-t}\sin 2t$, with $G(s) = 2/[(s+1)^2 + 4]$ for $s > -1$. Define the compressed signals $f(t) = g(3t)$ and $h(t) = 3g(3t)$.

Tasks

1. Derive the time-scaling identity for $g(at)$, $a > 0$, by substitution in the defining integral. Track the convergence parameter.
2. Find the transforms of f and h, and their full real convergence intervals. Explain why replacing $G(s)$ by $G(3s)$ gives the wrong scaling.
3. Compute the signed total integrals of g, f, h. Which compression preserves the total integral?
4. Use parameter differentiation to derive the first-moment identity. Compute $\int_0^\infty tf(t)\,dt$ and $\int_0^\infty th(t)\,dt$. Explain their different scaling factors.

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Question 3 – Solution

Strategy. Time compression changes both the exponential weight and the integration measure; amplitude compensation affects area separately.

Step 1: Derive the scaling rule. For $a > 0$, substituting $u = at$ gives

$$\int_0^\infty e^{-st} g(at)\,dt = \frac{1}{a}\int_0^\infty e^{-(s/a)u} g(u)\,du = \boxed{\frac{1}{a}G(s/a)}.$$

The effective parameter is s/a, so $s/a > -1$ here. Both the reciprocal parameter scaling and the factor $1/a$ are essential.

Step 2: Apply the rule to both signals. For $a = 3$,

$$\boxed{F(s) = \frac{6}{(s+3)^2+36}, \qquad H(s) = \frac{18}{(s+3)^2+36}, \qquad s > -3.}$$

These are the transforms of $e^{-3t}\sin 6t$ and three times that signal. At the boundary the undamped sine integral fails to converge; below it the exponentially growing sine fails the Cauchy criterion on positive half-wave subintervals. Thus $s > -3$ is exact. $G(3s)$ would stretch the parameter in the opposite direction and omit the measure factor.

Step 3: Compare signed areas. Since zero lies in the convergence intervals, substitution of $s = 0$ gives

$$\int_0^\infty g = \frac{2}{5}, \qquad \int_0^\infty f = \frac{2}{15}, \qquad \int_0^\infty h = \frac{2}{5}.$$

Compression alone reduces signed area by three; multiplying the amplitude by three restores it. These are signed integrals, not integrals of absolute value.

Step 4: Track first moments. Exponential decay justifies differentiation near $s = 0$, so $G'(s) = -\int_0^\infty te^{-st}g(t)\,dt$. As $G'(0) = -4/25$, the original first moment is $4/25$. Differentiating $F(s) = G(s/3)/3$ and $H(s) = G(s/3)$ yields

$$\boxed{\int_0^\infty tf(t)\,dt = \frac{4}{225}, \qquad \int_0^\infty th(t)\,dt = \frac{4}{75}.}$$

For f, one factor $1/3$ comes from dt and another from $t = u/3$. The amplitude factor in h cancels only one of them.

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Question 4

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Let $g(t) = e^{-t}\cos t$ and $y(t) = \int_0^t g(u)\,du$. The transform of g is $G(s) = (s+1)/[(s+1)^2+1]$ for $s > -1$.

Tasks

1. Derive the transform rule for y by integration by parts, showing where the zero initial value and the boundary at infinity enter.
2. Find an elementary expression for $y(t)$, its transform $Y(s)$, and the exact real convergence interval of Y.
3. Set $z(t) = y(t) - 1/2$. Find its transform and convergence interval, including its value at $s = 0$. Explain why cancellation can enlarge the interval.
4. Find the first maximum of y and its height. Does integration of this exponentially decaying input produce an exponentially decaying or monotone cumulative response?

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Question 4 – Solution

Strategy. A cumulative integral may retain a nonzero constant even when its input decays. Its transform domain must be checked anew.

Step 1: Derive the cumulative rule. Here $y(0) = 0$, $y' = g$, and y is bounded because g is absolutely integrable. For $s > 0$, integration by parts gives

$$G(s) = [e^{-st}y(t)]_0^\infty + s\int_0^\infty e^{-st}y(t)\,dt = sY(s).$$

The initial term is zero and the upper term vanishes because y is bounded. Thus $Y = G/s$ on this half-line.

Step 2: Determine the response and its domain. An antiderivative gives

$$y(t) = \frac{1 + e^{-t}(\sin t - \cos t)}{2}, \qquad \boxed{Y(s) = \frac{s+1}{s[(s+1)^2+1]}, \quad s > 0.}$$

The formula satisfies $y(0) = 0$ and $y' = g$. Since $y(t) \to 1/2$, it is eventually bounded below by a positive constant. Its transform therefore diverges for every $s \le 0$. The cumulative response has a smaller convergence interval than the input.

Step 3: Remove the persistent constant. Now $z = e^{-t}(\sin t - \cos t)/2$. Transforming these two terms directly gives

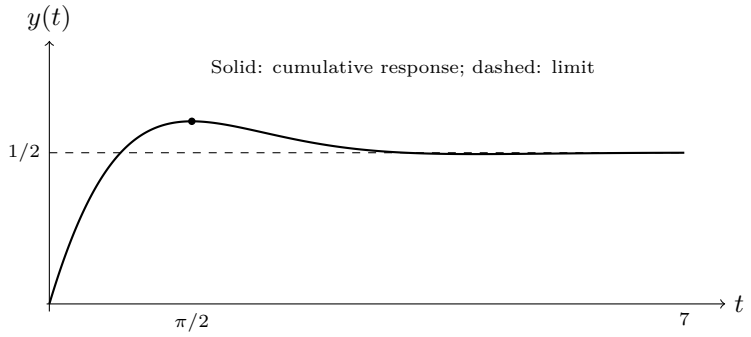
$$\boxed{Z(s) = -\frac{s}{2[(s+1)^2+1]}, \qquad s > -1.}$$

It converges absolutely there; at and below -1 the nondecaying or growing trigonometric tail prevents ordinary convergence. In particular, $Z(0) = 0$: its signed areas cancel. The identity $Z = Y - 1/(2s)$ is initially valid only for $s > 0$. The wider domain follows from the actual centered function, not by subtracting divergent integrals at $s = 0$.

Step 4: Locate the overshoot. Since $y' = e^{-t}\cos t$, the first positive turning time is $t = \pi/2$, with derivative changing from positive to negative. Its height is

$$\boxed{y(\pi/2) = \tfrac{1}{2}(1 + e^{-\pi/2}) > \tfrac{1}{2}.}$$

The response overshoots and oscillates about its limiting constant, so it is neither monotone nor decaying to zero. Its centered part does decay exponentially.



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Question 5

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Let $f(t) = e^{-t}(1+t)$. A student claims that the second derivative has transform $s^2F(s)$, without any initial terms.

Tasks

1. Derive the first- and second-derivative transform formulas using integration by parts on a finite interval before taking the limit.
2. Compute $F(s)$, $f'(t)$ and $f''(t)$, and obtain their transforms explicitly.
3. Identify the omitted term in the student's claim and verify the corrected result independently from the expression for f''.
4. Evaluate the transforms of f' and f'' at $s = 0$ and check them using endpoint differences. Explain how a nonzero second derivative can have zero total signed integral.

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Question 5 – Solution

Strategy. Differentiation in time creates boundary terms. Keep them until the initial data and exponential decay have been used.

Step 1: Retain the boundaries. Integration by parts on $[0, R]$ gives

$$\int_0^R e^{-st} f'(t)\,dt = e^{-sR} f(R) - f(0) + s\int_0^R e^{-st} f(t)\,dt.$$

Repeating for f'' and then taking $R \to \infty$ yields

$$\mathcal{L}\{f'\} = sF - f(0), \qquad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0),$$

provided the upper boundary terms vanish. Here they do for every $s > -1$, since f, f' are polynomials times e^{-t}.

Step 2: Compute the functions and transforms. Using elementary exponential integrals,

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} = \frac{s+2}{(s+1)^2}.$$

Direct differentiation gives $f' = -te^{-t}$ and $f'' = (t-1)e^{-t}$, with $f(0) = 1$ and $f'(0) = 0$. Hence

$$\boxed{\mathcal{L}\{f'\} = -\frac{1}{(s+1)^2}, \qquad \mathcal{L}\{f''\} = -\frac{s}{(s+1)^2}, \quad s > -1.}$$

These intervals are exact: at or below -1, the resulting polynomial or growing exponential tails do not have convergent improper integrals.

Step 3: Repair and check the claim. The omitted initial contribution is $-sf(0) - f'(0) = -s$. Thus the corrected expression is $s^2F - s = -s/(s+1)^2$. Independently, transforming $(t-1)e^{-t}$ gives

$$\frac{1}{(s+1)^2} - \frac{1}{s+1} = -\frac{s}{(s+1)^2}.$$

This agreement verifies both the boundary terms and the algebra.

Step 4: Check zero-parameter integrals. At zero, the first-derivative transform equals -1, consistent with $\int_0^\infty f' = f(\infty) - f(0) = 0 - 1$. The second-derivative transform is zero, consistent with $\int_0^\infty f'' = f'(\infty) - f'(0) = 0$. The function f'' is negative before $t = 1$ and positive afterward; its two signed contributions cancel. Zero integral is not the same as a zero function.

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Question 6

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

For $t > 0$, define

$$h(t) = \frac{e^{-t} - e^{-4t}}{t},$$

and extend it continuously to $t = 0$. This example concerns division by time; do not assume a transform rule for dividing an arbitrary function by t.

Tasks

1. Find $h(0)$ and express $h(t)$ as an integral of e^{-ut} over a finite interval in u.
2. Use that representation to derive the transform $H(s)$ and justify the order of integration.
3. Find the exact real convergence interval. Explain why transforming e^{-t}/t and e^{-4t}/t separately is invalid.
4. Find $H'(s)$ and $\lim_{s\to\infty} sH(s)$. Relate them to positivity of h and its value at zero.

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Question 6 – Solution

Strategy. Cancellation removes the origin singularity. Preserve that cancellation by integrating over an exponential parameter.

Step 1: Remove the apparent singularity. Expansion at zero, or l'Hopital's rule, gives $\boxed{h(0) = 3}$. Direct integration in u gives

$$h(t) = \int_1^4 e^{-ut}\, du,$$

including at $t = 0$ by continuity. In particular, h is positive and bounded above by three.

Step 2: Derive the logarithmic transform. For $s > -1$, $s + u > 0$ throughout $[1, 4]$. The absolute double integral is finite, as seen by integrating first in t:

$$H(s) = \int_1^4 \int_0^\infty e^{-(s+u)t}\, dt\, du = \int_1^4 \frac{du}{s+u} = \boxed{\ln \frac{s+4}{s+1}, \qquad s > -1.}$$

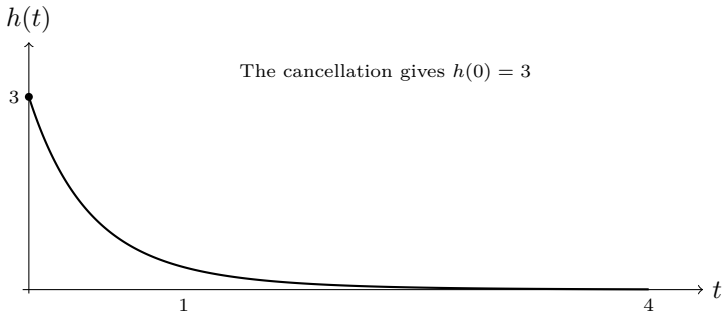
The finite value justifies interchange; both logarithm arguments are positive on this domain.

Step 3: Check the domain and invalid separation. At $s = -1$, the integrand is $(1 - e^{-3t})/t$, whose tail is at least $1/(2t)$ for sufficiently large t. It diverges. For smaller s, a larger positive exponential weight makes divergence persist. Thus $s > -1$ is the full real convergence interval. Each separate term e^{-t}/t or e^{-4t}/t has a nonintegrable origin singularity for every s. Their divergent transforms cannot be subtracted; it is the difference of the functions that is transformable.

Step 4: Interpret derivative and initial scale. Differentiating the logarithm gives

$$\boxed{H'(s) = \frac{1}{s+4} - \frac{1}{s+1} = -\frac{3}{(s+1)(s+4)} < 0.}$$

This agrees with $H'(s) = -\int_0^\infty te^{-st}h(t)\, dt$, since $h > 0$. Also $H(s) = \ln[1 + 3/(s+1)]$, so $\ln(1+x)/x \to 1$ implies $\boxed{sH(s) \to 3 = h(0)}$. The graph displays the continuous signal; its apparent division-by-zero point has been correctly filled in.



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Question 7

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

An unknown signal belongs to the family $f(t) = e^{-at}\cos bt$, with $a \geq 0$ and $b > 0$. Its transform satisfies the exact observations

$$\lim_{s\to\infty} s[sF(s) - 1] = -2, \qquad \lim_{s\to\infty} s^3[F(s) - 1/s + 2/s^2] = -5.$$

Tasks

1. Derive the transform in terms of a, b and expand it through the term in s^{-3}.
2. Recover a, b from the observations and state the resulting transform with its convergence interval.
3. Compute $f(0)$, $f'(0)$ and $f''(0)$ directly. Explain how the expansion coefficients encode these initial quantities.
4. For $q(t) = t^3 e^{-t}$, compute its transform and show that $f + \lambda q$ has the same two observed limits for every real λ. Explain the limit of the parameter recovery claim.

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Question 7 – Solution

Strategy. Large transform parameters emphasize early time. Within a specified family, the first few coefficients can determine parameters, but they do not determine an arbitrary signal.

Step 1: Transform and expand. Exponential shifting in the cosine pair gives

$$F(s) = \frac{s+a}{(s+a)^2+b^2}, \qquad s > -a.$$

Expanding this rational expression for large positive s yields

$$\boxed{F(s) = \frac{1}{s} - \frac{a}{s^2} + \frac{a^2-b^2}{s^3} + O(s^{-4}).}$$

For example, factor s^2 from the denominator and expand its reciprocal through order s^{-2} before multiplying by $1/s + a/s^2$.

Step 2: Recover the parameters. The first limit equals $-a$, so $a = 2$. With this value, the second equals $a^2 - b^2 = -5$, giving $b^2 = 9$. Since $b > 0$,

$$\boxed{a = 2, \quad b = 3, \qquad F(s) = \frac{s+2}{(s+2)^2+9}, \quad s > -2.}$$

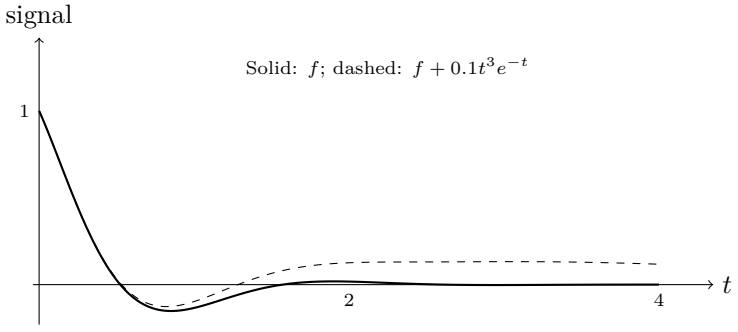
The boundary has an undamped cosine integral that does not converge; smaller parameters produce growing oscillations. Thus the interval is exact.

Step 3: Read the initial information. Direct differentiation gives

$$\boxed{f(0) = 1, \qquad f'(0) = -a = -2, \qquad f''(0) = a^2 - b^2 = -5.}$$

These are precisely the coefficients of $1/s$, $1/s^2$ and $1/s^3$ in the expansion. Repeated integration by parts explains the same pattern when the needed derivative bounds hold.

Step 4: Show what the measurements cannot identify. Three integrations by parts give $\mathcal{L}\{q\}(s) = 6/(s+1)^4$ for $s > -1$. Thus $f + \lambda q$ has transform $F(s) + 6\lambda/(s+1)^4$ on their common domain. The added term is $O(s^{-4})$, so it changes neither observed limit. Also $q(0) = q'(0) = q''(0) = 0$. Distinct λ therefore give different signals with the same recorded information. The parameters are uniquely recovered only within the stated exponential-cosine family.



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Question 8

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Suppose f is continuous, bounded and nonnegative on $[0, \infty)$. Consider the candidate expression

$$R(s) = \frac{s}{(s+1)^2}, \qquad s > 0.$$

Tasks

1. Derive the sign requirements on $F(s)$, $F'(s)$ and $F''(s)$ imposed by these hypotheses. State when $F(s) > 0$.
2. Although $R(s) > 0$ for every $s > 0$, prove that it cannot be the transform of such a nonnegative f.
3. Construct a continuous bounded sign-changing function g whose transform is R, using basic exponential pairs. Verify its sign change and full real convergence interval.
4. Compute $\int_0^\infty g(t)\,dt$ and explain why positivity of a transform throughout $s > 0$ does not imply positivity of its original function.

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Question 8 – Solution

Strategy. Positivity of the transform is only one consequence of a nonnegative signal; parameter derivatives supply stronger necessary tests.

Step 1: Derive the necessary signs. If $0 \le f \le M$, then the kernels obtained by differentiating once or twice are dominated locally in $s > 0$ by constant multiples of $te^{-s_0 t/2}$ and $t^2 e^{-s_0 t/2}$. Differentiation under the integral is therefore justified, and

$$F(s) \ge 0, \qquad F'(s) = -\int_0^\infty te^{-st} f(t)\,dt \le 0,$$

$$F''(s) = \int_0^\infty t^2 e^{-st} f(t)\,dt \ge 0.$$

If f is not identically zero, continuity gives a positive interval contribution and hence $F(s) > 0$ for every $s > 0$.

Step 2: Test the positive candidate. Although $R(s) > 0$,

$$R'(s) = \frac{1-s}{(s+1)^3} > 0 \qquad (0 < s < 1).$$

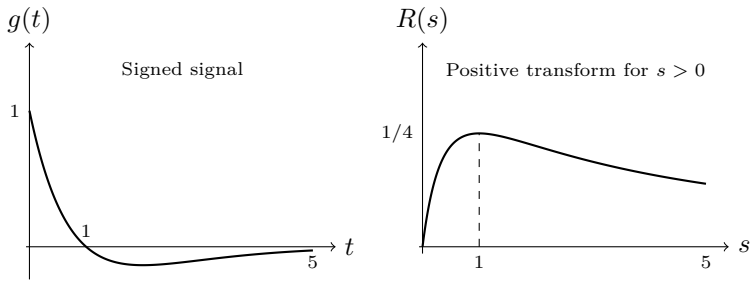
This contradicts the required nonpositive derivative. Thus no continuous bounded nonnegative signal can have this transform.

Step 3: Find a valid signed signal. Rewrite $R = 1/(s+1) - 1/(s+1)^2$. The basic exponential integrals give

$$\boxed{g(t) = e^{-t}(1-t), \qquad \mathcal{L}\{g\}(s) = R(s), \quad s > -1.}$$

The function is positive for $0 \le t < 1$, zero at one, and negative for $t > 1$. It is continuous and bounded. At or below $s = -1$, its eventually negative polynomial or growing tail has no finite improper integral, so the domain is exact.

Step 4: Explain the weighted cancellation. At $s = 0$, the integral exists and equals $R(0) = 0$, also verified by $\int e^{-t} = 1$ and $\int te^{-t} = 1$. For positive s, the extra weight suppresses the late negative portion more strongly, producing a positive transform. Consequently $F > 0$ alone is insufficient to infer $f \ge 0$. The two plots show a sign-changing signal and its everywhere-positive transform on $s > 0$.



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Question 9

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

A nonnegative density on $t \geq 0$ has the form $f(t) = cte^{-at}$ with $a > 0$, $c > 0$ and $\int_0^\infty f(t)\,dt = 1$. Define its mean by $\mu = \int_0^\infty tf(t)\,dt$ and variance by $\sigma^2 = \int_0^\infty t^2 f(t)\,dt - \mu^2$. An exact measurement gives $\mu = 4$.

Tasks

1. Determine the relation between a, c from normalization and compute the transform.
2. Derive the first two moment identities using parameter differentiation. Recover a, c and the variance from the measured mean.
3. Could this same family have mean 4 and variance 4? Give an exact consistency relation for arbitrary positive mean.
4. Find the location of the density maximum for the recovered parameters and compare it with the mean. Explain why those locations differ.

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Question 9 – Solution

Strategy. Normalization fixes one parameter, while derivatives of the transform at zero recover moments. A density's peak is not its mean.

Step 1: Normalize and transform. Integration by parts gives $\int_0^\infty te^{-at}\,dt = 1/a^2$. Thus $c = a^2$, and exponential shifting yields

$$\boxed{F(s) = \frac{a^2}{(s+a)^2}, \qquad s > -a.}$$

The nonnegative polynomial-exponential tail diverges at or below $-a$. In particular, $F(0) = 1$, as normalization requires.

Step 2: Extract moments and parameters. Since zero lies strictly inside the convergence interval, exponential decay justifies differentiation near zero:

$$-F'(0) = \int_0^\infty tf(t)\,dt, \qquad F''(0) = \int_0^\infty t^2 f(t)\,dt.$$

The rational expression gives $F'(s) = -2a^2/(s+a)^3$ and $F''(s) = 6a^2/(s+a)^4$. Hence

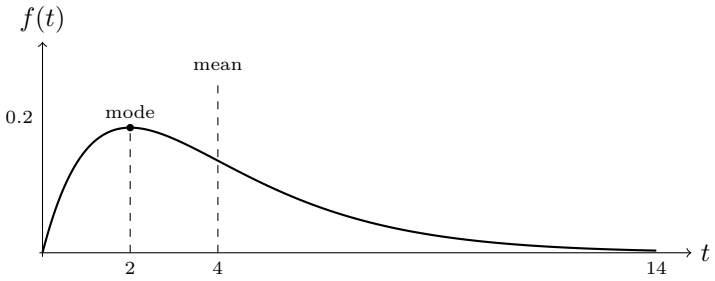
$$\mu = \frac{2}{a}, \qquad \sigma^2 = \frac{6}{a^2} - \frac{4}{a^2} = \frac{2}{a^2}.$$

From $\mu = 4$, we obtain

$$\boxed{a = \tfrac{1}{2}, \quad c = \tfrac{1}{4}, \quad \sigma^2 = 8, \qquad F(s) = \frac{1}{(2s+1)^2}.}$$

Step 3: Test consistency rather than refitting. Eliminating a gives $\boxed{\sigma^2 = \mu^2/2}$. Thus mean four forces variance eight, and the pair $(4, 4)$ is incompatible with this family. Changing c cannot repair it because normalization already fixes $c = a^2$.

Step 4: Locate the peak. For general a, $f'(t) = a^2e^{-at}(1 - at)$, positive before $t = 1/a$ and negative afterward. The unique density maximum is therefore at $t = 1/a = 2$, while the mean is four. The long right tail contributes to the mean; the point of largest density measures a different feature. The graph marks both the mode and the mean for $f(t) = te^{-t/2}/4$.



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Question 10

For real s, write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where the integral converges.

Let f be twice continuously differentiable, with f, f', f'' of exponential order. Work at sufficiently large real s so that all transforms below converge and all exponential boundary terms vanish. You may use

$$\mathcal{L}\{f'\} = sF - f(0), \qquad \mathcal{L}\{tf\} = -F', \qquad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0).$$

Tasks

1. Compute $\mathcal{L}\{(tf)'\}$ and $\mathcal{L}\{tf'\}$ separately. Explain their difference using the product rule in time.
2. Verify both results explicitly for $f(t) = e^{-t}$.
3. Derive $\mathcal{L}\{t^2 f''\}$, carefully differentiating every s-dependent factor.
4. A student keeps only s^2F'' in the last result. Identify the missing terms and verify the correct formula for $f = e^{-t}$ by transforming t^2e^{-t} directly.

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Question 10 – Solution

Strategy. Multiplication by time and differentiation in time correspond to operations in s that must be applied in the correct order.

Step 1: Compare the two orders. Since $(tf)(0) = 0$, applying the time-derivative rule after time multiplication gives

$$\boxed{\mathcal{L}\{(tf)'\} = -sF'(s).}$$

In the opposite order, parameter differentiation applies to the entire transform of f':

$$\boxed{\mathcal{L}\{tf'\} = -\frac{d}{ds}[sF(s) - f(0)] = -F(s) - sF'(s).}$$

Their difference is $F(s)$, exactly as required by $(tf)' - tf' = f$. The initial value $f(0)$ is constant in s and differentiates to zero.

Step 2: Check a concrete example. For $f = e^{-t}$, $F = 1/(s+1)$ and $F' = -1/(s+1)^2$. The formulas predict

$$\mathcal{L}\{(1-t)e^{-t}\} = \frac{s}{(s+1)^2}, \qquad \mathcal{L}\{-te^{-t}\} = -\frac{1}{(s+1)^2}.$$

Direct use of the exponential and te^{-t} integrals verifies both, for $s > -1$. Their difference is $1/(s+1) = F$.

Step 3: Differentiate the complete expression twice. Multiplication by t^2 corresponds to two parameter derivatives with positive sign. Thus

$$\mathcal{L}\{t^2 f''\} = \frac{d^2}{ds^2}[s^2F - sf(0) - f'(0)] = \boxed{2F + 4sF' + s^2F''.}$$

The first derivative of s^2F is $2sF + s^2F'$; differentiating again produces both product-rule contributions. The initial terms are at most linear in s, so their second derivatives vanish.

Step 4: Diagnose and verify. The proposed s^2F'' omits $2F + 4sF'$. For $f = e^{-t}$, the correct result simplifies to

$$\frac{2}{s+1} - \frac{4s}{(s+1)^2} + \frac{2s^2}{(s+1)^3} = \boxed{\frac{2}{(s+1)^3}}.$$

Since $f'' = e^{-t}$, direct integration of $t^2e^{-(s+1)t}$ gives this same value for $s > -1$. Differentiating only F while treating its prefactors as constants is the source of the error.

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Question 1

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$F(s) = \frac{s+5}{(s+1)(s+3)}.$$

The basic pair $\mathcal{L}\{e^{-at}\} = 1/(s+a)$ holds for $s > -a$.

Tasks

1. Find the inverse by partial fractions and verify the decomposition.
2. Determine the full real convergence interval of the recovered function.
3. Prove that the inverse is positive for all $t \geq 0$. Does its negative partial-fraction coefficient contradict positivity?
4. Find its initial value, its unique maximum and its limit at infinity. Explain why a sum of decaying modes need not decrease from the start.

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Question 1 – Solution

Strategy. Resolve the two poles, then factor the resulting time expression before deciding its sign or shape.

Step 1: Resolve and invert. Writing $F = A/(s+1) + B/(s+3)$ gives $A + B = 1$ and $3A + B = 5$. Thus $A = 2$, $B = -1$, and

$$\boxed{f(t) = 2e^{-t} - e^{-3t}.}$$

Transforming these two terms gives $2/(s+1) - 1/(s+3)$, whose numerator is $2(s+3) - (s+1) = s+5$. This verifies the entire rational expression, not just its pole locations.

Step 2: Retain the actual domain. For $s > -1$, both exponential integrals converge absolutely. Since $f = e^{-t}(2 - e^{-2t}) \geq e^{-t}$ on $t \geq 0$, its weighted integral diverges for $s \leq -1$. Hence the exact real domain is $\boxed{s > -1}$; other algebraic values of F are not ordinary transform values.

Step 3: Determine the sign. The factor $2 - e^{-2t}$ lies in $[1, 2)$, so $f(t) > 0$ everywhere. The negative coefficient belongs to one term in a decomposition; it does not determine the sign of the whole function. Here the faster-decaying negative term never dominates the positive term.

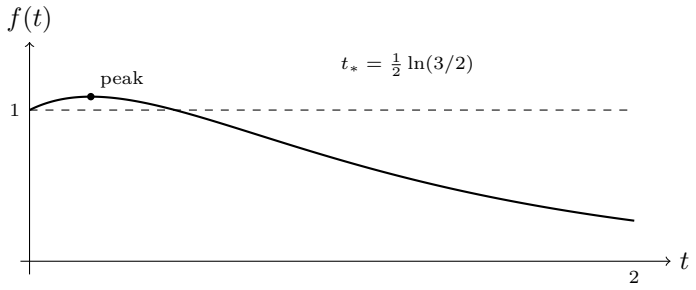
Step 4: Locate the peak. We have $f(0) = 1$, $f(t) \to 0$, and

$$f'(t) = e^{-t}(-2 + 3e^{-2t}).$$

The bracket decreases through zero once. Therefore f increases up to, then decreases after,

$$\boxed{t_* = \frac{1}{2}\ln(3/2), \qquad f(t_*) = \frac{4}{3}\sqrt{2/3}.}$$

In particular $f'(0) = 1 > 0$: decay of each mode does not force monotonic decay of their signed combination. The graph marks the small but real overshoot.



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Question 2

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{s^2 + 3s + 4}{(s+1)^3}.$$

You may use $\mathcal{L}\{t^n e^{-t}\} = n!/(s+1)^{n+1}$ for nonnegative integers n and $s > -1$.

Tasks

1. Find the complete partial-fraction expansion at the repeated pole.
2. Invert every term, explaining the factorial factors. Verify the transform of your answer.
3. A student includes only a term $C/(s+1)^3$. Show why this cannot represent F, and identify the missing time terms.
4. Compute $f(0), f'(0), f''(0)$ and the exact convergence interval. Check the three initial quantities against the first three large-s coefficients of F.

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Question 2 – Solution

Strategy. A pole of order three allows all three reciprocal powers, and each power has its own factorial normalization.

Step 1: Expand around the pole. Set $p = s + 1$. Then $s^2 + 3s + 4 = p^2 + p + 2$, so

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Equivalently, multiplying by $(s+1)^3$ gives the polynomial identity $s^2 + 3s + 4 = (s+1)^2 + (s+1) + 2$.

Step 2: Apply the correctly normalized pairs. The inverse of $(s+1)^{-3}$ is $t^2e^{-t}/2!$. Thus

$$\boxed{f(t) = e^{-t}(1 + t + t^2).}$$

Its three forward transforms are $1/(s+1)$, $1/(s+1)^2$ and $2/(s+1)^3$, confirming the result. Forgetting 2! would double the quadratic term incorrectly.

Step 3: Explain the incomplete ansatz. An expression $C/(s+1)^3$ has constant numerator after multiplication by the denominator. It cannot equal the nonconstant polynomial $s^2 + 3s + 4$ for every s. The proposed inverse $Ct^2e^{-t}/2$ lacks both e^{-t} and te^{-t}; no choice of C can repair that omission.

Step 4: Check initial data and convergence. Differentiation gives

$$f' = e^{-t}(t - t^2), \qquad f'' = e^{-t}(1 - 3t + t^2),$$

so $\boxed{f(0) = 1,\ f'(0) = 0,\ f''(0) = 1}$. Independently, expansion at large positive s gives

$$F(s) = \frac{1}{s} + \frac{0}{s^2} + \frac{1}{s^3} + O(s^{-4}).$$

These are the initial-value coefficients obtained by repeated integration by parts; this polynomial-exponential function satisfies all needed derivative bounds. Its defining integral converges absolutely for $s > -1$. At $s = -1$ the integrand is the positive polynomial $1 + t + t^2$, and for smaller s it grows still faster. The exact domain is $\boxed{s > -1}$.

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Question 3

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Recover the inverse of

$$F(s) = \frac{3s + 1}{s^2 + 4s + 13}.$$

Use the pairs for $e^{-at}\cos bt$ and $e^{-at}\sin bt$, whose numerators are $s + a$ and b, respectively, over $(s + a)^2 + b^2$.

Tasks

1. Complete the square and rewrite the numerator in terms of the shifted parameter.
2. Find and forward-check the inverse. Diagnose the candidate $e^{-2t}(3\cos 3t + \frac{1}{3}\sin 3t)$.
3. Express the inverse as $Re^{-2t}\cos(3t + \phi)$ with $R > 0$ and $0 < \phi < \pi/2$. Find its first positive zero.
4. Give its exact real convergence interval and explain the roles of the decay envelope and oscillation frequency.

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Question 3 – Solution

Strategy. Shift the numerator as well as the denominator; then distinguish phase, amplitude and exponential decay.

Step 1: Match the shifted pairs. The denominator is $(s+2)^2+9$, while $3s+1 = 3(s+2)-5$. Therefore

$$F(s) = 3\frac{s+2}{(s+2)^2+9} - \frac{5}{3}\frac{3}{(s+2)^2+9}.$$

Step 2: Invert and diagnose the sign error. The matching time function is

$$\boxed{f(t) = e^{-2t}[3\cos 3t - \tfrac{5}{3}\sin 3t].}$$

Its forward transform reproduces $3(s+2)-5 = 3s+1$. The proposed candidate instead has numerator $3(s+2)+1 = 3s+7$. Completing the square without adjusting the numerator loses a constant contribution.

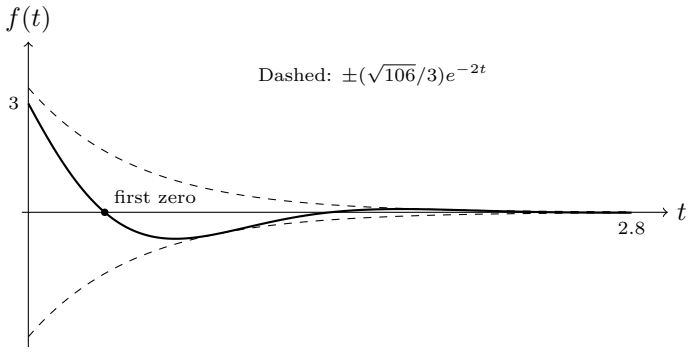
Step 3: Recover phase and the first zero. In $R\cos(3t+\phi) = R\cos\phi\cos 3t - R\sin\phi\sin 3t$, compare coefficients:

$$R = \frac{\sqrt{106}}{3}, \qquad \phi = \arctan(5/9).$$

Both sine and cosine of ϕ are positive, so the specified quadrant removes any phase ambiguity. Starting at $f(0) = 3 > 0$, the first zero occurs when $3t+\phi = \pi/2$:

$$\boxed{t_0 = \frac{\pi/2 - \arctan(5/9)}{3} = \frac{1}{3}\arctan(9/5).}$$

Step 4: Separate envelope from oscillation. The bound $|f(t)| \le Re^{-2t}$ gives absolute convergence for $s > -2$. At $s = -2$, the weighted signal is a nonzero sinusoid, whose primitive oscillates without a limit. For $s < -2$, positive fixed-length subintervals centered at successive cosine peaks have integrals growing exponentially, contradicting the Cauchy criterion. Thus $\boxed{s > -2}$ is exact. The decay rate is two and the angular frequency is three; neither gives the phase by itself. The graph shows both signed envelopes and the first zero.



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Question 4

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{5}{(s+1)(s^2+4)}.$$

The required basic pairs are e^{-t}, $\cos 2t$ and $\sin 2t$.

Tasks

1. Choose a complete real partial-fraction form and solve for its coefficients.
2. Find the inverse and verify its forward transform by recombining numerators.
3. Compute the first three initial quantities $f(0), f'(0), f''(0)$. Relate their cancellations to the leading large-s behavior of F.
4. Determine the exact real convergence interval. Does the inverse approach a limit as $t \to \infty$? Justify your answer using explicit sequences.

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Question 4 – Solution

Strategy. An irreducible quadratic requires a linear numerator. The transient and persistent parts then have different long-time roles.

Step 1: Resolve the real factors. Use

$$F = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Clearing denominators gives $5 = A(s^2+4) + (Bs+C)(s+1)$, so $A+B=0$, $B+C=0$, $4A+C=5$. Hence $A=1$, $B=-1$, $C=1$.

Step 2: Invert each numerator term. Since the sine pair has numerator two,

$$\boxed{f(t) = e^{-t} - \cos 2t + \tfrac{1}{2}\sin 2t.}$$

The forward numerator is $(s^2+4) - s(s+1) + (s+1) = 5$, giving the required transform. A constant numerator over the quadratic alone would have omitted the cosine contribution.

Step 3: Check the onset cancellations. Direct differentiation gives

$$f' = -e^{-t} + 2\sin 2t + \cos 2t, \qquad f'' = e^{-t} + 4\cos 2t - 2\sin 2t.$$

Therefore $\boxed{f(0)=0,\ f'(0)=0,\ f''(0)=5}$. Also $F(s) = 5/s^3 + O(s^{-4})$: the $1/s$ and $1/s^2$ terms vanish. Repeated integration by parts connects precisely these coefficients to the displayed initial values. In time, the first nonzero Taylor term is $5t^2/2$.

Step 4: Identify the persistent oscillation. The exponential term decays, but $-\cos 2t + \frac{1}{2}\sin 2t$ does not. In fact,

$$f(n\pi) \longrightarrow -1, \qquad f(n\pi + \pi/2) \longrightarrow 1.$$

Thus f has no limit at infinity, although it is bounded. Boundedness gives absolute convergence for $s > 0$. At $s = 0$, the primitive contains the nonconstant periodic term $-\sin 2R/2 + (1-\cos 2R)/4$, so it has no limit. For $s < 0$, the persistent sinusoid has positive fixed-length intervals with exponentially growing weighted integrals; the decaying time term cannot cancel it on late such intervals. Consequently the exact domain is $\boxed{s > 0}$.

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Question 5

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider the repeated quadratic factor

$$F(s) = \frac{1}{(s^2+4)^2}.$$

You may use $\mathcal{L}\{\sin 2t\} = 2/(s^2+4)$, $\mathcal{L}\{\cos 2t\} = s/(s^2+4)$ and $\mathcal{L}\{tg(t)\} = -G'(s)$ for $s > 0$.

Tasks

1. Derive the transform of $t\cos 2t$ by differentiating the cosine transform.
2. Find constants A, B such that the inverse is $A\sin 2t + Bt\cos 2t$. Verify the resulting rational numerator.
3. Show that the inverse is unbounded despite having no positive exponential factor. Give an explicit sequence witnessing growth.
4. Find its leading term near $t = 0$ and its exact real convergence interval. Compare the local term with the large-s scale $F(s) \sim s^{-4}$.

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Question 5 – Solution

Strategy. A repeated quadratic can be handled with parameter differentiation and linear combinations, without assuming a product rule for inverses.

Step 1: Produce the repeated denominator. Differentiating the cosine pair gives

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\frac{s}{s^2+4} = \frac{s^2-4}{(s^2+4)^2}.$$

Polynomial factors times the damping kernel are integrable for $s > 0$, justifying the differentiation.

Step 2: Cancel the unwanted numerator. The transform of $A\sin 2t + Bt\cos 2t$ has numerator $2A(s^2+4) + B(s^2-4)$. Requiring it to equal one gives $2A + B = 0$, $8A - 4B = 1$, hence

$$\boxed{f(t) = \frac{\sin 2t - 2t\cos 2t}{16}.}$$

Substitution gives $(s^2+4)/8 - (s^2-4)/8 = 1$, verifying the numerator exactly. Inverting a product by multiplying inverses would not give this result.

Step 3: Exhibit growth, not just an envelope. The bound $|f(t)| \le (1+2t)/16$ is only an upper bound. Actual unboundedness follows from

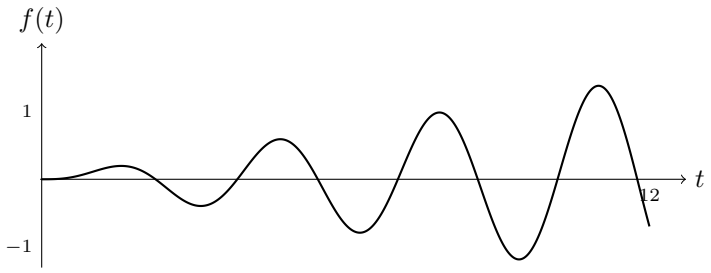
$$\boxed{f(n\pi) = -n\pi/8 \longrightarrow -\infty.}$$

The factor t produced by the repeated poles causes increasing oscillation size. The graph displays several oscillations at their actual scale.

Step 4: Check onset and convergence. Taylor expansion gives $\sin 2t - 2t\cos 2t = (8/3)t^3 + O(t^5)$, so $f(t) = t^3/6 + O(t^5)$. Since $\mathcal{L}\{t^3/6\} = s^{-4}$, the initial scale agrees with $F(s) \sim s^{-4}$. Absolute convergence holds for $s > 0$ by the linear envelope. At $s = 0$, direct integration gives

$$\int_0^R f(t)\,dt = \frac{1 - \cos 2R - R\sin 2R}{16},$$

which has no limit. For $s < 0$, intervals near successive positive peaks of $-\cos 2t$ have eventually positive integrand of order $te^{|s|t}$; their integrals fail the Cauchy criterion. Thus $\boxed{s > 0}$ is exact.



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Question 6

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

For $a > 0$, let

$$F_a(s) = \frac{1}{(s+1)(s+a)}.$$

The two poles coincide when $a = 1$.

Tasks

1. Find the inverse for $a \neq 1$ and separately for $a = 1$. Explain why substitution into the unsimplified partial fractions fails at $a = 1$.
2. Prove the inverse is positive for every $t > 0$, and give its exact real convergence interval for all $a > 0$.
3. Derive the representation $f_a(t) = te^{-t} \int_0^1 e^{-(a-1)tv}\, dv$ directly from your formula, including the case $a = 1$.
4. Use the representation to prove uniform convergence $f_a \to f_1$ on the whole half-line as $a \to 1$. Obtain an explicit bound proportional to $|a - 1|$ for $1/2 \leq a \leq 3/2$.

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Question 6 – Solution

Strategy. Combine the two large opposite coefficients before taking the coincident-pole limit. An integral representation controls the difference uniformly.

Step 1: Invert before and at collision. For $a \neq 1$, partial fractions give

$$F_a = \frac{1}{a-1}\left(\frac{1}{s+1} - \frac{1}{s+a}\right), \qquad \boxed{f_a(t) = \frac{e^{-t} - e^{-at}}{a-1}.}$$

At $a = 1$, $F_1 = (s+1)^{-2}$, so $\boxed{f_1(t) = te^{-t}}$. Both forward transforms verify the original expression. The two individual partial-fraction coefficients diverge as $a \to 1$; their combined inverse has a finite limit.

Step 2: Check sign and the slowest tail. For $a > 1$, numerator and denominator of f_a are positive when $t > 0$; for $0 < a < 1$, both are negative. Also $te^{-t} > 0$. For unequal poles the slower exponential has a positive, nonzero coefficient, while the other term decays faster. Thus its tail is bounded below by a positive multiple of $e^{-\min(1,a)t}$. Together with direct absolute convergence, this proves the exact domain

$$\boxed{s > -\min(1, a).}$$

At $a = 1$, the same statement follows from the positive polynomial-exponential tail.

Step 3: Remove the cancellation analytically. Integrating $e^{-(a-1)tv}$ in v gives

$$te^{-t}\int_0^1 e^{-(a-1)tv}\,dv = \frac{e^{-t} - e^{-at}}{a-1}$$

when $a \neq 1$ and $t > 0$. At $t = 0$ both sides extend to zero; at $a = 1$ the integral is one, yielding te^{-t}. This representation avoids subtracting nearly equal exponentials.

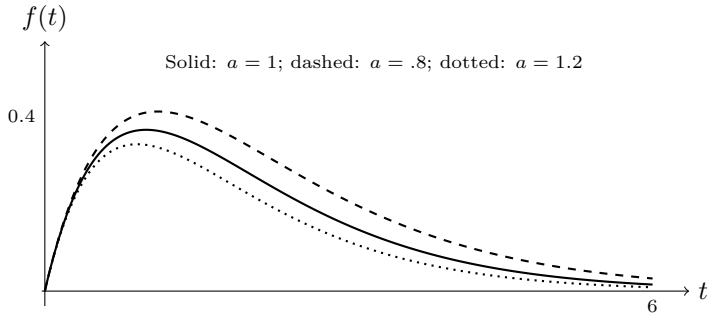
Step 4: Prove uniform half-line convergence. The mean value theorem gives $|e^{-x} - 1| \leq |x|e^{\max(0,-x)}$. With $m = \min(1, a)$, the representation yields

$$|f_a(t) - te^{-t}| \leq \frac{|a-1|}{2}t^2e^{-mt}.$$

The maximum of t^2e^{-mt} on $t \geq 0$ occurs at $t = 2/m$ and equals $4/(m^2e^2)$. Therefore, for $1/2 \leq a \leq 3/2$,

$$\boxed{\sup_{t\geq 0}|f_a(t) - f_1(t)| \leq \frac{8}{e^2}|a-1| \longrightarrow 0.}$$

This is uniform on the entire half-line, stronger than checking each fixed time. The plotted nearby curves meet at the same zero initial value.



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Question 7

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A parameter-dependent expression is

$$F_\lambda(s) = \frac{s - 2 + \lambda}{(s-2)(s+1)}, \qquad \lambda \in \mathbb{R}.$$

When a factor cancels, interpret its algebraic value at the canceled point by continuous extension.

Tasks

1. Find the inverse for every real λ and verify its transform.
2. Determine exactly which parameter values give a bounded inverse. State the full real convergence domain in every case.
3. Explain why the printed factor $s - 2$ alone does not establish a growing mode when $\lambda = 0$. What transform value exists at $s = 2$ in that case?
4. For $\lambda = 0.03$, find the time at which the growing and decaying contributions are equal. Explain how a small coefficient can change the eventual behavior.

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Question 7 – Solution

Strategy. Residues, including their possible zeros, decide which modes are actually present. An unreduced denominator can mislead.

Step 1: Recover both mode coefficients. Write $F_\lambda = A/(s-2) + B/(s+1)$. Clearing denominators gives $A + B = 1$ and $A - 2B = -2 + \lambda$, so

$$\boxed{f_\lambda(t) = \frac{\lambda}{3}e^{2t} + (1 - \tfrac{\lambda}{3})e^{-t}.}$$

Forward transformation recombines to $(A+B)s + (A-2B) = s - 2 + \lambda$, as required, for sufficiently large s.

Step 2: Classify boundedness and domains. If $\lambda = 0$, the function is e^{-t}, bounded and decaying, with exact transform domain $s > -1$. If $\lambda \neq 0$, then $e^{-2t}f_\lambda(t) \to \lambda/3 \neq 0$. Thus it is unbounded and its weighted tail at $s \leq 2$ has an eventual fixed sign and nonintegrable magnitude. For $s > 2$, all remaining terms converge absolutely. Consequently

$$\boxed{\text{bounded exactly when } \lambda = 0;} \qquad \boxed{\begin{array}{ll} s > -1, & \lambda = 0, \\ s > 2, & \lambda \neq 0. \end{array}}$$

This includes $\lambda = 3$, when the decaying term disappears.

Step 3: Interpret the canceled point. At $\lambda = 0$, cancellation gives $F_0 = 1/(s+1)$. The apparent pole at two was never a pole of the reduced transform. The actual integral at $s = 2$ is

$$\boxed{\int_0^\infty e^{-2t}e^{-t}\,dt = 1/3.}$$

The continuous algebraic extension agrees with it. The inverse is determined by the whole expression, not by a list of uncanceled denominator factors.

Step 4: Measure the delayed dominance. For $\lambda = .03$, the inverse is $.01e^{2t} + .99e^{-t}$. Equality of the contributions requires $e^{3t} = 99$, so

$$\boxed{t_c = \tfrac{1}{3}\ln 99 \approx 1.5317.}$$

The ratio of growing to decaying contributions is $e^{3t}/99$, strictly increasing through one. The initially small growing coefficient eventually dominates. At fixed finite time the inverse approaches e^{-t} as $\lambda \to 0$, but for any nonzero λ its eventual boundedness is different. That distinction does not justify discarding a small nonzero residue.

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Question 8

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

An inverse is to be selected from the rational family

$$F_a(s) = \frac{as + 1}{(s + 1)^2}, \qquad a \in \mathbb{R}.$$

The design requires a nonnegative continuous function with total integral one. Among such functions, one may also require it to be nonincreasing.

Tasks

1. Find the inverse for all a and verify its total integral and initial value.
2. Determine the exact set of a for which the inverse is nonnegative on $[0, \infty)$. Prove necessity as well as sufficiency.
3. Within that set, classify precisely when it is nonincreasing. For the other admissible parameters, locate the unique maximum.
4. The additional measurement $\int_0^\infty t f_a(t)\, dt = 3/2$ is exact. Determine a and decide whether all design requirements hold.

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Question 8 – Solution

Strategy. Invert first, then turn global sign and monotonicity requirements into inequalities for a linear time factor.

Step 1: Separate the numerator at its pole. Since $as + 1 = a(s + 1) + (1 - a)$,

$$\boxed{f_a(t) = e^{-t}[a + (1 - a)t].}$$

Forward transformation gives $a/(s+1)+(1-a)/(s+1)^2 = F_a$. The exact domain is $s > -1$ for every a: the nonzero linear or constant polynomial tail has eventual fixed sign and diverges at or below -1. Thus $s = 0$ is allowed, and

$$\int_0^\infty f_a(t)\,dt = F_a(0) = 1, \qquad f_a(0) = a.$$

Unit signed integral alone does not establish nonnegativity.

Step 2: Classify nonnegative inverses. Because $e^{-t} > 0$, the sign is that of $a+(1-a)t$. Nonnegativity at zero requires $a \geq 0$. If $a > 1$, the negative slope makes the factor negative for sufficiently large t. Conversely, $0 \leq a \leq 1$ makes both its intercept and slope nonnegative. Therefore the exact admissible set is $\boxed{0 \leq a \leq 1}$.

Step 3: Distinguish monotone and peaked designs. For admissible a,

$$f_a'(t) = e^{-t}[1 - 2a - (1 - a)t].$$

The bracket is nonincreasing in t, so it is nonpositive everywhere exactly when its initial value is nonpositive. Hence the nonincreasing designs have $\boxed{1/2 \leq a \leq 1}$. For $0 \leq a < 1/2$, the derivative changes from positive to negative at

$$\boxed{t_* = \frac{1 - 2a}{1 - a}, \qquad f_a(t_*) = (1 - a)\exp(-\frac{1 - 2a}{1 - a}).}$$

This also covers $a = 0$, whose maximum occurs at one.

Step 4: Use the extra integral to select a design. Direct exponential moments give

$$\int_0^\infty t f_a(t)\,dt = a + 2(1 - a) = 2 - a.$$

Equating this to $3/2$ yields $\boxed{a = 1/2}$ and $f(t) = (1 + t)e^{-t}/2$. This is nonnegative, has unit integral, and is nonincreasing: $f'(t) = -te^{-t}/2 \leq 0$. Its derivative vanishes only at the initial endpoint. The measurement selects one member, rather than allowing the initial value to be prescribed independently.

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Question 9

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$G(s) = \frac{s^2 + 2s + 2}{(s+1)^2}.$$

Work only with ordinary integrals and continuous functions of exponential order. You may use that such a function has a global bound $|g(t)| \leq Me^{ct}$ on $t \geq 0$ for some $M > 0$ and real c.

Tasks

1. Divide the polynomials to separate G into its polynomial part and a strictly proper rational part.
2. Prove from the stated bound that every allowed ordinary transform tends to zero as $s \to \infty$. Decide whether G has an inverse in the stated class.
3. Find the inverse h of the strictly proper remainder and verify it directly. Explain why reporting h as the inverse of the whole G would be incorrect.
4. Define $k(t) = h(t)$ except that $k(1) = 7$. Do k and h have the same ordinary transform? Explain why this does not contradict uniqueness in the stated class.

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Question 9 – Solution

Strategy. Check whether an ordinary inverse can exist before matching a rational remainder to a table. Keep the continuity hypothesis in any uniqueness claim.

Step 1: Divide before attempting inversion. Because $(s+1)^2 = s^2 + 2s + 1$,

$$\boxed{G(s) = 1 + \frac{1}{(s+1)^2}.}$$

The nonzero constant part cannot be silently discarded. In particular $G(s) \to 1$, rather than zero, at large positive s.

Step 2: Apply a necessary existence test. For an allowed g and $s > c$, absolute convergence and the global bound yield

$$|\mathcal{L}\{g\}(s)| \le \int_0^\infty e^{-st}|g(t)|\,dt \le M\int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c} \longrightarrow 0.$$

A continuous function of exponential order admits this global bound by enlarging M on an initial compact interval. Since G has limit one, $\boxed{G \text{ has no inverse in the stated class.}}$ This conclusion concerns ordinary continuous functions under the explicit hypotheses.

Step 3: Invert only the actual remainder. The strictly proper part has inverse $\boxed{h(t) = te^{-t}}$. Direct integration by parts gives

$$\int_0^\infty te^{-(s+1)t}\,dt = \frac{1}{(s+1)^2}, \qquad s > -1.$$

The integral diverges at or below -1. Its transform is $G - 1$, not G. The missing constant does not arise from changing $h(0)$: changing one finite point value changes no ordinary integral.

Step 4: State uniqueness with its hypothesis. Here $h(1) = e^{-1} \neq 7$. Changing that one value leaves every finite-interval integral, and hence every convergent improper transform, unchanged. Thus k and h have identical transforms on $s > -1$. However, k is discontinuous at one, so it is not in the continuous class specified in the question. There is no contradiction: uniqueness within that class compares two continuous functions. The example separates equality of transforms from unrestricted pointwise equality. No generalized transform is needed to reach any of these conclusions.

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Question 10

Use the ordinary one-sided Laplace transform for real s. Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A transform has the form

$$F(s) = \frac{As^2 + Bs + C}{(s+1)(s+2)(s+3)}.$$

Its inverse satisfies $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 6$. Assume f is three times continuously differentiable and $f, f', f'', f^{(3)}$ are of exponential order, so repeated integration by parts is available.

Tasks

1. Derive the expansion $F(s) = f(0)/s + f'(0)/s^2 + f''(0)/s^3 + O(s^{-4})$. Use it to determine A, B, C.
2. Find the inverse by partial fractions and verify all three prescribed initial quantities.
3. Factor the inverse to prove its sign on $t \geq 0$ and determine its exact real convergence interval.
4. Find its sharp maximum value and the unique time at which it is attained. Explain why the initial data determine the numerator here but would not determine an arbitrary function.

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Question 10 – Solution

Strategy. Initial data determine large-parameter coefficients. The fixed denominator converts those coefficients into a finite algebraic reconstruction.

Step 1: Recover the missing numerator. Three integrations by parts, with vanishing upper boundary terms for sufficiently large s, give

$$F(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{1}{s^3}\mathcal{L}\{f^{(3)}\}(s).$$

The last transform is $O(1/s)$ by an exponential bound. Since the denominator is $s^3+6s^2+11s+6$, expansion of the rational expression gives

$$F(s) = \frac{A}{s} + \frac{B-6A}{s^2} + \frac{C-6B+25A}{s^3} + O(s^{-4}).$$

The prescribed values imply $\boxed{A=0,\ B=0,\ C=6}$.

Step 2: Resolve the three poles and check. Evaluation of the cleared identity at $s=-1,-2,-3$ gives coefficients $3,-6,3$. Hence

$$F = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}, \qquad \boxed{f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}.}$$

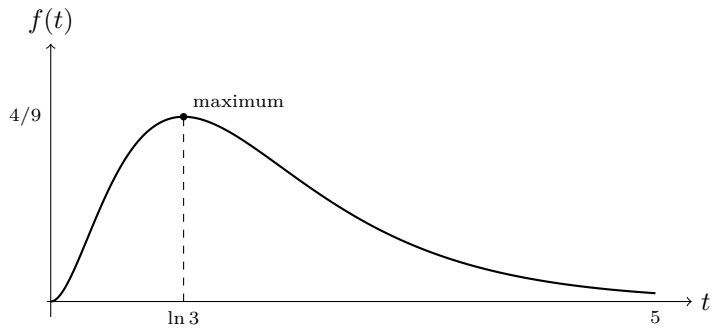
Their forward numerator is six. The initial derivative sums are $3-6+3=0$, $-3+12-9=0$, and $3-24+27=6$, verifying all data.

Step 3: Factor to determine the sign and tail. The useful form is $\boxed{f(t) = 3e^{-t}(1-e^{-t})^2}$. It is zero at $t=0$ and strictly positive for $t>0$. Its tail is asymptotic to $3e^{-t}$, so the exact real convergence interval is $\boxed{s>-1}$: above it all terms converge; at or below it the eventually positive weighted tail has infinite integral.

Step 4: Find the sharp peak and scope of reconstruction. Put $u=e^{-t}$, so $0<u\le 1$ and $f=3u(1-u)^2$. Its derivative with respect to u is $3(1-u)(1-3u)$, so the unique interior maximum occurs at $u=1/3$:

$$\boxed{t_* = \ln 3, \qquad \max_{t\ge 0} f(t) = 4/9.}$$

The endpoint values are zero (at $t=0$ and as $t\to\infty$). The fixed three-parameter numerator made the reconstruction unique. Adding t^3e^{-t} to this inverse preserves all three initial data but changes its transform and leaves the specified rational family. Thus three initial quantities alone do not determine an arbitrary function.



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Question 1

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

A signal follows the schedule

$$f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ -1, & 1 \leq t < 3, \\ 4, & 3 \leq t < 5, \\ 0, & t \geq 5. \end{cases}$$

Tasks

1. Write f as an initial level plus a linear combination of u_1, u_3, u_5. Explain how to choose the coefficients.
2. Find its transform first for $s > 0$, then determine its full real convergence set.
3. Compute $F(0)$ and $F'(0)$ directly from the finite time intervals. Check the removable singularity in your formula.
4. A student uses the new levels themselves as the step coefficients. Diagnose the error by finding the resulting levels after each switch.

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Question 1 – Solution

Strategy. A step adds a change to the existing level; it does not replace that level. Compact support must be assessed after the changes are combined.

Step 1: Encode the changes. The changes at times one, three and five are $-3, +5, -4$. Therefore

$$\boxed{f(t) = 2 - 3u_1(t) + 5u_3(t) - 4u_5(t).}$$

The convention $u_a(a) = 1$ gives the prescribed value at every switch. Successively subtracting adjacent levels determines these coefficients uniquely for the specified switch times.

Step 2: Transform and check support. For $s > 0$, $\mathcal{L}\{u_a\} = e^{-as}/s$ by direct integration from a to infinity. Thus

$$\boxed{F(s) = \frac{2 - 3e^{-s} + 5e^{-3s} - 4e^{-5s}}{s} \quad (s \neq 0).}$$

Although the separate step integrals require $s > 0$, the actual signal vanishes after five. Integrating on its three finite intervals proves the same formula for every real $s \neq 0$; the original integral exists for every real s.

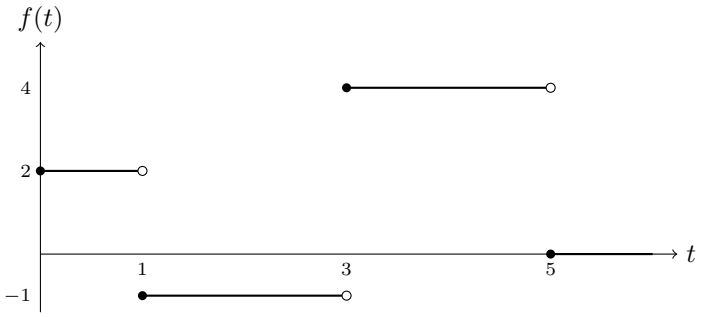
Step 3: Evaluate the removable value and first moment. Directly,

$$F(0) = 2(1) - 1(2) + 4(2) = \boxed{8},$$

$$F'(0) = -\int_0^5 t f(t)\, dt = -[1 - 4 + 32] = \boxed{-29}.$$

Finite support justifies differentiation under the integral. Expanding the numerator gives $8s - 29s^2 + O(s^3)$, so $F(s) = 8 - 29s + O(s^2)$, confirming both results. Zero is removable, not a divergent transform point.

Step 4: Diagnose replacement versus addition. The mistaken expression $2 - u_1 + 4u_3 + 0u_5$ produces levels $2, 1, 5, 5$. It neither reaches the intended levels nor shuts off. Each step must carry the difference between the new and old levels. The graph uses open and filled endpoints to record the specified right-hand values.



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Question 2

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

Compare the switched-on and delayed signals

$$p(t) = t^2 u_2(t), \qquad q(t) = (t-2)^2 u_2(t).$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{t\} = 1/s^2$ and $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.

Tasks

1. Derive the shift formula for $u_a(t)g(t-a)$ by substitution in the defining integral.
2. Find the transforms of p and q. Rewrite the switched-on polynomial in its local time before using the formula.
3. A student assigns $2e^{-2s}/s^3$ to both functions. Identify the missing terms and check the values immediately after the switch.
4. Give the exact real convergence domains. Determine the leading large-s behavior of $e^{2s}P(s)$ and $e^{2s}Q(s)$ and relate it to the local onset.

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Question 2 – Solution

Strategy. The factor e^{-as} shifts the time origin of the entire accompanying inverse, not just its domain of activity.

Step 1: Derive the delayed-shape rule. For parameters where the integral converges, substitute $v = t - a$:

$$\int_a^\infty e^{-st} g(t-a)\,dt = e^{-as}\int_0^\infty e^{-sv} g(v)\,dv.$$

Hence $\boxed{\mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}G(s)}$. The effective shape is a function of elapsed time since the switch.

Step 2: Rewrite the unshifted polynomial. For $t \geq 2$, write $t = v + 2$. Then p has local shape $(v+2)^2 = v^2 + 4v + 4$, whereas q has local shape v^2. Therefore

$$\boxed{P(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right), \qquad Q(s) = \frac{2e^{-2s}}{s^3}, \quad s > 0.}$$

Direct substitution in each defining integral verifies these expressions.

Step 3: Identify the omitted local terms. The difference is

$$p - q = [4(t-2) + 4]u_2(t), \qquad P - Q = e^{-2s}(4/s^2 + 4/s).$$

At the switch, $p(2) = 4$ while $q(2) = 0$, equal to their respective right limits. The switched-on signal jumps from zero to four; the delayed square starts continuously at zero with zero right slope. The student's expression describes only the latter.

Step 4: Check domains and onset scales. Both signals have nonnegative quadratic tails, so they converge absolutely for $s > 0$ and diverge for $s \leq 0$. Their exact domain is $\boxed{s > 0}$. Removing the delay factor gives

$$e^{2s}P(s) \sim 4/s, \qquad e^{2s}Q(s) = 2/s^3 \quad (s \to \infty).$$

The constant local onset of p produces the $1/s$ term; the quadratic local onset of q produces $2/s^3$. A common switch time does not imply a common transformed shape.

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Question 3

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

An oscillator is observed only during the window

$$f(t) = \sin t \, [u_{\pi/2}(t) - u_{3\pi/2}(t)].$$

Use $\mathcal{L}\{\cos t\} = s/(s^2 + 1)$ for $s > 0$.

Tasks

1. Write the exact piecewise function, including its values at both window endpoints.
2. Express each switched sine using its local clock and derive $F(s)$ for $s > 0$.
3. Determine the full real convergence set, $F(0)$ and $F'(0)$. Verify them by direct finite-interval integration.
4. Explain why $F(s) > 0$ for $s > 0$ even though the signal has both signs and zero unweighted integral. Use the window symmetry to justify the sign.

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Question 3 – Solution

Strategy. Re-express the phase at each switch separately. The second switch occurs at a different oscillator phase.

Step 1: Identify the window and endpoint values. The function equals $\sin t$ on $[\pi/2, 3\pi/2)$ and zero elsewhere. In particular $f(\pi/2) = 1$, while $f(3\pi/2) = 0$. At the latter time the left limit is -1; the graph distinguishes it from the assigned value.

Step 2: Shift the two phases correctly. With $a = \pi/2$ and $b = 3\pi/2$, we have $\sin(v + a) = \cos v$ and $\sin(v + b) = -\cos v$. Therefore

$$f = u_a(t)\cos(t - a) + u_b(t)\cos(t - b),$$

$$\boxed{F(s) = \frac{s(e^{-\pi s/2} + e^{-3\pi s/2})}{s^2 + 1}.}$$

The plus sign in the transform reflects the phase reversal at shutoff. After b, the two shifted cosine terms cancel exactly, as the original window requires.

Step 3: Use finite support to evaluate zero. The defining integral has finite support, so it converges for every real s; finite-interval integration gives the displayed expression on the whole real line. Directly,

$$F(0) = \int_a^b \sin t\, dt = 0,$$

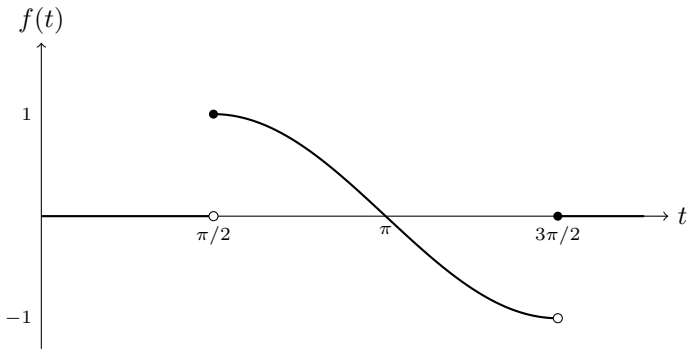
$$F'(0) = -\int_a^b t \sin t\, dt = -[-t\cos t + \sin t]_a^b = \boxed{2}.$$

The formula also gives $F(0) = 0$ and expands as $2s + O(s^2)$, verifying the derivative.

Step 4: Pair the equal and opposite lobes. Write $t = \pi + x$ with $-\pi/2 \le x \le \pi/2$. Pair x with $-x$. For $0 < x < \pi/2$, the earlier value is $\sin x > 0$ and the later value is $-\sin x$. Their weighted contribution is

$$e^{-\pi s}\sin x\,[e^{sx} - e^{-sx}] > 0 \qquad (s > 0).$$

Integrating these pairs proves strict positivity, despite cancellation at $s = 0$. This also agrees with the positive numerator and denominator of the formula for $s > 0$.



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Question 4

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

Let $r_a(t) = (t-a)u_a(t)$. A continuous triangular signal is

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 3, \\ (7-t)/2, & 3 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$$

Tasks

1. Express f as a linear combination of r_1, r_3, r_7 by tracking slope changes.
2. Find $F(s)$ using shifted ramp transforms, and establish its full real convergence set.
3. Explain algebraically why the ramp combination is zero after time seven. State the two cancellation conditions for a general sum $\sum c_j r_{a_j}(t)$ to vanish after its last switch.
4. Compute the total area and the first moment. Confirm the values from the expansion of F at zero.

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Question 4 – Solution

Strategy. A ramp changes slope rather than level. Cancel both the final slope and the final intercept to create compact support.

Step 1: Encode slope changes. The slopes are $0, 1, -1/2, 0$, giving changes $1, -3/2, 1/2$. Thus

$$\boxed{f = r_1 - \tfrac{3}{2}r_3 + \tfrac{1}{2}r_7.}$$

Each ramp is zero at its own switch, so the combination is continuous. At the peak time three its value is two; it then decreases to zero at seven.

Step 2: Transform the ramps. For $s > 0$, substitution gives $\mathcal{L}\{r_a\} = e^{-as}/s^2$. Hence

$$\boxed{F(s) = \frac{e^{-s} - \frac{3}{2}e^{-3s} + \frac{1}{2}e^{-7s}}{s^2} \quad (s \neq 0).}$$

The actual function has finite support, so its integral exists for all real s. Direct integration of the two linear pieces extends the formula to all $s \neq 0$; zero is removable.

Step 3: Cancel the tail, not just the slope. For $t \geq 7$, the ramp sum is

$$(1 - \tfrac{3}{2} + \tfrac{1}{2})t - [1 - \tfrac{3}{2}(3) + \tfrac{1}{2}(7)] = 0.$$

In general, after the last switch a finite ramp sum equals $t\sum c_j - \sum c_j a_j$. It vanishes identically there exactly when

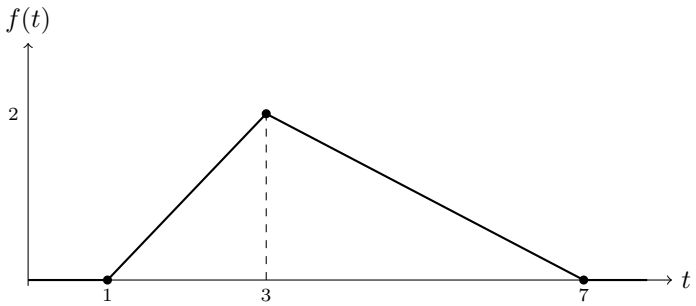
$$\boxed{\sum c_j = 0, \qquad \sum c_j a_j = 0.}$$

Canceling only the slope could leave a nonzero constant tail and a different convergence domain.

Step 4: Check area and moment. The triangle has base six and height two, so $\boxed{F(0) = 6}$. Direct integration gives

$$\int_1^3 t(t-1)\,dt = \frac{14}{3}, \qquad \frac{1}{2}\int_3^7 t(7-t)\,dt = \frac{52}{3}.$$

Thus the first moment is $\boxed{22}$ and $F'(0) = -22$. Expanding the numerator of F gives $6s^2 - 22s^3 + O(s^4)$, agreeing with both results. The horizontal time coordinate of the area centroid is $22/6 = 11/3$, consistent with the three triangle vertices.



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Question 5

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

Find a piecewise continuous inverse of

$$F(s) = (e^{-2s} - e^{-5s})\frac{s+3}{(s+1)^2+4}.$$

Use the step convention above to specify values at switches. Basic shifted sine and cosine pairs may be used.

Tasks

1. Invert the rational factor, then apply the two delays correctly.
2. Write the result on $0 \leq t < 2$, $2 \leq t < 5$ and $t \geq 5$. Find the jumps at two and five.
3. Does the inverse vanish after five? Compare it with genuinely cutting off the first delayed signal at five.
4. Determine the exact real convergence interval and verify the transform on it.

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Question 5 – Solution

Strategy. Subtracting a delayed copy restarts that copy's clock. It generally does not switch off the earlier copy.

Step 1: Invert the common shape. Since $s+3=(s+1)+2$, the rational factor is the transform of

$$g(v)=e^{-v}(\cos 2v+\sin 2v), \qquad v\geq 0.$$

Thus one inverse with the stipulated endpoint convention is

$$\boxed{f(t)=u_2(t)g(t-2)-u_5(t)g(t-5).}$$

Step 2: Inspect the clocks and jumps. The piecewise values are zero before two, $g(t-2)$ on $[2,5)$, and $g(t-2)-g(t-5)$ on $[5,\infty)$. As $g(0)=1$, the jumps are $\boxed{+1 \text{ at } 2,\ -1 \text{ at } 5}$. In particular $f(2)=1$ and $f(5)=g(3)-1$, while the left limit at five is $g(3)$.

Step 3: Distinguish subtraction from shutoff. At $t=5$, $|g(3)|\leq\sqrt{2}e^{-3}<1$, so $g(3)-1\neq 0$. In fact the final piece is a nonzero exponentially decaying sinusoid, not zero. A genuinely truncated signal is instead

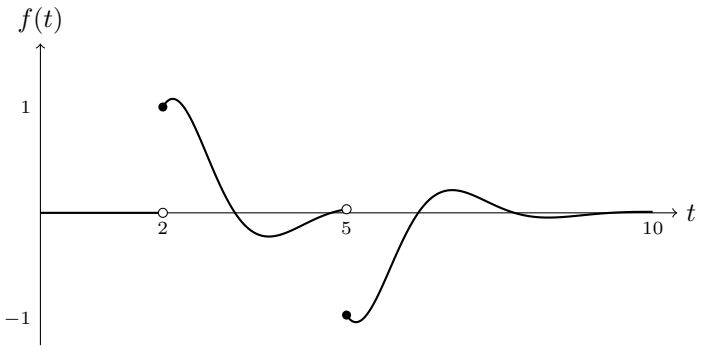
$$h(t)=[u_2(t)-u_5(t)]g(t-2).$$

Its shutoff term has the same global argument $t-2$. To transform that term at time five, its local shape must be $g(v+3)$, not $g(v)$. The graph shows the actual continuing tail of f.

Step 4: Verify the transform and domain. On $t\geq 5$, put $v=t-5$. The tail is

$$e^{-v}\{[e^{-3}(\cos 6+\sin 6)-1]\cos 2v+[e^{-3}(\cos 6-\sin 6)-1]\sin 2v\}.$$

The cosine coefficient is nonzero by the preceding bound. Hence absolute convergence holds for $s>-1$; at $s=-1$ a nonzero undamped sinusoid has no convergent primitive, and below it fixed-sign lobe intervals violate the Cauchy criterion. The exact domain is $\boxed{s>-1}$. On that domain each delayed copy converges and transforms to $e^{-2s}G(s)$ or $e^{-5s}G(s)$ by substitution, proving the given expression. The truly truncated h would instead converge for every real s.



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Question 6

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

A function has a deliberately exceptional point value:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 7, & t = 1, \\ e^{-(t-1)}, & t > 1. \end{cases}$$

Compare it with $g(t) = u_1(t)e^{-(t-1)}$.

Tasks

1. Compare f and g pointwise, including their left and right limits at one.
2. Find their ordinary transforms and exact convergence intervals.
3. Give a pointwise formula for f using g and the indicator of the singleton $\{1\}$. Explain why a finite sum of right-continuous step terms times continuous functions cannot reproduce this exceptional point value.
4. If a step convention instead assigns value $1/2$ at the switch, what changes in the formula for g and what stays the same in its transform? State precisely the uniqueness conclusion justified here.

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Question 6 – Solution

Strategy. Separate equality at every point from equality except at isolated points. An ordinary integral cannot recover the latter point values.

Step 1: Locate the discrepancy. For $t \neq 1$, $f(t) = g(t)$. At one, $g(1) = 1$ but $f(1) = 7$. Both functions have left limit zero and right limit one. Thus g is right-continuous, whereas the chosen value of f agrees with neither limit.

Step 2: Transform the interval contribution. A single changed value contributes nothing to an integral. Substituting $v = t - 1$ in the tail gives, for both functions,

$$\boxed{F(s) = G(s) = \frac{e^{-s}}{s+1}, \qquad s > -1.}$$

At $s = -1$, the weighted tail is the positive constant e; below -1 it grows exponentially. Thus the stated convergence interval is exact. The isolated value seven cannot change this conclusion.

Step 3: Give a pointwise correction and prove the limitation. If $\mathbf{1}_{\{1\}}$ is one only at $t = 1$ and zero elsewhere, then

$$\boxed{f(t) = g(t) + 6\mathbf{1}_{\{1\}}(t).}$$

Each u_a under our convention is right-continuous. Multiplying it by a continuous function preserves right-continuity, and finite sums preserve that property. Such a sum must equal its right limit at one, so it cannot have the specified value seven while its right limit is one. This is a representational obstruction, not a transform-calculation error.

Step 4: Change the convention without changing the integral. With switch value $1/2$, the corresponding expression for g equals $1/2$ at one and is unchanged elsewhere. Its ordinary transform is still $e^{-s}/(s+1)$. A pointwise correction to recover f would now be $(13/2)\mathbf{1}_{\{1\}}$. Thus transform data alone do not specify the value at an isolated discontinuity. Here the right-hand continuous representative is fixed by the chosen convention; an unrestricted pointwise uniqueness claim would be false. The usual uniqueness statement at continuity points is consistent with all these examples.

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Question 7

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

Let $f(t) = u_2(t)[1 + (t - 2)]$. Define its ordinary derivative away from the switch as $h(t) = 0$ for $t < 2$ and $h(t) = 1$ for $t > 2$; its value at two is irrelevant to integration. A student asserts $H(s) = sF(s) - f(0)$.

Tasks

1. Calculate $F(s)$ and $H(s)$ directly for $s > 0$, and test the assertion.
2. Apply integration by parts separately on $[0, 2)$ and $(2, R]$ to identify the missing contribution.
3. For a piecewise C^1 function with finitely many jumps $J_j = f(c_j+) - f(c_j-)$ at positive times, derive the corresponding formula for the transform of its ordinary derivative on the smooth pieces. State the needed boundary assumptions.
4. Apply your formula to the continuous delayed ramp $k(t) = (t - 2)u_2(t)$. Explain why the usual derivative rule works for k despite its corner.

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Question 7 – Solution

Strategy. Integration by parts across smooth pieces leaves internal boundary terms. A jump is different from a continuous corner.

Step 1: Compute both ordinary integrals. The local shape of f is $1+v$, and the local shape of h is one. Thus

$$F(s) = e^{-2s}(1/s + 1/s^2), \qquad H(s) = e^{-2s}/s, \qquad s > 0.$$

Since $f(0) = 0$, the student's expression is $sF = e^{-2s}(1 + 1/s)$, larger than H by $\boxed{e^{-2s}}$. Both actual integrals have exact domain $s > 0$ because their tails are a positive linear function and a positive constant, respectively.

Step 2: Retain the internal boundary. Only the second piece contributes. For $R > 2$, integration by parts gives

$$\int_2^R e^{-st} f'(t)\,dt = e^{-sR} f(R) - e^{-2s} f(2+) + s\int_2^R e^{-st} f(t)\,dt.$$

Here $f(2+) = 1$. The upper term vanishes for $s > 0$, leaving $H = sF - e^{-2s}$. The omitted term records the jump from zero to one, not a value assigned to h at the single point two.

Step 3: Sum the piecewise boundary terms. At an internal time c_j, the left piece contributes $e^{-sc_j} f(c_j-)$ and the right piece contributes $-e^{-sc_j} f(c_j+)$. Their sum is $-J_j e^{-sc_j}$. Consequently

$$\boxed{\mathcal{L}\{f'_{\mathrm{pw}}\}(s) = sF(s) - f(0+) - \sum_j J_j e^{-sc_j}.}$$

Use finite one-sided limits, convergence of the piecewise function and derivative integrals, and $e^{-sR} f(R) \to 0$. Exponential-order bounds on the pieces and their derivatives suffice for large s. If f is right-continuous at zero, $f(0+) = f(0)$. The formula concerns ordinary piecewise derivatives only.

Step 4: Check a continuous corner. For k, both one-sided values at two are zero, so its jump is zero. We have $K = e^{-2s}/s^2$ and its ordinary piecewise derivative is u_2 except possibly at two. Thus

$$\mathcal{L}\{k'_{\mathrm{pw}}\} = e^{-2s}/s = sK - k(0).$$

A corner changes slopes but introduces no internal value jump. Splitting the integral proves the usual rule here without pretending the derivative exists at the corner.

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Question 8

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

A three-stage pulse is one on $[0, 1)$, has unknown height B on $[1, 2)$ and unknown height C on $[2, 3)$, and is zero for $t \geq 3$. It must satisfy

$$\int_0^\infty f(t)\,dt = 0, \qquad \int_0^\infty t f(t)\,dt = 0.$$

Tasks

1. Determine B, C and prove that these heights are uniquely determined.
2. Write the resulting pulse using steps and derive its transform in factored form.
3. Find the full real convergence set, the first nonzero term of $F(s)$ near zero, and the second time moment.
4. Explain why the two vanishing moments do not make the pulse or its transform identically zero. Determine the sign of $F(s)$ for $s > 0$.

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Question 8 – Solution

Strategy. Moment constraints form a small linear system. The step differences then expose cancellation in the transform numerator.

Step 1: Solve the two moment equations. The unit-width intervals give

$$1 + B + C = 0, \qquad \tfrac{1}{2} + \tfrac{3}{2}B + \tfrac{5}{2}C = 0.$$

Subtracting the first equation from twice the second yields $2B + 4C = 0$. Combining this with $B + C = -1$ gives $\boxed{B = -2,\ C = 1}$. The coefficient determinant is $5/2 - 3/2 = 1 \neq 0$, so the solution is unique.

Step 2: Encode changes and factor the transform. The levels $1, -2, 1, 0$ have jumps $-3, +3, -1$. Hence

$$f(t) = 1 - 3u_1(t) + 3u_2(t) - u_3(t),$$

$$\boxed{F(s) = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s} = \frac{(1 - e^{-s})^3}{s} \quad (s \neq 0).}$$

This follows first by transforming steps for $s > 0$, or directly on the three finite intervals for any $s \neq 0$.

Step 3: Interpret the removable zero. Compact support gives convergence for every real s. Since $1 - e^{-s} = s + O(s^2)$,

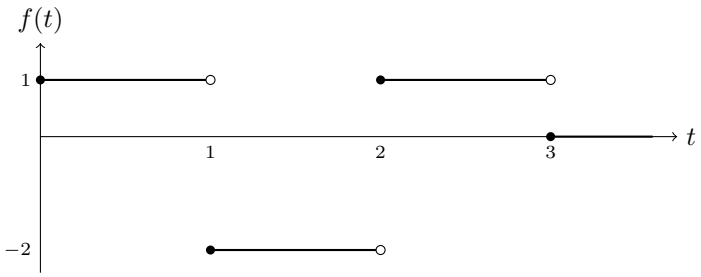
$$\boxed{F(s) = s^2 + O(s^3), \quad F(0) = F'(0) = 0, \quad F''(0) = 2.}$$

Directly, the second moment is

$$\int_0^3 t^2 f(t)\, dt = \tfrac{1}{3} - 2(\tfrac{7}{3}) + \tfrac{19}{3} = \boxed{2},$$

consistent with differentiating the finite-support transform twice.

Step 4: Separate two constraints from identity. The pulse is one throughout its first interval, so it is not zero. For $s > 0$, both $1 - e^{-s}$ and s are positive, giving $\boxed{F(s) > 0}$. The zero area and first moment eliminate only the constant and linear terms near $s = 0$. They do not eliminate all weighted integrals. The second moment already witnesses the remaining information. The graph makes the signed cancellation visible without replacing it by a pointwise sign claim.



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Question 9

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

For an integer $n \geq 2$, define a right-continuous staircase

$$f_n(t) = \frac{k}{n} \quad (\frac{k}{n} \leq t < \frac{k+1}{n},\ k = 0, \ldots, n-1), \qquad f_n(t) = 0 \quad (t \geq 1).$$

The target is $r(t) = t$ on $0 \leq t < 1$ and $r(t) = 0$ for $t \geq 1$.

Tasks

1. Express f_n using steps and derive a finite-sum formula for its transform.
2. Find $F_n(0)$ and the target transform $R(s)$, including $R(0)$. Give the full real convergence sets.
3. For $s > 0$, derive a bound on $0 \leq R(s) - F_n(s)$ from the pointwise staircase error.
4. Using that bound, find the least integer n that certifies $R(s) - F_n(s) \leq 0.01$ simultaneously for every $s \geq 1$. Distinguish this certificate from an exact minimal-error calculation.

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Question 9 – Solution

Strategy. Encode each upward increment and the final reset. A pointwise error bound becomes an integral certificate valid for a whole range of parameters.

Step 1: Include the final shutoff. There are increments $1/n$ at k/n, $1 \le k \le n-1$, followed by a drop $(n-1)/n$ at one. Hence

$$f_n(t) = \frac{1}{n}\sum_{k=1}^{n-1} u_{k/n}(t) - \frac{n-1}{n}u_1(t),$$

$$\boxed{F_n(s) = \frac{\sum_{k=1}^{n-1} e^{-ks/n} - (n-1)e^{-s}}{ns} \quad (s \neq 0).}$$

Transforming steps gives the formula for $s > 0$; finite-interval integration gives it for all $s \neq 0$.

Step 2: Evaluate area and the target. Each stair has width $1/n$, so

$$\boxed{F_n(0) = \frac{1}{n^2}\sum_{k=0}^{n-1} k = \frac{n-1}{2n}.}$$

Integration by parts on $[0,1]$ gives

$$\boxed{R(s) = \frac{1-(1+s)e^{-s}}{s^2} \ (s \neq 0), \qquad R(0) = \tfrac{1}{2}.}$$

Both original functions have finite support, so both transform domains are the whole real line. Their apparent singularities at zero are removable.

Step 3: Convert pointwise error to weighted error. On each half-open stair, $0 \le r - f_n < 1/n$, and both functions vanish from one onward. For $s > 0$,

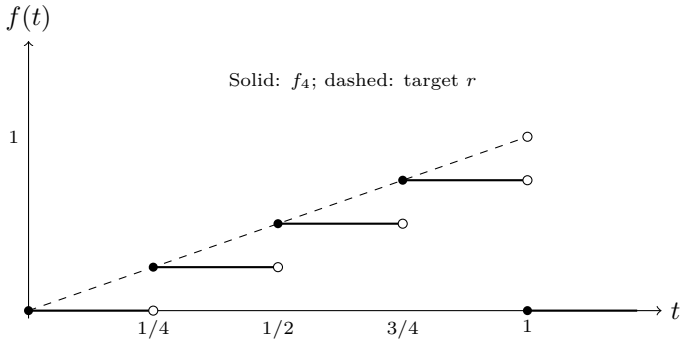
$$\boxed{0 \le R(s) - F_n(s) \le \frac{1}{n}\int_0^1 e^{-st}\,dt = \frac{1-e^{-s}}{ns}.}$$

At zero the actual error is $1/(2n)$, while the continuous limiting bound is $1/n$. The bound is sufficient, not an equality.

Step 4: Certify a simultaneous tolerance. The function $b(s) = \int_0^1 e^{-st}\,dt$ is decreasing for $s > 0$, since its derivative is $-\int_0^1 te^{-st}\,dt < 0$. Its largest value on $s \ge 1$ is $1 - e^{-1}$. Thus the certificate requires

$$n \ge 100(1-e^{-1}) \approx 63.2121, \qquad \boxed{n = 64}.$$

For $n = 63$ this bound exceeds .01 at $s = 1$; that does not prove the actual error exceeds .01. The graph shows $n = 4$ only to display the staircase geometry; it is not the certified design.



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Question 10

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s; a value at one isolated point does not change an integral.

An unknown rectangular pulse has height $A > 0$ on $[a, b)$ and is zero elsewhere, where $0 \leq a < b$. Let F be its transform and set $H(s) = sF(s)$. Exact data are

$$H(1) = \frac{3}{8}, \qquad H(2) = \frac{15}{64}, \qquad H(3) = \frac{63}{512}.$$

Tasks

1. Write the pulse using steps and derive $H(s)$ in terms of A, a, b.
2. Set $x = e^{-a}$ and $y = e^{-b}$. Use the ratios $H(2)/H(1)$ and $H(3)/H(1)$ to determine $x + y$ and xy.
3. Recover A, a, b, prove uniqueness within the stated pulse family, and check all three data.
4. Find the pulse area and full real transform domain. Explain, with a construction or a dimension argument, why these three data do not determine an arbitrary piecewise continuous signal.

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Question 10 – Solution

Strategy. Consecutive exponential measurements determine symmetric functions of the two endpoints. The ordering of the endpoints then removes the root ambiguity.

Step 1: Encode the window. The pulse is $f = A(u_a - u_b)$. Integrating on its finite support gives

$$F(s) = A\frac{e^{-as} - e^{-bs}}{s} \ (s \neq 0), \qquad H(s) = A(e^{-as} - e^{-bs}).$$

With $x = e^{-a}$ and $y = e^{-b}$, the restrictions become $0 < y < x \leq 1$.

Step 2: Recover the two symmetric quantities. Since $H(1) = A(x - y) > 0$,

$$\frac{H(2)}{H(1)} = x + y = \frac{5}{8}, \qquad \frac{H(3)}{H(1)} = x^2 + xy + y^2 = \frac{21}{64}.$$

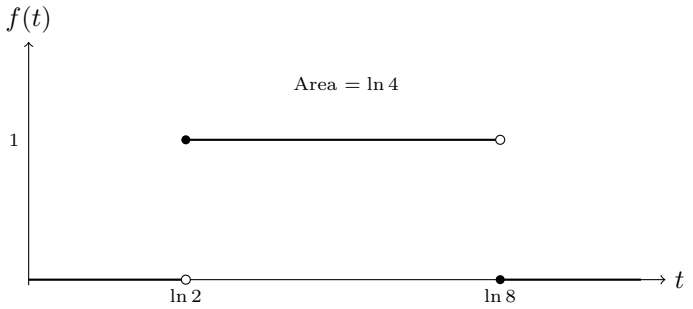
Therefore $xy = (5/8)^2 - 21/64 = \boxed{1/16}$. The numbers x, y are the roots of $z^2 - (5/8)z + 1/16 = 0$.

Step 3: Order the roots and find the height. The roots are $1/2$ and $1/8$. The condition $x > y$ gives

$$\boxed{a = \ln 2, \quad b = \ln 8, \quad A = \frac{3/8}{1/2 - 1/8} = 1.}$$

Both endpoint restrictions hold. Direct checks yield $x - y = 3/8$, $x^2 - y^2 = 15/64$ and $x^3 - y^3 = 63/512$. The two roots, their ordering and the nonzero first measurement fix all three parameters, proving uniqueness within this family.

Step 4: State what the data do and do not determine. The area is $\boxed{F(0) = A(b - a) = \ln 4}$. Finite support gives convergence for every real s, with zero removable in the displayed formula. To see the limitation of three measurements, take four disjoint unit pulses on $[4 + j, 5 + j)$, $j = 0, 1, 2, 3$. Requiring their linear combination to have zero transform at $s = 1, 2, 3$ gives three homogeneous linear equations in four coefficients. A nonzero coefficient vector exists; disjoint supports make its signal nonzero. Adding it preserves all three data but changes the original signal. This signed perturbation need not belong to the rectangular family, so it does not contradict the uniqueness just proved.



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Question 1

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider the IVP

$$y' + 2y = 3e^{-t}, \qquad y(0) = 1.$$

A student writes $Y(s) = 3/[(s+1)(s+2)]$.

Tasks

1. Derive the correct transformed equation, and identify which IVP the student actually solved.
2. Find the correct inverse using partial fractions.
3. Check the original differential equation and initial value directly. State the exact real convergence interval of Y.
4. Find the unique maximum of y and its time. Explain why a decaying forcing does not force the response to decrease from the start.

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Question 1 – Solution

Strategy. The derivative transform carries the initial value into the algebra. Keep that contribution before resolving the two poles.

Step 1: Include the initial term. Using $\mathcal{L}\{y'\} = sY - y(0)$ gives

$$(s+2)Y - 1 = \frac{3}{s+1}, \qquad Y = \frac{1}{s+2} + \frac{3}{(s+1)(s+2)}.$$

The student omitted the first term. Their expression solves the same equation with initial value zero, not one.

Step 2: Resolve and invert. Combining the terms yields

$$Y = \frac{s+4}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{2}{s+2}, \qquad \boxed{y(t) = 3e^{-t} - 2e^{-2t}.}$$

Each reciprocal factor matches an elementary exponential transform.

Step 3: Verify the IVP and domain. We have $y(0) = 3 - 2 = 1$ and $y' = -3e^{-t} + 4e^{-2t}$. Consequently

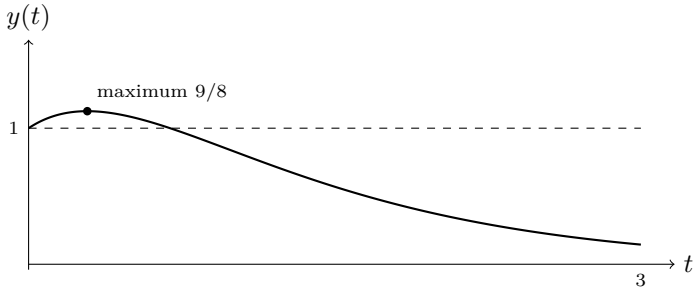
$$y' + 2y = (-3+6)e^{-t} + (4-4)e^{-2t} = 3e^{-t}.$$

The coefficients and forcing are continuous, so the usual uniqueness theorem for this linear IVP makes the verified solution the unique one. Its tail is asymptotic to $3e^{-t}$ and is positive; hence its exact real transform domain is $\boxed{s > -1}$. The student instead obtains $3e^{-t} - 3e^{-2t}$, whose initial value is zero.

Step 4: Locate the overshoot. Since $y' = e^{-t}(-3 + 4e^{-t})$, the derivative is positive before and negative after $e^{-t} = 3/4$. Thus

$$\boxed{t_* = \ln(4/3), \qquad y(t_*) = 9/8.}$$

Here $y'(0) = 1$ although the forcing decreases. The input initially exceeds the loss term $2y$, so the state first rises. It later decays to zero. The graph marks the overshoot above the initial level one.



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Question 2

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Solve

$$y'' + 3y' + 2y = 4, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Call the solution with the same initial data and zero forcing the initial-data response. Call the solution with forcing four and zero initial data the forced response.

Tasks

1. Derive Y and split it into the two responses defined above.
2. Invert both parts and obtain the complete solution.
3. Verify both initial data and the original equation. Check that the two parts satisfy their respective data and forcing.
4. Find the limiting value and the exact real transform domain. Explain why solving only the forced response would not solve the stated IVP.

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Question 2 – Solution

Strategy. Linearity separates the numerator contribution of initial data from that of the input, but the final answer needs both.

Step 1: Transform all derivative terms. The derivative transforms are $s^2Y - s + 1$ and $sY - 1$. Thus

$$(s^2 + 3s + 2)Y - s - 2 = 4/s,$$

$$Y = \underbrace{\frac{s+2}{(s+1)(s+2)}}_{Y_{\text{initial}}} + \underbrace{\frac{4}{s(s+1)(s+2)}}_{Y_{\text{forced}}}.$$

The first part simplifies to $1/(s+1)$; the second resolves as $2/s - 4/(s+1) + 2/(s+2)$.

Step 2: Invert and combine. Hence

$$y_{\text{initial}} = e^{-t}, \qquad y_{\text{forced}} = 2 - 4e^{-t} + 2e^{-2t},$$

$$\boxed{y(t) = 2 - 3e^{-t} + 2e^{-2t}.}$$

The initial-data response satisfies the homogeneous equation, and the constant term in the forced response supplies the constant forcing.

Step 3: Verify each contribution. For the sum, $y' = 3e^{-t} - 4e^{-2t}$ and $y'' = -3e^{-t} + 8e^{-2t}$. Then

$$y'' + 3y' + 2y = 4 + (-3 + 9 - 6)e^{-t} + (8 - 12 + 4)e^{-2t} = 4.$$

Also $y(0) = 1$ and $y'(0) = -1$. The initial-data part has values $1, -1$ and zero equation residual. The forced part has values $0, 0$ and residual four. These checks establish the intended decomposition; uniqueness of the linear IVP establishes the final solution.

Step 4: Interpret the steady value and domain. The exponentials vanish, so $\boxed{y(t) \to 2}$. This agrees with the constant balance $2y = 4$. Its ordinary transform converges exactly for $s > 0$: boundedness gives convergence above zero, while the eventually positive nonzero constant tail causes divergence at or below zero. Although the initial-data response has the wider domain $s > -1$, the complete response does not. Reporting only the forced response gets the limiting value right but misses both prescribed initial data.

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Question 3

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider a repeated-root operator with matching exponential forcing:

$$y'' + 2y' + y = e^{-t}, \qquad y(0) = 2, \qquad y'(0) = -2.$$

Tasks

1. Find Y and identify the order of the pole produced by the forcing.
2. Invert Y, keeping the factorial normalization of the repeated pole.
3. Verify the equation and both initial data directly. You may simplify the residual by setting $y = e^{-t}v$.
4. Determine whether y decreases to zero and whether $e^t y$ stays bounded. Explain why these conclusions are compatible and state the transform domain.

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Question 3 – Solution

Strategy. A repeated operator root and a matching input pole combine. Pole order controls the polynomial factor, not necessarily growth of the full time response.

Step 1: Form the transformed equation. The initial terms give

$$(s^2Y - 2s + 2) + 2(sY - 2) + Y = \frac{1}{s+1},$$

so

$$(s+1)^2Y = 2(s+1) + \frac{1}{s+1}, \qquad Y = \frac{2}{s+1} + \frac{1}{(s+1)^3}.$$

The forcing contributes a third-order pole at $s = -1$, although the differential equation has order two.

Step 2: Invert the repeated pole correctly. Since $\mathcal{L}\{t^2e^{-t}\} = 2/(s+1)^3$,

$$\boxed{y(t) = e^{-t}(2 + t^2/2).}$$

The factor $1/2$ is essential. Transforming the two terms back gives exactly the displayed Y.

Step 3: Check the residual and data. With $v = 2 + t^2/2$, the product rule gives $y' = e^{-t}(v' - v)$ and $y'' = e^{-t}(v'' - 2v' + v)$. Therefore

$$y'' + 2y' + y = e^{-t}v'' = e^{-t}.$$

At zero, $v(0) = 2$, $v'(0) = 0$, so $y(0) = 2$ and $y'(0) = -2$. These checks and linear-IVP uniqueness verify the solution.

Step 4: Distinguish a scaled response from the actual response. We have

$$y' = e^{-t}(-2 + t - t^2/2) = -\tfrac{1}{2}e^{-t}[(t-1)^2 + 3] < 0.$$

Thus y decreases strictly and tends to zero, since exponential decay dominates its quadratic factor. Meanwhile $\boxed{e^t y = 2 + t^2/2 \to \infty}$. Dividing out the decaying envelope changes the quantity being studied; there is no contradiction. The positive polynomial-exponential tail gives the exact domain $\boxed{s > -1}$, with divergence at and below the boundary.

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Question 4

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An undamped oscillator starts at rest:

$$y'' + 4y = \sin 2t, \qquad y(0) = 0, \qquad y'(0) = 0.$$

Define its energy by $E(t) = [y'(t)^2 + 4y(t)^2]/2$.

Tasks

1. Derive Y and invert it. You may use $\mathcal{L}\{t\cos 2t\} = (s^2 - 4)/(s^2 + 4)^2$.
2. Check the equation and both initial data directly.
3. Prove that the response is unbounded by evaluating a suitable explicit time sequence. State its exact real transform domain.
4. Derive the energy identity and calculate $E(n\pi)$ for positive integers n. Explain why bounded forcing can supply unbounded accumulated energy.

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Question 4 – Solution

Strategy. Matching forcing and natural frequencies produce repeated transform poles. Explicit values and the work identity distinguish real growth from a mere upper bound.

Step 1: Transform and invert. Zero initial data give

$$(s^2+4)Y = \frac{2}{s^2+4}, \qquad Y = \frac{2}{(s^2+4)^2}.$$

Combining the sine pair with the supplied time-cosine pair yields

$$\boxed{y(t) = \frac{\sin 2t - 2t\cos 2t}{8}.}$$

Indeed the forward numerator is $(s^2+4)/4 - (s^2-4)/4 = 2$.

Step 2: Verify the IVP. Direct differentiation gives

$$y' = \tfrac{t}{2}\sin 2t, \qquad y'' = \tfrac{1}{2}\sin 2t + t\cos 2t.$$

Adding $4y = \frac{1}{2}\sin 2t - t\cos 2t$ gives $\sin 2t$. Both initial values are zero, proving the claimed solution by uniqueness.

Step 3: Exhibit unbounded values and the domain. At $t = n\pi$,

$$\boxed{y(n\pi) = -n\pi/4 \longrightarrow -\infty.}$$

The actual response is therefore unbounded; a growing envelope alone would not prove that. Its linear envelope gives absolute transform convergence for $s > 0$. At zero, its finite integral is $[1-\cos 2R - R\sin 2R]/8$, which has no limit. Below zero, intervals around successive peaks of the $t\cos 2t$ term violate the Cauchy criterion. Hence the exact domain is $\boxed{s > 0}$.

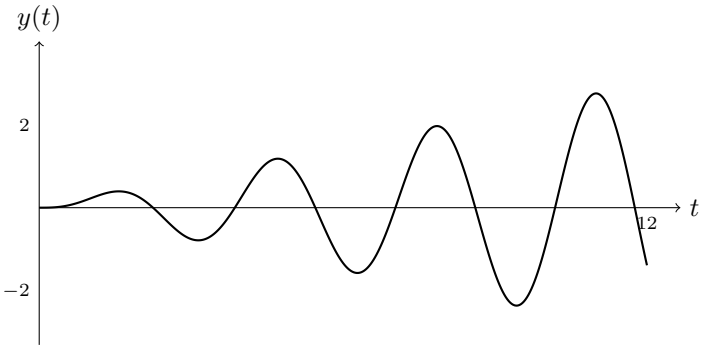
Step 4: Account for the energy input. Multiplying the equation by y' gives

$$E' = y'(y'' + 4y) = y'\sin 2t = \tfrac{t}{2}\sin^2 2t \geq 0.$$

At $n\pi$, the velocity is zero, so

$$\boxed{E(n\pi) = 2y(n\pi)^2 = n^2\pi^2/8.}$$

Although the forcing amplitude is at most one, its work accumulates over an increasing time interval. Resonance makes that accumulated work unbounded. The graph shows the growing displacement oscillations, not the energy itself.



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Question 5

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An unstable equation has adjustable initial velocity:

$$y'' - y = e^{-2t}, \qquad y(0) = 1, \qquad y'(0) = v.$$

Tasks

1. Derive and invert Y for arbitrary real v.
2. Find the unique v that gives a bounded solution, and verify the resulting IVP.
3. If the chosen velocity is perturbed by δ, derive the exact error in the solution. Determine the largest allowed $|\delta|$ that guarantees an error at most 0.01 for every $0 \leq t \leq 5$.
4. State the exact transform domains for the bounded choice and for every other v. Explain why the uncanceled operator factor alone does not determine the response domain.

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Question 5 – Solution

Strategy. Initial data control the coefficient of the unstable mode. Exact cancellation can give a bounded trajectory but does not remove sensitivity to an initial error.

Step 1: Transform and resolve the modes. The transformed equation is

$$(s^2-1)Y = s + v + \frac{1}{s+2}.$$

Partial fractions give

$$Y = \frac{v/2+2/3}{s-1} + \frac{-v/2}{s+1} + \frac{1/3}{s+2},$$

$$\boxed{y(t) = (v/2+2/3)e^{t} - (v/2)e^{-t} + \tfrac{1}{3}e^{-2t}.}$$

Recombining the fractions reproduces the transformed equation.

Step 2: Cancel the growing coefficient. Boundedness requires $v/2 + 2/3 = 0$, so the unique choice is

$$\boxed{v = -4/3, \qquad y_b(t) = \tfrac{2}{3}e^{-t} + \tfrac{1}{3}e^{-2t}.}$$

Then $y_b(0) = 1$, $y_b'(0) = -2/3 - 2/3 = -4/3$, and $y_b'' - y_b = e^{-2t}$. If the growing coefficient is nonzero, it eventually dominates both decaying terms, so no other velocity works.

Step 3: Quantify the initial-error tolerance. Replacing v by $-4/3 + \delta$ changes the solution by

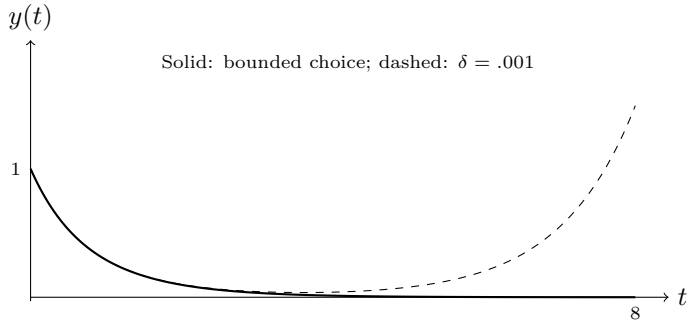
$$\boxed{y - y_b = \tfrac{\delta}{2}(e^{t} - e^{-t}) = \delta \sinh t.}$$

Since $\sinh t$ increases on $t \geq 0$, the maximum absolute error on $[0, 5]$ is $|\delta| \sinh 5$. The exact, necessary and sufficient tolerance is

$$\boxed{|\delta| \leq \frac{0.01}{\sinh 5} \approx 1.3477 \times 10^{-4}.}$$

Equality is permitted and attains the error limit at time five. The graph compares the bounded choice with a small positive velocity error.

Step 4: Check the actual tails. For the bounded choice, the nonzero slow tail is $(2/3)e^{-t}$, so the exact domain is $s > -1$. For every other v, a nonzero e^{t} term gives exact domain $s > 1$. The operator factor $s - 1$ cancels from the selected response numerator. Its presence in the operator does not mean it survives in every solution; the initial data determine the residue.



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Question 6

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Two parameters are unknown in the IVP

$$y'' + 2y' + 5y = Ae^{-t}, \qquad y(0) = 0, \qquad y'(0) = v.$$

Exact measurements are $e^{\pi/2}y(\pi/2) = 2$ and $e^{\pi/4}y(\pi/4) = 2$.

Tasks

1. Solve by Laplace transforms in terms of A, v.
2. Use the two measurements to recover A, v and prove uniqueness of this parameter recovery.
3. Verify the calibrated equation and both initial data. State the transform domain of the calibrated solution.
4. Would measurements taken only at $t = n\pi$, for positive integers n, identify these parameters? Justify your answer from the exact response, not from a count of samples.

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Question 6 – Solution

Strategy. Solve once with symbolic parameters, then evaluate the resulting basis functions at the measurement times.

Step 1: Transform with unknown initial velocity. The transformed equation is

$$[(s+1)^2+4]Y = v + \frac{A}{s+1}.$$

Writing $p = s+1$ and using $1/[p(p^2+4)] = (1/4)[1/p - p/(p^2+4)]$ gives

$$\boxed{y(t) = e^{-t}[\tfrac{v}{2}\sin 2t + \tfrac{A}{4}(1-\cos 2t)].}$$

Transforming the sine and constant-minus-cosine terms verifies the algebraic equation for sufficiently large s.

Step 2: Recover the parameters. At $t = \pi/2$, the local sine is zero and the cosine is -1, so $A/2 = 2$, giving $A = 4$. At $t = \pi/4$, the sine is one and the cosine is zero, so $v/2 + A/4 = 2$, giving $v = 2$. Thus

$$\boxed{A = 4, \quad v = 2, \qquad y = e^{-t}(1 - \cos 2t + \sin 2t).}$$

The first measurement fixes A independently, then the second fixes v with a nonzero coefficient. This proves uniqueness for the specified two-parameter model.

Step 3: Verify calibration and the IVP. Set $w = 1 - \cos 2t + \sin 2t$, so $y = e^{-t}w$. The product rule gives

$$y'' + 2y' + 5y = e^{-t}(w'' + 4w) = 4e^{-t}.$$

Also $w(0) = 0$ and $w'(0) = 2$, so $y(0) = 0$ and $y'(0) = 2$. The exact transform domain is $s > -1$. At its boundary the weighted signal is the nonzero periodic function w, with nonzero mean; below it fixed-sign lobe intervals fail the Cauchy criterion. Above it the exponential envelope gives absolute convergence.

Step 4: Identify blind sampling times. At every $t = n\pi$, both $\sin 2t = 0$ and $1 - \cos 2t = 0$. Therefore

$$\boxed{y(n\pi) = 0 \quad \text{for every } A, v.}$$

Even infinitely many such samples carry no information about either parameter. The issue is where the response basis vanishes, not merely how many measurements are collected. The two given times avoid this degeneracy.

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Question 7

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

The data are prescribed at time one, not at zero:

$$y'' + y = t, \qquad y(1) = 2, \qquad y'(1) = 0, \qquad t \geq 1.$$

Use a new time origin before applying a one-sided transform.

Tasks

1. Define $\tau = t - 1$ and $z(\tau) = y(1 + \tau)$. Derive the transformed IVP for z.
2. Find $Z(s) = \mathcal{L}_\tau\{z\}(s)$ and invert it.
3. Convert back to t and verify the equation and the data at time one.
4. A student treats the supplied values as $y(0), y'(0)$ and transforms the forcing as $1/s^2$. Explain both changes needed to repair that calculation. State the exact domain of Z.

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Question 7 – Solution

Strategy. Shift the independent variable and the forcing together. A Laplace transform beginning at the new origin must use data at that origin.

Step 1: Move the initial time. With $\tau = t - 1$, the derivatives have the same form but the forcing becomes $\tau + 1$. Thus

$$z'' + z = \tau + 1, \qquad z(0) = 2, \qquad z'(0) = 0.$$

Writing $Z = \mathcal{L}_\tau\{z\}$ gives

$$(s^2 + 1)Z - 2s = \frac{1}{s^2} + \frac{1}{s}.$$

Step 2: Resolve and invert. A useful decomposition is

$$Z = \frac{2s + 1/s^2 + 1/s}{s^2 + 1} = \frac{1}{s^2} + \frac{1}{s} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.$$

Therefore $\boxed{z(\tau) = \tau + 1 + \cos\tau - \sin\tau}$. Transforming each term back checks the decomposition, including the constant part of the shifted forcing.

Step 3: Return to the original clock. The solution on the requested domain is

$$\boxed{y(t) = t + \cos(t - 1) - \sin(t - 1), \qquad t \geq 1.}$$

Its first derivative is $1 - \sin(t - 1) - \cos(t - 1)$ and its second is $-\cos(t - 1) + \sin(t - 1)$. Hence $y'' + y = t$, while $y(1) = 2$ and $y'(1) = 0$. These are the actual specified data, so uniqueness on $[1, \infty)$ verifies the result.

Step 4: Diagnose the two clock errors. The values two and zero belong to $z(0), z'(0)$, not to values of y at an unspecified earlier time. Once that clock change is made, the forcing is $\tau + 1$ and its transform is $1/s^2 + 1/s$, not just $1/s^2$. The transform being inverted is Z, defined using $\tau \geq 0$. Since $z(\tau) = \tau + O(1)$ is eventually positive, it has exact domain $\boxed{s > 0}$. No extension of y to times before one was needed.

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Question 8

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An accumulated quantity enters an IVP:

$$y'(t) + y(t) + \int_0^t y(u)\,du = 1, \qquad y(0) = 0.$$

You may use $\mathcal{L}\{\int_0^t y(u)\,du\} = Y(s)/s$ where the integrals converge.

Tasks

1. Solve directly by Laplace transforms, including the integral term.
2. Find an explicit expression for $I(t) = \int_0^t y(u)\,du$ and verify the original equation.
3. Differentiate the original equation and determine the extra initial condition needed for the resulting second-order equation. Explain why omitting that condition enlarges the solution set.
4. Find the first positive zero of y, its limit, the limit of I, and the exact transform domain of Y. Explain why a constant right-hand side does not force a nonzero limiting y here.

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Question 8 – Solution

Strategy. Transform the accumulated state as well as the instantaneous state. Verify the original integral equation after any differentiation.

Step 1: Transform and complete the square. For sufficiently large s, the initial value gives

$$sY + Y + Y/s = 1/s, \qquad Y = \frac{1}{s^2 + s + 1}.$$

Let $b = \sqrt{3}/2$. Completing the square gives

$$\boxed{y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin(bt).}$$

This inverts $Y = 1/[(s + 1/2)^2 + b^2]$ with the required sine normalization.

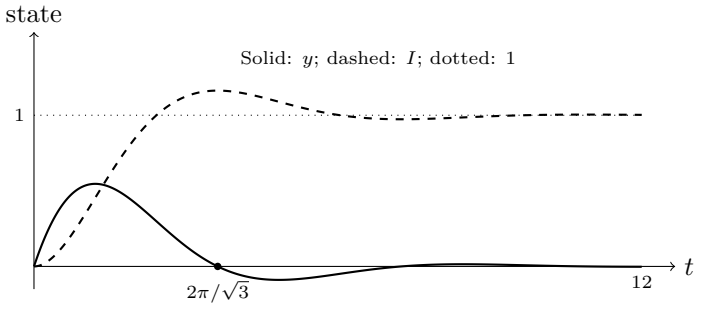
Step 2: Reconstruct the integral and verify. Integration, or differentiation of the following expression, gives

$$\boxed{I(t) = 1 - e^{-t/2}[\cos(bt) + \tfrac{1}{\sqrt{3}} \sin(bt)].}$$

It satisfies $I(0) = 0$ and $I' = y$. Also $y' = e^{-t/2}[\cos(bt) - \sin(bt)/\sqrt{3}]$. Substitution yields $y' + y + I = 1$ identically, and $y(0) = 0$. This checks the undifferentiated problem.

Step 3: Retain the compatibility condition. Differentiation gives $y'' + y' + y = 0$. Evaluating the original equation at zero gives $\boxed{y'(0) = 1}$. Thus the equivalent second-order IVP has data $y(0) = 0, y'(0) = 1$. If only $y(0) = 0$ is retained, a free initial slope remains; it sets the constant value of $y' + y + I$, which need not be one. Conversely, a solution of the differentiated equation with both correct data makes that quantity constant and equal to one, proving equivalence and uniqueness.

Step 4: Interpret the limiting balance. The first positive zero is $\boxed{t = \pi/b = 2\pi/\sqrt{3}}$. The function changes sign there and tends to zero, while $I(t) \to 1$. The accumulated state supplies the limiting balance, rather than a nonzero instantaneous state. The exact domain of Y is $\boxed{s > -1/2}$, from the damped sine and its nonconvergent boundary integral. The integral state's transform needs $s > 0$ because $I \to 1$; the initial derivation used this common domain. The wider domain of Y follows from the recovered function itself.



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Question 9

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Use Laplace transforms for the third-order IVP

$$y^{(3)} + 3y'' + 3y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Tasks

1. Derive the transformed equation, displaying the contributions of all three initial data.
2. Invert the result at the repeated pole, and check all three data.
3. Verify the third-order differential equation directly, using $y = e^{-t}v$ if useful.
4. Prove positivity and strict decrease for $t > 0$. Compute the total integral and exact transform domain, and explain why the initial flatness does not make the solution constant.

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Question 9 – Solution

Strategy. Higher derivatives introduce more initial terms. A repeated transform pole packages the polynomial-exponential solution compactly.

Step 1: Transform the derivative hierarchy. The general third-derivative rule is $\mathcal{L}\{y^{(3)}\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$. Here the first three derivatives transform to $sY - 1$, $s^2Y - s$ and $s^3Y - s^2$. Hence

$$(s^3Y - s^2) + 3(s^2Y - s) + 3(sY - 1) + Y = 0,$$

$$\boxed{Y = \frac{s^2 + 3s + 3}{(s+1)^3}.}$$

Even the zero initial values must be placed in the correct derivative formulas before simplifying.

Step 2: Expand at the repeated pole. Writing $p = s + 1$ makes the numerator $p^2 + p + 1$, so

$$Y = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}, \qquad \boxed{y = e^{-t}(1 + t + t^2/2).}$$

We have $y' = -t^2e^{-t}/2$ and $y'' = e^{-t}(t^2/2 - t)$. Thus the three initial values are exactly $1, 0, 0$.

Step 3: Verify the operator identity. For $y = e^{-t}v$, repeated product rules give

$$y^{(3)} + 3y'' + 3y' + y = e^{-t}v^{(3)}.$$

Here $v = 1 + t + t^2/2$ has third derivative zero. This verifies the equation everywhere. The linear third-order IVP with all three data has a unique solution.

Step 4: Check shape, area and domain. The polynomial factor is positive on $t \geq 0$, and $y' < 0$ for every $t > 0$, so y is positive and strictly decreasing after its initially flat point. It tends to zero. Direct exponential moments give

$$\boxed{\int_0^\infty y(t)\,dt = 1 + 1 + 1 = 3 = Y(0).}$$

The exact transform domain is $s > -1$; at or below -1 the positive polynomial tail makes the integral diverge. The data $y'(0) = y''(0) = 0$ describe only the initial point. The equation gives $y^{(3)}(0) = -1$, so the first change is cubic rather than absent.

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Question 10

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An input $q(t)$ is to produce a prescribed smooth rise in

$$y'' + 3y' + 2y = q(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

The target family is $y_k(t) = 1 - (1 + kt)e^{-kt}$, with $k > 0$. Allowed inputs have the form $q(t) = 2 + Ce^{-kt}$, with no term involving te^{-kt}.

Tasks

1. Find the target transform $Y_k(s)$ and derive the input transform required by the IVP.
2. Invert that input transform and determine exactly which $k > 0$ satisfy the allowed input form.
3. An additional requirement is $y(0) = 4$. Select k, C and verify the complete IVP with the resulting input.
4. Prove that the selected response increases to one without overshoot. Explain why the required input is unique once the full target trajectory is fixed.

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Question 10 – Solution

Strategy. Transform the target first, then use the differential operator to recover the necessary input. Check the permitted input family before selecting parameters.

Step 1: Transform the prescribed trajectory. The exponential and time-exponential pairs give

$$Y_k = \frac{1}{s} - \frac{1}{s+k} - \frac{k}{(s+k)^2} = \boxed{\frac{k^2}{s(s+k)^2}}.$$

All targets have $y_k(0) = y_k'(0) = 0$, so their required input transform is

$$Q_k = (s^2 + 3s + 2)Y_k = \frac{k^2(s+1)(s+2)}{s(s+k)^2}.$$

Step 2: Recover the input and impose the restriction. Partial fractions, verified by clearing denominators, give

$$Q_k = \frac{2}{s} + \frac{k^2 - 2}{s+k} - \frac{k(k-1)(k-2)}{(s+k)^2},$$

$$q_k(t) = 2 + (k^2 - 2)e^{-kt} - k(k-1)(k-2)te^{-kt}.$$

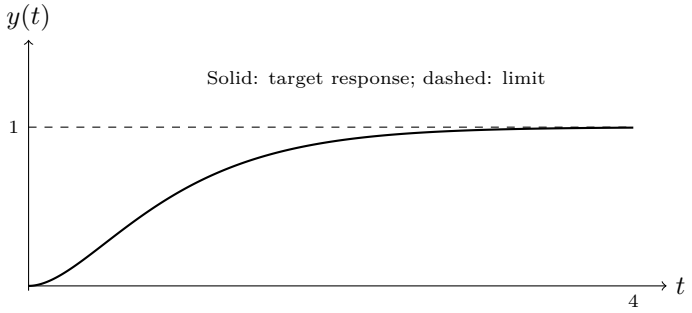
For $k > 0$, the last term vanishes exactly for $\boxed{k = 1 \text{ or } k = 2}$. Their respective values of C are -1 and 2. A nonzero te^{-kt} coefficient cannot be absorbed into a constant multiple of e^{-kt} on an interval.

Step 3: Select and verify the desired acceleration. Differentiation gives $y_k' = k^2te^{-kt}$ and $y_k'' = k^2(1-kt)e^{-kt}$, so $y_k''(0) = k^2$. The added requirement selects

$$\boxed{k = 2, \quad C = 2, \quad q = 2 + 2e^{-2t}, \quad y = 1 - (1+2t)e^{-2t}.}$$

Here $y' = 4te^{-2t}$, $y'' = (4-8t)e^{-2t}$, and $y'' + 3y' + 2y = 2 + 2e^{-2t}$. The initial values are $0, 0$ and the acceleration is four, as required. Both Y and Q have exact domain $s > 0$ because their original functions approach positive constants.

Step 4: Prove the rise and uniqueness of the input. Since $y'(t) > 0$ for $t > 0$, the response increases from zero. Also $1 - y = (1+2t)e^{-2t} > 0$ for every finite t, and this difference tends to zero. Thus $\boxed{0 < y(t) < 1 \ (t > 0), \quad y(t) \to 1}$ with no overshoot. Once the entire twice-differentiable target is fixed, the equation forces $q = y'' + 3y' + 2y$ pointwise. The input is unique, rather than merely one successful choice; linear-IVP uniqueness then verifies that it produces this trajectory.



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Question 1

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the regular IVP

$$(1+t)y' + y = 1, \qquad y(0) = 0, \qquad t \geq 0.$$

An answer in terms of a convergent parameter integral is acceptable for Y.

Tasks

1. Derive the differential equation satisfied by Y. Differentiate the entire transform of y' when handling ty'.
2. Solve that equation using an integrating factor and a condition at $s = \infty$ that a bounded time function must satisfy.
3. Identify the inverse and verify the original IVP. Check your parameter-integral expression directly against its forward transform.
4. State the exact real transform domain and derive the leading large-s behavior from the initial slope.

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Question 1 – Solution

Strategy. Multiplication by time makes a differential equation in the transform variable. Its integration constant must be selected using transform behavior.

Step 1: Differentiate the full derivative transform. Here $\mathcal{L}\{y'\} = sY$, so $\mathcal{L}\{ty'\} = -(Y + sY')$. The transformed equation is

$$sY - Y - sY' + Y = 1/s, \qquad \boxed{Y' - Y = -1/s^2}, \quad s > 0.$$

Treating $\mathcal{L}\{ty'\}$ as only $-sY'$ would introduce an erroneous extra term.

Step 2: Select the integration constant. Multiply by e^{-s}. A bounded function has $Y = O(1/s)$ as $s \to \infty$, hence $e^{-s}Y \to 0$. Integration from s to infinity gives

$$\boxed{Y(s) = e^s \int_s^\infty \frac{e^{-u}}{u^2}\, du.}$$

The other homogeneous term is Ce^s, incompatible with that bound unless $C = 0$. The inverse identified below is bounded, so this selection is self-consistent and verified rather than assumed without a check.

Step 3: Identify and forward-check the inverse. The original left side is $[(1+t)y]'$, giving $(1+t)y = t$ and

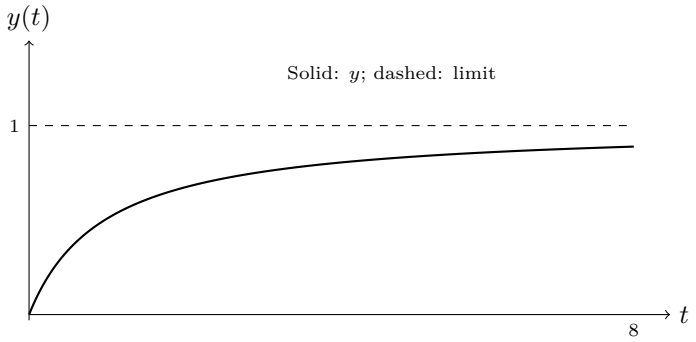
$$\boxed{y(t) = \frac{t}{1+t}.}$$

Its derivative is $(1 + t)^{-2}$, so $(1 + t)y' + y = 1$ and $y(0) = 0$. For $s > 0$, substitution $u = s(1 + t)$ gives $\mathcal{L}\{(1 + t)^{-1}\} = e^s \int_s^\infty e^{-u}/u\, du$. Thus the forward transform is $1/s - e^s \int_s^\infty e^{-u}/u\, du$, equal to the boxed parameter integral by integration by parts. The regular first-order IVP is unique.

Step 4: Check domain and onset. Since $y \to 1$ and is nonnegative, its exact domain is $\boxed{s > 0}$. At the origin $y'(0) = 1$, and the substitution $v = st$ gives

$$s^2 Y(s) = \int_0^\infty e^{-v} \frac{v}{1 + v/s}\, dv \longrightarrow 1.$$

The integrand is dominated by ve^{-v}, so dominated convergence justifies the limit. Thus $Y \sim 1/s^2$, matching the zero initial value and unit initial slope.



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Question 2

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve

$$y' + 2ty = 0, \qquad y(0) = 1, \qquad t \geq 0.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-v^2}\, dv = \sqrt{\pi}/2$.

Tasks

1. Derive the first-order equation for Y and solve it using an integrating factor.
2. Select the admissible transform branch by its behavior as $s \to \infty$.
3. Identify the time solution, verify the IVP, and prove that the Gaussian-integral expression is its forward transform for every real s.
4. Evaluate $Y(0)$ and prove $1/s - 2/s^3 \leq Y(s) \leq 1/s$ for $s > 0$. Interpret the resulting initial-value limit.

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Question 2 – Solution

Strategy. The variable damping gives a growing homogeneous solution in transform space; a genuine transform selects the complementary integral tail.

Step 1: Transform and apply the integrating factor. The equation becomes $sY - 1 - 2Y' = 0$, or

$$Y' - \tfrac{s}{2}Y = -\tfrac{1}{2}, \qquad (e^{-s^2/4}Y)' = -\tfrac{1}{2}e^{-s^2/4}.$$

Step 2: Impose the transform condition at infinity. For a bounded candidate, $Y = O(1/s)$, so $e^{-s^2/4}Y \to 0$. Integrating to infinity yields

$$\boxed{Y(s) = \tfrac{1}{2}e^{s^2/4}\int_s^\infty e^{-u^2/4}\,du = e^{s^2/4}\int_{s/2}^\infty e^{-v^2}\,dv.}$$

Any additional $Ce^{s^2/4}$ violates the bound. The recovered inverse will verify boundedness and the entire transform calculation.

Step 3: Identify the inverse and its full domain. Separation in time gives $\boxed{y = e^{-t^2}}$, with $y' = -2te^{-t^2}$ and $y(0) = 1$. Completing the square directly gives

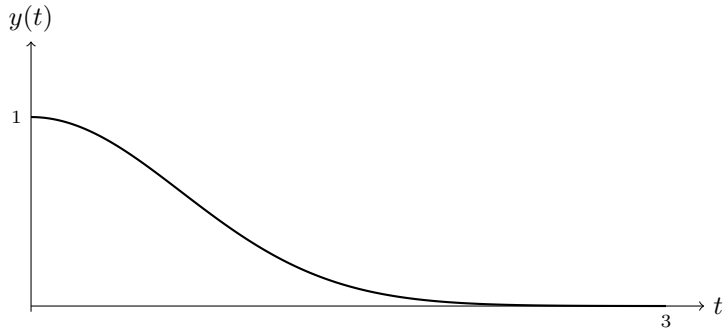
$$\int_0^\infty e^{-st-t^2}\,dt = e^{s^2/4}\int_{s/2}^\infty e^{-v^2}\,dv.$$

The Gaussian tail is integrable for every real s, so this is the exact real domain. This direct verification also justifies the selected transformed branch. Uniqueness follows from the regular first-order equation.

Step 4: Check the area and large-parameter bound. At zero, $\boxed{Y(0) = \sqrt{\pi}/2}$. For $t \geq 0$, the elementary inequality $e^{-x} \geq 1 - x$ gives $1 - t^2 \leq e^{-t^2} \leq 1$. Integrating with the positive kernel yields

$$\boxed{\frac{1}{s} - \frac{2}{s^3} \leq Y(s) \leq \frac{1}{s}, \qquad s > 0.}$$

The lower bound may be negative for small s but remains valid. Multiplying by s and taking $s \to \infty$ gives $sY(s) \to 1 = y(0)$. The graph shows the actual rapidly decaying time solution.



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Question 3

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Investigate the validity of the Laplace method for

$$y' - 2ty = 0, \qquad y(0) = 1, \qquad t \geq 0.$$

Do not assume that every global smooth IVP solution has an ordinary Laplace transform.

Tasks

1. Solve the time-domain IVP and determine whether its ordinary transform exists for any real s.
2. Formally apply the transform rules and solve the resulting equation for a candidate Y, without yet claiming it is a transform.
3. Show that every member of this formal family satisfies $Y(s) \to 0$ and $sY(s) \to 1$ as $s \to \infty$.
4. Explain why neither those limits nor solving the formal equation validates the calculation. Identify the failed hypothesis and contrast it with the sign-reversed damping equation.

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Question 3 – Solution

Strategy. Check the actual time solution first. Necessary transform limits can hold for functions of s that are not transforms of the IVP solution.

Step 1: Establish nonexistence of an ordinary transform. Separation and the initial datum give $\boxed{y = e^{t^2}}$, which is smooth and global. For any fixed real s, however, $t^2 - st \geq t^2/2$ for sufficiently large t. Thus

$$\int_0^\infty e^{-st} y(t)\, dt = \int_0^\infty e^{t^2 - st}\, dt = \infty.$$

The exact real transform domain is empty.

Step 2: Solve the formal transformed equation. Pretending the rules apply gives $sY - 1 + 2Y' = 0$, so

$$Y' + \tfrac{s}{2} Y = \tfrac{1}{2}, \qquad \boxed{Y_C(s) = e^{-s^2/4}[C + \tfrac{1}{2} \int_0^s e^{u^2/4}\, du].}$$

This is a valid family of solutions to the differential equation in s. It is not yet a family of Laplace transforms.

Step 3: Check the deceptively correct limits. As $s \to \infty$, l'Hopital's rule gives

$$\frac{\int_0^s e^{u^2/4}\, du}{2e^{s^2/4}/s} \longrightarrow 1,$$

because the derivative ratio is $1/(1 - 2/s^2) \to 1$. Hence the integral contribution to Y_C is asymptotic to $1/s$, while $Ce^{-s^2/4}$ decays faster. Every real C therefore satisfies

$$\boxed{Y_C(s) \to 0, \qquad sY_C(s) \to 1.}$$

These two necessary-looking conditions do not even select a unique formal branch here.

Step 4: Identify the invalid step. The ordinary integrals defining Y, the transform of ty, and the derivative transform do not converge for any real s. The time solution is not of exponential order, and direct growth proves failure, not merely absence of a sufficient hypothesis. Applying transform identities to these divergent integrals was unjustified. The limits in Step3 cannot repair that step. In contrast, changing the sign to $y' + 2ty = 0$ gives e^{-t^2}, whose transform exists for every real s and can be verified directly. A differential equation in transform space is useful only after its relationship to convergent time integrals is established.

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Question 4

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider a singular initial point:

$$ty' + 2y = e^{-t}, \qquad y(0) = \tfrac{1}{2}.$$

Seek a solution continuously differentiable on $[0, \infty)$ and satisfying the equation also at zero.

Tasks

1. Check compatibility at zero and determine whether boundedness near zero leaves a free integration constant in the time equation.
2. Derive and solve the equation for Y, selecting a branch that tends to zero at large positive s.
3. Identify and verify the inverse, including its limiting value and derivative at zero.
4. Find the full real convergence set and $Y(0)$. Explain why a finite transform value at zero need not imply a finite derivative there.

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Question 4 – Solution

Strategy. A vanishing leading coefficient imposes compatibility. Boundedness at the singular endpoint can replace a free constant that would exist away from it.

Step 1: Inspect the singular endpoint. At zero, the equation requires $2y(0) = 1$, so the given datum is compatible. For $t > 0$, multiplying by t gives $(t^2y)' = te^{-t}$. Its general integral has an additive constant in t^2y. Boundedness of y forces that constant to be zero, hence

$$\boxed{y(t) = \frac{1 - (1+t)e^{-t}}{t^2} \quad (t > 0).}$$

This proves uniqueness among bounded solutions near zero, without invoking a regular-IVP theorem at a singular point.

Step 2: Solve in transform space. The full rule $\mathcal{L}\{ty'\} = -Y - sY'$ gives $Y - sY' = 1/(s+1)$. Therefore

$$(Y/s)' = -\frac{1}{s^2(s+1)}, \qquad Y = s\int_s^\infty \frac{du}{u^2(u+1)} = \boxed{1 - s\ln\frac{s+1}{s}}, \quad s > 0.$$

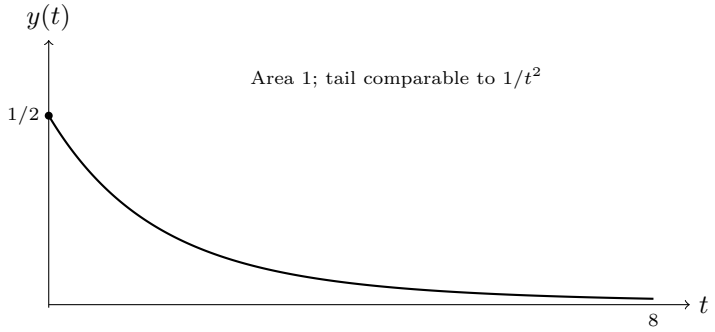
The omitted homogeneous term is Cs and cannot tend to zero. The inverse below verifies boundedness and transform existence.

Step 3: Verify the endpoint and forward transform. A nonsingular representation is $y(t) = \int_0^1 ue^{-ut}\,du$, including at zero. Thus $y(0) = 1/2$, $y'(0) = -1/3$, and differentiation verifies the original equation (or use $(t^2y)' = te^{-t}$ for $t > 0$ and compatibility at zero). For $s > 0$, integrating first in time gives

$$Y(s) = \int_0^1 \frac{u}{s+u}\,du = 1 - s\ln\frac{s+1}{s}.$$

The finite absolute double integral justifies interchange.

Step 4: Check the boundary rather than just the formula. The function is positive and $y(t) \sim 1/t^2$. Therefore the exact real convergence set is $\boxed{[0,\infty)}$, with divergence for negative s. At zero the same nonnegative double integral gives $\boxed{Y(0) = \int_0^1 1\,du = 1}$. But $ty(t) \sim 1/t$, so its first moment diverges. Indeed $Y'(s) = -\ln[(s+1)/s] + 1/(s+1) \to -\infty$ as $s \downarrow 0$. A finite boundary area does not imply a finite boundary moment.



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Question 5

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Seek a twice continuously differentiable solution of

$$ty'' + 2y' + ty = 0, \qquad y(0) = 1,$$

including at $t = 0$. When checking the transform boundary, you may use the estimate $|\int_R^M e^{-st} \sin t/t \, dt| \leq 2/R$ for $M > R > 0$, uniformly for $s \geq 0$, obtained by integration by parts.

Tasks

1. Determine the initial slope forced by the equation at zero, then derive the differential equation for Y.
2. Solve for $Y(s)$ on $s > 0$ using its decay at infinity and identify the inverse.
3. Use $w(t) = ty(t)$ to verify the equation and prove uniqueness in the stated smooth class. Explain why the initial slope disappeared from the transformed equation.
4. Find the exact real transform domain, including the ordinary boundary value at zero. Use the supplied uniform tail estimate to justify passage to that value.

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Question 5 – Solution

Strategy. The singular equation fixes a missing initial slope. Transform differentiation removes that slope algebraically, so compatibility must be checked separately.

Step 1: Recover compatibility and transform. At zero the equation requires $\boxed{y'(0) = 0}$. Since

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2Y' + 1,$$

adding $2\mathcal{L}\{y'\}$ and $\mathcal{L}\{ty\}$ gives

$$\boxed{(s^2+1)Y' = -1.}$$

Step 2: Select and identify the inverse. The condition $Y \to 0$ as $s \to \infty$ gives

$$Y(s) = \int_s^\infty \frac{du}{1+u^2} = \arctan(1/s), \quad s > 0.$$

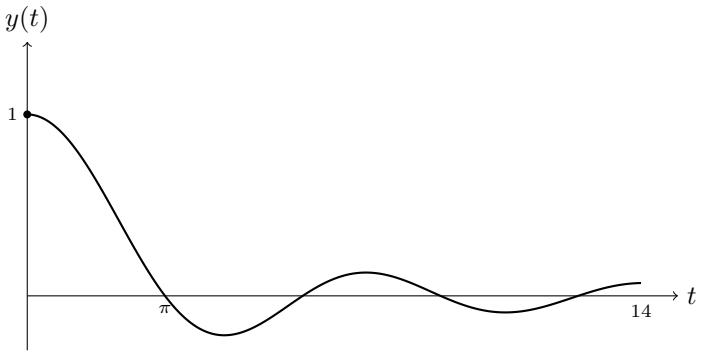
The time function $\boxed{y(t) = \sin t/t \ (t > 0), \quad y(0) = 1}$ has representation $y = \int_0^1 \cos(ut)\,du$. Transforming this integral for $s > 0$ yields $\int_0^1 s/(s^2+u^2)\,du = \arctan(1/s)$. Absolute double integrability justifies the interchange.

Step 3: Verify smoothness and uniqueness. The expansion $y = 1 - t^2/6 + O(t^4)$ supplies a smooth extension with slope zero. Set $w = ty$. Then $w'' + w = ty'' + 2y' + ty = 0$, with $w(0) = 0$, $w'(0) = 1$. The regular oscillator IVP has the unique solution $w = \sin t$, proving the original equation and uniqueness of its smooth extension. The term $-y'(0)$ is constant in s, so it disappears when differentiating the transform of y''; arbitrary initial slopes are nevertheless not admissible.

Step 4: Establish the boundary value. The supplied estimate makes the tails uniformly small for $s \geq 0$. On each fixed finite interval the damped integrals converge to the undamped one as $s \downarrow 0$. Taking the interval long first and then taking $s \downarrow 0$ proves

$$\boxed{Y(0) = \int_0^\infty \frac{\sin t}{t}\,dt = \pi/2.}$$

The boundary integral is conditional: intervals where $|\sin t| \geq 1/2$ give a divergent harmonic lower bound for its absolute integral. For $s < 0$, fixed-length positive lobe intervals have weighted integrals growing like $e^{|s|t}/t$, violating the Cauchy criterion. The exact real domain is $\boxed{[0, \infty)}$.



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Question 6

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve the regular second-order IVP

$$(1+t)y'' + y' = 0, \qquad y(0) = 0, \qquad y'(0) = 1.$$

A convergent parameter integral may be retained in the formula for Y.

Tasks

1. Derive the differential equation for Y, including all initial contributions.
2. Solve it with an integrating factor and explain the condition that removes its homogeneous term.
3. Identify and verify the time solution, then check the parameter-integral transform directly.
4. Prove $0 < Y(s) < 1/s^2$ for $s > 0$ and determine the exact real convergence interval. Explain why a growing time solution can still be transformable.

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Question 6 – Solution

Strategy. Keep the initial derivative term from the unmultiplied second derivative. The coefficient $1 + t$ never vanishes on the time domain.

Step 1: Transform with both data. The three contributions are

$$\mathcal{L}\{ty''\} = -2sY - s^2Y', \quad \mathcal{L}\{y''\} = s^2Y - 1, \quad \mathcal{L}\{y'\} = sY.$$

Hence $-s^2Y' + (s^2 - s)Y = 1$, or

$$\boxed{Y' - (1 - 1/s)Y = -1/s^2}, \qquad s > 0.$$

Step 2: Select the parameter integral. The integrating factor is se^{-s}, so $(se^{-s}Y)' = -e^{-s}/s$. An exponential-order continuous time solution has $Y = O(1/s)$ for large s, and thus $se^{-s}Y \to 0$. Therefore

$$\boxed{Y(s) = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u}\, du.}$$

An additional term Ce^s/s violates the large-parameter bound unless $C = 0$. Existence and the growth condition are verified by the recovered solution.

Step 3: Identify and check the inverse. In time, the left side is $[(1 + t)y']'$. Integrating with the data gives $y' = 1/(1 + t)$ and

$$\boxed{y(t) = \ln(1 + t).}$$

It has $y(0) = 0$, $y'(0) = 1$, $y'' = -1/(1 + t)^2$, and zero equation residual. The coefficient is regular, so uniqueness applies. Integrating its transform by parts gives

$$Y(s) = \frac{1}{s} \int_0^\infty \frac{e^{-st}}{1 + t}\, dt = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u}\, du,$$

where the upper boundary vanishes for $s > 0$.

Step 4: Compare growth rates and domains. For $t > 0$, $0 < \ln(1+t) < t$. Multiplying by the positive kernel and integrating gives

$$\boxed{0 < Y(s) < \frac{1}{s^2}, \qquad s > 0.}$$

The exact domain is $s > 0$: logarithmic growth is dominated by the exponential kernel above zero, whereas its positive time integral diverges at zero and for negative parameters. Unboundedness by itself does not prevent a Laplace transform; growth relative to the kernel is what matters.

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Question 7

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the local IVP

$$(t-1)y' + y = 0, \qquad y(0) = 1.$$

For $0 < T < 1$, define the finite transform $F_T(s) = \int_0^T e^{-st} y(t)\, dt$.

Tasks

1. Solve the IVP up to its first singularity and decide whether a global classical solution on $[0, \infty)$ exists.
2. Formally applying the infinite-transform rules gives $Y' + Y = 1/s$. Identify why this equation cannot be used as an ordinary transform solution of the IVP.
3. Derive the correct differential equation for F_T by retaining the upper boundary term during integration by parts.
4. Check that finite-transform equation directly, including its value at $s = 0$. Explain why a finite cutoff repairs the calculation but does not remove the singularity.

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Question 7 – Solution

Strategy. An interior zero of the leading coefficient can stop the classical solution. A finite transform before that point must retain its endpoint term.

Step 1: Locate the obstruction in time. Since $[(t-1)y]' = 0$, the datum gives $(t-1)y = -1$. Thus

$$\boxed{y(t) = \frac{1}{1-t}, \qquad 0 \le t < 1.}$$

This diverges as $t \uparrow 1$ and admits no continuous classical continuation through one. There is no global classical IVP solution on the full half-line. The integral up to one already diverges for every real s, because its positive weight stays bounded below near one while $1/(1-t)$ has a logarithmic divergence.

Step 2: Diagnose the formal infinite equation. The formal substitutions $\mathcal{L}\{ty'\} = -Y - sY'$ and $\mathcal{L}\{y'\} = sY - 1$ would give $Y' + Y = 1/s$. But the required infinite time integrals do not exist and no global classical solution satisfies the starting hypotheses. Solving this formal equation cannot provide an ordinary transform of the local IVP solution.

Step 3: Retain the actual cutoff boundary. For $T < 1$, integration by parts gives

$$0 = [e^{-st}(t-1)y(t)]_0^T + s\int_0^T e^{-st}(t-1)y(t)\,dt.$$

The first term is $1 - e^{-sT}$, using $(T-1)y(T) = -1$. Since $\int_0^T te^{-st}y = -F_T'$, the result is

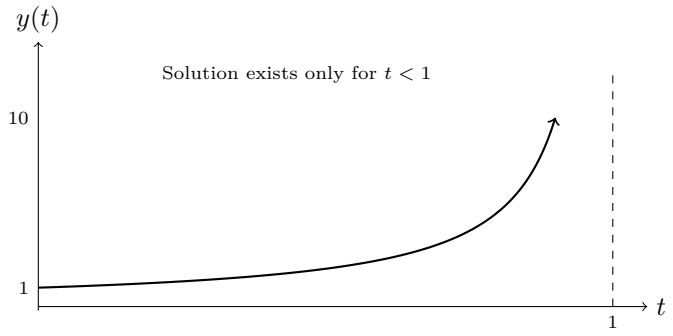
$$\boxed{F_T'(s) + F_T(s) = \frac{1 - e^{-sT}}{s} \quad (s \neq 0).}$$

Discarding the upper endpoint would lose the term $-e^{-sT}/s$.

Step 4: Verify the finite identity and its limits. Direct differentiation on the compact interval gives

$$F_T' + F_T = \int_0^T (1-t)e^{-st}y(t)\,dt = \int_0^T e^{-st}\,dt.$$

This is the stated expression for $s \neq 0$ and equals $\boxed{T}$ at $s = 0$. Also $F_T(0) = -\ln(1-T)$ and $F_T'(0) = T + \ln(1-T)$, whose sum is T. Every finite F_T exists for all real s, but it diverges as $T \uparrow 1$ for each fixed s. A valid finite-window calculation does not supply a missing continuation through the pole. The plot stops short of the singular time.



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Question 8

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Classify the data for which a twice continuously differentiable solution on $[0, \infty)$ can satisfy

$$t^2y'' + 2ty' - 2y = 0, \qquad y(0) = a, \qquad y'(0) = b.$$

For a smooth exponential-order function with $a = 0$, you may use $\lim_{s\to\infty} s^2Y(s) = b$.

Tasks

1. Find the compatibility condition at zero and solve the time equation for $t > 0$ using trial powers t^m.
2. Derive the second-order differential equation for Y, accounting for the effect of two parameter derivatives on the initial terms.
3. Solve the transformed equation using trial powers s^m. Determine what the condition $Y \to 0$ fixes and what initial information is still needed.
4. Classify all admissible pairs (a, b), verify the resulting solutions, and state their exact transform domains. Is specifying only $y(0)$ enough for uniqueness?

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Question 8 – Solution

Strategy. Singular coefficients can erase initial data from the transformed equation. Recover admissibility and the remaining amplitude separately.

Step 1: Check compatibility and time powers. At zero, the equation requires $\boxed{a = 0}$. For $t > 0$, trial powers give $m(m-1) + 2m - 2 = (m-1)(m+2) = 0$. The two independent solutions are t and t^{-2}, so $y = Ct + Dt^{-2}$. They span the solutions on $(0, \infty)$ because the normalized equation is regular there. Continuity at zero forces $D = 0$; the initial slope then sets $C = b$.

Step 2: Transform the polynomial coefficients. For sufficiently large s,

$$\mathcal{L}\{t^2 y''\} = \frac{d^2}{ds^2}[s^2 Y - sa - b] = 2Y + 4sY' + s^2 Y'',$$

while $2\mathcal{L}\{ty'\} = -2Y - 2sY'$. Adding $-2Y$ gives

$$\boxed{s^2 Y'' + 2sY' - 2Y = 0.}$$

Both a and b vanished under differentiation; their disappearance is not permission to ignore the time-domain data.

Step 3: Select the transform and its amplitude. Trial powers of s have exponents one and -2, yielding $Y = As + B/s^2$ on $s > 0$. A genuine transform of an exponential-order continuous function tends to zero at large s, so $A = 0$. This still leaves B arbitrary. The supplied initial-slope limit gives $\boxed{B = b}$ and hence $Y = b/s^2$. The initial-value limit $sY \to 0$ also agrees with the necessary datum $a = 0$.

Step 4: Verify and classify uniqueness. All and only the data $\boxed{a = 0,\ b \in \mathbb{R}}$ are admissible, with unique solution $\boxed{y = bt}$. Substitution gives $0 + 2tb - 2bt = 0$, and both data hold. For $b \neq 0$, the exact domain is $s > 0$; the signed linear tail diverges at or below zero. For $b = 0$, the zero function has transform zero for every real s. Specifying only $y(0) = 0$ leaves the entire family bt, so it does not determine a unique solution at this singular point.

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Question 9

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

An unknown parameter $\alpha > 0$ appears in

$$(1 + t)y' + \alpha y = 0, \qquad y(0) = 1.$$

An exact measurement gives $\int_0^\infty y(t)\,dt = 1/2$. A second proposed measurement is $\int_0^\infty ty(t)\,dt = 1/3$.

Tasks

1. Derive the differential equation for Y and solve it as a convergent parameter integral for $s > 0$.
2. Identify and verify the time solution for general $\alpha > 0$.
3. Determine which parameters admit a finite area, then recover α from the first measurement.
4. Test the proposed first moment for consistency, and state the exact real transform domain of the recovered solution.

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Question 9 – Solution

Strategy. Solve the parameter family before fitting measurements. A boundary transform value is meaningful only when the actual time integral converges.

Step 1: Derive and solve the transform equation. Using $\mathcal{L}\{y'\} = sY - 1$ and $\mathcal{L}\{ty'\} = -Y - sY'$ gives

$$-sY' + (s + \alpha - 1)Y = 1.$$

The integrating factor for the divided equation is $e^{-s}s^{1-\alpha}$, so its product with Y has derivative $-e^{-s}s^{-\alpha}$. The transform bound at infinity selects

$$\boxed{Y(s) = e^s s^{\alpha-1} \int_s^\infty e^{-u}u^{-\alpha}\,du, \qquad s > 0.}$$

The alternative homogeneous term $Ce^s s^{\alpha-1}$ violates decay as $s \to \infty$.

Step 2: Verify the parameter family in time. Separation gives $\boxed{y(t) = (1+t)^{-\alpha}}$. Its derivative is $-\alpha(1+t)^{-\alpha-1}$, so the equation and initial value hold. The coefficient $1+t$ is nonzero on $t \geq 0$, establishing regular-IVP uniqueness. Substitution $u = s(1+t)$ in the forward integral gives exactly the parameter expression above, verifying its inversion.

Step 3: Use the convergent area to identify the parameter. The positive power tail has finite area exactly when $\alpha > 1$, and then

$$\int_0^\infty (1+t)^{-\alpha}\,dt = \frac{1}{\alpha - 1}.$$

Equating this to $1/2$ gives the unique value $\boxed{\alpha = 3}$. This uses convergence of the integral, not a formal substitution into an expression initially derived only for $s > 0$.

Step 4: Test the second datum rather than refit. A finite first moment requires $\alpha > 2$. Substituting $x = 1 + t$ gives

$$\int_0^\infty \frac{t}{(1+t)^\alpha}\,dt = \frac{1}{\alpha - 2} - \frac{1}{\alpha - 1} = \frac{1}{(\alpha - 1)(\alpha - 2)}.$$

At $\alpha = 3$, it is $\boxed{1/2}$, not $1/3$, so the proposed data are inconsistent with this family. The recovered positive tail is $(1+t)^{-3}$; its exact transform domain is $\boxed{[0, \infty)}$. Positive s gives exponential damping, zero gives the finite area $1/2$, and negative s makes the weighted power tail diverge.

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Question 10

Use ordinary one-sided Laplace integrals for real s. Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Choose constants A, B so that a twice continuously differentiable solution on $[0, \infty)$ satisfies

$$ty' + (t+2)y = (At + B)e^{-t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

The equation must hold at the singular initial point as well as for $t > 0$.

Tasks

1. Derive the differential equation for Y without inserting unsupported free initial terms.
2. Solve it for general A, B and select the branch that can be an ordinary transform.
3. Use the recovered time solution and the two initial conditions to determine A, B. Verify the resulting equation at and away from zero.
4. Prove uniqueness in the stated smooth class, determine the exact transform domain, and describe the sign and monotonicity of the selected response.

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Question 10 – Solution

Strategy. The data lost from the transform equation return as compatibility conditions on the forcing. Boundedness removes the singular time-domain homogeneous mode.

Step 1: Transform the time multipliers. The three left-hand terms transform to $-Y - sY'$, $-Y'$ and $2Y$. Thus

$$\boxed{-(s+1)Y' + Y = \frac{A}{(s+1)^2} + \frac{B}{s+1}.}$$

The initial value in $sY - y(0)$ differentiates to zero in $\mathcal{L}\{ty'\}$. It must still be imposed on the inverse, not added back as a fictitious term.

Step 2: Solve and remove the inadmissible branch. Writing $p = s + 1$, divide by p^2 to obtain

$$(Y/p)' = -A/p^4 - B/p^3.$$

Integration gives

$$Y = \frac{A/3}{(s+1)^2} + \frac{B/2}{s+1} + C(s+1).$$

Decay requires $C = 0$. The remaining terms invert to $\boxed{y = e^{-t}(At/3 + B/2)}$, whose transform can now be verified directly.

Step 3: Recover the forcing constants and verify. The initial value requires $B/2 = 1$, and the initial slope requires $A/3 - B/2 = 0$. Hence

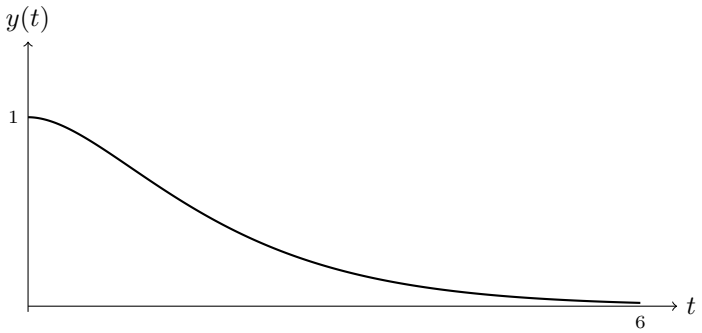
$$\boxed{A = 3, \quad B = 2, \qquad y = (1+t)e^{-t}.}$$

Its derivative is $-te^{-t}$, so

$$ty' + (t+2)y = e^{-t}[-t^2 + (t+2)(1+t)] = (3t+2)e^{-t}.$$

At zero the equation gives $2y(0) = B$, and both specified data hold. The smooth extension also has $y''(0) = -1$.

Step 4: Prove uniqueness and check the response. For $t > 0$, multiplying the original first-order equation by te^t gives $(t^2e^ty)' = At^2 + Bt$. Thus every local solution differs from the selected one by Ke^{-t}/t^2. This mode is unbounded at zero unless $K = 0$, proving uniqueness in the stated smooth class. The selected function is positive, with $y' < 0$ for $t > 0$, and decreases from one to zero after an initially horizontal tangent. Its positive polynomial-exponential tail gives exact transform domain $\boxed{s > -1}$. The parameter restrictions arise from compatibility, not from a regular existence theorem where the leading coefficient vanishes.



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Question 1

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A first-order system starts above zero and receives a finite pulse:

$$y' + 2y = 3[H_1(t) - H_3(t)], \qquad y(0) = 1, \quad t \geq 0.$$

Tasks

1. Find $Y(s)$ and invert it in step-function form, retaining the initial response.
2. Write the solution on the three time intervals and verify the equation and matching values.
3. Find the jumps in y' at both switches. Explain whether y jumps.
4. Locate the global maximum on $[0, \infty)$, prove it is global, and state the exact real transform domain.

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Question 1 – Solution

Strategy. Keep the initial response separate from the two delayed step responses, then compare the three monotonic pieces.

Step 1: Transform and invert. The initial term gives $(s+2)Y - 1 = 3(e^{-s} - e^{-3s})/s$, hence

$$Y = \frac{1}{s+2} + \frac{3(e^{-s} - e^{-3s})}{s(s+2)}, \qquad s > 0.$$

Since $\mathcal{L}^{-1}\{1/[s(s+2)]\} = (1 - e^{-2t})/2$,

$$\boxed{y = e^{-2t} + \tfrac{3}{2}H_1(t)(1 - e^{-2(t-1)}) - \tfrac{3}{2}H_3(t)(1 - e^{-2(t-3)}).}$$

Step 2: Match the pieces. Set $M = \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-4}$. The same solution is

$$y(t) = \begin{cases} e^{-2t}, & 0 \le t < 1, \\ \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-2(t-1)}, & 1 \le t < 3, \\ Me^{-2(t-3)}, & t \ge 3. \end{cases}$$

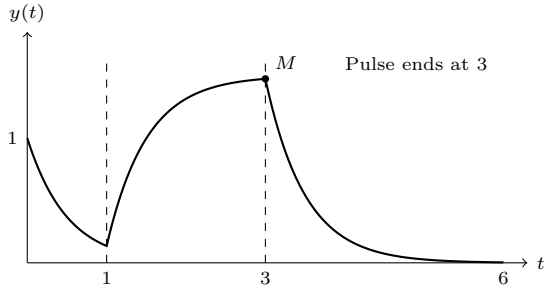
The matching values are $y(1) = e^{-2}$ and $y(3) = M$. Each piece satisfies $y' + 2y = 0, 3, 0$, respectively, and $y(0) = 1$. These matching conditions also give uniqueness interval by interval.

Step 3: Identify the corners. Writing $[v]_a = v(a+) - v(a-)$, the equation and continuity imply

$$[y']_1 = 3, \qquad [y']_3 = -3, \qquad [y]_1 = [y]_3 = 0.$$

The step factors multiply responses that vanish at their activation times, so no jump in y is introduced.

Step 4: Compare all possible maxima. The solution decreases on $[0, 1]$, increases on $[1, 3]$ toward $3/2$, then decreases to zero. Moreover $M > \frac{3}{2} - \frac{3}{2}e^{-4} > 1$, since $e^4 > 3$. Thus the unique global maximum is $\boxed{y(3) = M}$. Its nonzero tail is exactly a positive multiple of e^{-2t}, so the exact real transform domain is $\boxed{s > -2}$; the displayed formula extends removably across $s = 0$.



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Question 2

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A ramp is switched on without resetting its clock, then switched off:

$$y' + y = f(t), \qquad y(0) = 0, \qquad f(t) = \begin{cases} t, & 1 \leq t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

A proposed input transform is $(e^{-s} - e^{-3s})/s^2$.

Tasks

1. Explain the error in the proposed transform and derive the correct $F(s)$.
2. Find $Y(s)$ and invert it, rewriting each active input in its own delayed time.
3. Give the piecewise solution, its maximum, and the two jumps in y'.
4. Compute $\int_0^\infty y(t)\,dt$ directly and from the differential equation. Explain the apparent singularities in the transform formula.

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Question 2 – Solution

Strategy. A window does not reset the ramp's clock. At each endpoint rewrite t as delayed time plus the endpoint.

Step 1: Shift the whole input. The input is $H_1(t)[(t-1)+1] - H_3(t)[(t-3)+3]$, so

$$\boxed{F = e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right).}$$

The proposed expression instead transforms $H_1(t)(t-1) - H_3(t)(t-3)$, which has a constant tail of 2 after 3. It describes neither the given ramp window nor its shutoff.

Step 2: Invert the response. The zero datum gives $Y = F/(s+1)$. For a delayed input $\tau + c$, the zero-start response solves $r_c' + r_c = \tau + c$, and is

$$r_c(\tau) = \tau + c - 1 + (1-c)e^{-\tau}.$$

Its transform is $(1/s^2 + c/s)/(s+1)$, as direct transformation confirms. In particular $r_1 = \tau$ and $r_3 = \tau + 2 - 2e^{-\tau}$. Thus

$$\boxed{y = H_1(t)(t-1) - H_3(t)[(t-3) + 2 - 2e^{-(t-3)}].}$$

Step 3: Read the intervals and switches. Simplifying after the second switch gives

$$y(t) = \begin{cases} 0, & 0 \le t < 1, \\ t-1, & 1 \le t < 3, \\ 2e^{-(t-3)}, & t \ge 3. \end{cases}$$

The pieces meet at 0 and 2, and satisfy $y' + y = 0, t, 0$. Their slopes are $0, 1, -2e^{-(t-3)}$, so $[y']_1 = 1$ and $[y']_3 = -3$. The unique global maximum is $\boxed{y(3) = 2}$, by increase during the window and strict decay afterward.

Step 4: Check the total response. Directly,

$$\int_0^\infty y\,dt = \int_1^3 (t-1)\,dt + \int_3^\infty 2e^{-(t-3)}\,dt = 2 + 2 = \boxed{4}.$$

Integrating the equation gives the same answer because $y(\infty) - y(0) = 0$ and $\int_1^3 t\,dt = 4$. The compact input has a transform for every real s, with its apparent $s = 0$ poles canceling. The output's nonzero exponential tail gives exact domain $\boxed{s > -1}$, and the removable value $Y(0) = 4$ agrees with the area.

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Question 3

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An undamped oscillator receives a rectangular force of selectable duration:

$$y'' + \omega^2 y = F[1 - H_T(t)], \qquad y(0) = y'(0) = 0, \qquad F > 0,\ \omega > 0,\ T > 0.$$

Use the energy $E = (y'^2 + \omega^2 y^2)/2$.

Tasks

1. Derive $Y(s)$ and the delayed step-response formula for y.
2. Find the displacement and velocity at shutoff, and the residual oscillation amplitude.
3. Classify every duration T for which the system remains at rest after shutoff; explain why $y(T) = 0$ alone is generally insufficient.
4. Compute the residual energy and classify the exact real transform domain, including the rest-producing durations.

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Question 3 – Solution

Strategy. Removing the force subtracts a delayed step response. The two surviving state coordinates determine whether motion persists.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{F(1 - e^{-Ts})}{s(s^2 + \omega^2)}, \qquad \boxed{y = \frac{F}{\omega^2}\left[1 - \cos\omega t - H_T(t)(1 - \cos\omega(t - T))\right].}$$

Before T this has zero initial data and satisfies the forced equation. The subtracted response and its first derivative vanish at activation, so y, y' match at T; after T the equation is homogeneous.

Step 2: Recover the shutoff state and amplitude. The matching state is

$$y(T) = \frac{F}{\omega^2}(1 - \cos\omega T), \qquad y'(T) = \frac{F}{\omega}\sin\omega T.$$

For $t \geq T$, subtraction and the cosine difference identity give

$$y(t) = \frac{2F}{\omega^2}\sin(\omega T/2)\sin(\omega(t - T/2)).$$

Its amplitude is therefore $\boxed{R = 2F|\sin(\omega T/2)|/\omega^2}$.

Step 3: Classify complete return to rest. A homogeneous oscillator remains zero exactly when both state coordinates are zero. Here that happens precisely for

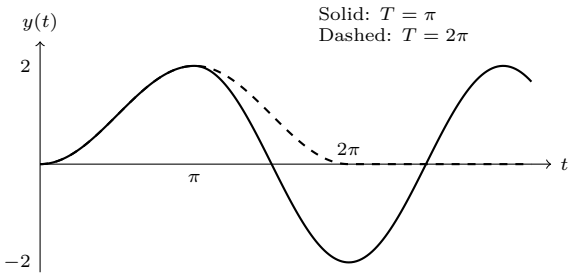
$$\boxed{T = \frac{2\pi n}{\omega}, \qquad n = 1, 2, \ldots.}$$

In this particular pulse family, $y(T) = 0$ forces $\cos\omega T = 1$ and hence also $y'(T) = 0$. For a general forcing history, zero displacement alone is insufficient: a nonzero velocity launches a new oscillation.

Step 4: Compute energy and convergence. After shutoff,

$$\boxed{E = \tfrac{1}{2}\omega^2 R^2 = \frac{2F^2}{\omega^2}\sin^2(\omega T/2).}$$

If $R > 0$, the oscillatory tail has an ordinary Laplace transform exactly for $s > 0$: at zero its primitive does not converge, and negative s gives growing integrals over fixed sign intervals. If $R = 0$, the output has compact support, so its transform exists for every real s. Apparent poles in the displayed formula then cancel. The graph compares $F = \omega = 1$ at two durations.



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Question 4

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A critically damped system receives a unit pulse:

$$y'' + 2y' + y = 1 - H_2(t), \qquad y(0) = y'(0) = 0.$$

The force is already zero for $t > 2$.

Tasks

1. Find $Y(s)$ and derive the unit-step response needed to invert it.
2. Write the post-shutoff response explicitly and verify continuity of y and y' at 2.
3. Find the unique global peak time and height. Prove whether the peak occurs before or after shutoff.
4. Determine the jump in acceleration at shutoff and the total area under y. State the exact real transform domain.

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Question 4 – Solution

Strategy. A decaying stable system can continue moving upward after its input stops. Differentiate the actual tail to locate the peak.

Step 1: Transform the pulse. The zero data give

$$Y = \frac{1-e^{-2s}}{s(s+1)^2}, \qquad \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Thus $r(t) = 1-(1+t)e^{-t}$ is the unit-step response, with $r(0) = r'(0) = 0$ and $r'' + 2r' + r = 1$. The solution is $\boxed{y = r(t) - H_2(t)r(t-2)}$.

Step 2: Simplify the tail and match. For $t \geq 2$,

$$y(t) = e^{-t}[(e^2-1)t - (e^2+1)].$$

At 2 this gives $1-3e^{-2} = r(2)$; its derivative there is $2e^{-2} = r'(2)$. Hence y, y' are continuous, with positive velocity even after the forcing is removed.

Step 3: Locate and certify the peak. For $0 < t < 2$, $y' = te^{-t} > 0$. On the tail,

$$y' = e^{-t}[2e^2 - (e^2-1)t].$$

This derivative changes sign exactly once, at

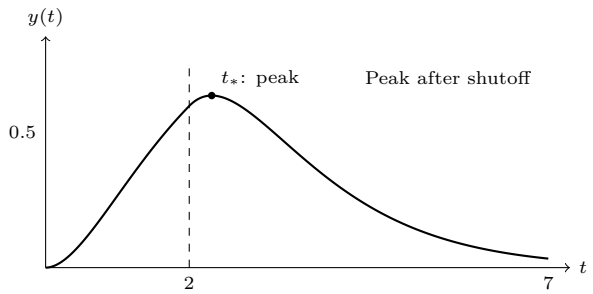
$$\boxed{t_* = \frac{2e^2}{e^2-1} > 2, \qquad y(t_*) = (e^2-1)e^{-t_*}.}$$

Thus the peak is unique and global. The tail remains positive because its linear factor is already positive at 2 and has positive slope, while $y(t) \to 0$. Damping does not imply an immediate drop in displacement at shutoff.

Step 4: Check the switch, area and domain. Since y, y' are continuous, we have $\boxed{[y'']_2 = -1}$. Both y, y' start at zero and tend to zero. Integrating the equation over the half-line, including the matching boundary terms, yields

$$\int_0^\infty y\,dt = \int_0^2 1\,dt = \boxed{2}.$$

The nonzero te^{-t} tail gives exact domain $\boxed{s > -1}$. The $s = 0$ singularity in the transform expression is removable, with $Y(0) = 2$.



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Question 5

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A resonant input is windowed without restarting its phase:

$$y'' + y = \sin t\,[H_a(t) - H_b(t)], \quad y(0) = y'(0) = 0, \qquad a = \frac{\pi}{2}, \quad b = \frac{3\pi}{2}.$$

Tasks

1. Rewrite the forcing in delayed coordinates and derive its transform. Pay particular attention to the sign at the second switch.
2. Find and invert $Y(s)$, deriving the required repeated-quadratic transform pair.
3. Find the exact motion after b and its displacement, velocity and energy at shutoff.
4. Determine the acceleration jumps and the exact real transform domain. Explain why a zero displacement at shutoff does not mean rest.

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Question 5 – Solution

Strategy. The two endpoints have opposite sine phases. Rewriting both phases first prevents a sign error in the delayed inverse.

Step 1: Shift the phases. For delayed time τ, $\sin(\tau + a) = \cos\tau$ and $\sin(\tau + b) = -\cos\tau$. Therefore

$$f = H_a(t)\cos(t-a) + H_b(t)\cos(t-b), \qquad F = \frac{s(e^{-as} + e^{-bs})}{s^2+1}.$$

The plus sign at b cancels the continuing negative-phase input; it does not add a second physical forcing window.

Step 2: Derive and invert the response pair. The zero-data equation gives

$$Y = \frac{s(e^{-as} + e^{-bs})}{(s^2+1)^2}, \qquad s > 0.$$

Differentiating $\mathcal{L}\{\sin t\} = 1/(s^2+1)$ with respect to s gives $\mathcal{L}\{t\sin t/2\} = s/(s^2+1)^2$. With $q(\tau) = \tau\sin\tau/2$,

$$\boxed{y = H_a(t)q(t-a) + H_b(t)q(t-b).}$$

Here $q(0) = q'(0) = 0$ and $q'' + q = \cos\tau$, directly verifying the equation, the zero initial data and both matching conditions.

Step 3: Evaluate the residual motion. For $t = b + v$, $v \geq 0$, the two ages are $v + \pi$ and v. Hence

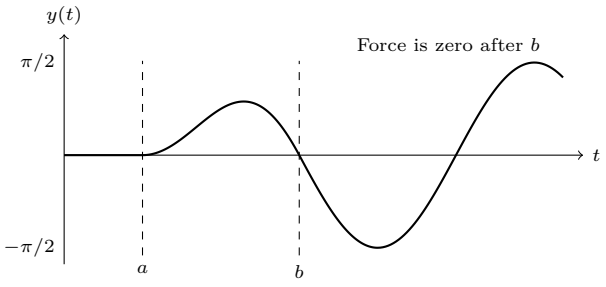
$$y(b+v) = q(v+\pi) + q(v) = \boxed{-\tfrac{\pi}{2}\sin v}.$$

In particular $y(b) = 0$, $y'(b) = -\pi/2$, and the post-shutoff energy is

$$\boxed{E = \tfrac{1}{2}(y'^2 + y^2) = \frac{\pi^2}{8}.}$$

The pulse ends precisely at a zero crossing with nonzero velocity, so the system continues oscillating with amplitude $\pi/2$.

Step 4: Inspect the switches and convergence. The input jumps from 0 to 1 at a, and from -1 to 0 at b. Continuity of y gives $\boxed{[y'']_a = [y'']_b = 1}$. There are no jumps in y or y'. The nonzero sinusoidal tail has exact real transform domain $\boxed{s > 0}$: its undamped primitive fails to converge at zero, and exponential growth of fixed-sign tail integrals rules out negative s.



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Question 6

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Choose two constant force levels to bring an oscillator exactly to rest:

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \qquad f(t) = \begin{cases} A, & 0 \leq t < a, \\ B, & a \leq t < T, \\ 0, & t \geq T, \end{cases} \quad a = \frac{\pi}{4}, \quad T = \frac{\pi}{2}.$$

Require $y(t) = 0$ for every $t \geq T$. The levels A, B may have either sign.

Tasks

1. Express f using steps and derive $Y(s)$, including the initial response.
2. Invert $Y(s)$ and derive the two linear equations that the terminal conditions impose on A, B.
3. Solve for both force levels and prove that they give a unique motion which remains at rest after T.
4. Decide whether nonnegative levels could meet the target. Describe the smoothness at both switches and the exact real transform domain of the designed output.

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Question 6 – Solution

Strategy. Exact rest requires two terminal state conditions. Use the step response to convert them into two equations for the two force levels.

Step 1: Transform with the initial displacement. The level changes give $f = A + (B - A)H_a(t) - BH_T(t)$, so

$$(s^2 + 1)Y - s = \frac{A + (B - A)e^{-as} - Be^{-Ts}}{s}.$$

Writing $r(t) = 1 - \cos t$, the inverse is

$$\boxed{y = \cos t + Ar(t) + (B - A)H_a(t)r(t - a) - BH_T(t)r(t - T).}$$

The unforced term $\cos t$ supplies the initial data. Since $r(0) = r'(0) = 0$, each added response preserves y, y' at activation, and $r'' + r = 1$ verifies the forcing on each open interval.

Step 2: Impose both terminal conditions. Set $c = \cos a = \sin a = 1/\sqrt{2}$. At $T = \pi/2$ the last response and its derivative still vanish. The terminal values are

$$y(T) = cA + (1 - c)B, \qquad y'(T) = -1 + (1 - c)A + cB.$$

Thus rest requires

$$cA + (1 - c)B = 0, \qquad (1 - c)A + cB = 1.$$

Step 3: Solve and prove complete rest. The determinant is $c^2 - (1 - c)^2 = 2c - 1 = \sqrt{2} - 1 > 0$. Consequently

$$A = -\frac{1 - c}{2c - 1} = \boxed{-\frac{1}{\sqrt{2}}}, \qquad B = \frac{c}{2c - 1} = \boxed{1 + \frac{1}{\sqrt{2}}}.$$

These are the unique levels in the stated two-stage family. The formula satisfies the original IVP and has zero displacement and velocity at T. For $t > T$ the forcing is zero, so uniqueness of the homogeneous oscillator with these terminal data implies $y(t) = 0$ there. Matching gives a unique solution on the entire half-line.

Step 4: Check feasibility and regularity. Both weights in $cA + (1 - c)B = 0$ are positive. If $A, B \geq 0$, that equation forces $A = B = 0$, contradicting the velocity equation. A negative first level is therefore necessary within this family. The motion is C^1, while the acceleration jumps are

$$[y'']_a = B - A = 1 + \sqrt{2}, \qquad [y'']_T = -B = -1 - \frac{1}{\sqrt{2}}.$$

After T the output is identically zero, so its transform exists for $\boxed{\text{every real } s}$, despite the apparent denominators in Y.

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Question 7

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An unstable first-order system is to be stopped by a two-stage input:

$$y' - y = f_B(t), \quad y(0) = 1, \qquad f_B(t) = \begin{cases} -1, & 0 \leq t < 1, \\ B, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Choose $B = B_*$ so that $y(t) = 0$ for all $t \geq 2$. A realized level may instead be $B_* + \delta$, where δ is a constant error.

Tasks

1. Find $Y(s)$ and invert it using the unstable system's step response.
2. Determine B_* and the resulting piecewise output. Verify the stopping condition.
3. Derive the exact output error for $t \geq 2$ and the sharp bound on $|\delta|$ that ensures $|y(t)| \leq 0.01$ for every $2 \leq t \leq 5$.
4. Compare the exact real transform domains for perfect and imperfect levels. Explain why exact cancellation does not make the system stable.

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Question 7 – Solution

Strategy. A finite input can cancel one unstable trajectory exactly. Keep the cancellation coefficient explicit to measure its sensitivity.

Step 1: Transform and invert. Here $f_B = -1 + (B+1)H_1(t) - BH_2(t)$ and

$$Y = \frac{1 + [-1 + (B+1)e^{-s} - Be^{-2s}]/s}{s-1}, \qquad s > 1.$$

The zero-start response to a unit step for $y' - y = 1$ is $r(t) = e^t - 1$. Adding the initial response gives

$$y = e^t - r(t) + (B+1)H_1(t)r(t-1) - BH_2(t)r(t-2).$$

This formula satisfies $y(0) = 1$ and is continuous at both switches.

Step 2: Select the exact cancellation. Before 1, $y = 1$. On $[1,2]$, $y = (B+1)e^{t-1} - B$, so $y(2) = e + B(e-1)$. Consequently

$$\boxed{B_* = -\frac{e}{e-1}}, \qquad \boxed{y_*(t) = \begin{cases} 1, & 0 \le t < 1, \\ (e - e^{t-1})/(e-1), & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}}$$

Direct substitution gives $y_*' - y_* = -1, B_*, 0$ on the three intervals, and the values match at 1 and 2. The zero terminal state remains zero under the homogeneous equation after 2.

Step 3: Quantify the error sharply. The error is zero before 1, then equals $\delta(e^{t-1} - 1)$ until 2. After 2 the homogeneous equation amplifies its terminal value:

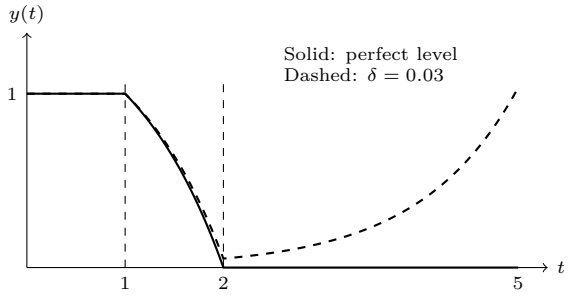
$$\boxed{y(t) - y_*(t) = \delta(e-1)e^{t-2}, \qquad t \ge 2.}$$

Its magnitude increases on $[2,5]$ when $\delta \neq 0$. Thus the required necessary and sufficient tolerance is

$$\boxed{|\delta| \le \frac{0.01\, e^{-3}}{e-1}.}$$

Equality attains the limit at $t = 5$.

Step 4: Distinguish trajectory cancellation from stability. The perfect output has compact support and a transform for every real s. For any $\delta \neq 0$, a nonzero e^t tail remains and the exact real domain is $s > 1$. At the perfect setting the apparent singularity at $s = 1$ is removable, as is the $s = 0$ expression after cancellations. Small level errors eventually grow without bound; selecting one compact response does not change the unstable homogeneous equation.



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Question 8

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An initially unexcited system is driven by one unknown positive step:

$$y' + 2y = AH_a(t), \qquad y(0) = 0, \qquad A > 0, \quad a > 0.$$

Exact measurements give $y(1) = 0$, $y(2) = 1$, and $y(3) = 2$. A fourth proposed measurement is $y(4) = 3$.

Tasks

1. Derive the transformed and time-domain response for arbitrary A, a.
2. Use positivity and the measurements to locate the switch interval, then determine A and a exactly.
3. Verify that the recovered parameters satisfy all three original measurements and are unique in this one-step family.
4. Test the fourth measurement and find the limiting response. Explain the scope of the identification.

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Question 8 – Solution

Strategy. The zero measurement locates the delay. Once the input is on, equally spaced samples obey a simple exponential approach to the same level.

Step 1: Solve the parameterized IVP. For $s > 0$,

$$Y = \frac{Ae^{-as}}{s(s+2)}, \qquad \boxed{y(t) = \frac{A}{2} H_a(t)(1 - e^{-2(t-a)}).}$$

The response is zero through a, positive for $t > a$, and continuous at a. Its derivative jumps by A, as the equation requires.

Step 2: Locate and recover the switch. Since $A > 0$, the first two measurements imply $1 \le a < 2$. Put $r = e^{-2}$ and $C = A/2$. On each full unit interval after activation,

$$y(t+1) = r\, y(t) + C(1 - r).$$

Thus $2 = r + C(1 - r)$, and

$$\boxed{A = \frac{2(2-r)}{1-r}}.$$

Using $y(2) = 1$ gives $e^{-2(2-a)} = 1 - 1/C = 1/(2-r)$, hence

$$\boxed{a = 2 - \tfrac{1}{2}\ln(2 - e^{-2}).}$$

Step 3: Verify admissibility and uniqueness. Because $1 < 2 - r < 2 < e^2$, one has $0 < \ln(2-r) < 2$, so $1 < a < 2$. Also $C = (2-r)/(1-r) > 0$. The recovered values give $y(1) = 0$ and

$$y(2) = C\left(1 - \frac{1}{2-r}\right) = 1, \qquad y(3) = r + C(1-r) = 2.$$

The sample recurrence fixes C uniquely. Then the exponential expression for $y(2)$ is strictly monotone in a, fixing the delay uniquely. Direct substitution into the response verifies the IVP on both sides of the switch.

Step 4: Test the extra datum and interpret. The same recurrence predicts

$$\boxed{y(4) = 2r + C(1-r) = 2 + e^{-2} < 3.}$$

The fourth datum is inconsistent with the stated model. The limiting level is $\boxed{C = (2 - e^{-2})/(1 - e^{-2})}$, approached from below after activation. Its positive constant tail gives exact real transform domain $s > 0$. Uniqueness here concerns the specified single positive step with known system coefficient; finitely many samples do not identify an arbitrary forcing history.

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Question 9

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Consider two qualitatively different input changes:

$$y'' + 3y' + 2y = H_1(t) + (t-2)H_2(t), \qquad y(0) = y'(0) = 0.$$

One claim says that the jump at 1 makes y jump; another says that because the ramp is continuous at 2, the solution must be smooth to every order there.

Tasks

1. Find $Y(s)$ and invert it explicitly using the step and ramp responses.
2. Verify the original IVP and determine the jumps in y, y', y'', y''' at 1 and 2.
3. Derive the leading small-τ behavior of each newly activated response. Relate the orders of onset to the jump calculations.
4. Evaluate both claims and explain whether a different assigned value of $H_a(a)$ changes the response.

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Question 9 – Solution

Strategy. A second-order system smooths a step more than the input itself, but only by a finite number of derivatives. Check one-sided limits directly.

Step 1: Transform and invert. With zero data,

$$Y = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{e^{-2s}}{s^2(s+1)(s+2)}.$$

Partial fractions give the two zero-start responses

$$r(\tau) = \tfrac{1}{2} - e^{-\tau} + \tfrac{1}{2}e^{-2\tau}, \qquad p(\tau) = \tfrac{\tau}{2} - \tfrac{3}{4} + e^{-\tau} - \tfrac{1}{4}e^{-2\tau},$$

so $\boxed{y = H_1(t)r(t-1) + H_2(t)p(t-2)}$. Their transforms are the two unshifted rational factors displayed above.

Step 2: Verify the equation and all jumps. Direct differentiation gives $r'' + 3r' + 2r = 1$ and $p'' + 3p' + 2p = \tau$. Moreover $r(0) = r'(0) = p(0) = p'(0) = 0$, verifying the initial data and C^1 matching. For the next derivatives,

$$r''(0) = 1, \quad r'''(0) = -3, \qquad p''(0) = 0, \quad p'''(0) = 1.$$

The previously active response is smooth at the next switch. Thus

switch	$[y]$	$[y']$	$[y'']$	$[y''']$
1	0	0	1	-3
2	0	0	0	1

These are jumps of one-sided ordinary derivatives, not assigned values at the switch.

Step 3: Relate onset orders to regularity. Taylor expansion of the newly added responses gives

$$r(\tau) = \tfrac{1}{2}\tau^2 - \tfrac{1}{2}\tau^3 + O(\tau^4), \qquad p(\tau) = \tfrac{1}{6}\tau^3 + O(\tau^4), \qquad \tau \downarrow 0.$$

Joining the first to zero preserves one derivative but creates a second- derivative jump. Joining the second to zero preserves two derivatives but creates a third-derivative jump. These leading coefficients reproduce the jump sizes above.

Step 4: Resolve both claims and the convention. The first claim is false: y and y' stay continuous at 1, while y'' jumps. The second is also false: y is C^2 near 2 but not C^3 there. Changing an input at one isolated time does not change its Laplace integral or the solution obtained by continuous state matching. At switches the equation is interpreted through its one-sided limits, not by demanding an unavailable classical derivative. Away from them the solution is smooth. The growing linear tail gives exact real transform domain $s > 0$.

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Question 10

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A unit input is switched on and off repeatedly:

$$y' + y = f(t), \quad y(0) = 0, \qquad f(t) = \begin{cases} 1, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2, \end{cases} \qquad k = 0, 1, 2, \ldots.$$

Use only step functions and a geometric series; no periodic-transform formula is needed.

Tasks

1. Represent f by a locally finite sum of steps, derive $F(s)$ for $s > 0$, and obtain $Y(s)$ and a delayed-response series.
2. For $x_k = y(2k)$, derive and solve a recurrence for the samples.
3. Find the attracting two-periodic response explicitly on one cycle and prove the exact transient error for all $t \geq 0$.
4. Find its cycle minimum, maximum and mean. Does $y(t)$ have a limit? Find the earliest $T \geq 0$ for which the error from the periodic response is at most 0.01 for every $t \geq T$.

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Question 10 – Solution

Strategy. The step series gives the transient solution, while the cycle map identifies a periodic state and its exact attraction rate.

Step 1: Transform the repeated switches. At each fixed time only finitely many terms are active:

$$f=\sum_{k=0}^{\infty}(H_{2k}(t)-H_{2k+1}(t)),\qquad F=\frac{1-e^{-s}}{s}\sum_{k=0}^{\infty}e^{-2ks}=\frac{1}{s(1+e^{-s})},\quad s>0.$$

Integration of the nonnegative disjoint pulses justifies the sum. With $r(t)=1-e^{-t}$ for $t\geq 0$,

$$Y=\frac{1}{s(s+1)(1+e^{-s})},\qquad y=\sum_{k=0}^{\infty}[H_{2k}(t)r(t-2k)-H_{2k+1}(t)r(t-2k-1)].$$

This time series is locally finite; each activated response starts at zero and satisfies the required first-order equation on its active intervals.

Step 2: Solve the cycle recurrence. During the on-half, $y(2k+1)=1+(x_k-1)e^{-1}$. During the off-half this value is multiplied by e^{-1}, so

$$x_{k+1}=e^{-2}x_k+e^{-1}(1-e^{-1}),\qquad \boxed{x_k=m(1-e^{-2k}),\quad m=\frac{1}{e+1}},$$

using $x_0=0$. The cycle map has the unique fixed point m.

Step 3: Build the periodic state and exact error. Write $t=2k+\tau$, $0\leq\tau<2$, and put $M=e/(e+1)$. Then

$$y_{\mathrm{per}}(t)=\begin{cases}1-(1-m)e^{-\tau}, & 0\leq\tau<1,\\ Me^{-(\tau-1)}, & 1\leq\tau<2.\end{cases}$$

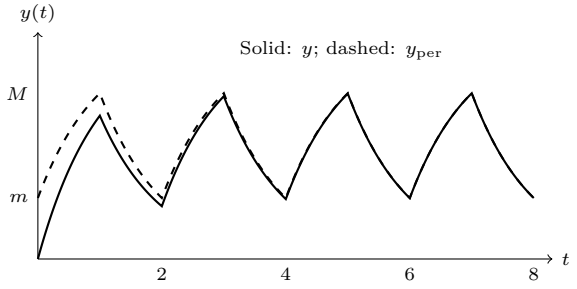
The pieces match at M, and the value at the end of the cycle is m. Both solutions have the same forcing, so their difference satisfies $d'+d=0$ with $d(0)=m$. Continuity propagates this identity across every switch:

$$\boxed{y_{\mathrm{per}}(t)-y(t)=me^{-t},\qquad t\geq 0.}$$

Step 4: Interpret the limiting cycle and tolerance. The periodic minimum is m and maximum is M. Integrating its equation over a cycle gives $\int_0^2 y_{\mathrm{per}}\,dt=1$, so its mean is $1/2$. The even and odd integer samples of y tend to the different values m, M; there is no single limit. The error decreases strictly, giving the sharp time

$$\boxed{T=\ln\left(\frac{100}{e+1}\right).}$$

Both responses have exact real transform domain $s>0$, since their positive persistent cycles prevent convergence at or below zero.



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Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Let $a, J > 0$ and $\varepsilon > 0$. A finite pulse and its impulse idealization drive

$$y_\varepsilon' + 2y_\varepsilon = \frac{J}{\varepsilon}[H_a(t) - H_{a+\varepsilon}(t)], \qquad y' + 2y = J\delta(t-a), \qquad y_\varepsilon(0) = y(0) = 0.$$

Tasks

1. Prove that the finite pulse tends to $J\delta(t-a)$ when integrated against a continuous test function. Derive the limiting input transform.
2. Find the impulsive response, including its jump, and solve the finite-pulse IVP explicitly.
3. For $t \geq a + \varepsilon$, prove $0 \leq y_\varepsilon(t) - y(t) \leq J\varepsilon$.
4. Describe pointwise convergence, including the exceptional time a, and decide whether convergence can be uniform on an interval containing a in its interior.

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Question 1 – Solution

Strategy. Preserve pulse area rather than height. Compare the two exact responses before making any claim about convergence.

Step 1: Test the pulse and transform. Its action on a continuous ϕ is

$$\frac{J}{\varepsilon}\int_a^{a+\varepsilon}\phi(t)\,dt \longrightarrow J\phi(a).$$

The error is at most $J\sup_{[a,a+\varepsilon]}|\phi(t)-\phi(a)|$, which tends to zero. Taking $\phi = e^{-st}$ gives $F_\varepsilon = Je^{-as}(1-e^{-\varepsilon s})/(\varepsilon s) \to Je^{-as}$, with the removable value $F_\varepsilon(0) = J$.

Step 2: Solve both equations. The impulse gives $Y = Je^{-as}/(s+2)$, so $\boxed{y = JH_a(t)e^{-2(t-a)}}$ and $[y]_a = J$. The finite-pulse solution is continuous and equals

$$y_\varepsilon(t) = \begin{cases} 0, & t < a, \\ \dfrac{J}{2\varepsilon}(1-e^{-2(t-a)}), & a \le t < a+\varepsilon, \\ \dfrac{J}{2\varepsilon}(1-e^{-2\varepsilon})e^{-2(t-a-\varepsilon)}, & t \ge a+\varepsilon. \end{cases}$$

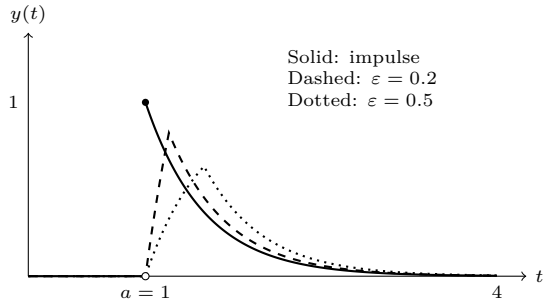
Each piece satisfies its ordinary equation, and the endpoints match.

Step 3: Bound the tail error. Put $\tau = t-a \ge \varepsilon$. The explicit tail can be rewritten as $y_\varepsilon = (J/\varepsilon)\int_0^\varepsilon e^{-2(\tau-u)}\,du$. Since $x \mapsto e^{-2x}$ is decreasing and has derivative magnitude at most 2 for $x \ge 0$,

$$0 \le y_\varepsilon - y = \frac{J}{\varepsilon}\int_0^\varepsilon [e^{-2(\tau-u)} - e^{-2\tau}]\,du \le \frac{J}{\varepsilon}\int_0^\varepsilon 2u\,du = \boxed{J\varepsilon}.$$

This representation follows directly from the finite-pulse formula.

Step 4: Distinguish the kinds of convergence. For each fixed $t \ne a$, $y_\varepsilon(t) \to y(t)$. At a, $y_\varepsilon(a) = 0$ but the right-hand representative has $y(a) = J$. Uniform convergence on an interval containing a is impossible: the approximants are continuous and the limit has a jump. Changing only $y(a)$ does not remove that jump or change its transform. The graph uses $a = J = 1$; open and filled dots mark the impulsive response.



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Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Consider a weighted impulse and a reversed impulse argument:

$$y' + y = (t^2 - 1)\delta(2t - 4) + 3\delta(3 - t), \qquad y(0) = 2.$$

You may use $\delta(c(t-a)) = \delta(t-a)/|c|$ for $c \neq 0$.

Tasks

1. Reduce the forcing to impulses at named times with constant weights. Explain the absolute value in the scaling rule.
2. Find $Y(s)$ and the complete time response, including the initial state.
3. Determine both jumps and verify the ordinary equation on each interval.
4. Compute the response area and its exact real transform domain. Explain why the response area is not just the sum of the impulse weights.

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Question 2 – Solution

Strategy. Scale the delta first, then evaluate its smooth multiplier at the impulse time. The initial state contributes its own response area.

Step 1: Normalize and sift. Here $\delta(2t-4) = \delta(t-2)/2$ and $\delta(3-t) = \delta(t-3)$. The sampling property gives

$$\boxed{f(t) = \tfrac{3}{2}\delta(t-2) + 3\delta(t-3).}$$

The first weight is $(2^2-1)/2 = 3/2$. Reversing an argument does not turn a positive unit mass into a negative one: changing integration variables reverses the limits as well as introducing the negative derivative. Thus the scale factor is $1/|c|$.

Step 2: Transform and invert. The initial term is retained:

$$(s+1)Y - 2 = \tfrac{3}{2}e^{-2s} + 3e^{-3s}, \qquad \boxed{Y = \frac{2 + \frac{3}{2}e^{-2s} + 3e^{-3s}}{s+1}.}$$

Consequently

$$\boxed{y(t) = 2e^{-t} + \tfrac{3}{2}H_2(t)e^{-(t-2)} + 3H_3(t)e^{-(t-3)}.}$$

Step 3: Verify the jumps and intervals. Every active exponential satisfies $y' + y = 0$ away from 2 and 3. The newly activated terms give

$$[y]_2 = \tfrac{3}{2}, \qquad [y]_3 = 3, \qquad y(0) = 2.$$

Equivalently, integrating the equation across either impulse makes the bounded y integral vanish as the interval shrinks, leaving the specified jump. Ordinary uniqueness on each interval and these jumps determine the whole response.

Step 4: Account for the initial stored response. Each shifted exponential of coefficient C has integral C. Hence

$$\boxed{\int_0^\infty y(t)\,dt = 2 + \tfrac{3}{2} + 3 = \tfrac{13}{2}.}$$

Integrating the impulsive equation over the half-line gives $y(\infty) - y(0) + \int_0^\infty y\,dt = 9/2$ and the same result. The derivative contribution is -2, not zero. The positive nonzero e^{-t} tail gives exact real domain $\boxed{s > -1}$.

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Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An oscillator moving through equilibrium receives an impulse:

$$y'' + 4y = 6\delta(t-a), \qquad y(0) = 1, \quad y'(0) = 0, \qquad a = \frac{\pi}{4}.$$

For unit mass, use $E = (y'^2 + 4y^2)/2$. Ordinary impulses in a second-order equation keep displacement continuous but may jump velocity.

Tasks

1. Derive the velocity jump and find $Y(s)$ and $y(t)$.
2. Compute the states immediately before and after the impulse and simplify the subsequent motion.
3. Find the energy change and derive the general energy-change formula for an impulse of strength J at this same time.
4. For $J > 0$, classify when the impulse lowers the energy, preserves it, or raises it. Identify the unique strength that brings this oscillator completely to rest.

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Question 3 – Solution

Strategy. The impulse changes velocity, not displacement. Its work depends on the velocity immediately before the kick as well as its strength.

Step 1: Jump, transform and invert. Integrating across a gives $[y']_a = 6$, because the integral of $4y$ tends to zero. With $[y]_a = 0$, the transformed equation is

$$Y = \frac{s + 6e^{-as}}{s^2 + 4}, \qquad \boxed{y(t) = \cos 2t + 3H_a(t)\sin(2(t-a)).}$$

The new term vanishes at activation, while its derivative starts at 6. Both pieces satisfy $y'' + 4y = 0$ away from a and retain the initial data.

Step 2: Evaluate the state at impact. Since $2a = \pi/2$,

$$(y(a-), y'(a-)) = (0, -2), \qquad (y(a+), y'(a+)) = \boxed{(0, 4)}.$$

For $t = a + \tau$, the pre-existing response is $-\sin 2\tau$, so the total post-impact motion is $\boxed{y(a+\tau) = 2\sin 2\tau}$. Thus the impulse reverses the direction and doubles the amplitude.

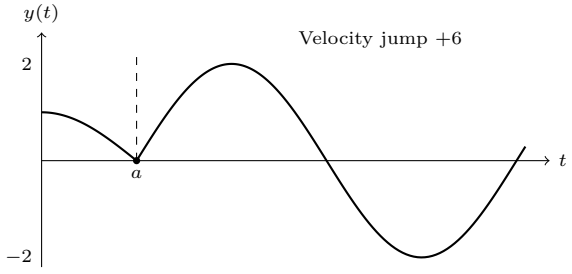
Step 3: Compute the energy transfer. The energies are $E_- = 2$ and $E_+ = 8$, giving $\boxed{\Delta E = 6}$. For any impulse J, continuity of position and $v_+ = v_- + J$ imply

$$\Delta E = \tfrac{1}{2}[(v_- + J)^2 - v_-^2] = Jv_- + \tfrac{1}{2}J^2.$$

At this impact time $v_- = -2$, hence

$$\boxed{\Delta E = \tfrac{1}{2}J(J-4), \qquad E_+ = \tfrac{1}{2}(J-2)^2.}$$

Step 4: Classify the positive kicks. For $0 < J < 4$ the energy decreases; at $J = 4$ it is unchanged; for $J > 4$ it increases. A positive impulse can remove energy when it opposes the incoming velocity. Complete rest requires both post-impact state coordinates to vanish. Position is already zero here, so the unique strength is $\boxed{J = 2}$, which makes velocity zero. A zero velocity kick at a nonzero displacement would not in general leave an oscillator at rest. The plot shows the stated $J = 6$ case, with continuous displacement and a corner.



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Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A critically damped system starts from rest and receives two impulses:

$$y'' + 2y' + y = \delta(t-1) + K\delta(t-b), \qquad y(0) = y'(0) = 0, \qquad b > 1.$$

You may choose b and K. The energy is $E = (y'^2 + y^2)/2$.

Tasks

1. Find $Y(s)$ and the response in terms of the age since each impulse.
2. Determine whether any finite $b > 1$ and K can make $y(t) = 0$ for every $t \geq b$.
3. For a fixed b, find the unique K minimizing the energy immediately after the second impulse, and find that minimum.
4. Derive the motion after this minimizing kick and explain why zero post-impact velocity does not imply complete rest.

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Question 4 – Solution

Strategy. One impulse can adjust velocity only. Inspect the displacement at the chosen impact time before attempting to cancel the entire state.

Step 1: Find the impulse response. The zero-data transform is

$$Y = \frac{e^{-s} + Ke^{-bs}}{(s+1)^2}.$$

Since $\mathcal{L}\{te^{-t}\} = 1/(s+1)^2$, set $h(t) = te^{-t}$ for $t \geq 0$. Then

$$\boxed{y = H_1(t)h(t-1) + KH_b(t)h(t-b).}$$

Here $h(0) = 0$, $h'(0) = 1$, and $h'' + 2h' + h = 0$ for positive age. Thus displacement is continuous, and velocity jumps by 1 and K.

Step 2: Test the terminal displacement. Put $d = b - 1 > 0$. Immediately before the second kick,

$$y_b = de^{-d} > 0, \qquad v_- = (1-d)e^{-d}.$$

The second impulse leaves y_b unchanged. A solution identically zero after b must have zero displacement there, so $\boxed{\text{no finite } b > 1 \text{ and } K \text{ can produce complete rest.}}$ The strict positivity of $h(d)$ is the obstruction, not a shortage of algebraic manipulation.

Step 3: Minimize the post-impact energy. For fixed b,

$$E_+(K) = \tfrac{1}{2}[(v_- + K)^2 + y_b^2].$$

This strictly convex quadratic has the unique minimizer and value

$$\boxed{K_* = (d-1)e^{-d}, \qquad E_{\min} = \tfrac{1}{2}d^2e^{-2d} > 0.}$$

The minimizing kick makes $v_+ = 0$ but leaves positive stored spring energy.

Step 4: Follow the resulting motion. For $\tau = t - b \geq 0$, the critical homogeneous solution with state $(y_b, 0)$ is

$$\boxed{y(b+\tau) = de^{-d}(1+\tau)e^{-\tau}.}$$

Its derivative is $-de^{-d}\tau e^{-\tau}$: it starts horizontal and then moves strictly downward. Its acceleration at $\tau = 0+$ is $-y_b \neq 0$. Substitution verifies the equation and the terminal state. Thus the system eventually approaches rest, but is not at rest throughout the post-impact interval. The positive polynomial-exponential tail has transform domain $s > -1$.

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Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Two impulses act on an initially resting undamped oscillator:

$$y'' + \omega^2 y = I\delta(t-a) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad I, \omega > 0, \quad 0 < a < b.$$

Require complete rest after the second impulse.

Tasks

1. Find the transformed response and the time-domain solution.
2. Classify all delays $b - a$ and strengths K that meet the rest condition.
3. If K must be positive, find the earliest possible second impulse. Explain why the net impulse need not be zero.
4. Keep this earliest positive strength, but deliver it at $b+\eta$, where $|\eta| < \pi/\omega$. Find the residual amplitude and its leading behavior as $\eta \to 0$.

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Question 5 – Solution

Strategy. The second kick must occur at a zero of displacement, then cancel the incoming velocity. Timing errors leave a measurable residual oscillation.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{Ie^{-as} + Ke^{-bs}}{s^2 + \omega^2}, \qquad \boxed{y = \frac{I}{\omega} H_a(t) \sin \omega(t-a) + \frac{K}{\omega} H_b(t) \sin \omega(t-b).}$$

Each activated sine is zero initially, with derivative equal to its impulse strength. Thus y is continuous and $[y']_a = I$, $[y']_b = K$.

Step 2: Impose both state conditions. For $d = b - a$, the pre-impact state is

$$y(b-) = \frac{I}{\omega} \sin \omega d, \qquad y'(b-) = I \cos \omega d.$$

Displacement cannot jump. Therefore complete rest requires and is ensured by

$$\boxed{d = \frac{n\pi}{\omega}, \qquad K = (-1)^{n+1} I, \qquad n = 1, 2, \ldots.}$$

After these jumps the homogeneous oscillator has zero data, so it remains zero. The corresponding output has compact support and a transform for every real s.

Step 3: Restrict the second impulse to be positive. Positive K requires odd n. The earliest choice is $\boxed{b = a + \pi/\omega,\ K = I}$. The net applied impulse is $2I$, yet the oscillator stops: during the intervening motion the restoring force also changes momentum. Indeed $\int_a^b \omega^2 y \, dt = 2I$, balancing the applied impulses in the integrated equation. Ignoring this force would give an incorrect momentum test.

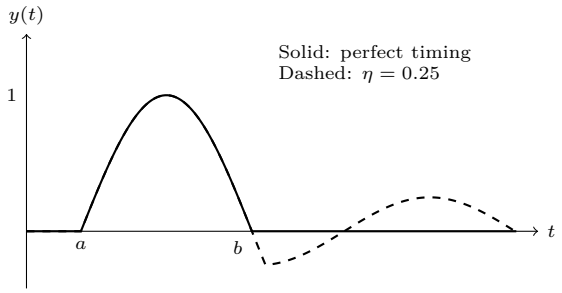
Step 4: Quantify a timing error. At the actual second kick the age is $\pi/\omega + \eta > 0$. The state immediately afterward is

$$y_+ = -\frac{I}{\omega} \sin \omega\eta, \qquad v_+ = I(1 - \cos \omega\eta).$$

The subsequent homogeneous amplitude is

$$R = \sqrt{y_+^2 + (v_+/\omega)^2} = \boxed{\frac{2I}{\omega} |\sin(\omega\eta/2)|} = I|\eta| + O(|\eta|^3).$$

For every nonzero η in the stated interval this amplitude is positive, so the tail's exact real transform domain is $s > 0$. The plot compares perfect timing and $\eta = 0.25$ for $I = \omega = a = 1$.



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Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A single unknown impulse drives a first-order system:

$$y' + \lambda y = J\delta(t-a), \qquad y(0) = 0, \qquad \lambda, J > 0, \quad 0 < a < 2.$$

Exact observations give $y(2) = 4$ and $y(3) = 2$. An additional measurement is the total response area $\int_0^\infty y(t)\,dt = 8$.

Tasks

1. Derive the transform and response for general λ, J, a.
2. Use only the two point observations to determine what is identifiable. Parameterize every admissible triple consistent with them.
3. Use the area measurement to recover all three parameters uniquely, and verify the delay restriction.
4. Check all measurements directly and explain why the extra area measurement resolves an ambiguity that another later point sample would not.

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Question 6 – Solution

Strategy. Post-impulse samples determine a decay rate and an effective amplitude. The area determines the impulse strength independently of its delay.

Step 1: Solve the parameterized problem. The zero datum gives

$$Y = \frac{Je^{-as}}{s+\lambda}, \qquad \boxed{y = JH_a(t)e^{-\lambda(t-a)}.}$$

The response has jump J at a, solves $y' + \lambda y = 0$ afterward, and has area J/λ. Its exact real transform domain is $s > -\lambda$.

Step 2: Identify the point-sample information. Both observations occur after the impulse. Taking their ratio gives $e^{-\lambda} = 1/2$, so $\lambda = \ln 2$. The first observation then gives $J2^{a-2} = 4$, or

$$\boxed{\lambda = \ln 2, \qquad J = 16\,2^{-a}, \qquad 0 < a < 2.}$$

This is a full one-parameter family, not a unique impulse time. The strength ranges over $4 < J < 16$.

Step 3: Use the area and prove admissibility. The area condition now fixes $J/\ln 2 = 8$. Thus

$$\boxed{J = 8\ln 2, \qquad a = \frac{\ln(2/\ln 2)}{\ln 2}, \qquad \lambda = \ln 2.}$$

Since $1/2 < \ln 2 < 1$, we have $1 < 2/\ln 2 < 4$. Taking logarithms to base 2 shows $0 < a < 2$. The area fixes J uniquely, and $J = 16\,2^{-a}$ is strictly decreasing in a, so this recovered triple is unique in the stated family.

Step 4: Verify and interpret. The recovered delay satisfies $2^a = 2/\ln 2$. Therefore

$$y(2) = (8\ln 2)\frac{2/\ln 2}{4} = 4, \qquad y(3) = \tfrac{1}{2}y(2) = 2, \qquad \int_0^\infty y\,dt = \frac{8\ln 2}{\ln 2} = 8.$$

For every member of the two-sample family, all later values are $y(t) = 16\,2^{-t}$ for $t \geq 2$. Thus another exact sample after 2 would either agree with every member or reject every member; it could not separate strength from delay. The total area includes the unsampled interval after the unknown impulse and supplies the missing information.

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Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An unstable second-order system is given one corrective impulse:

$$y'' - y = J\delta(t-a), \qquad y(0) = 1, \quad y'(0) = 0, \qquad a > 0.$$

Choose J to make the response decay to zero after the impulse. A realized impulse may differ by a constant error Δ.

Tasks

1. Derive $Y(s)$ and $y(t)$ for arbitrary J, including the initial response.
2. Find the unique decaying choice J_*, its explicit tail and the global maximum of the selected response.
3. Derive the response error for strength $J_* + \Delta$ and a sharp bound on $|\Delta|$ ensuring error at most $\epsilon > 0$ throughout $a \leq t \leq a + L$, where $L > 0$.
4. Compare exact real transform domains for the selected and perturbed strengths. Explain the effect of cancellation on the system's stability.

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Question 7 – Solution

Strategy. Cancel the growing mode's coefficient, then inspect the remaining mode and the error produced by imperfect cancellation.

Step 1: Transform and invert. For sufficiently large s,

$$(s^2-1)Y - s = Je^{-as}, \qquad Y = \frac{s + Je^{-as}}{s^2-1}, \qquad \boxed{y = \cosh t + JH_a(t)\sinh(t-a).}$$

The added term starts at zero with derivative J, so it preserves displacement and produces the correct velocity jump. Away from a both terms solve the homogeneous equation.

Step 2: Select and interpret the decaying mode. After a, the e^t coefficient is $(1 + Je^{-a})/2$. Decay requires

$$\boxed{J_* = -e^a, \qquad y_*(t) = \begin{cases} \cosh t, & 0 \le t < a, \\ \cosh a\, e^{-(t-a)}, & t \ge a. \end{cases}}$$

The pre-impact velocity is $\sinh a$ and the post-impact velocity is $-\cosh a$; their difference is $-e^a$, as required. The response increases strictly before a and decreases strictly after it. Its unique global maximum is $\boxed{y_*(a) = \cosh a}$.

Step 3: Quantify the sensitivity. Linearity or direct subtraction gives

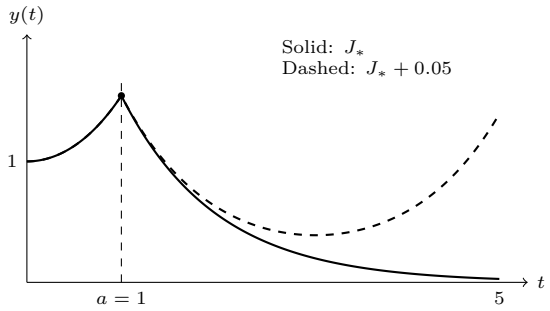
$$\boxed{y(t) - y_*(t) = \Delta H_a(t)\sinh(t-a).}$$

Because $\sinh\tau$ increases strictly on $[0, L]$, the desired error bound is equivalent to

$$\boxed{|\Delta| \le \frac{\epsilon}{\sinh L}.}$$

This is necessary as well as sufficient, and equality attains the bound at $a + L$. The selected solution still approaches zero only asymptotically; the impulse does not put it at the zero state.

Step 4: Check convergence and stability. The selected tail is a positive multiple of e^{-t}, so its exact real transform domain is $\boxed{s > -1}$. The apparent singularity at $s = 1$ in its transform is removable. For every $\Delta \neq 0$, the error contains a nonzero growing mode and the actual domain becomes $\boxed{s > 1}$. The differential equation's unstable homogeneous mode is unchanged. The plot uses $a = 1$, with a dashed curve for $\Delta = 0.05$.



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Question 8

Interpret the equation through ordinary motion away from the impulse and its jump conditions. For this question only, the initial-time transform convention is specified below; write $[v]_0 = v(0^+) - v(0^-)$ for a jump.

Use a *full-mass initial impulse* convention: the transform starts at 0^-, $\mathcal{L}\{\delta(t)\} = 1$, and

$$\mathcal{L}\{y'\} = sY - y(0^-), \qquad \mathcal{L}\{y''\} = s^2Y - sy(0^-) - y'(0^-).$$

Consider

$$2y'' + 4y' + 2y = J\delta(t), \qquad y(0^-) = 1, \quad y'(0^-) = 2.$$

These are pre-impact data, not post-impact data.

Tasks

1. Find $y(0^+)$ and $y'(0^+)$ by the impulse jump conditions.
2. Use the stated 0^- convention to find $Y(s)$ and the response for $t \geq 0$.
3. Reproduce the same response by solving a regular homogeneous IVP from 0^+. Explain why keeping the impulse again would double-count it.
4. Classify every J for which the post-impact response is nonnegative and nonincreasing for all $t \geq 0$. Describe the boundary cases.

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Question 8 – Solution

Strategy. An impulse at the initial time requires a declared convention. Either transform pre-impact data with the impulse or restart after the jump.

Step 1: Compute the post-impact data. There is no displacement jump under an ordinary force impulse. Integrating the equation across the origin gives $2[y']_0 = J$; the damping contribution is $4[y]_0 = 0$. Thus

$$\boxed{y(0^+) = 1, \qquad y'(0^+) = 2 + \frac{J}{2}.}$$

The factor 2 is the mass and must not be omitted.

Step 2: Transform the pre-impact formulation. The stipulated derivative rules give

$$2(s^2Y - s - 2) + 4(sY - 1) + 2Y = J,$$

so

$$Y = \frac{2s + 8 + J}{2(s+1)^2} = \frac{1}{s+1} + \frac{3 + J/2}{(s+1)^2}.$$

Writing $c = 3+J/2$, the inverse is $\boxed{y(t) = e^{-t}(1 + ct), \quad t \geq 0}$. It has $y(0^+) = 1$ and $y'(0^+) = c-1 = 2+J/2$, exactly as the jump calculation.

Step 3: Check the equivalent restart. For $t > 0$ the equation is $y'' + 2y' + y = 0$. Using the post-impact data $1, 2 + J/2$ gives the critical solution $e^{-t}[1 + (3 + J/2)t]$ again. This restart has no remaining initial impulse: its effect is already encoded in the velocity. Inserting $J\delta(t)$ again with those data would add a second velocity change $J/2$.

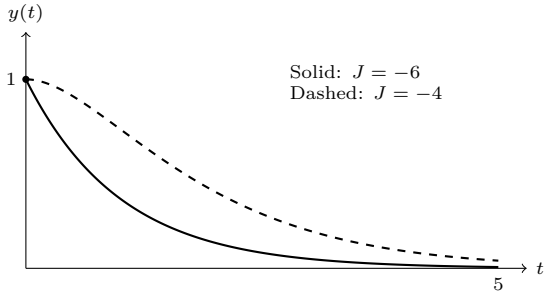
Step 4: Classify positivity and monotonicity. Since the exponential is positive, nonnegativity for every $t \geq 0$ requires and is ensured by $c \geq 0$. Also

$$y' = e^{-t}(c - 1 - ct).$$

For $c \geq 0$, this is nonpositive for all $t \geq 0$ exactly when $c \leq 1$. Therefore

$$\boxed{0 \leq c \leq 1 \quad \Longleftrightarrow \quad -6 \leq J \leq -4.}$$

At $J = -6$, $y = e^{-t}$; at $J = -4$, $y = (1+t)e^{-t}$ starts horizontal and then decreases strictly. For $J < -6$ the response eventually becomes negative, and for $J > -4$ it initially increases. All these nonzero polynomial-exponential responses have exact real transform domain $s > -1$.



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Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An actuator may use ordinary impulses at 1 and 3, with no other force:

$$y'' + 2y' + y = J_1\delta(t-1) + J_3\delta(t-3), \qquad y(0) = y'(0) = 0.$$

The prescribed response has the form

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ (t-1)e^{-(t-1)}, & 1 \leq t < 3, \\ Ce^{-(t-3)}, & t \geq 3. \end{cases}$$

An ordinary impulse in this equation can jump velocity but not displacement.

Tasks

1. Determine the only possible value of C under the permitted actuator.
2. Recover J_1, J_3 from the one-sided states, verifying that no ordinary forcing is needed between impulses.
3. Find $Y(s)$, invert it, and confirm the prescribed tail. Decide whether choosing $C = 0$ could shut the motion off instantaneously with an ordinary impulse.
4. Verify the total response area from the time formula and the integrated impulsive equation, and state the exact real transform domain.

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Question 9 – Solution

Strategy. Infer the required impulses from velocity jumps after enforcing displacement continuity. Then independently check the recovered forcing by inversion.

Step 1: Enforce continuity at the second impulse. The target displacement just before 3 is $2e^{-2}$, while the right-hand value is C. Hence the permitted actuator requires

$$\boxed{C = 2e^{-2}.}$$

The first junction is already continuous because $(t-1)e^{-(t-1)}$ vanishes at 1.

Step 2: Recover the impulse strengths. For $h(\tau) = \tau e^{-\tau}$, $h'(0) = 1$ and $h'(2) = -e^{-2}$. Thus the first velocity jump is 1. At the second switch the new tail has velocity $-C = -2e^{-2}$, so

$$\boxed{J_1 = 1, \qquad J_3 = -2e^{-2} - (-e^{-2}) = -e^{-2}.}$$

Because y is continuous, integrating the equation across a junction gives $J_j = [y']_j$. Direct differentiation of each piece gives $y'' + 2y' + y = 0$ on all three open intervals.

Step 3: Verify by transformation and inversion. The recovered input gives

$$Y = \frac{e^{-s} - e^{-2}e^{-3s}}{(s+1)^2}, \qquad y = H_1(t)(t-1)e^{-(t-1)} - e^{-2}H_3(t)(t-3)e^{-(t-3)}.$$

For $t = 3 + \tau$, the last expression reduces to

$$e^{-2}[(\tau + 2) - \tau]e^{-\tau} = 2e^{-2}e^{-\tau},$$

which is the prescribed tail. Setting $C = 0$ instead would require a jump of $-2e^{-2}$ in displacement at 3, forbidden for the stated impulse-only actuator. Canceling velocity cannot instantaneously erase nonzero displacement.

Step 4: Check the area balance. Direct integration gives

$$\int_0^\infty y\,dt = \int_0^2 \tau e^{-\tau}\,d\tau + 2e^{-2}\int_0^\infty e^{-\tau}\,d\tau = 1 - 3e^{-2} + 2e^{-2} = \boxed{1 - e^{-2}}.$$

The integrated derivative terms are zero when their impulse jumps are included: initial and limiting displacement and velocity are zero. Thus $\int y = J_1 + J_3 = 1 - e^{-2}$ independently confirms the result. The nonzero e^{-t} tail gives exact real domain $\boxed{s > -1}$.

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Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)\,dt = \phi(a)$ for continuous ϕ near a. Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A leaky system receives a sequence of equal positive impulses:

$$y' + \kappa y = I \sum_{n=1}^{\infty} \delta(t - nT), \qquad y(0) = 0, \qquad I, \kappa, T > 0.$$

The first impulse is at T, not at zero. Use right-hand response values at each impulse.

Tasks

1. Derive $Y(s)$ by a geometric series and give the locally finite time-domain sum.
2. For $x_n = y(nT^+)$ and $x_0 = 0$, derive and solve the recurrence for post-impulse values.
3. Find the attracting periodic profile and the exact transient error. State its peak, pre-impulse trough and cycle mean, distinguishing a left limit from an attained value.
4. For $I = \kappa = 1$, classify all T for which the steady peak is at most 2 while the steady cycle mean is at least 1. State whether $y(t)$ has a single long-time limit.

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Question 10 – Solution

Strategy. Each impulse adds I to the state, while the gap between impulses multiplies it by an exponential. The cycle map determines the limiting profile.

Step 1: Sum the transforms and invert. For $s > 0$, the impulse series converges geometrically:

$$(s+\kappa)Y = I\sum_{n=1}^{\infty} e^{-nTs} = \frac{Ie^{-Ts}}{1-e^{-Ts}}, \qquad \boxed{Y = \frac{Ie^{-Ts}}{(s+\kappa)(1-e^{-Ts})}.}$$

Thus

$$y(t) = I\sum_{n=1}^{\infty} H_{nT}(t)e^{-\kappa(t-nT)}.$$

Only finitely many terms are active at each time. The formula gives zero before T, jumps of I, and $y'+\kappa y = 0$ between impulses.

Step 2: Solve the post-impulse recurrence. Put $q = e^{-\kappa T} \in (0,1)$. Decay followed by a jump gives

$$x_n = qx_{n-1} + I, \qquad \boxed{x_n = \frac{I(1-q^n)}{1-q}, \quad n \geq 0.}$$

The steady post-impulse level is $P = I/(1-q)$.

Step 3: Construct the profile and compare. For $t = nT+\tau$, $0 \leq \tau < T$, define $y_{\text{per}}(t) = Pe^{-\kappa\tau}$. Then $y(t) = x_n e^{-\kappa\tau}$, including $n = 0$, and

$$\boxed{y_{\text{per}}(t) - y(t) = Pe^{-\kappa t}.}$$

The profile's peak P is attained just after each impulse. Its pre-impulse trough is the left limit qP; under the right-hand convention this infimum is not attained within a cycle. Its mean is

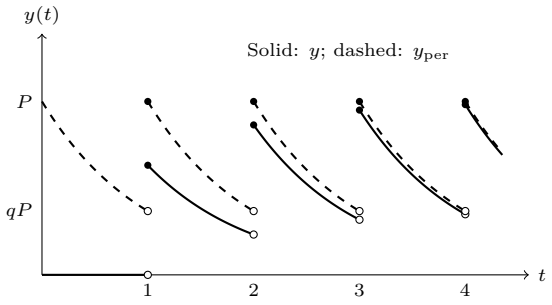
$$\frac{1}{T}\int_0^T Pe^{-\kappa\tau}\,d\tau = \boxed{\frac{I}{\kappa T}}.$$

The jump $P - qP = I$ is preserved in the limit, not smoothed away.

Step 4: Design a feasible spacing and interpret. With $I = \kappa = 1$, the peak constraint gives $1/(1-e^{-T}) \leq 2$, equivalent to $T \geq \ln 2$. The mean constraint gives $1/T \geq 1$, equivalent to $T \leq 1$. Therefore

$$\boxed{\ln 2 \leq T \leq 1.}$$

There is no single long-time limit: samples at nT tend to P, while samples at $nT+T/2$ tend to the different value $Pe^{-\kappa T/2}$. Both responses have exact real transform domain $s > 0$ because their persistent positive cycles prevent convergence at or below zero. The plot uses $I = \kappa = T = 1$ and marks the jump endpoints explicitly.



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Question 1

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ for $t \geq 0$. A student claims that multiplying their transforms gives the transform of their pointwise product.

Tasks

1. Compute $c = f * g$ directly and prove $f * g = g * f$ by a change of variable.
2. Find the pointwise product $p = fg$. Compare $c(0)$, $p(0)$ and their transforms to assess the claim.
3. Derive the convolution theorem for these functions by changing variables in a double integral. State a real range of s that justifies the interchange of integrals.
4. Find the unique global maximum of c, its total area, and its exact real transform domain. Explain why the convolution initially vanishes even though both factors are positive at zero.

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Question 1 – Solution

Strategy. Convolution integrates over a moving interval; pointwise multiplication does not.

Step 1: Integrate and reverse the factors. For $t \geq 0$,

$$c(t) = e^{-t} \int_0^t e^{-u}\, du = \boxed{e^{-t} - e^{-2t}}.$$

The substitution $v = t - u$ changes $\int_0^t f(t-u)g(u)\, du$ to $\int_0^t g(t-v)f(v)\, dv$, proving commutativity directly.

Step 2: Test the student's claim. Pointwise multiplication gives $p(t) = e^{-3t}$, so $p(0) = 1$ whereas $c(0) = 0$. Their transforms are

$$P(s) = \frac{1}{s+3}, \qquad C(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}.$$

The product $F(s)G(s)$ equals $C(s)$, not $P(s)$. For example, at $s = 0$ they have values $1/2$ and $1/3$ respectively.

Step 3: Derive the product formula. In the triangular region $0 \leq u \leq t < \infty$, put $v = t - u$. Its image is the quadrant $u, v \geq 0$, and its Jacobian has absolute value 1. For real $s > -1$, the resulting integrand has a finite absolute integral:

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty \int_0^t e^{-st} e^{-(t-u)} e^{-2u}\, du\, dt \\ &= \int_0^\infty \int_0^\infty e^{-(s+1)v} e^{-(s+2)u}\, du\, dv = F(s)G(s). \end{aligned}$$

Thus the interchange is justified by absolute integrability; it is not a formal rule for pointwise products.

Step 4: Locate the peak and inspect the tail. Since $c' = e^{-t}(2e^{-t} - 1)$, the derivative changes from positive to negative only at $t = \ln 2$. Consequently

$$\boxed{\max_{t \geq 0} c(t) = \tfrac{1}{4}, \qquad \int_0^\infty c(t)\, dt = 1 - \tfrac{1}{2} = \tfrac{1}{2}.}$$

The positive tail is asymptotic to e^{-t}, giving exact real domain $\boxed{s > -1}$; at $s = -1$ the transformed integrand tends to 1. At zero the integration interval has length zero. Finite endpoint values cannot produce a nonzero integral over that interval.

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Question 2

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two delayed rectangular pulses are

$$f = H_1 - H_3, \qquad g = H_2 - H_5,$$

and $w = f * g$.

Tasks

1. Express $w(t)$ as the length of an intersection of intervals, then derive a complete piecewise formula.
2. Find its transform and invert it as a sum of delayed ramps. Check agreement with the geometric formula.
3. Determine the support, all maximizing times, continuity, and the jumps of the first derivative.
4. Compute the area in two independent ways and give the exact real transform domain, including the value at $s = 0$.

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Question 2 – Solution

Strategy. For unit-height rectangles, the integral measures overlap length.

Step 1: Track the overlap. Using commutativity, integrate $f(u)g(t-u)$. Both factors are 1 on $[1,3] \cap [t-5, t-2]$, so

$$w = \max\{0, \min(3, t-2) - \max(1, t-5)\} = \begin{cases} 0, & t < 3, \\ t-3, & 3 \le t < 5, \\ 2, & 5 \le t < 6, \\ 8-t, & 6 \le t < 8, \\ 0, & t \ge 8. \end{cases}$$

Endpoints of the intervals do not change their length.

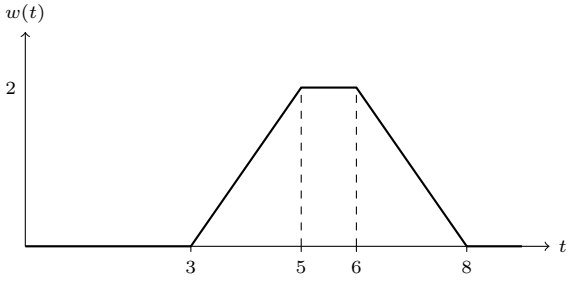
Step 2: Transform and recover the ramps. For $s \neq 0$,

$$W = \frac{e^{-3s}(1-e^{-2s})(1-e^{-3s})}{s^2}.$$

The inverse is $w = (t-3)H_3 - (t-5)H_5 - (t-6)H_6 + (t-8)H_8$. Activating these four ramps successively gives slopes $1, 0, -1, 0$, exactly as in the piecewise formula.

Step 3: Describe the geometry and regularity. The support (closure of the nonzero set) is $[3,8]$; $w > 0$ precisely on $(3,8)$. Its maximum is 2, attained at every $t \in [5,6]$. It is continuous everywhere. The first derivative has jumps $+1, -1, -1, +1$ at $3, 5, 6, 8$, respectively, and is undefined there in the ordinary two-sided sense.

Step 4: Check area and convergence. Two triangles and the plateau have total area $2 + 2 + 2 = 6$. Independently, the convolution area is $(\int f)(\int g) = 2 \cdot 3 = 6$, by integrating over the rectangle in the two integration variables. Compact support makes the transform finite for every real s. The displayed expression has removable value $W(0) = 6$.



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Question 3

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

An undamped oscillator receives a rapidly vanishing force:

$$y'' + y = e^{-t^2}, \qquad y(0) = 2, \quad y'(0) = -1.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}/2 < 1$.

Tasks

1. Find $Y(s)$ in terms of $G(s) = \mathcal{L}\{e^{-t^2}\}$ and write a convolution formula for y.
2. Verify the equation and both initial conditions by differentiating your convolution.
3. Find a limiting sinusoidal expression $y_\infty(t)$ with constant coefficients and prove $|y - y_\infty| \leq e^{-t^2}/(2t)$ for $t > 0$.
4. Prove that the response does not tend to zero, despite the vanishing force. Identify which feature of the homogeneous equation makes this possible.

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Question 3 – Solution

Strategy. Keep the Gaussian force inside an integral and separate the permanent oscillation from the remaining tail integral.

Step 1: Transform and invert. The initial terms give $(s^2+1)Y - 2s + 1 = G$, hence

$$Y = \frac{2s-1+G}{s^2+1}, \qquad \boxed{y = 2\cos t - \sin t + z(t), \quad z(t) = \int_0^t \sin(t-u)e^{-u^2}\,du.}$$

These transform operations are valid for real $s > 0$.

Step 2: Differentiate with the endpoint terms. The sine kernel vanishes at its endpoint, so $z' = \int_0^t \cos(t-u)e^{-u^2}\,du$. Differentiating again gives $z'' = e^{-t^2} - z$ because $\cos 0 = 1$. Also $z(0) = z'(0) = 0$. Thus the formula satisfies the full IVP.

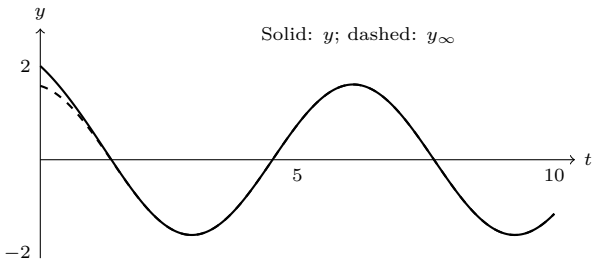
Step 3: Isolate the asymptotic oscillation. Set $A = \int_0^\infty e^{-u^2}\cos u\,du$ and $B = \int_0^\infty e^{-u^2}\sin u\,du$, both absolutely convergent. Expanding $\sin(t-u)$ gives

$$y_\infty = (2-B)\cos t + (A-1)\sin t, \qquad y - y_\infty = -\int_t^\infty \sin(t-u)e^{-u^2}\,du.$$

For $t > 0$, use $1 \le u/t$ on the tail:

$$|y - y_\infty| \le \int_t^\infty e^{-u^2}\,du \le \frac{1}{t}\int_t^\infty ue^{-u^2}\,du = \boxed{\frac{e^{-t^2}}{2t}}.$$

Step 4: Establish persistent motion. Since $|A| \le \sqrt{\pi}/2 < 1$, the coefficient $A - 1$ is nonzero. The limiting sinusoid therefore has positive amplitude and has recurring positive and negative extrema. Along their times the error tends to zero, so y cannot tend to zero (or to any single constant). The homogeneous modes $\cos t, \sin t$ have no damping. The figure compares the exact integral solution with y_∞; the two quickly become indistinguishable.



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Question 4

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let f be real and piecewise continuous on each finite interval, with $|f(t)| \leq 1$. Consider

$$y'' + 2y' + 2y = f(t), \qquad y(0) = y'(0) = 0.$$

We seek the smallest constant M such that $|y(t)| \leq M$ for every admissible input and every $t \geq 0$.

Tasks

1. Find the impulse-response kernel h and express y by convolution. Compute $\int_0^\infty h$.
2. Compute $\int_0^\infty |h|$ exactly by summing half-periods, and prove it is a uniform response bound.
3. Construct an input attaining $\int_0^T |h|$ at any prescribed $T > 0$. Prove that your uniform bound is sharp, and decide whether it can be attained at a finite time.
4. Give an explicit two-level input on $[0, 2\pi]$ whose response at 2π exceeds $1/2$. Explain the error in using the signed kernel integral as the universal bound.

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Question 4 – Solution

Strategy. A sign-changing kernel can reinforce a suitably chosen sign-changing input. Its signed area loses that information.

Step 1: Recover the kernel. The zero-data transform gives $Y = F/[(s+1)^2+1]$, so

$$h(t) = e^{-t}\sin t, \qquad y = h * f, \qquad \int_0^\infty h(t)\,dt = \tfrac{1}{2}.$$

For example, the last integral follows from the antiderivative $-e^{-t}(\sin t + \cos t)/2$.

Step 2: Sum the absolute areas. Put $q = e^{-\pi}$. The first positive lobe has area $A_0 = (1+q)/2$. Each later absolute lobe is q times its predecessor, so

$$\boxed{M = \int_0^\infty |h| = \frac{1+q}{2(1-q)}.}$$

For every admissible input, $|y(t)| \le \int_0^t |h(t-u)|\,du \le M$.

Step 3: Prove sharpness and distinguish attainment. For any $T > 0$, set

$$f_T(u) = \operatorname{sgn}(\sin(T-u)) \quad (0 \le u \le T), \qquad f_T(u) = 0 \quad (u > T),$$

with arbitrary values in $[-1,1]$ at zeros. This input is admissible and $y(T) = \int_0^T |h(T-u)|\,du = \int_0^T |h(v)|\,dv$. These values approach M as $T \to \infty$, so any smaller proposed constant fails for some finite T. However, the absolute kernel has a strictly positive tail after every finite T, so $|y(T)| < M$ for every input at every finite time. The universal supremum is sharp but is not attained.

Step 4: Refute the signed-area bound explicitly. Take $f = -1$ on $(0,\pi)$, $f = 1$ on $(\pi, 2\pi)$ and zero afterward. This is the preceding construction for $T = 2\pi$. Therefore

$$y(2\pi) = A_0(1+q) = \boxed{\frac{(1+q)^2}{2} > \frac{1}{2}}.$$

The signed integral describes cancellation within the kernel, as for a constant input's limiting response. An arbitrary bounded input can reverse its own sign so that both lobes contribute positively; the absolute area is the appropriate worst-case quantity.

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Question 5

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

For a real parameter λ, solve the Volterra integral equation

$$u(t) = 1 + \lambda \int_0^t e^{-(t-\tau)} u(\tau)\,d\tau, \qquad t \geq 0.$$

Seek a continuous solution on the entire half-line.

Tasks

1. Use transforms to derive the solution, treating every exceptional parameter value explicitly.
2. Derive an equivalent first-order IVP and prove the converse implication, establishing uniqueness among continuous solutions.
3. Classify boundedness, positivity and long-time behavior for all real λ, including $\lambda = 0$.
4. Give the exact real transform domain in every case. Explain why exponential decay of the memory kernel does not by itself guarantee a bounded solution.

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Question 5 – Solution

Strategy. Feedback changes the effective growth rate. Verify the transformed candidate through an equivalent IVP so no growth assumption is needed for uniqueness.

Step 1: Solve the transformed equation. Initially for sufficiently large s,

$$U = \frac{1}{s} + \frac{\lambda U}{s+1}, \qquad U = \frac{s+1}{s(s+1-\lambda)}.$$

Partial fractions give

$$\boxed{u(t) = \frac{1-\lambda e^{(\lambda-1)t}}{1-\lambda} \quad (\lambda \neq 1), \qquad u(t) = 1+t \quad (\lambda = 1).}$$

In particular $\lambda = 0$ gives $u = 1$; the factor $s+1$ then cancels in U.

Step 2: Prove equivalence without assuming a transform. For continuous u, define $v = e^{-t} * u$. Differentiation gives $v' = u - v$, $v(0) = 0$. The integral equation makes u continuously differentiable and yields

$$u' = (\lambda - 1)u + 1, \qquad u(0) = 1.$$

Conversely, let u solve this IVP and form v as above. The residual $w = u-1-\lambda v$ satisfies $w' = -u+1+\lambda v = -w$ and $w(0) = 0$. Hence $w = 0$ and the integral equation holds, including at $\lambda = 0$. The linear IVP has a unique global solution, which proves the asserted existence and uniqueness.

Step 3: Classify all parameters. For $\lambda < 1$, the solution is bounded and tends to $1/(1-\lambda) > 0$. For $\lambda = 1$ it grows linearly; for $\lambda > 1$ it grows exponentially with positive leading coefficient $\lambda/(\lambda-1)$. For $\lambda \neq 1$, $u' = \lambda e^{(\lambda-1)t}$: it decreases from 1 to a positive limit when $\lambda < 0$, is constant when $\lambda = 0$, and increases when $\lambda > 0$. Thus it is positive for every real λ and every $t \geq 0$.

Step 4: Inspect the actual transform domain. For $\lambda < 1$ the positive nonzero limiting constant forces $s > 0$; for $\lambda = 1$ the linear term also requires $s > 0$. For $\lambda > 1$ the nonzero exponential requires $s > \lambda - 1$. Consequently

$$\boxed{s > 0 \ (\lambda \leq 1), \qquad s > \lambda - 1 \ (\lambda > 1).}$$

Positive feedback can offset or exceed the kernel's decay. The rate in the equivalent equation is $\lambda - 1$, not simply the kernel's rate -1.

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Question 6

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Three identical smoothing stages each have causal kernel $k_a(t) = ae^{-at}$, where $a > 0$. Their cascade has kernel $r_a = k_a * k_a * k_a$.

Tasks

1. Compute the two-stage and three-stage kernels by direct integration.
2. Find the transform of r_a and justify why the grouping of the convolutions does not matter.
3. Find the unique peak of r_a, its height and its initial value and slope. Compare these with a single stage.
4. Compute the mass and mean time of both kernels. Explain what is preserved by cascading and what changes. You may derive needed exponential moments by integration by parts.

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Question 6 – Solution

Strategy. Equal exponential factors leave polynomial integrals inside the convolution. Unit mass is preserved while the response is spread over time.

Step 1: Integrate the cascade. Directly,

$$(k_a * k_a)(t) = a^2 e^{-at} \int_0^t 1\, du = a^2 t e^{-at}, \qquad r_a(t) = a^3 e^{-at} \int_0^t u\, du = \boxed{\tfrac{1}{2} a^3 t^2 e^{-at}}.$$

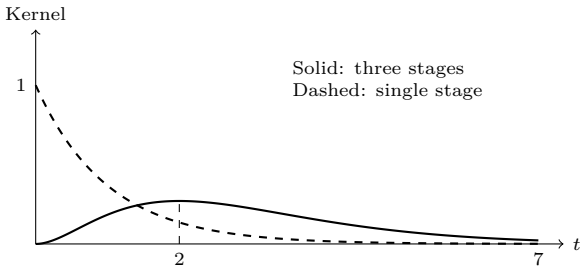
Step 2: Check transforms and grouping. For real $s > -a$, $R_a(s) = [a/(s+a)]^3$. Both groupings integrate the same product $k_a(t-u-v)k_a(u)k_a(v)$ over $u, v \geq 0$, $u + v \leq t$. The continuous integrand is integrable on this bounded triangle, so reversing the integration order proves associativity directly.

Step 3: Compare the peak and onset. For $t > 0$, $r_a' = \frac{1}{2} a^3 e^{-at} t(2 - at)$. Thus the unique maximum is at $t = 2/a$, with height $2ae^{-2}$. Also $r_a(0) = r_a'(0) = 0$. A single stage has maximum a at zero and right derivative $-a^2$ there. The cascade begins more smoothly, reaches its peak later and has a lower peak.

Step 4: Compute mass and mean time. Integration by parts yields $\int_0^\infty t^n e^{-at}\, dt = n!/a^{n+1}$ for nonnegative integers n. Hence

$$\int k_a = 1, \quad \int r_a = 1, \qquad \int t k_a(t)\, dt = \frac{1}{a}, \quad \int t r_a(t)\, dt = \frac{3}{a}.$$

Since both masses are one, these first moments are their mean times. The total area is preserved; the mean time triples. The cascade remains positive at every $t > 0$, so its later peak is not a literal waiting interval with zero output. The figure uses $a = 1$ and the same vertical scale for both.



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Question 7

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let the unstable kernel be $h(t) = e^t$. First allow a piecewise continuous input f supported in $[0, T]$, where $T > 0$. Then specialize to

$$f_c(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -c, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \qquad c > 0,$$

with output $y = h * f_c$.

Tasks

1. For the general input, derive a necessary and sufficient integral condition for the output to vanish for every $t \geq T$.
2. For f_c, compare the choice that makes the ordinary input area zero with the choice that eliminates the output tail.
3. For the tail-eliminating choice, derive the complete piecewise output and find its maximum.
4. Find the transformed output for general c and compare its exact real transform domain before and after cancellation. Explain why the common transform domain of the factors need not be the output's full domain.

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Question 7 – Solution

Strategy. An unstable kernel remembers a weighted input area. Equal positive and negative ordinary areas need not cancel that memory.

Step 1: Factor the growing exponential. For an input supported in $[0, T]$,

$$(h * f)(t) = e^t \int_0^{\min(t,T)} e^{-u} f(u)\, du.$$

Thus the output is zero for every $t \geq T$ if and only if $\boxed{\int_0^T e^{-u} f(u)\, du = 0}$, since e^t never vanishes.

Step 2: Distinguish the two cancellations. The ordinary area is $1 - c$, so it vanishes at $c = 1$. The weighted area is $(1 - e^{-1}) - c(e^{-1} - e^{-2})$; its unique zero is $\boxed{c = e}$. At $c = 1$ it equals $(1 - e^{-1})^2 > 0$, leaving a growing tail.

Step 3: Recover the selected response. Using the partial weighted integrals with $c = e$ gives

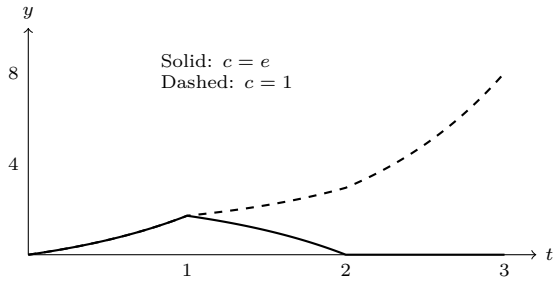
$$\boxed{y_*(t) = \begin{cases} e^t - 1, & 0 \leq t < 1, \\ e - e^{t-1}, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}}$$

The pieces agree at both junctions. It increases up to 1 and decreases from 1 to 2, so its unique maximum is $e - 1$ at $t = 1$.

Step 4: Track the transform domain. The convolution theorem initially gives, for $s > 1$,

$$Y(s) = \frac{1 - (1 + c)e^{-s} + ce^{-2s}}{s(s - 1)}.$$

If $c \neq e$, the output has a nonzero e^t tail and exact real domain $s > 1$. If $c = e$, the output is compactly supported, so its transform exists for every real s; both apparent singularities at 0 and 1 are removable. The product formula agrees with it on $s > 1$ and extends by removal of these singularities. Cancellation changes this output, not the unstable kernel. The dashed comparison uses $c = 1$, which cancels only the ordinary input area.



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Question 8

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Suppose an ordinary continuous input f produces $z = e^{-t} * f$. No impulses and no nonzero initial state are allowed. Consider the desired output $z_0(t) = te^{-t}$ and the perturbed outputs

$$z_{\varepsilon,N}(t) = te^{-t} + \varepsilon e^{-t}\sin(Nt), \qquad \varepsilon > 0, \quad N \in \{1, 2, \ldots\}.$$

Tasks

1. Characterize all continuously differentiable outputs obtainable from such inputs, and prove uniqueness of the recovered input.
2. Recover the input for z_0. Decide whether the target $z(t) = e^{-t}$ is obtainable, explaining why differentiating alone can give a misleading answer.
3. Recover the input for $z_{\varepsilon,N}$ and bound its output error in the uniform norm on $[0, \infty)$. Compute the input error at zero.
4. Choose a sequence $\varepsilon = \varepsilon_N \to 0$ for which the output errors tend uniformly to zero but the input errors grow without bound. Interpret what this means for reconstruction from approximate measurements.

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Question 8 – Solution

Strategy. Inversion differentiates the output. Check the initial-value restriction before using that derivative formula.

Step 1: Derive and prove the inverse rule. Differentiation under the integral gives $z' = f - z$, $z(0) = 0$. Therefore a C^1 target is obtainable exactly when it starts at zero, and its only possible continuous input is $\boxed{f = z' + z}$. Conversely, for this input, the target and the convolution solve the same IVP $w' + w = f$, $w(0) = 0$. Uniqueness proves they agree.

Step 2: Check two targets. For $z_0 = te^{-t}$, $z_0' + z_0 = e^{-t}$, so the input is $f_0 = e^{-t}$. The target e^{-t} is impossible because its value at zero is 1. Although its derivative plus itself is zero, a zero input with zero initial state produces the zero output. Differentiation discards the missing initial condition.

Step 3: Differentiate the perturbation. The two exponential-derivative terms cancel, leaving

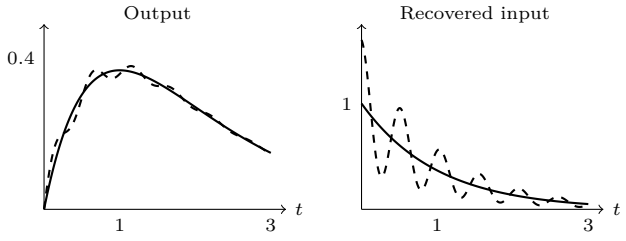
$$\boxed{f_{\varepsilon,N} = e^{-t} + \varepsilon N e^{-t}\cos(Nt).}$$

All these inputs are continuous, and the target outputs vanish at zero. The uniform output error is at most ε because $e^{-t}|\sin Nt| \le 1$. The input error at zero is εN; in fact this is its uniform norm since $e^{-t}|\cos Nt| \le 1$.

Step 4: Exhibit the instability. Take $\varepsilon_N = N^{-1/2}$. Then

$$\|z_{\varepsilon_N,N} - z_0\|_\infty \le N^{-1/2} \to 0, \qquad \|f_{\varepsilon_N,N} - f_0\|_\infty = \sqrt{N} \to \infty.$$

Thus the inverse is not continuous in the uniform output norm, even on smooth attainable outputs. Small high-frequency measurement errors can produce large reconstruction errors. The figure uses $\varepsilon = 0.05$, $N = 12$; each panel compares its own exact and perturbed functions.



Solid: exact; dashed: perturbed

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Question 9

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two normalized stages $k_\alpha(t) = \alpha e^{-\alpha t}$ and $k_\beta(t) = \beta e^{-\beta t}$, with $\alpha, \beta > 0$, are followed by a delay $d \geq 0$:

$$r(t) = H_d(t)(k_\alpha * k_\beta)(t - d).$$

The observed response is positive precisely for $t > 1$, has mass 1, mean $\mu = 5/2$ and variance $v = 5/4$. For a nonnegative unit-mass kernel p, $\mu_p = \int_0^\infty tp(t)\,dt$ and $v_p = \int_0^\infty (t - \mu_p)^2 p(t)\,dt$.

Tasks

1. Prove that convolution adds the means and variances of two nonnegative unit-mass kernels with finite second moments.
2. Compute the mean and variance of k_α, and explain the effect of a delay on mass, mean and variance.
3. Recover d and the two rates from the observations. State precisely what is and is not uniquely identifiable.
4. For general prescribed delay d, mean μ and variance v, give necessary and sufficient conditions for two finite positive rates to exist. Interpret the equality case and the excluded upper boundary.

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Question 9 – Solution

Strategy. Work with reciprocal rates. Convolution moment identities reduce identification to a quadratic equation.

Step 1: Prove the moment identities. In the convolution integral use $t = u+w$, $u, w \geq 0$. Nonnegativity permits interchange, and finite second moments make the following values finite:

$$\int (p * q) = (\int p)(\int q) = 1,$$
$$\mu_{p*q} = \iint (u + w)p(u)q(w)\,du\,dw = \mu_p + \mu_q,$$
$$\int t^2(p * q)(t)\,dt = \int u^2 p(u)\,du + 2\mu_p\mu_q + \int w^2 q(w)\,dw.$$

Subtracting $(\mu_p + \mu_q)^2$ from the last line proves $\boxed{v_{p*q} = v_p + v_q}$.

Step 2: Calculate the stage statistics. Integration by parts gives mass 1, first moment $1/\alpha$ and second moment $2/\alpha^2$ for k_α. Its variance is therefore $1/\alpha^2$. A delay replaces time by $t = d + u$: mass stays 1, the mean increases by d, and the centered second moment is unchanged. Hence

$$\mu = d + \frac{1}{\alpha} + \frac{1}{\beta}, \qquad v = \frac{1}{\alpha^2} + \frac{1}{\beta^2}.$$

Step 3: Recover the parameters. Both stage kernels are positive for positive times, so their convolution is positive at every positive time and zero at zero. The observed onset thus forces $d = 1$. Put $x = 1/\alpha$, $z = 1/\beta$. Then

$$x + z = \tfrac{3}{2}, \quad x^2 + z^2 = \tfrac{5}{4}, \quad xz = \tfrac{1}{2}, \quad (X - x)(X - z) = X^2 - \tfrac{3}{2}X + \tfrac{1}{2}.$$

Its roots are 1 and 1/2, giving $\boxed{\{\alpha, \beta\} = \{1, 2\}}$. The delay and unordered rate pair are unique. Stage order cannot be identified even from the entire output because convolution is commutative.

Step 4: Give the full feasibility condition. Write $m = \mu - d$. Positive x, z require and are possible exactly when

$$\boxed{m > 0, \qquad \frac{m^2}{2} \leq v < m^2.}$$

Indeed $xz = (m^2 - v)/2 > 0$ and $(x - z)^2 = 2v - m^2 \geq 0$; conversely these conditions make $(m \pm \sqrt{2v - m^2})/2$ real and strictly positive. At the lower equality the rates coincide. At the upper boundary one reciprocal rate would be zero, requiring an infinite rate, excluded here.

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Question 10

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t - u)g(u)\,du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f \in C^1[0, \infty)$ satisfy $|f'(t)| \leq L$, with $L \geq 0$. For $\varepsilon > 0$ define

$$k_\varepsilon(t) = \frac{1}{\varepsilon} e^{-t/\varepsilon}, \qquad y_\varepsilon = k_\varepsilon * f.$$

This causal averaging kernel becomes concentrated near zero as ε decreases.

Tasks

1. Derive an IVP for y_ε and use integration by parts to express $y_\varepsilon - f$ in terms of $f(0)$ and f'.
2. Prove an explicit error bound containing both the initial transient and a term at most $L\varepsilon$.
3. Distinguish uniform convergence on $[0, \infty)$ from uniform convergence on $[t_0, \infty)$ for $t_0 > 0$. Show the $L\varepsilon$ bound is sharp when $f(0) = 0$.
4. For $f(t) = 1 + t$, decide separately whether $\varepsilon = 1/20$ and $\varepsilon = 1/40$ ensure $|y_\varepsilon(t) - f(t)| \leq 1/20$ for every $t \geq 1$. Justify the strict inequalities involved.

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Question 10 – Solution

Strategy. Separate the error due to starting from rest from the error due to variation of the input.

Step 1: Obtain the equation and exact error identity. Differentiation gives $\varepsilon y_\varepsilon' + y_\varepsilon = f$, $y_\varepsilon(0) = 0$. Integrating the convolution by parts in u gives

$$y_\varepsilon(t) - f(t) = -e^{-t/\varepsilon} f(0) - \int_0^t e^{-(t-u)/\varepsilon} f'(u)\, du.$$

Step 2: Bound the two contributions. Taking absolute values yields

$$\boxed{|y_\varepsilon - f| \le |f(0)| e^{-t/\varepsilon} + L\varepsilon(1 - e^{-t/\varepsilon}).}$$

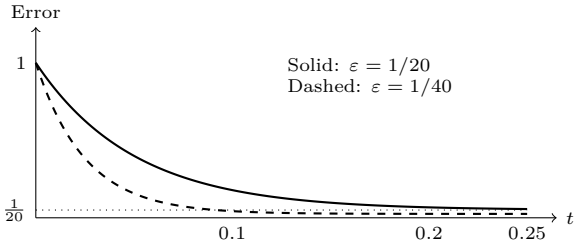
This accounts for the kernel mass missing before time zero as well as the variation over its recent history.

Step 3: Distinguish the convergence statements. If $f(0) = 0$, the error is uniformly at most $L\varepsilon$ on the half-line. For the ramp $f = Lt$ it equals $L\varepsilon(1 - e^{-t/\varepsilon})$ in magnitude, whose supremum is $L\varepsilon$ (approached as $t \to \infty$ when $L > 0$). Thus the constant is sharp. If $f(0) \neq 0$, the error at zero is always $|f(0)|$, preventing uniform convergence on $[0, \infty)$. On $[t_0, \infty)$ it is bounded by $|f(0)| e^{-t_0/\varepsilon} + L\varepsilon \to 0$ for every fixed $t_0 > 0$.

Step 4: Test the two proposed averaging scales. For $f = 1 + t$, $y_\varepsilon = t + 1 - \varepsilon - (1 - \varepsilon)e^{-t/\varepsilon}$. For $0 < \varepsilon < 1$ the error magnitude is $E_\varepsilon = \varepsilon + (1 - \varepsilon)e^{-t/\varepsilon}$. At $\varepsilon = 1/20$ this is strictly greater than $1/20$ at every finite time, so that choice fails. At $\varepsilon = 1/40$ and $t \ge 1$,

$$E_\varepsilon \le \frac{1}{40} + \frac{39}{40} e^{-40} < \frac{1}{20},$$

since $e^{40} > 1 + 40 = 41 > 39$. Thus the smaller choice succeeds. The figure shows the initial error layer and the $1/20$ tolerance.



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Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A draft transform table contains these three proposed pairs, with $a > 0$:

$$t^3 e^{-2t} \longleftrightarrow \frac{1}{(s+2)^4}, \qquad e^{-at}\sin(3t) \longleftrightarrow \frac{s+a}{(s+a)^2+9}, \qquad f(2t) \longleftrightarrow F(s/2).$$

In the third pair $F = \mathcal{L}\{f\}$, where f is continuous and of exponential order.

Tasks

1. Identify and correct every error. Derive the power entry by integration by parts and the scaling entry by substitution.
2. State the exact real convergence domains for the first two corrected entries. State the sufficient transformed domain for the third if F converges absolutely for $s > c$.
3. Use the corrected rules to find the transform of $t^3 e^{-2t} + e^{-at}\sin(3t)$, without a fresh integration.
4. Check your answer independently using the leading large-s behavior and the value at $s = 0$. Explain what each check can and cannot establish.

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Question 1 – Solution

Strategy. A table entry includes its normalization and domain, not only its denominator.

Step 1: Repair the entries. For $p > 0$, integration by parts gives $I_n(p) = nI_{n-1}(p)/p$, $I_0(p) = 1/p$, with vanishing boundary terms. Thus $I_3(p) = 6/p^4$. The sine entry has frequency in its numerator. Substitution $u = 2t$ in the transform supplies the missing Jacobian. The corrected pairs are

$$t^3e^{-2t} \leftrightarrow \frac{6}{(s+2)^4}, \quad e^{-at}\sin 3t \leftrightarrow \frac{3}{(s+a)^2+9}, \quad f(2t) \leftrightarrow \frac{1}{2}F(s/2).$$

The proposed sine numerator instead represents $e^{-at}\cos 3t$.

Step 2: Include convergence information. The first domain is $s > -2$. At its boundary the integrand is t^3. The second is $s > -a$; at the boundary the sine integral has no improper limit, and below it the oscillations grow. For the scaled general function, $s > 2c$ ensures absolute convergence. This is sufficient and need not be exact if the supplied c is merely an exponential-order bound.

Step 3: Combine the corrected entries. Linearity gives

$$\boxed{Y(s) = \frac{6}{(s+2)^4} + \frac{3}{(s+a)^2+9}, \qquad s > \max(-2, -a).}$$

The polynomial-exponential and sinusoidal tails cannot cancel one another.

Step 4: Make independent checks. Near zero, the sum is $3t + O(t^2)$, so its leading transform behavior is $3/s^2$, consistent with the displayed expression. This would immediately reject a cosine numerator, which would introduce a $1/s$ term. Direct areas are $6/2^4 = 3/8$ and $3/(a^2+9)$, giving $Y(0) = 3/8+3/(a^2+9)$. This catches the missing factorial. These checks are necessary consistency tests, not substitutes for the integration and operational derivations: many different transforms can share one asymptotic term and one value.

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Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $a > 0$, define the delay and time-multiplication operations

$$(D_a f)(t) = H_a(t) f(t-a), \qquad (Mf)(t) = t f(t).$$

Take $f(t) = e^{-t}$, and compare $p = D_a(Mf)$ with $q = M(D_a f)$.

Tasks

1. Derive the transform rules for D_a and M from their defining integrals. State a domain where the derivations are justified for this f.
2. Find p, q and their transforms. Explain exactly why applying the two operations in opposite orders changes the answer.
3. Prove the general identity $MD_a f - D_a M f = a D_a f$ whenever the expressions are defined, and verify its transform.
4. For $a = 2$, invert $e^{-2s}/(s+1)^2$ and $-\frac{d}{ds}[e^{-2s}/(s+1)]$. Compute the area of each inverse and use it to distinguish them.

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Question 2 – Solution

Strategy. A delayed clock measures $t-a$; multiplication after the delay still uses the original clock t.

Step 1: Derive the rules. The substitution $u=t-a$ gives $\mathcal{L}\{D_a f\}=e^{-as}F(s)$. Differentiating the transform integral with respect to s gives $\mathcal{L}\{Mf\}=-F'(s)$. For $f=e^{-t}$ and its delayed polynomial multiples, $s>-1$ makes these integrals absolutely convergent and permits this differentiation, locally dominated by an integrable exponential times a polynomial.

Step 2: Keep the clocks explicit. Here

$$p=H_a(t)(t-a)e^{-(t-a)}, \qquad q=H_a(t)te^{-(t-a)},$$

so

$$P=\frac{e^{-as}}{(s+1)^2}, \qquad Q=-\frac{d}{ds}\frac{e^{-as}}{s+1}=e^{-as}\left[\frac{a}{s+1}+\frac{1}{(s+1)^2}\right].$$

The extra term comes from differentiating the delay factor itself.

Step 3: Prove the operation identity. For $t<a$ both sides vanish. For $t\geq a$, subtraction gives $tf(t-a)-(t-a)f(t-a)=af(t-a)$. Therefore

$$\boxed{MD_af-D_aMf=aD_af.}$$

In transform language, $-(e^{-as}F)'-e^{-as}(-F')=ae^{-as}F$, the same identity by the product rule.

Step 4: Invert and distinguish the two expressions. For $a=2$ the first inverse is $H_2(t)(t-2)e^{-(t-2)}$, while the second is $H_2(t)te^{-(t-2)}$. Their areas, after putting $u=t-2$, are

$$\int_0^\infty ue^{-u}\,du=\boxed{1}, \qquad \int_0^\infty (u+2)e^{-u}\,du=\boxed{3}.$$

Their transforms at zero have these same values. The first starts continuously from zero at the delay; the second has right-hand value 2 there. Isolated endpoint conventions do not change either area.

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Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A short table gives only $\mathcal{L}\{\sin(at)\} = a/(s^2 + a^2)$, where $a > 0$. You need inverses of

$$A_a(s) = \frac{s}{(s^2 + a^2)^2}, \qquad B_a(s) = \frac{1}{(s^2 + a^2)^2}.$$

Tasks

1. Derive the inverse of A_a by differentiating a transform with respect to s.
2. Derive the inverse of B_a by differentiating the given sine entry with respect to a. Explain why the parameter differentiation is justified for real $s > 0$.
3. Check the first nonzero Taylor term at $t = 0$ against the leading large-s term of each transform.
4. Find both pointwise limits as $a \to 0^+$ and explain why substituting $a = 0$ into an unsimplified inverse formula is invalid.

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Question 3 – Solution

Strategy. Generate missing table rows through operations whose normalization can be checked independently.

Step 1: Differentiate with respect to s. Since $-\frac{d}{ds}[a/(s^2+a^2)] = 2as/(s^2+a^2)^2$,

$$\boxed{\mathcal{L}^{-1}\{A_a\} = \frac{t\sin(at)}{2a}.}$$

The factor $2a$ must be divided out.

Step 2: Differentiate with respect to a. The parameter derivative gives $\mathcal{L}\{t\cos(at)\} = (s^2-a^2)/(s^2+a^2)^2$. For fixed $s>0$, te^{-st} dominates the differentiated integrand uniformly in a, justifying differentiation under the integral. Subtracting this row from $\mathcal{L}\{\sin(at)/a\}$ yields

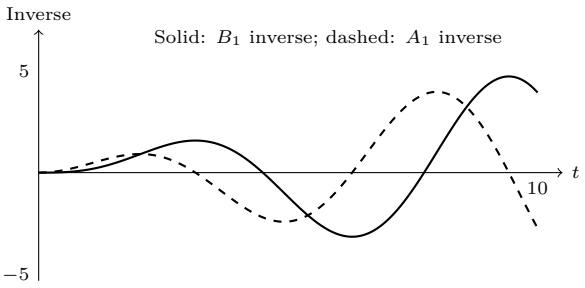
$$\frac{1}{s^2+a^2} - \frac{s^2-a^2}{(s^2+a^2)^2} = \frac{2a^2}{(s^2+a^2)^2}.$$

Thus

$$\boxed{\mathcal{L}^{-1}\{B_a\} = \frac{\sin(at) - at\cos(at)}{2a^3}.}$$

Step 3: Check the onset powers. The inverses start as $t^2/2 + O(t^4)$ and $t^3/6 + O(t^5)$. Their leading transforms are respectively $1/s^3$ and $1/s^4$, matching A_a and B_a. This checks both factorials and the cancellation of the linear term in the second numerator.

Step 4: Remove the apparent parameter singularities. At each fixed t, the Taylor expansions give limits $t^2/2$ and $t^3/6$. Their transforms are $1/s^3$ and $1/s^4$, agreeing with the limits of the rational functions for $s>0$. Directly inserting $a=0$ into the inverse quotients produces $0/0$; the limits require cancellation first. The figure uses $a=1$ and shows that the repeated denominator can produce oscillations with increasing size, unlike a single sine entry.



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Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

An unknown function has transform

$$F(s) = \frac{As + B}{s^2 + 4}, \qquad s > 0,$$

where A, B are real. Measurements give $f(0) = 1$ and $f'(0) = -2$.

Tasks

1. Invert the general transform by matching numerators to table entries, then recover A, B from the data.
2. For the recovered function, find its amplitude, all positive zeros, and the conserved quantity $f'(t)^2 + 4f(t)^2$.
3. Classify all real pairs (A, B) for which the inverse is nonnegative for every $t \geq 0$. Prove the classification.
4. A student argues that $\lim_{s\to 0^+} sF(s) = 0$ proves $\lim_{t\to\infty} f(t) = 0$. Refute the conclusion with two sequences of times and identify the missing hypothesis in that use of the final-value theorem.

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Question 4 – Solution

Strategy. Matching a denominator is insufficient: the numerator determines the phase, initial state and sign changes.

Step 1: Match and use the data. The table gives $f = A\cos 2t + (B/2)\sin 2t$, so $f(0) = A$ and $f'(0) = B$. Therefore $A = 1$, $B = -2$ and

$$\boxed{f(t) = \cos 2t - \sin 2t = \sqrt{2}\cos(2t + \pi/4).}$$

Step 2: Determine the motion. The amplitude is $\sqrt{2}$. The positive zeros are $t = \pi/8 + k\pi/2$, $k = 0, 1, 2, \ldots$. Direct differentiation gives $f'' + 4f = 0$, so $(f'^2 + 4f^2)' = 2f'(f'' + 4f) = 0$. Its initial value is $(-2)^2 + 4(1)^2 = \boxed{8}$.

Step 3: Prove the positivity classification. If $(A, B) \neq (0, 0)$, the amplitude $\sqrt{A^2 + B^2/4}$ is positive. The sinusoid attains both that amplitude and its negative at arbitrarily large nonnegative times. It cannot be nonnegative on the whole half-line. Only $\boxed{A = B = 0}$ works. An alternative proof uses zero integral over one period: a continuous nonnegative function with zero integral must vanish.

Step 4: Check the claimed final value. Although $sF(s) \to 0$ as $s \to 0^+$, the selected solution obeys $f(n\pi) = 1$ and $f(n\pi + \pi/2) = -1$. Hence it has no time limit. The usual rational final-value theorem requires the poles of $sF(s)$ to lie strictly in the left half-plane after cancellation. Here the uncanceled poles are $s = \pm 2i$, supporting persistent oscillations. An algebraic limit of the transform does not by itself establish a limit of the function.

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Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A table contains the infinite-tail entry $te^{-t} \leftrightarrow 1/(s+1)^2$. A finite experiment instead uses

$$w(t) = \begin{cases} te^{-t}, & 0 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

A student subtracts $e^{-2s}/(s+1)^2$ to remove the tail.

Tasks

1. Explain why the proposed subtraction is incorrect, and express the actual tail as a delayed function with its internal clock made explicit.
2. Derive $W(s)$ from table operations and verify it by integration over the finite experiment.
3. Find the exact real transform domain and the removable value at $s = -1$. Explain why the infinite-tail table domain does not restrict this finite-window transform.
4. Find the area, the global maximum and the one-sided values at the cutoff. Draw the finite-window response accurately on the solution page.

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Question 5 – Solution

Strategy. Removing an old tail requires its original age, not a fresh copy starting at zero.

Step 1: Identify the tail that must be removed. For $t = 2 + u$, the old signal is $e^{-2}(u+2)e^{-u}$. Thus

$$w = te^{-t} - e^{-2}H_2(t)[(t-2)+2]e^{-(t-2)}.$$

The proposed subtraction instead removes $H_2(t)(t-2)e^{-(t-2)}$, which has both the wrong age polynomial and the wrong exponential scale.

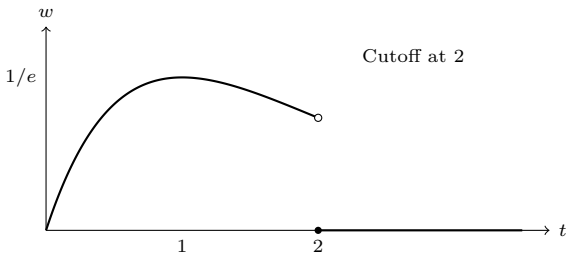
Step 2: Transform and check directly. For $s \neq -1$, the correct expression is

$$\boxed{W(s) = \frac{1 - e^{-2(s+1)}[1 + 2(s+1)]}{(s+1)^2}.}$$

Initially table operations justify it for $s > -1$. Independently put $p = s + 1$: $W = \int_0^2 te^{-pt}\,dt$. Integration by parts gives $[1 - e^{-2p}(1+2p)]/p^2$, agreeing with the table calculation and extending its formula to $p \neq 0$ of either sign.

Step 3: Remove the singularity and identify the domain. A bounded finite interval makes this transform finite for every real s. At $s = -1$ the defining integral is $\int_0^2 t\,dt = \boxed{2}$. The numerator's Taylor expansion is $2p^2 + O(p^3)$, so the apparent pole is removable. The two infinite-tail terms need only be transformed separately on their common domain; their difference has a larger actual domain.

Step 4: Record the geometric checks. The area is $W(0) = \boxed{1 - 3e^{-2}}$. Since $(te^{-t})' = (1-t)e^{-t}$, the unique global maximum is $1/e$ at $t = 1$. At the cutoff, $w(2^-) = 2e^{-2}$ while $w(2) = w(2^+) = 0$. The open dot marks the excluded left-hand value and the filled dot the actual value at 2; no slanted segment bridges the jump.



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Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Let p be the continuous periodic triangular signal of period 2 with

$$p(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 \leq t < 2, \end{cases} \qquad p(t+2) = p(t).$$

A standard table lists delayed ramps but does not list this periodic function.

Tasks

1. Derive the periodic-transform formula by splitting the half-line into cycles. Apply it to this signal.
2. Write a locally finite series of delayed ramps for p and use it to obtain the same transform independently.
3. Find the exact real transform domain. Compute $\lim_{s \to 0^+} sP(s)$ and interpret it using the mean over one period.
4. Explain why the cycle mean is not a pointwise final value, and determine the first two leading terms of $P(s)$ as $s \to 0^+$.

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Question 6 – Solution

Strategy. A one-cycle transform and a geometric series supply the periodic row; delayed ramps independently encode its slope changes.

Step 1: Sum the cycles. For $s > 0$, boundedness gives absolute convergence. Splitting into intervals $[2n, 2n+2]$ and translating yields

$$P(s) = \frac{\int_0^2 e^{-st} p(t)\,dt}{1 - e^{-2s}} = \boxed{\frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})}.}$$

The one-cycle integral is $(1-e^{-s})^2/s^2$, obtained by integration by parts on $[0,1]$ and $[1,2]$.

Step 2: Recover the slope changes. Write $R_a = (t-a)H_a(t)$. The series

$$p(t) = \sum_{n=0}^{\infty} [R_{2n} - 2R_{2n+1} + R_{2n+2}]$$

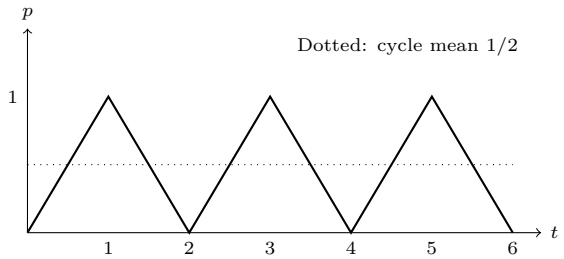
is locally finite; each bracket is a unit triangular pulse on $[2n, 2n+2]$. Its transform is $e^{-2ns}(1-2e^{-s}+e^{-2s})/s^2$. Summing for $s > 0$ gives the same P. These brackets, rather than the individual growing ramps across all cycles, also make absolute convergence clear.

Step 3: Check convergence and the mean. The signal is bounded, nonnegative and has area 1 per period. Thus the exact real domain is $s > 0$: at $s = 0$ the areas sum to infinity, and for $s < 0$ the weighted cycle contributions grow. Using $(1-e^{-s})/(1+e^{-s}) = \tanh(s/2)$, $\lim_{s\to 0^+} sP(s) = \boxed{1/2}$, the cycle mean $1/2$.

Step 4: Distinguish the limit notions. The samples $p(2n) = 0$ and $p(2n+1) = 1$ rule out a pointwise final value. Taylor expansion of the last quotient gives

$$\boxed{P(s) = \frac{1}{2s} - \frac{s}{24} + O(s^3).}$$

The leading pole records the mean, while the constant term vanishes. The figure displays continuous corners and equal successive cycles.



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Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $t > 0$, let

$$g(t) = \frac{e^{-t} - e^{-3t}}{t}.$$

A rational-only table offers no obvious row for this function.

Tasks

1. Extend g continuously to zero and represent it as an integral of exponential functions over a finite parameter interval.
2. Use that representation to derive its transform, carefully justifying interchange and specifying a real domain.
3. Prove positivity and strict decrease on $[0, \infty)$, and compute the total area.
4. Find the exact real transform domain, including a proof of divergence at its boundary. Explain why an algebraically real continuation of a logarithm is not sufficient evidence of transform convergence.

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Question 7 – Solution

Strategy. Integrating a familiar exponential row over its rate parameter generates a nonrational row.

Step 1: Remove the time singularity. Direct integration in a gives

$$\boxed{g(t) = \int_1^3 e^{-at}\,da, \qquad g(0) = 2.}$$

This formula also proves continuity at zero without dividing by t.

Step 2: Integrate the table row. For real $s > -1$, the nonnegative double integral is finite, and interchange gives

$$G(s) = \int_1^3 \int_0^\infty e^{-(s+a)t}\,dt\,da = \int_1^3 \frac{da}{s+a} = \boxed{\ln\frac{s+3}{s+1}}.$$

Both logarithm arguments are positive in this domain. Finiteness follows also from $s + a \geq s + 1 > 0$ on the finite parameter interval.

Step 3: Establish shape and area. The integrand is positive, so $g(t) > 0$. Parameter integration on a bounded interval permits differentiation at every $t \geq 0$:

$$g'(t) = -\int_1^3 ae^{-at}\,da < 0.$$

Thus the extension decreases strictly from 2, approaching zero. Since 0 lies in the transform domain, its area is $G(0) = \boxed{\ln 3}$.

Step 4: Check the boundary from the original integral. At $s = -1$, the transformed integrand is $(1 - e^{-2t})/t$ for $t > 0$. For $t \geq 1$ it is at least $(1 - e^{-2})/t$, whose integral diverges. For $s < -1$ the positive integrand is even larger on the tail. Therefore $\boxed{s > -1}$ is the exact real domain. For example, the logarithm of $(s + 3)/(s + 1)$ is real again when $s < -3$, but that algebraic expression cannot represent the divergent defining integral. The table entry must travel with its domain and the continuous value at zero.

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Question 8

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Consider the transform $F(s) = s/(s+1)$. Allow a full-mass impulse at the origin, with $\mathcal{L}\{\delta(t)\} = 1$ and pre-impact data at 0^-. For ordinary candidates, restrict to functions continuous near zero and of exponential order. The input with transform F drives

$$y' + 2y = f, \qquad y(0^-) = 0.$$

Tasks

1. Explain why F cannot be the transform of an ordinary candidate in the stated class. Recover its distributional inverse.
2. Find $Y(s)$ and the complete response for $t \geq 0$, including the jump at zero.
3. Find the positive zero, the global minimum, and the total response area. Explain how a nonzero response can have zero area.
4. Verify the post-impact equation and the equivalent ordinary restart. State the exact real transform domain of the response.

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Question 8 – Solution

Strategy. Divide an improper rational expression before consulting ordinary-function rows. Its polynomial part may represent an impulse.

Step 1: Identify the input class. For an ordinary candidate continuous near zero and of exponential order, its Laplace transform tends to zero as $s \to \infty$: the bounded initial piece has integral $O(1/s)$ and the exponentially bounded tail tends to zero. But $s/(s+1) \to 1$. Polynomial division gives

$$F = 1 - \frac{1}{s+1}, \qquad \boxed{f = \delta(t) - e^{-t}}.$$

The impulse supplies the nonvanishing constant term.

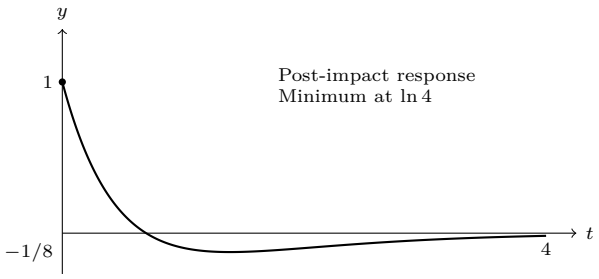
Step 2: Transform the equation and account for the jump. With the stated pre-impact convention, $(s+2)Y = F$, so

$$Y = \frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2}, \qquad \boxed{y = 2e^{-2t} - e^{-t} \quad (t \geq 0).}$$

Write $[y]_0 = y(0^+) - y(0^-)$. Integrating across zero gives $[y]_0 = 1$, consistent with $y(0^+) = 1$. The negative exponential input has no impulse mass.

Step 3: Find the sign changes and area. The unique positive zero is $t = \ln 2$. The derivative is $y' = e^{-t} - 4e^{-2t}$, negative before $\ln 4$ and positive afterward. Thus the global minimum is $\boxed{y(\ln 4) = -1/8}$. The area is $2/2 - 1 = \boxed{0}$: positive and negative parts cancel. For comparison, the signed input area is $1 - 1 = 0$ when the full initial impulse is included.

Step 4: Verify the ordinary restart and convergence. For $t > 0$, direct substitution gives $y' + 2y = -e^{-t}$. The same response solves this ordinary equation with $y(0^+) = 1$. Adding another initial impulse to that restart would count it twice. The tail is asymptotic to $-e^{-t}$, so the exact real transform domain is $\boxed{s > -1}$. The figure shows only the post-impact branch on $t \geq 0$; the filled initial point records the prescribed right-hand value.



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Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A measured response is known to lie in the three-mode family

$$f(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

The available data are $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 2$.

Tasks

1. Recover the three coefficients and prove uniqueness within the stated family.
2. Construct $F(s)$ by table lookup and simplify it to one rational expression. Check its leading large-s term against the data.
3. Prove that the recovered function is positive for every $t > 0$ despite a negative coefficient. Find its unique global maximum.
4. Compute its area and exact real transform domain. Explain why the three measurements would not identify a unique response without the three-mode assumption.

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Question 9 – Solution

Strategy. The assumed modes turn derivative data into a finite linear system. A negative coefficient need not make their sum negative.

Step 1: Solve and prove uniqueness. The data give

$$c_1 + c_2 + c_3 = 0, \quad -c_1 - 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 2.$$

Adding the first two equations gives $c_2 = -2c_3$, and the first then gives $c_1 = c_3$. Substitution into the third gives $2c_3 = 2$. Every coefficient is forced, proving uniqueness: $\boxed{(c_1, c_2, c_3) = (1, -2, 1)}$.

Step 2: Assemble the rational transform. The table yields

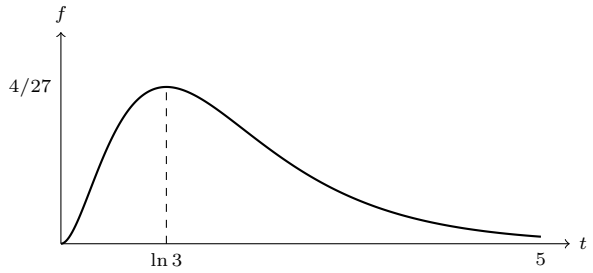
$$F = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} = \boxed{\frac{2}{(s+1)(s+2)(s+3)}}.$$

Near zero, $f = t^2 + O(t^3)$, so its leading transform is $2/s^3$, matching the quotient. The $1/s$ and $1/s^2$ terms cancel exactly.

Step 3: Establish positivity and the peak. Factor the function as $f = e^{-t}(1 - e^{-t})^2$. It is positive for every $t > 0$ and zero at 0. Put $x = e^{-t} \in (0, 1]$. The function $x(1-x)^2$ has derivative $(1-x)(1-3x)$ with respect to x; along increasing t its unique interior maximum is at $x = 1/3$. Therefore

$$\boxed{t_{\max} = \ln 3, \qquad f(t_{\max}) = 4/27.}$$

Step 4: Check area, domain and identifiability. The area is $1 - 2/2 + 1/3 = \boxed{1/3}$. The nonzero e^{-t} tail gives exact real domain $s > -1$. Without the mode restriction, adding any multiple of $t^3 e^{-t}$ would preserve all three measured derivatives but change the response. Thus the identification is unique within the stated model, not among all smooth functions.



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Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Use the chapter's table and operational rules to solve

$$y'' + 2y' + y = H_1(t)(t-1)e^{-(t-1)} + 2\delta(t-3), \qquad y(0) = 1, \quad y'(0) = -1.$$

Ordinary force impulses preserve displacement and may jump velocity. All figures belong on the solution page.

Tasks

1. Find $Y(s)$, carefully retaining both initial terms and the two different forcing operations.
2. Invert the transform and give the response on the three intervals separated by 1 and 3.
3. Verify the initial state, the equation between switches, and the one-sided displacement and velocity changes at each switch.
4. Compute the total response area in two ways and determine the exact real transform domain. Explain why this finite list of table operations gives a unique solution.

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Question 10 – Solution

Strategy. Separate the initial response, delayed ordinary forcing and impulse response before inversion.

Step 1: Transform with all data. The left side transforms to $(s^2Y - s + 1) + 2(sY - 1) + Y = (s+1)^2Y - s - 1$. Consequently

$$\boxed{Y = \frac{1}{s+1} + \frac{e^{-s}}{(s+1)^4} + \frac{2e^{-3s}}{(s+1)^2}.}$$

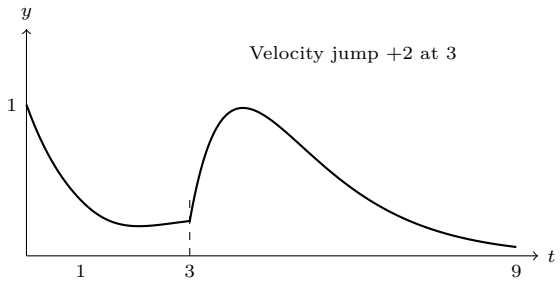
The ordinary forcing has transform $e^{-s}/(s+1)^2$; the impulse has transform $2e^{-3s}$.

Step 2: Invert and make the intervals explicit. Define $v(\tau) = \tau^3 e^{-\tau}/6$ and $h(\tau) = \tau e^{-\tau}$ for $\tau \geq 0$. The factorial in the fourth-power table row gives

$$\boxed{y(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-t} + v(t-1), & 1 \leq t < 3, \\ e^{-t} + v(t-1) + 2h(t-3), & t \geq 3. \end{cases}}$$

Step 3: Verify each kind of activation. The initial branch gives $(y(0), y'(0)) = (1, -1)$. For $D = d/d\tau$ and any polynomial P, $(D+1)^2[e^{-\tau}P(\tau)] = e^{-\tau}P''(\tau)$. Thus v produces $\tau e^{-\tau}$ and h produces zero between switches. At 1, $v(0) = v'(0) = 0$, so displacement and velocity remain continuous. At 3, $h(0) = 0$, $h'(0) = 1$, so $[y]_3 = 0$ and $[y']_3 = 2$. Here $[z]_a = z(a^+) - z(a^-)$. These are exactly the required jump conditions.

Step 4: Check area, convergence and uniqueness. Direct areas are 1, $3!/6 = 1$, and 2, giving $\int_0^\infty y = 4$. Independently, integrating the equation, including the velocity jump, gives $1 - 2 + \int y = 1 + 2$, again yielding $\boxed{\int y = 4}$. The tail is a nonzero cubic polynomial times e^{-t}, so the exact real domain is $s > -1$. Linear IVP uniqueness on each ordinary interval, together with the forced matching and jump conditions, fixes the whole solution.



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Question 1

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Consider the parameter-dependent system

$$x + y + z = 2, \qquad 2x + 3y + z = 5, \qquad 3x + 4y + 2z = k,$$

where k is real.

Tasks

1. Classify every k according to whether the system has no solution, one solution or infinitely many solutions.
2. For each consistent case, give the entire solution set with one free parameter and verify it in all three original equations.
3. Add the measurement $x - y = 0$. Determine all solutions of the enlarged system for every k.
4. Explain why three equations in three unknowns need not determine a unique answer. Compare the information supplied by the original third equation with that supplied by the new measurement.

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Question 1 – Solution

Strategy. Inspect relationships between whole equations before committing to a long elimination.

Step 1: Test consistency. The third left-hand side is the sum of the first two. Any common solution must therefore satisfy $k = 2 + 5 = 7$. If $k \neq 7$, subtracting the first two equations from the third gives the contradiction $0 = k - 7$.

Step 2: Find and verify the full family. When $k = 7$, the third equation contributes no new restriction. Subtract twice the first equation from the second to get $y - z = 1$. Setting $z = t$ gives

$$\boxed{(x, y, z) = (1 - 2t,\, 1 + t,\, t), \qquad t \in \mathbb{R}.}$$

The first left side is 2, the second is 5, and the third is 7 for every t. Conversely, every solution must have $y = 1 + z$ and $x = 1 - 2z$, so no solutions have been omitted. Thus $k = 7$ gives infinitely many solutions; no value of k gives exactly one solution in the original system.

Step 3: Use the additional measurement. On this family, $x - y = -3t$. The new measurement forces $t = 0$, giving

$$\boxed{(x, y, z) = (1, 1, 0) \quad \text{if } k = 7.}$$

If $k \neq 7$, the contradiction from the original equations remains, so adding a measurement cannot create a solution.

Step 4: Distinguish equation count from information. For $k = 7$, the third equation merely repeats a consequence already known. For $k \neq 7$, it contradicts that consequence. It never removes just the one remaining freedom. By contrast, $x - y$ varies as $-3t$ along the family, so its measured value selects exactly one member. Counting equations and unknowns alone cannot distinguish these situations; their relationships matter.

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Question 2

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

For real λ, solve

$$(\lambda - 1)x + y = 1, \qquad x + (\lambda - 1)y = 1.$$

Tasks

1. Use the sum and difference of the equations to classify the solution set for every λ.
2. Explain why dividing by λ or $\lambda - 2$ before checking their zeros can lose entire cases. Verify the exceptional cases directly.
3. Find the limits of the unique solution as $\lambda \to 2$ and as $\lambda \to 0$ from either side. Compare these limits with the solution sets at the exceptional values.
4. Classify all λ for which a solution with $x, y \geq 0$ exists, giving the full nonnegative solution set whenever it is not a single point.

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Question 2 – Solution

Strategy. Adding and subtracting separates the sum of the unknowns from their difference, exposing both exceptional parameter values.

Step 1: Reduce reversibly. The sum and difference equations are

$$\lambda(x+y) = 2, \qquad (\lambda-2)(x-y) = 0.$$

They are equivalent to the originals: taking half their sum and half their difference recovers the two original equations. If $\lambda \neq 0, 2$, then

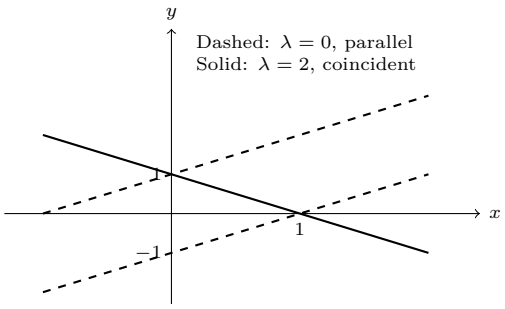
$$\boxed{x = y = 1/\lambda.}$$

At $\lambda = 0$, the sum is $0 = 2$, giving no solution. At $\lambda = 2$, only $x + y = 1$ remains, giving infinitely many solutions.

Step 2: Retain the exceptional equations. Directly, $\lambda = 0$ gives $-x+y = 1$ and $x-y = 1$, incompatible parallel lines. At $\lambda = 2$, both equations become $x + y = 1$. Dividing by a factor that can be zero discards these cases rather than solving them. The figure displays the two exceptional geometries.

Step 3: Compare limits with exact parameter values. As $\lambda \to 2$ through the unique-solution cases, $(x, y) \to (1/2, 1/2)$. This is one point of the entire line present at $\lambda = 2$, not the whole limiting parameter's solution set. As $\lambda \to 0^+$ both coordinates increase without bound; as $\lambda \to 0^-$ both decrease without bound. There is no finite limiting solution, consistent with the contradiction at zero.

Step 4: Enforce nonnegativity. For $\lambda \neq 0, 2$, the solution is nonnegative exactly when $\lambda > 0$. At $\lambda = 2$, the nonnegative solutions are $(x, y) = (u, 1-u)$, $0 \leq u \leq 1$. Therefore a nonnegative solution exists exactly for $\boxed{\lambda > 0}$, with the entire stated segment at 2.



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Question 3

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A system begins with an equation whose x coefficient is zero:

$$2y - z = 1, \qquad x - y + 2z = 3, \qquad 2x + y + z = 6 + \delta,$$

where δ is a real perturbation of the last measurement.

Tasks

1. Solve the system for $\delta = 0$ using valid elimination. Explain why the initial zero coefficient is not evidence of nonuniqueness.
2. Find the exact solution for arbitrary δ, keeping fractions exact.
3. Verify the perturbed solution in all original equations, and decide whether any δ makes the system inconsistent or nonunique.
4. Compute $x + 3y$ and recover δ from an exact measurement of z. Explain why measuring $x + 3y$ alone cannot detect the perturbation.

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Question 3 – Solution

Strategy. Choose an equation with a nonzero coefficient for elimination. An inconvenient ordering does not change the information in a system.

Step 1: Eliminate without dividing by zero. The second equation gives $x = 3 + y - 2z$. Substituting into the third gives $3y - 3z = \delta$, or $y - z = \delta/3$. Together with $2y - z = 1$, this determines y, z, then x. For $\delta = 0$ the result is $\boxed{(x, y, z) = (2, 1, 1)}$. Swapping the first two equations would also provide an immediate nonzero x coefficient; neither approach requires a zero division.

Step 2: Solve for the perturbation exactly. Subtract $y - z = \delta/3$ from $2y - z = 1$ to obtain $y = 1 - \delta/3$. It follows that

$$\boxed{x = 2 + \delta, \qquad y = 1 - \frac{\delta}{3}, \qquad z = 1 - \frac{2\delta}{3}.}$$

Only the nonzero constants 1 and 3 were used as divisors, so this calculation applies to every real δ.

Step 3: Check the original measurements and uniqueness. Substitution gives

$$\begin{aligned} 2y - z &= 2 - 2\delta/3 - 1 + 2\delta/3 = 1, \\ x - y + 2z &= 2 + \delta - 1 + \delta/3 + 2 - 4\delta/3 = 3, \\ 2x + y + z &= 4 + 2\delta + 1 - \delta/3 + 1 - 2\delta/3 = 6 + \delta. \end{aligned}$$

Every solution was forced by reversible substitutions and nonzero divisions. Thus exactly one solution exists for every real δ.

Step 4: Identify a blind measurement. The combination $x + 3y = 2 + \delta + 3 - \delta = \boxed{5}$ is independent of δ. In contrast, z determines $\boxed{\delta = \frac{3}{2}(1 - z)}$ uniquely. The first two equations fix certain combinations even when the last datum changes; a measurement that stays constant along that change cannot reveal it.

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Question 4

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three unknown quantities are measured through pairwise sums and a total:

$$x + y = a, \qquad y + z = b, \qquad x + z = c, \qquad x + y + z = d.$$

The recorded values are $(a, b, c, d) = (2, 3, 4, 5)$.

Tasks

1. Derive a necessary and sufficient condition on arbitrary real a, b, c, d for consistency. Give the unique solution when it exists.
2. For the recorded data, solve the first three equations and identify the conflict with the fourth.
3. Assume the total $d = 5$ is exact and exactly one of the three pairwise readings is wrong. Find every possible corrected reading and corresponding triple (x, y, z).
4. Decide whether these measurements identify which pairwise reading is wrong, even if x, y, z must be nonnegative. Explain the difference between detecting an error and locating it.

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Question 4 – Solution

Strategy. Redundant measurements can reveal a contradiction without identifying its source.

Step 1: Derive the consistency condition and solution. Adding the three pairwise equations gives $2(x+y+z) = a+b+c$. Thus consistency requires $\boxed{a+b+c=2d}$. The pairwise equations alone uniquely give

$$\boxed{x = \frac{a+c-b}{2}, \quad y = \frac{a+b-c}{2}, \quad z = \frac{b+c-a}{2}.}$$

These values satisfy all three pairwise equations, and their total is $(a+b+c)/2$. Therefore the displayed condition is also sufficient.

Step 2: Diagnose the recorded discrepancy. For $(a,b,c) = (2,3,4)$, the unique triple is $(3/2, 1/2, 5/2)$. Its total is $9/2$, not 5. Hence the full recorded system has no solution; rounding the computed triple would not repair an exact contradiction.

Step 3: Enumerate every single-reading correction. The exact total requires the sum of the pairwise readings to be 10 rather than 9. If exactly one pairwise reading changes, its correction must be $+1$. There are precisely three possibilities:

Changed reading	(a,b,c)	(x,y,z)
$a : 2 \to 3$	$(3,3,4)$	$(2,1,2)$
$b : 3 \to 4$	$(2,4,4)$	$(1,1,3)$
$c : 4 \to 5$	$(2,3,5)$	$(2,0,3)$

Each triple satisfies its corrected pairwise readings and has total 5.

Step 4: State the identification limit. All three corrected triples are nonnegative. Thus even with nonnegativity, the measurements cannot select the faulty reading. The redundancy detects that the four recorded equations cannot all be true. Locating the error requires another independent measurement or an additional justified restriction; it does not follow merely from detecting the inconsistency.

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Question 5

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Two nearly parallel measurement equations are

$$x + y = 2, \qquad x + (1 + \varepsilon)y = 2 + \delta,$$

where $\varepsilon > 0$ is known and δ is an unknown measurement error satisfying $|\delta| \leq \eta$, with $\eta > 0$.

Tasks

1. Find the exact solution for every admissible δ and explain what would happen if $\varepsilon = 0$ were allowed.
2. Find the smallest uniform bound on the largest absolute coordinate error relative to the nominal solution at $\delta = 0$. Prove sharpness.
3. For $\varepsilon = \eta = 10^{-3}$, find the full range of possible solutions. Compare the small equation residual of the nominal answer with its possible coordinate error.
4. Find a necessary and sufficient condition on ε ensuring both coordinate errors are at most 0.1 for every $|\delta| \leq \eta$. Interpret the condition geometrically.

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Question 5 – Solution

Strategy. Subtracting almost identical equations exposes the small coefficient that amplifies uncertainty.

Step 1: Solve and retain the excluded case. Subtract the first equation from the second to get $\varepsilon y = \delta$. For $\varepsilon > 0$,

$$\boxed{(x, y) = \left(2 - \frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right).}$$

If $\varepsilon = 0$ and $\delta \neq 0$, there is no solution; if both are zero, the two equations coincide and every point on $x + y = 2$ is a solution.

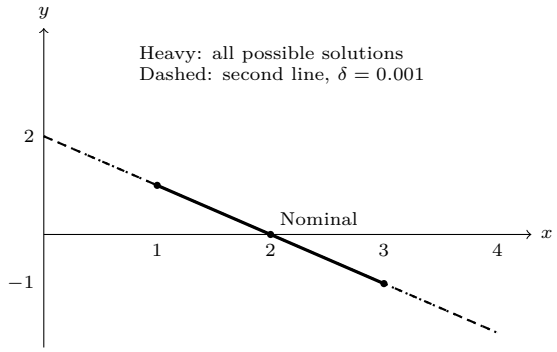
Step 2: Derive the exact uncertainty bound. The nominal solution is $(2, 0)$. Each coordinate changes in magnitude by $|\delta|/\varepsilon$. Therefore

$$\boxed{\max(|x - 2|, |y|) \leq \eta/\varepsilon.}$$

Equality occurs when $\delta = \eta$ or $\delta = -\eta$, so no smaller universal bound is possible.

Step 3: Quantify the nearly parallel example. For $\varepsilon = \eta = 10^{-3}$, let $u = \delta/\varepsilon \in [-1, 1]$. The full possible solution set is the segment $\boxed{(x, y) = (2 - u, u),\ -1 \leq u \leq 1}$, from $(3, -1)$ to $(1, 1)$. The nominal answer satisfies the first equation exactly and has residual $-\delta$ in the second, of magnitude at most 0.001. Nevertheless its coordinate error can be 1. Small residuals do not always certify small solution errors. The graph uses true scales; the two measured lines are nearly indistinguishable, while the possible solution segment is visible.

Step 4: Give the exact design condition. The requested guarantee holds exactly when $\eta/\varepsilon \leq 0.1$, or $\boxed{\varepsilon \geq 10\eta}$. Necessity follows by taking an endpoint error; sufficiency follows from the bound. Increasing ε separates the line directions and reduces the effect of shifting one measurement's right-hand side.



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Question 6

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A homogeneous system is

$$2x - y + z = 0, \qquad x + y - 2z = 0.$$

Two proposed ways to choose a scale are $x + y + z = 9$ and $x^2 + y^2 + z^2 = 35$. These are alternative added conditions, not simultaneous unless explicitly stated.

Tasks

1. Find and verify the entire solution set of the homogeneous system, explaining the role of the zero solution.
2. Apply the linear normalization $x + y + z = 9$ and determine whether it selects a unique solution.
3. Apply the quadratic normalization $x^2 + y^2 + z^2 = 35$ instead. Explain its sign ambiguity and resolve it if nonnegative coordinates are additionally required.
4. For arbitrary real a, b, c, m, classify the effect of adding $ax + by + cz = m$: give exact conditions for no solution, one solution or infinitely many solutions.

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Question 6 – Solution

Strategy. A homogeneous system can fix ratios while leaving the overall scale free. A normalization must actually measure that remaining freedom.

Step 1: Solve the homogeneous equations. From the second equation, $y = 2z - x$. Substitution into the first gives $3x - z = 0$, hence $z = 3x$ and $y = 5x$. Thus

$$\boxed{(x, y, z) = t(1, 5, 3), \qquad t \in \mathbb{R}.}$$

Substitution gives $2t - 5t + 3t = 0$ and $t + 5t - 6t = 0$. Every solution has this form, including the zero solution at $t = 0$. Dividing by an unknown coordinate would risk losing that solution.

Step 2: Apply the linear normalization. The coordinate sum is $9t$. The condition $9t = 9$ fixes $t = 1$, giving $\boxed{(1, 5, 3)}$ uniquely.

Step 3: Compare the quadratic normalization. The squared length is $(1+25+9)t^2 = 35t^2$. Therefore the quadratic condition gives $t^2 = 1$ and exactly two solutions:

$$\boxed{(1, 5, 3) \quad \text{and} \quad (-1, -5, -3).}$$

Squaring loses the overall sign. If nonnegative coordinates are also required, only the positive triple remains. The zero triple does not satisfy either stated normalization.

Step 4: Classify an arbitrary linear measurement. On the homogeneous family, the extra left side is $(a + 5b + 3c)t$. Set $K = a + 5b + 3c$. Then

$$\boxed{\begin{array}{ll} K \neq 0: & t = m/K \text{ is the unique solution scale,} \\ K = 0,\ m \neq 0: & \text{no solution,} \\ K = 0,\ m = 0: & \text{every real } t \text{ remains possible.} \end{array}}$$

A nonzero-looking added equation can still vanish on the entire family. What matters is its value on the remaining freedom, not the visual presence of three coefficients.

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Question 7

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A sensor's response is assumed to be a quadratic polynomial $p(t) = a + bt + ct^2$. Exact readings at three times are

$$p(0) = 1, \qquad p(1) = 2, \qquad p(2) = 5.$$

Later, consider a perturbation of only the last reading to $5 + \epsilon$, where $|\epsilon| \leq \eta$ and $\eta > 0$.

Tasks

1. Find the unique polynomial from the exact readings using a system of equations. Predict $p(3)$.
2. Find the exact perturbed coefficients and the response error at an arbitrary time $T \geq 0$.
3. Identify all nonnegative times whose predicted value is insensitive to this perturbation. Find sharp uniform error bounds on $0 \leq T \leq 2$ and at $T = 10$.
4. Explain which conclusions depend on the quadratic model. Construct a nonquadratic smooth response with the same three original readings but a different value at 3.

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Question 7 – Solution

Strategy. A model assumption converts function recovery into coefficient recovery, while extrapolation can amplify measurement errors.

Step 1: Solve the exact coefficient equations. The data give $a = 1$, $a+b+c = 2$ and $a+2b+4c = 5$. Thus $b+c = 1$ and $2b+4c = 4$, yielding $c = 1$ and $b = 0$. Every coefficient is forced, so

$$\boxed{p(t) = 1 + t^2, \qquad p(3) = 10.}$$

Step 2: Retain the measurement perturbation. The first two equations are unchanged, while $2b+4c = 4+\epsilon$. Solving gives

$$a = 1, \quad b = -\epsilon/2, \quad c = 1 + \epsilon/2, \qquad \boxed{p_\epsilon(T) - p(T) = \tfrac{\epsilon}{2} T(T-1).}$$

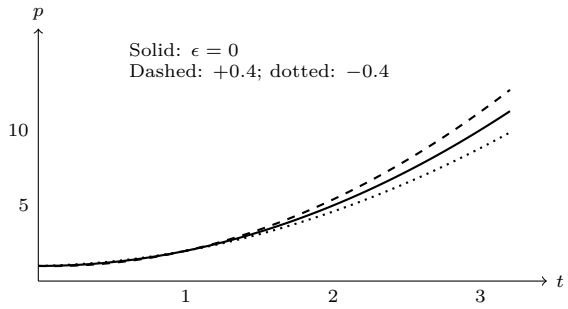
The error vanishes at both unchanged measurement times and equals ϵ at $T = 2$, as required.

Step 3: Compare interpolation and extrapolation. The error is independent of ϵ exactly when $T(T-1) = 0$, so the nonnegative insensitive times are 0 and 1. On $[0,1]$, $|T(T-1)| \le 1/4$; on $[1,2]$, $T(T-1)$ increases from 0 to 2. Therefore

$$\boxed{\sup_{0 \le T \le 2} |p_\epsilon - p| \le \eta, \qquad |p_\epsilon(10) - p(10)| \le 45\eta.}$$

Both bounds are sharp at $|\epsilon| = \eta$, the first at $T = 2$. The figure shows the true curves for $\epsilon = 0, \pm 0.4$.

Step 4: State the model's limitation. For any real K, the smooth function $q_K(t) = 1 + t^2 + Kt(t-1)(t-2)$ matches all three original readings. For $K \neq 0$ it is not quadratic, and $q_K(3) = 10 + 6K \neq 10$. Thus three readings uniquely determine a member of the quadratic family, not an arbitrary smooth response. Extrapolation bounds also rely on the quadratic family and on the stated location of the measurement error.



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Question 8

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A steady flow network has directed internal flows p from A to B, q from B to C, and r from C to A. External flow enters A at rate 3; rates 1 and 2 leave B and C, respectively. There is no storage. All rates use the same units, and internal flows must be nonnegative.

Tasks

1. Write the three node-balance equations and identify any redundancy.
2. Find every nonnegative steady flow and explain why the external data do not determine the internal circulation uniquely.
3. Impose capacity bounds $p \leq 5$, $q \leq 4$ and $r \leq 3$. Find the entire feasible set and identify which capacities limit it.
4. Among feasible flows, minimize the cost rate $2p + q + 3r$. Prove uniqueness of the minimizer and interpret it in terms of circulation.

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Question 8 – Solution

Strategy. Conservation fixes net transfers but can leave an arbitrary circulating flow around a closed loop.

Step 1: Balance every node. Inflow equals outflow, giving

$$A:\ r+3=p, \qquad B:\ p=q+1, \qquad C:\ q=r+2.$$

Equivalently $p-r=3$, $q-p=-1$, $r-q=-2$. The three left sides sum to zero, as do their right sides. Any two balances imply the third because total external inflow equals total external outflow.

Step 2: Recover the free circulation. Set $r=t$. Then

$$\boxed{(p,q,r)=(t+3,t+2,t), \qquad t\geq 0.}$$

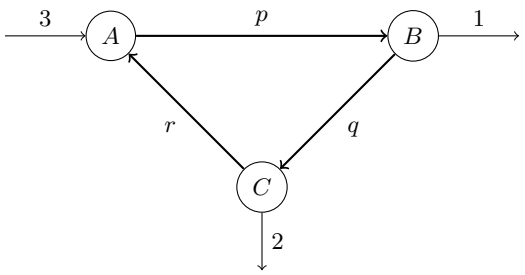
The condition $t\geq 0$ is necessary from $r\geq 0$ and sufficient for all three flows to be nonnegative. Adding the same amount to all three internal flows preserves every node balance: it is an additional circulation around the loop.

Step 3: Apply all capacities. The bounds give $t\leq 2$, $t\leq 2$ and $t\leq 3$, respectively. Thus the entire feasible family has $\boxed{0\leq t\leq 2}$. The capacities on p and q both limit the upper endpoint; at $t=2$ they are saturated, while $r=2<3$ leaves unused capacity on the third link.

Step 4: Minimize the cost on the feasible family. Substituting the flow formulas yields

$$2p+q+3r=2(t+3)+(t+2)+3t=8+6t.$$

This increases strictly on $[0,2]$. Therefore the unique minimizer is $\boxed{(p,q,r)=(3,2,0)}$, with cost rate $\boxed{8}$. It sends the required external transfers without an added loop circulation. The diagram shows directions, not arrows scaled in proportion to flow.



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Question 9

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

You may design a second linear measurement to accompany $x + y = 1$:

$$ax + by = c.$$

It must select the target $(x, y) = (1/3, 2/3)$ uniquely. In a normalized design, require $a, b \geq 0$ and $a + b = 3$.

Tasks

1. Classify all real triples (a, b, c) that select the target uniquely. Explain the excluded case geometrically.
2. Under the normalization, describe every admissible design using one parameter and identify the excluded parameter value.
3. If the second right-hand side changes from c to $c + \delta$ while the first equation remains exact, derive the coordinate errors for an admissible design.
4. For $|\delta| \leq \eta$, where $\eta > 0$, find all normalized designs minimizing the worst possible absolute coordinate error. Give that minimum and verify the proposed designs directly.

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Question 9 – Solution

Strategy. Passing through a target is necessary, but the measurement must also distinguish points along the first equation's line.

Step 1: Require both passage and uniqueness. The target lies on the second line exactly when $c = (a + 2b)/3$. Substituting $y = 1 - x$ gives $(a - b)x = c - b$. Hence unique selection requires

$$\boxed{c = (a + 2b)/3, \qquad a \neq b.}$$

If $a = b$, the target condition forces $c = a$ and the second equation is a multiple of $x + y = 1$ (possibly $0 = 0$). The whole first line remains possible.

Step 2: List all normalized designs. Put $b = 3 - a$, with $0 \leq a \leq 3$. Then

$$\boxed{b = 3 - a, \quad c = 2 - a/3, \quad a \in [0, 3] \setminus \{3/2\}.}$$

The midpoint is excluded because both coefficients would equal $3/2$.

Step 3: Derive the exact measurement sensitivity. Subtract the target equations from the perturbed equations. With $\Delta x = x - 1/3$ and $\Delta y = y - 2/3$, one obtains $\Delta x + \Delta y = 0$ and $a\Delta x + b\Delta y = \delta$. Thus

$$\boxed{\Delta x = \frac{\delta}{a - b}, \qquad \Delta y = -\frac{\delta}{a - b}.}$$

The worst coordinate error over $|\delta| \leq \eta$ is exactly $\eta/|a - b|$, attained at either endpoint of the allowed error interval.

Step 4: Optimize within the allowed normalization. Since $a + b = 3$ and both are nonnegative, $|a - b| \leq 3$, with equality exactly at $(a, b) = (3, 0)$ or $(0, 3)$. Therefore the minimum worst error is $\boxed{\eta/3}$, attained by exactly two designs:

$$\boxed{3x = 1 \quad \text{or} \quad 3y = 2.}$$

Perturbing these readings directly changes the measured coordinate by $\delta/3$ and the other by its negative. Both attain the claimed bound. Without the normalization, arbitrary rescaling would change the meaning of an absolute right-hand-side error, making this comparison ill-defined.

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Question 10

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three nonnegative production amounts satisfy two exact resource totals:

$$x + y + z = 6, \qquad x + 2y + 3z = 10, \qquad x, y, z \geq 0.$$

The return is $P = 4x + 2y + z$. Fractional production amounts are permitted.

Tasks

1. Find all solutions of the two equalities, then determine the exact parameter interval allowed by nonnegativity.
2. Find the minimum and maximum return over the feasible set, giving every allocation at which either is attained.
3. Add the capacity constraint $x \leq 3$. Find the new feasible set and the new maximizing allocation and return.
4. Explain why solving the equalities alone does not solve the allocation problem. Verify the optimizing allocations against both original totals and all applicable inequalities.

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Question 10 – Solution

Strategy. Solve the equalities first, then optimize over the remaining feasible interval rather than over an unrestricted algebraic family.

Step 1: Parameterize and restrict. Subtract the first equality from the second to obtain $y + 2z = 4$. Set $z = t$, giving

$$(x, y, z) = (2 + t, 4 - 2t, t).$$

Nonnegativity imposes $t \geq 0$, $t \leq 2$ and $t \geq -2$. The complete feasible set is therefore

$$\boxed{(x, y, z) = (2 + t, 4 - 2t, t), \qquad 0 \leq t \leq 2.}$$

Each member satisfies both original totals; every feasible member must arise from this parameterization.

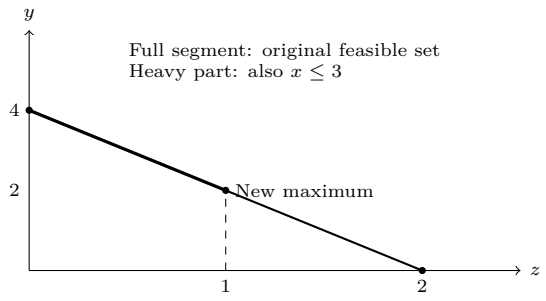
Step 2: Optimize on the feasible interval. The return becomes $P = 4(2+t)+2(4-2t)+t = 16+t$. It is strictly increasing, so the unique minimum is $\boxed{16}$ at $(2, 4, 0)$, and the unique maximum is $\boxed{18}$ at $(4, 0, 2)$. Neither an interior stationary-point calculation nor an integer search is needed; the stated amounts may be fractional.

Step 3: Include the new capacity. The constraint $x \leq 3$ becomes $2 + t \leq 3$, or $t \leq 1$. The new feasible interval is $[0, 1]$, with unique maximizing allocation

$$\boxed{(x, y, z) = (3, 2, 1), \qquad P = 17.}$$

The minimum remains at $t = 0$. In the figure, the feasible sets are shown in the (z, y) plane; $x = 2 + z$ is determined by the equalities.

Step 4: Verify the optimizers and the role of inequalities. The endpoint triples $(2, 4, 0)$ and $(4, 0, 2)$ have totals $(6, 10)$ and are nonnegative. The new optimizer $(3, 2, 1)$ also has totals $(6, 10)$ and satisfies $x = 3$. The former maximizer has $x = 4$, so it is no longer allowed. Over the unrestricted equality family, $P = 16+t$ has no finite maximum or minimum. The inequalities make the allowed interval bounded and turn this into a well-posed allocation problem.



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Question 1

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Consider the rectangular matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Tasks

1. Check the dimensions of AB and BA, compute both products, and identify which identity matrix occurs.
2. For an arbitrary $b = (b_1, b_2)^T$, describe every solution of $Ax = b$ and explain which solution Bb selects.
3. Prove that no matrix C can satisfy $CA = I_3$. Use a nonzero vector sent to zero by A, rather than only counting equations.
4. For $P = BA$, prove $P^2 = P$, describe all vectors fixed by P, and compute Pv for $v = (2, -1, 3)^T$. Explain what information this operation discards.

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Question 1 – Solution

Strategy. A rectangular matrix can have a right inverse without having a left inverse. Check what happens to vectors, not just products.

Step 1: Multiply with dimensions attached. The sizes are $A : 2 \times 3$ and $B : 3 \times 2$. Thus

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \qquad BA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

The two products even act on spaces of different dimensions.

Step 2: Recover the whole solution family. The equation $Ax = b$ says $x_1 + x_3 = b_1$ and $x_2 + x_3 = b_2$. Hence

$$\boxed{x = (b_1 - t, b_2 - t, t)^T, \qquad t \in \mathbb{R}.}$$

Every such vector works, and every solution has this form. The vector $Bb = (b_1, b_2, 0)^T$ selects $t = 0$; it is a choice, not the only inverse image.

Step 3: Rule out a left inverse. The nonzero vector $n = (-1, -1, 1)^T$ satisfies $An = 0$. If $CA = I_3$, then $n = I_3 n = CAn = C0 = 0$, a contradiction. Thus no left inverse exists, even though $AB = I_2$.

Step 4: Interpret the repeated operation. Associativity gives $P^2 = B(AB)A = BI_2A = P$. For $x = (x_1, x_2, x_3)^T$, $Px = (x_1 + x_3, x_2 + x_3, 0)^T$. Consequently $Px = x$ exactly when $x_3 = 0$, and

$$\boxed{P(2, -1, 3)^T = (5, 2, 0)^T.}$$

The discarded difference is $x - Px = x_3(-1, -1, 1)^T$, invisible to A. Indeed $APx = Ax$. This is an idempotent projection onto the plane $x_3 = 0$, but it is not the perpendicular projection: it changes the first two coordinates.

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Question 2

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In the plane, apply the shear and quarter-turn

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Composition acts on column vectors from right to left.

Tasks

1. Compute RS and SR. Apply each to $v = (1, 1)^T$ and explain the difference in operation order.
2. Find every vector w for which $RSw = SRw$. Prove that your list is complete.
3. Find $(RS)^{-1}$ by reversing the operations, and verify the inverse by multiplication.
4. Map the four vertices of the unit square in their cyclic order under both products. Compare areas and orientation, and explain why equal determinants do not imply equal transformations.

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Question 2 – Solution

Strategy. Matrix multiplication records the order of geometric operations. Determinants retain only part of that information.

Step 1: Compute both compositions. Direct multiplication gives

$$RS = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \qquad SR = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $RSv = (-1, 3)^T$ and $SRv = (1, 1)^T$. The first shears then rotates; the second rotates then shears.

Step 2: Test agreement on an arbitrary vector. Subtracting gives

$$(RS - SR)w = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

This is zero exactly when $w_1 = w_2 = 0$. Therefore the two compositions agree only on the zero vector, despite each being invertible.

Step 3: Undo the operations in reverse order. Since $S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $R^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

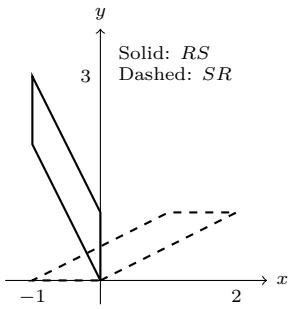
$$\boxed{(RS)^{-1} = S^{-1}R^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.}$$

Multiplication with RS on either side gives I_2.

Step 4: Compare the images of the square. For vertices $(0, 0), (1, 0), (1, 1), (0, 1)$ in that order, the images are

RS	$(0, 0)$	$(0, 1)$	$(-1, 3)$	$(-1, 2)$
SR	$(0, 0)$	$(2, 1)$	$(1, 1)$	$(-1, 0)$

Both determinants are 1, so both images have area 1 and preserve the counterclockwise orientation. The parallelograms nevertheless differ. A determinant encodes signed area scaling, not the full image of each vector. The plot uses equal scales on both axes.



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Question 3

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$A_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \tau & 1 \\ 1 & 1 & \tau \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \tau \in \mathbb{R}.$$

Tasks

1. Find $\det A_\tau$ by determinant-preserving row operations and classify invertibility.
2. For the invertible cases, derive A_τ^{-1} by solving $A_\tau x = c$ for a general column c. Verify the formula.
3. Solve $A_\tau x = b$ for every τ. At the singular parameter, classify consistency for an arbitrary right-hand side c.
4. Compare the behavior near $\tau = 1$ for right-hand sides $(1, 2, 3)^T$ and $(1, 1, 1)^T$. Explain why invertibility for each nearby parameter does not imply a uniform bound on recovered vectors.

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Question 3 – Solution

Strategy. Subtracting the first row exposes the parameter that controls both inversion and the exceptional case.

Step 1: Compute the determinant. Replace rows two and three by themselves minus row one. These operations do not change the determinant and produce

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \tau - 1 & 0 \\ 0 & 0 & \tau - 1 \end{pmatrix}.$$

Therefore $\boxed{\det A_\tau = (\tau - 1)^2}$, and inversion is possible exactly when $\tau \neq 1$.

Step 2: Solve for a general right-hand side. Put $d = \tau - 1 \neq 0$. The subtracted equations give $x_2 = (c_2 - c_1)/d$, $x_3 = (c_3 - c_1)/d$, and $x_1 = c_1 - x_2 - x_3$. Thus

$$\boxed{A_\tau^{-1} = \begin{pmatrix} 1 + 2/d & -1/d & -1/d \\ -1/d & 1/d & 0 \\ -1/d & 0 & 1/d \end{pmatrix}.}$$

Substitution in the three original equations returns c_1, c_2, c_3 for every c, so multiplication $A_\tau A_\tau^{-1} = I_3$ is verified. The displayed matrices are symmetric, and transposing that equality also verifies multiplication in the opposite order.

Step 3: Handle the prescribed and singular systems. For $\tau \neq 1$,

$$\boxed{x = \left(1 - \frac{3}{d}, \frac{1}{d}, \frac{2}{d}\right)^T.}$$

At $\tau = 1$, all three left sides are $x_1 + x_2 + x_3$, while the entries of b differ, so there is no solution. For a general c, consistency at 1 is equivalent to $c_1 = c_2 = c_3$. If they coincide, the whole plane $x_1 + x_2 + x_3 = c_1$ is the solution set; otherwise it is empty.

Step 4: Distinguish invertibility from bounded recovery. For $b = (1, 2, 3)^T$, the coordinate $x_2 = 1/(\tau - 1)$ becomes unbounded near 1. For $c = (1, 1, 1)^T$, the unique nearby solution is always $(1, 0, 0)^T$. Thus some data remain harmless while other fixed data yield arbitrarily large solutions. The factors $1/d$ in the inverse prevent a uniform bound near the singular parameter, even though every matrix with $d \neq 0$ is invertible.

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Question 4

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Use the ordered basis $u = (1, 1)^T$, $v = (1, -1)^T$ of the plane. Let P have columns u, v, so a coordinate column c represents the physical vector Pc. A transformation in standard coordinates is

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Tasks

1. Find P^{-1} and the coordinates of $w = (3, 1)^T$ in the ordered basis.
2. Derive the matrix representing T in this basis, explaining the order of all three operations.
3. Apply the coordinate matrix to your coordinate column and check the result against direct standard-coordinate multiplication.
4. For arbitrary coordinate columns, derive the relation between coordinate length and physical length. Compare the determinants of the two representations and interpret the physical area factor.

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Question 4 – Solution

Strategy. A vector and its coordinate column describe the same object in different languages. Convert in the correct order.

Step 1: Build and invert the basis matrix. Here

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad [w]_{u,v} = P^{-1}w = \boxed{(2,1)^T}.$$

Thus $w = 2u + v$, as the figure illustrates.

Step 2: Express the transformation in the new coordinates. Start with a coordinate column c. Convert it to Pc, apply T, and convert the result back with P^{-1}. Consequently

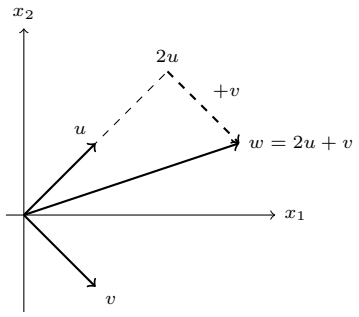
$$C = P^{-1}TP = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Step 3: Verify both routes. The coordinate route gives $C(2,1)^T = (4,3)^T$, representing $P(4,3)^T = (7,1)^T$. Directly, $T(3,1)^T = (7,1)^T$ as well. Equivalently, $PC = TP$ verifies the agreement for every input, not only w.

Step 4: Compare lengths and area factors. Since $P^TP = 2I_2$,

$$\boxed{\|Pc\|^2 = c^TP^TPc = 2\|c\|^2.}$$

The basis vectors are perpendicular but have length $\sqrt{2}$, so physical length is $\sqrt{2}$ times coordinate length. Also $\det C = \det(P^{-1})\det T \det P = \det T = 2$. The physical transformation doubles areas and preserves orientation, regardless of which basis is used to describe it. The negative determinant of P describes the basis orientation; it does not change this area factor.



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Question 5

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let $a = (1, 2)^T$ and $b = (2, 1)^T$. We seek the point on the line through a closest to b. All points on that line have the form ta, $t \in \mathbb{R}$.

Tasks

1. Minimize $\|b - ta\|^2$ exactly, giving the unique closest point and the minimum distance.
2. Construct a matrix Q that sends any vector to its closest point on this same line. Compute Q^T and Q^2.
3. Describe all vectors sent to zero and all vectors fixed by Q. Verify that the error $b - Qb$ is perpendicular to the line.
4. Prove $\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2$ for every x, and deduce a sharp length inequality with its equality condition.

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Question 5 – Solution

Strategy. Complete the square to find the best coefficient, then express that coefficient as a linear function of the input.

Step 1: Minimize the distance. Using $a^T a = 5$, $a^T b = 4$ and $b^T b = 5$,

$$\|b - ta\|^2 = 5t^2 - 8t + 5 = 5(t - 4/5)^2 + 9/5.$$

The unique minimizer is $t = 4/5$, so

$$\boxed{Qb = (4/5, 8/5)^T, \qquad \text{minimum distance} = 3/\sqrt{5}.}$$

Step 2: Build the projection matrix. For any input x, the minimizing coefficient is $(a^T x)/(a^T a)$. Therefore

$$Q = \frac{aa^T}{5} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \qquad Q^T = Q, \qquad Q^2 = Q.$$

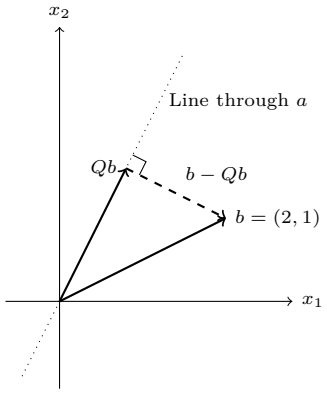
The last equality follows from $aa^T aa^T = a(a^T a)a^T = 5aa^T$.

Step 3: Describe the two perpendicular directions. The condition $Qx = 0$ is equivalent to $x_1 + 2x_2 = 0$, so the vectors sent to zero form the line through $(2, -1)^T$. The fixed vectors are exactly the multiples of a: the image lies on that line, and $Q(ta) = ta$. For the stated input, $b - Qb = (6/5, -3/5)^T$, whose dot product with a is zero.

Step 4: Prove the length identity and equality case. In general $a^T(x - Qx) = 0$, so Qx is perpendicular to $x - Qx$. Expanding the squared length of their sum gives

$$\boxed{\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2, \qquad \|Qx\| \le \|x\|.}$$

Equality holds exactly when $x - Qx = 0$, that is, when x lies on the line through a, including the zero vector. For b, the squared lengths are $5 = 16/5 + 9/5$. The plot uses equal axis scales and marks the right angle.



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Question 6

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In three dimensions, let

$$u = (1, 1, 0)^T, \qquad v = (1, 0, 1)^T, \qquad w_\tau = (\tau, 1, 1)^T.$$

The plane of interest passes through $p = (1, 2, 3)^T$ and has direction vectors u, v. Use the scalar triple product $(u \times v) \cdot w$ as signed volume.

Tasks

1. Find a normal vector and a Cartesian equation for the plane. Verify that the specified point and directions satisfy the required conditions.
2. Find the area of the parallelogram spanned by u, v, and the signed and unsigned volumes spanned by u, v, w_τ.
3. Classify every τ for which the three direction vectors are linearly independent. At each dependent value, give an explicit nontrivial relation.
4. Explain how reversing the order of u, v affects the normal, plane equation, signed volume and unsigned volume. Distinguish a direction lying in the plane from a point lying on this translated plane.

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Question 6 – Solution

Strategy. A cross product encodes both a perpendicular direction and an orientation. Translation affects a plane's points, not its direction space.

Step 1: Find and verify the normal. Direct calculation gives

$$n = u \times v = (1, -1, -1)^T.$$

Both dot products $n \cdot u$ and $n \cdot v$ vanish. The plane equation is $n \cdot (x - p) = 0$, or

$$\boxed{x_1 - x_2 - x_3 = -4.}$$

The point p satisfies it, and adding any combination of u, v to p leaves its left side unchanged.

Step 2: Compute area and volume. The parallelogram area is $\|n\| = \boxed{\sqrt{3}}$. The scalar triple product is

$$\boxed{n \cdot w_\tau = \tau - 2, \qquad \text{unsigned volume} = |\tau - 2|.}$$

For example, $\tau = 0$ gives signed volume -2 and volume 2.

Step 3: Prove the independence classification. If $\alpha u + \beta v + \gamma w_\tau = 0$, dotting with n gives $\gamma(\tau - 2) = 0$. For $\tau \neq 2$, this forces $\gamma = 0$. The second and third coordinates of $\alpha u + \beta v = 0$ then give $\alpha = \beta = 0$. Thus the vectors are independent exactly for $\tau \neq 2$. At $\tau = 2$, $\boxed{w_2 = u + v}$ is a nontrivial dependence, and the volume collapses to zero.

Step 4: Separate orientation from translation. Reversing the order gives $v \times u = -n$. The plane equation becomes $-x_1 + x_2 + x_3 = 4$, describing the same plane. Signed volume changes sign, while unsigned volume does not. As a direction, w_τ lies in the plane's direction space exactly when $n \cdot w_\tau = 0$, or $\tau = 2$. As a point with those coordinates, it lies on the translated plane exactly when $\tau - 2 = -4$, or $\tau = -2$. These are different questions because the plane does not pass through the origin.

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Question 7

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For real t, let

$$M(t) = \begin{pmatrix} 1 & t \\ 0 & 2 \end{pmatrix}, \qquad v(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}, \qquad y(t) = M(t)v(t).$$

Differentiate matrices entry by entry.

Tasks

1. Derive the matrix-vector product rule from component sums. Compute y and y' in this example by two routes.
2. Find $N(t) = M(t)^{-1}$. Derive the inverse differentiation rule from $MN = I_2$ and verify it here.
3. Compute $(M^2)'$, $M'M + MM'$ and $2MM'$. Decide whether the scalar shortcut for differentiating a square is valid.
4. Compute $\det M(t)$ and examine $\|M(t)(0,1)^T\|$. Explain why a constant nonzero determinant does not force a constant matrix or constant vector lengths.

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Question 7 – Solution

Strategy. Scalar differentiation still applies to each entry, but matrix factors must retain their multiplication order.

Step 1: Derive and apply the product rule. For component i, $y_i = \sum_j M_{ij} v_j$. Differentiating the finite sum gives $y_i' = \sum_j (M_{ij}' v_j + M_{ij} v_j')$, hence $y' = M'v + Mv'$. Direct multiplication and differentiation give

$$y = \begin{pmatrix} t^2 + te^t \\ 2e^t \end{pmatrix}, \qquad \boxed{y' = \begin{pmatrix} 2t + (1+t)e^t \\ 2e^t \end{pmatrix}.}$$

The product-rule route gives $M'v = (e^t, 0)^T$ and $Mv' = (2t + te^t, 2e^t)^T$, confirming the same derivative.

Step 2: Differentiate the inverse with the correct order. The inverse is $N = \begin{pmatrix} 1 & -t/2 \\ 0 & 1/2 \end{pmatrix}$. Differentiating $MN = I_2$ gives $M'N + MN' = 0$; multiplying on the left by N yields

$$\boxed{N' = -NM'N = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}.}$$

This agrees with direct entrywise differentiation of N.

Step 3: Test the square rule explicitly. Since $M^2 = \begin{pmatrix} 1 & 3t \\ 0 & 4 \end{pmatrix}$,

$$(M^2)' = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{M'M} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{MM'}.$$

But $2MM' = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, which is different. The shortcut works when M and M' commute; they do not here.

Step 4: Interpret the constant determinant. The determinant is always 2, whereas $M(t)(0,1)^T = (t,2)^T$ has length $\sqrt{t^2+4}$. Thus signed area scaling remains fixed while shear and some vector lengths change. Determinant information alone cannot determine every entry or every geometric effect of a matrix.

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Question 8

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For $n = (1, 2)^T$, define

$$H = I_2 - \frac{2nn^T}{n^T n}.$$

Consider its action on the triangle with ordered vertices $O = (0, 0)$, $a = (3, 1)$ and $b = (1, 2)$.

Tasks

1. Compute H. Prove from a perpendicular decomposition that it reflects the plane across a named line.
2. Verify $H^T H = I_2$ and $H^2 = I_2$. Deduce preservation of lengths and dot products and describe the inverse operation.
3. Find the reflected triangle, its area and orientation relative to the original. Check the midpoint and displacement of a and its image.
4. Find every vector fixed by H and every vector sent to its negative. Derive these directions directly from the reflection formula.

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Question 8 – Solution

Strategy. Separate a vector into components parallel and perpendicular to the normal. The formula changes the sign of only the normal component.

Step 1: Identify the reflection axis. Here $n^T n = 5$, so

$$\boxed{H = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}.}$$

Write $x = x_\perp + x_\parallel$, where $x_\parallel = n(n^T x)/5$ and $n^T x_\perp = 0$. Then $Hx = x_\perp - x_\parallel$: it reflects across the line $\boxed{x_1 + 2x_2 = 0}$, whose direction is $(2, -1)^T$.

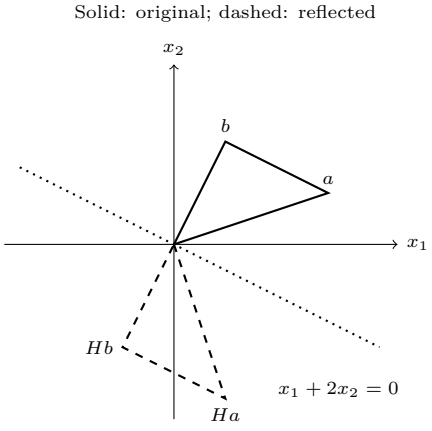
Step 2: Verify the metric and inverse identities. The matrix is symmetric, and its numerator squared is $\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$. Thus $H^2 = H^T H = I_2$. For any x, z, $(Hx)^T(Hz) = x^T H^T H z = x^T z$, preserving dot products and hence lengths. Also $H^{-1} = H$: reflecting twice restores the original vector.

Step 3: Transform and inspect the triangle. The images are

$$HO = O, \qquad Ha = (1, -3)^T, \qquad Hb = (-1, -2)^T.$$

The original ordered edge determinant is $3 \cdot 2 - 1 \cdot 1 = 5$, so its area is $5/2$. The reflected edge determinant is $1(-2) - (-1)(-3) = -5$. Its area remains $5/2$ but its orientation reverses, consistent with $\det H = -1$. The midpoint of a and Ha is $(2, -1)$ on the axis; the displacement $Ha - a = (-2, -4) = -2n$ is perpendicular to that axis.

Step 4: Classify fixed and reversed vectors. From the formula, $Hx = x$ exactly when $n^T x = 0$, giving the axis. For $Hx = -x$, the decomposition gives $x_\perp = 0$, so x is a multiple of n. The zero vector belongs to both sets. These conclusions follow from the components directly, without a characteristic-polynomial calculation. The plot uses equal axis scales.



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Question 9

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

An unknown real 2×2 matrix A satisfies

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}, \qquad A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}.$$

Tasks

1. Recover A uniquely, determine whether it is invertible, and verify both supplied measurements.
2. Predict $A(3,1)^T$ without multiplying by the recovered matrix. Decide whether an additional reported image $(4,5)^T$ is compatible.
3. In a separate experiment, replace the second input by $(2,2)^T$ and its image by $(4,0)^T$, keeping the first measurement. Find every matrix compatible with this experiment.
4. Classify the invertible matrices in the second experiment. Explain why one more image of an input outside the line through $(1,1)^T$ would determine a unique matrix.

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Question 9 – Solution

Strategy. Images of a basis determine a linear map, while repeated information along one input direction leaves freedom.

Step 1: Recover the columns from independent inputs. The first and second measurements are respectively the sum and difference of the columns of A. Half their sum gives column one; half their difference gives column two. Therefore

$$\boxed{A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \qquad \det A = -4 \neq 0.}$$

The column sum is $(2,0)^T$ and the difference is $(0,4)^T$, checking both measurements. Both columns are forced, so recovery is unique.

Step 2: Predict by linearity and test a new report. Because $(3,1)^T = 2(1,1)^T + (1,-1)^T$, linearity predicts $\boxed{A(3,1)^T = 2(2,0)^T + (0,4)^T = (4,4)^T}$. Thus a report of $(4,5)^T$ is inconsistent with the two original measurements for any linear map, not just with a particular computation of its entries.

Step 3: Describe the dependent-input experiment. Now the second input and output are exactly twice the first, so the second measurement adds no restriction. Write the first column as $(\alpha, \beta)^T$. The required column sum $(2,0)^T$ forces the second column, giving

$$\boxed{A_{\alpha,\beta} = \begin{pmatrix} \alpha & 2-\alpha \\ \beta & -\beta \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R}.}$$

Every member gives both required outputs; every compatible matrix has this form.

Step 4: Classify inversion and the missing information. The determinant is $-\alpha\beta - (2-\alpha)\beta = -2\beta$. Thus exactly the members with $\boxed{\beta \neq 0}$ are invertible. If another input w is outside the line through $(1,1)^T$, those two inputs form a basis. Each standard coordinate vector has a unique expression in that basis, so the two measured images uniquely determine both columns. Any assigned image for this independent input defines one linear map; invertibility would additionally require independent output images.

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Question 10

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = I_3 + N.$$

A discrete update is $v_{m+1} = Av_m$, starting from $v_0 = (0, 0, 1)^T$. The index m is a nonnegative integer, and each update uses the entire old vector.

Tasks

1. Compute N^2, N^3, $\det A$ and A^{-1}. Verify the inverse without a general inverse algorithm.
2. Prove a finite formula for A^m for all nonnegative integers m, using induction or the binomial identity with justification.
3. Find v_m and describe its coordinate growth. Explain why $\det A = 1$ does not imply bounded iterates.
4. A program first updates $y \leftarrow y + z$ and then $x \leftarrow x + y$, using the already updated y. Find the matrix it actually applies and its trajectory from the same initial vector. Compare the two after two steps.

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Question 10 – Solution

Strategy. The powers of the strictly upper triangular part terminate. A matrix update is simultaneous unless a different order is explicitly encoded.

Step 1: Exploit the terminating powers. Multiplication gives $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N^3 = 0$. The upper triangular matrix A has determinant 1. Moreover

$$\boxed{A^{-1} = I_3 - N + N^2}$$

because $(I_3 + N)(I_3 - N + N^2) = I_3 + N^3 = I_3$; the reverse product is the same.

Step 2: Derive the finite power formula. The identity matrix commutes with N, so the binomial expansion terminates:

$$\boxed{A^m = I_3 + mN + \frac{m(m-1)}{2} N^2.}$$

Alternatively, the formula is true at $m = 0$. Multiplication by $I_3 + N$ increases the coefficient of N to $m+1$ and that of N^2 to $m(m-1)/2 + m = m(m+1)/2$, establishing the induction step.

Step 3: Find the exact iterates. Applying the power formula to v_0 gives

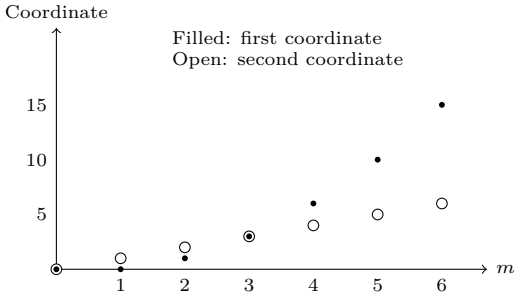
$$\boxed{v_m = \big(m(m-1)/2,\ m,\ 1\big)^T.}$$

The first coordinate grows quadratically, the second linearly and the third stays fixed. Determinant 1 preserves oriented volumes of three-dimensional sets, not individual lengths or boundedness under repetition. The plot marks integer-time values only; coincident coordinate values share a location.

Step 4: Diagnose the sequential-update error. The program computes $y_{\text{new}} = y + z$ and then $x_{\text{new}} = x + y + z$. It actually applies

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I_3 + N + N^2.$$

Starting from v_0, its third coordinate remains 1, its second becomes m, and its first accumulates $1 + 2 + \cdots + m = m(m+1)/2$. After two steps the correct vector is $(1, 2, 1)^T$, while the program gives $(3, 2, 1)^T$. Both matrices are invertible with determinant 1, so checking only the determinant would not expose this order-of-update mistake.



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Question 1

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad u = (1,1)^T, \quad v = (1,-1)^T,$$

and write a real starting vector as $x_0 = \alpha u + \beta v$.

Tasks

1. Find the eigenvalues, their full eigenspaces and both multiplicities. Explain why arbitrary nonzero multiples of an eigenvector are allowed but the zero vector is not.
2. Express $w = (3,1)^T$ in the eigenvector basis and decide whether w itself is an eigenvector.
3. Find $A^m x_0$ for every nonnegative integer m, and specialize to w. Verify the result without multiplying out m matrices.
4. For every nonzero real x_0, determine the limit of $A^m x_0 / \|A^m x_0\|$. Include the cases $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$.

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Question 1 – Solution

Strategy. Resolve the starting vector into invariant directions before analyzing repeated multiplication.

Step 1: Compute the eigenspaces. The characteristic polynomial is $(\lambda-2)^2-1=(\lambda-3)(\lambda-1)$. Solving the two singular systems gives

$$\boxed{E_3=\operatorname{span}\{u\},\qquad E_1=\operatorname{span}\{v\}.}$$

Both algebraic and geometric multiplicities are 1. Scaling a nonzero eigenvector preserves its eigenvalue equation. Zero solves every homogeneous eigenvalue equation and supplies no direction, so it is excluded as an eigenvector.

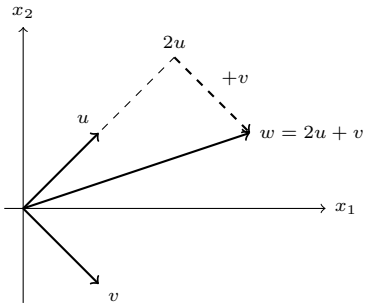
Step 2: Decompose and test the given vector. The equations $\alpha+\beta=3$, $\alpha-\beta=1$ give $w=2u+v$. Both components are nonzero and their eigenvalues differ. Directly, $Aw=(7,5)^T$ is not a scalar multiple of $(3,1)^T$, so w is not an eigenvector.

Step 3: Compute all iterates. Linearity and $Au=3u$, $Av=v$ give, by induction,

$$\boxed{A^m x_0=3^m\alpha u+\beta v,\qquad A^m w=(2\cdot 3^m+1,\,2\cdot 3^m-1)^T.}$$

The formula holds at $m=0$, and one multiplication advances each coefficient by its eigenvalue, proving the next case.

Step 4: Classify normalized direction limits. If $\alpha\neq 0$, divide the iterate by $3^m|\alpha|$ before normalizing. The remaining v coefficient tends to zero, so the limit is $\boxed{u/\sqrt{2}}$ for $\alpha>0$ and $\boxed{-u/\sqrt{2}}$ for $\alpha<0$. If $\alpha=0$, then $\beta\neq 0$ and the iterate is the unchanged vector βv, giving limit $\boxed{\operatorname{sgn}(\beta)v/\sqrt{2}}$. The dominant eigenvalue determines the limit only when its component is present. The diagram shows the two directions and the decomposition $w=2u+v$.



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Question 2

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real a, consider

$$B_a = \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}.$$

A student claims that a repeated eigenvalue automatically provides two independent eigenvectors.

Tasks

1. Find the eigenvalue and both multiplicities for every a, treating $a = 0$ separately.
2. Classify diagonalizability over $\mathbb{R}$ and over $\mathbb{C}$. Explain whether allowing complex eigenvectors repairs any failure.
3. Find an exact formula for B_a^m for every nonnegative integer m. Derive it rather than assuming diagonalization.
4. For a fixed vector $x = (x_1, x_2)^T$, classify when $2^{-m} B_a^m x$ stays bounded as $m \to \infty$. Compare this result with what the repeated eigenvalue alone suggests.

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Question 2 – Solution

Strategy. Root multiplicity and the number of independent eigenvectors answer different questions.

Step 1: Solve the eigenspace equation. The characteristic polynomial is $(\lambda - 2)^2$ for every a. The equation $(B_a - 2I)x = 0$ reduces to $ax_2 = 0$. Thus the algebraic multiplicity is always 2, while

$$\boxed{a = 0 :\ E_2 = \mathbb{R}^2,\ \dim E_2 = 2; \qquad a \neq 0 :\ E_2 = \operatorname{span}\{(1,0)^T\},\ \dim E_2 = 1.}$$

A repeated root does not itself produce a second independent vector.

Step 2: Classify diagonalization in both fields. At $a = 0$, $B_a = 2I$ is already diagonal. At $a \neq 0$, there is only one independent eigenvector, so no eigenvector basis exists. The same equation $ax_2 = 0$ holds over $\mathbb{C}$, with the same dimension count there. Hence $\boxed{B_a \text{ is diagonalizable exactly when } a = 0}$ over either field.

Step 3: Derive the powers from a terminating product. Let $N = B_a - 2I = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, so $N^2 = 0$. The commuting binomial expansion gives

$$\boxed{B_a^m = 2^m \begin{pmatrix} 1 & ma/2 \\ 0 & 1 \end{pmatrix}.}$$

For $m = 0$ this is I. Multiplication by B_a adds $a/2$ to the scaled upper-right entry, confirming the formula inductively without an eigenvector basis.

Step 4: Identify the remaining growth. The scaled iterate is

$$2^{-m} B_a^m x = (x_1 + max_2/2,\ x_2)^T.$$

It is bounded exactly when $\boxed{ax_2 = 0}$, and then is constant. For every $a \neq 0$ and $x_2 \neq 0$, its first coordinate grows in magnitude linearly. Dividing out the common exponential factor 2^m does not remove this extra growth. The missing eigenvector permits behavior that the list of eigenvalues, even with multiplicity, does not describe by itself.

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Question 3

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Allow complex eigenvalues and eigenvectors for the real matrix

$$C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Write $i^2 = -1$, and let $e_1 = (1, 0)^T$.

Tasks

1. Find both complex eigenvalues and their eigenspaces, and verify a representative eigenvector for each directly.
2. Prove that there is no nonzero real eigenvector, despite the existence of a complex eigenvector basis.
3. Write an eigenvector for $1 + 2i$ as $p + iq$ with p, q real, and derive the two real equations obtained from its eigenvalue equation.
4. Express C as a positive scale times a real rotation. Find $C^m e_1$ and its length for every nonnegative integer m, with the rotation angle specified unambiguously.

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Question 3 – Solution

Strategy. Complex eigenvectors encode a real invariant plane; they need not describe a real line fixed in direction.

Step 1: Find and verify the complex pairs. The polynomial is $(\lambda-1)^2+4$, so

$$\boxed{\lambda_\pm = 1 \pm 2i, \qquad E_{1+2i} = \operatorname{span}_{\mathbb{C}}\{(1,-i)^T\}, \quad E_{1-2i} = \operatorname{span}_{\mathbb{C}}\{(1,i)^T\}.}$$

Indeed $C(1,-i)^T = (1+2i, 2-i)^T = (1+2i)(1,-i)^T$. Conjugation verifies the other pair. The distinct eigenvalues give independent complex eigenvectors.

Step 2: Exclude a real eigenvector. If a real nonzero x satisfied $Cx = \lambda x$, a nonzero coordinate of x would make λ the ratio of two real numbers. Thus λ would be real, but the characteristic polynomial has no real root. No such vector exists.

Step 3: Extract the real plane relations. For $(1,-i)^T = p + iq$, take $p = (1,0)^T$, $q = (0,-1)^T$. Equating real and imaginary parts of $C(p+iq) = (1+2i)(p+iq)$ gives

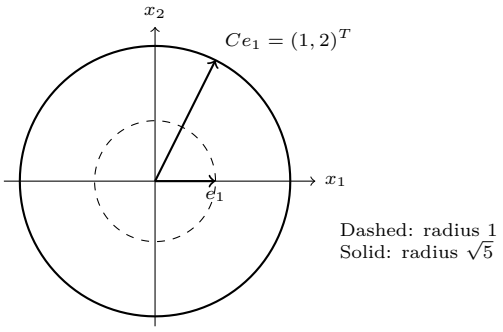
$$\boxed{Cp = p - 2q, \qquad Cq = 2p + q.}$$

Direct multiplication verifies both. These independent real vectors span the plane even though neither is a real eigenvector.

Step 4: Recover the rotation and scale. Let $\theta \in (0, \pi/2)$ satisfy $\cos\theta = 1/\sqrt{5}$ and $\sin\theta = 2/\sqrt{5}$; equivalently $\theta = \arctan(2)$. Then

$$C = \sqrt{5}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \boxed{C^m e_1 = 5^{m/2}(\cos m\theta, \sin m\theta)^T.}$$

Its length is $5^{m/2}$. The figure shows e_1 and its image $(1,2)^T$, with unit and image circles drawn on equal scales.



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Question 4

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Consider the real symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Tasks

1. Find an orthonormal eigenvector basis and an orthogonal diagonalization $S = QDQ^T$. Verify the columns directly.
2. Prove that eigenvectors of a real symmetric matrix belonging to distinct real eigenvalues are perpendicular, and connect the proof to your basis.
3. Find the sharp minimum and maximum of $x^T Sx$ over real vectors with $\|x\| = 1$, giving every equality case.
4. Use the spectral decomposition to find S^{-1} and verify it. Explain why a negative eigenvalue does not prevent invertibility but does prevent positive definiteness.

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Question 4 – Solution

Strategy. An orthonormal eigenvector basis turns both a quadratic form and an inverse into scalar calculations.

Step 1: Construct and check the basis. Take

$$q_1 = \frac{1}{\sqrt{2}}(1,1,0)^T, \quad q_2 = \frac{1}{\sqrt{2}}(1,-1,0)^T, \quad q_3 = (0,0,1)^T.$$

Direct multiplication gives eigenvalues $3, 1, -1$, respectively. The columns are orthonormal, so $Q = (q_1\ q_2\ q_3)$ has $Q^TQ = I$. Thus $SQ = QD$ with $D = \operatorname{diag}(3,1,-1)$, and $\boxed{S = QDQ^T}$.

Step 2: Prove perpendicularity. For a real symmetric matrix, if $Sa = \lambda a$ and $Sb = \mu b$, then

$$\lambda a^T b = (Sa)^T b = a^T S^T b = a^T Sb = \mu a^T b.$$

When $\lambda \neq \mu$, this forces $a^T b = 0$. Normalizing the three perpendicular directions gives exactly the orthonormal basis above.

Step 3: Derive the sharp quadratic-form bounds. Write $x = c_1q_1 + c_2q_2 + c_3q_3$. For a unit vector, $c_1^2 + c_2^2 + c_3^2 = 1$, and

$$x^T Sx = 3c_1^2 + c_2^2 - c_3^2.$$

This is a weighted average of $3, 1, -1$, so $\boxed{-1 \leq x^T Sx \leq 3}$. The minimum occurs exactly at $x = \pm q_3$; the maximum exactly at $x = \pm q_1$. The coefficients of all other directions must vanish at an endpoint.

Step 4: Invert the nonzero eigenvalues. Since no eigenvalue is zero,

$$\boxed{S^{-1} = Q\operatorname{diag}(1/3,1,-1)Q^T = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.}$$

Multiplying by S gives I_3. Invertibility excludes zero eigenvalues, not negative ones. But $q_3^T Sq_3 = -1 < 0$, so S is not positive definite; in fact its quadratic form takes both positive and negative values.

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Question 5

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

A real 2×2 matrix A is required to have

$$Au = 2u, \qquad Av = -v, \qquad u = (1, 1)^T, \quad v = (2, -1)^T.$$

Tasks

1. Construct the unique matrix satisfying these specifications and verify both eigenpairs.
2. Explain the effect of rescaling the two eigenvectors or swapping their order together with the corresponding eigenvalues. Contrast this with assigning the eigenvalues to the opposite directions.
3. Can the same matrix also have $(1, 0)^T$ as an eigenvector for some eigenvalue? Prove your answer.
4. Keeping the two prescribed directions, replace their eigenvalues by arbitrary real r, s. Find the full matrix formula and explain how its eigenspaces change when $r = s$.

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Question 5 – Solution

Strategy. Independent eigenvectors form an input basis, so their prescribed images determine every column of the transformation.

Step 1: Build the spectral representation. The basis matrix and inverse are

$$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$\boxed{A = P\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}P^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.}$$

Its products with u, v are $(2,2)^T = 2u$ and $(-2,1)^T = -v$. Since u, v span the plane, these images determine A uniquely.

Step 2: Separate notation changes from changed assignments. Multiplying either basis column by a nonzero scalar leaves its eigenvalue relation unchanged, so the reconstructed matrix stays the same. Reordering columns and the matching diagonal entries also leaves the map unchanged. Assigning -1 to u and 2 to v instead produces $\boxed{I - A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}}$, a different map. The correspondence between each direction and its scalar matters.

Step 3: Test a proposed third direction. Directly, $A(1,0)^T = (0,1)^T$, not a multiple of $(1,0)^T$. Thus no choice of eigenvalue makes the additional request possible. Equivalently, $(1,0)^T = (u+v)/3$ combines both distinct eigendirections.

Step 4: Recover the general two-scalar family. The same basis computation yields

$$\boxed{A_{r,s} = \frac{1}{3}\begin{pmatrix} r+2s & 2r-2s \\ r-s & 2r+s \end{pmatrix}.}$$

For $r \neq s$, the eigenspaces are exactly the two specified lines. For $r = s$, the matrix becomes rI_2 and every nonzero vector is an eigenvector; the eigenspace is the whole plane. Repeating an eigenvalue here enlarges the eigenspace, unlike a defective repeated-root example.

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Question 6

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Two matrices commute when $AB = BA$.

Tasks

1. Find the full family of real matrices B commuting with A, by equating entries of the two products.
2. Find a common eigenvector basis for every matrix in this family. Explain why simultaneous diagonalization always implies commutation.
3. For $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find the eigenvalues of AB and $A + B$ using the common basis. Explain how two invertible matrices can have a singular sum.
4. Within the commuting family, classify every invertible B for which $A + B$ is singular. Identify when the sum has a one-dimensional kernel and when it is the zero matrix.

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Question 6 – Solution

Strategy. Commutation forces preservation of this matrix's two distinct eigendirections, allowing scalar tests in a common basis.

Step 1: Equate the products. Subtracting the products gives

$$AB - BA = \begin{pmatrix} r-q & s-p \\ p-s & q-r \end{pmatrix}.$$

Thus $r = q$ and $s = p$, with no further restrictions. Thus the entire commuting family is $\boxed{B = \begin{pmatrix} p & q \\ q & p \end{pmatrix}}$.

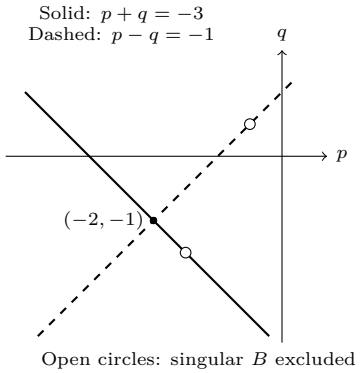
Step 2: Use the common directions. For $u = (1,1)^T$ and $v = (1,-1)^T$, the eigenvalues of A are $3, 1$, and those of B are $p+q, p-q$, respectively. These independent vectors form a common eigenvector basis, even if the two eigenvalues of B coincide. Conversely, if $A = PDP^{-1}$ and $B = PEP^{-1}$ with D, E diagonal, then $AB = PDEP^{-1} = PEDP^{-1} = BA$.

Step 3: Analyze the concrete sum and product. For $p = 1, q = 2$, the eigenvalues of B on u, v are $3, -1$. Therefore the eigenvalues of AB are $9, -1$ and those of $A + B$ are $6, 0$. Both A and B are invertible, but their actions on v cancel in the sum. The sum has kernel $\operatorname{span}\{v\}$.

Step 4: Classify all cancellation cases. Invertibility of B requires $(p+q)(p-q) \neq 0$. Singularity of the sum requires $(p+q+3)(p-q+1) = 0$. Equivalently the full allowed set is

$$\boxed{\{p+q=-3,\ p-q\neq 0\}\ \cup\ \{p-q=-1,\ p+q\neq 0\}.}$$

On the first line only, the kernel is the line through u; on the second only, it is the line through v. Their intersection is $(p,q) = (-2,-1)$, where $B = -A$ and $A + B = 0$, so the kernel is the entire plane. The excluded points are $(-3/2, -3/2)$ and $(-1/2, 1/2)$, where B itself is singular. The diagram marks these exclusions with open circles.



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Question 7

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For a real parameter K, let

$$T_K = \begin{pmatrix} 1/2 & K \\ 0 & 1/4 \end{pmatrix}.$$

Both eigenvalues have magnitude less than one. We compare long-term iteration with the stronger requirement $\|T_K x\| \leq \|x\|$ for every real vector x.

Tasks

1. Find an eigenvector basis for every K and determine whether the matrix is diagonalizable.
2. Derive T_K^m for every nonnegative integer m and prove that every fixed starting vector tends to zero under iteration.
3. For the starting vector $e_2 = (0, 1)^T$, find the first-step length. Determine exactly when that first step increases length, and explain why this does not contradict the long-term limit.
4. Find the necessary and sufficient condition on K for $\|T_K x\| \leq \|x\|$ to hold for every real x. Prove it by completing a square, rather than testing only e_2.

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Question 7 – Solution

Strategy. Decaying eigencomponents need not make Euclidean length decrease at every step when the eigenvectors are not orthogonal.

Step 1: Find the distinct-eigenvalue basis. The eigenvalues are $1/2$ and $1/4$, with eigenvectors $u = (1, 0)^T$ and $v = (-4K, 1)^T$. These are independent for every K, so T_K is always diagonalizable over $\mathbb{R}$.

Step 2: Compute all powers. For $m \geq 1$, the upper-right entry in the product is

$$K \sum_{j=0}^{m-1} (1/2)^{m-1-j}(1/4)^j = 4K(2^{-m} - 4^{-m}).$$

This finite geometric sum yields

$$\boxed{T_K^m = \begin{pmatrix} 2^{-m} & 4K(2^{-m} - 4^{-m}) \\ 0 & 4^{-m} \end{pmatrix}.}$$

The formula is I at $m = 0$. Every entry tends to zero for each fixed K, so $T_K^m x \to 0$ for every fixed starting vector x.

Step 3: Exhibit a transient length increase. The first image of e_2 is $(K, 1/4)^T$, of length $\sqrt{K^2 + 1/16}$. It exceeds 1 exactly when $\boxed{|K| > \sqrt{15}/4}$. An early increase and eventual decay concern different times, so there is no contradiction. The plot uses $K = 2$ and shows the actual lengths at integer iteration counts; the initial increase is followed by decay.

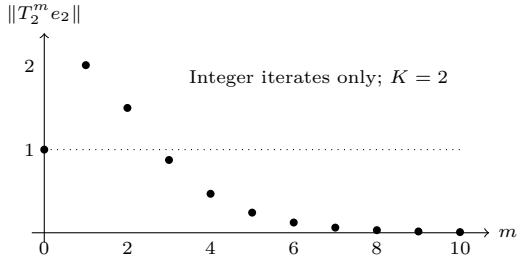
Step 4: Test all directions, not just one. For $x = (a, b)^T$, direct expansion and completion of the square give

$$\|T_K x\|^2 - \|x\|^2 = -\frac{3}{4}\left(a - \frac{2K}{3}b\right)^2 + \left(\frac{4K^2}{3} - \frac{15}{16}\right) b^2.$$

This is nonpositive for every a, b exactly when the last coefficient is nonpositive: necessity follows by choosing $a = 2Kb/3$ with $b \neq 0$. Hence the exact condition is

$$\boxed{|K| \leq \frac{3\sqrt{5}}{8}.}$$

This stricter threshold cannot be recovered by checking only e_2 or only the eigenvalues. At equality, nonzero vectors on $a = 2Kb/3$ retain their length in one step; outside that line they decrease.



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Question 8

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real τ, consider

$$A_\tau = \begin{pmatrix} 0 & 1 \\ -\tau & 2 \end{pmatrix}.$$

Investigate diagonalization over both $\mathbb{R}$ and $\mathbb{C}$.

Tasks

1. Find the characteristic polynomial and classify its roots for $\tau < 1$, $\tau = 1$ and $\tau > 1$. Check trace and determinant against the roots.
2. Find every eigenspace and classify diagonalizability in both fields, including the repeated-root case.
3. For $\tau > 1$, write a complex eigenvector as $p + iq$ and construct a real basis in which A_τ has a real 2×2 block representing its complex pair.
4. As $\tau \to 1^-$, examine a real eigenvector basis and its inverse. Explain why the limiting matrix can fail to be diagonalizable even though every matrix before the limit is diagonalizable.

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Question 8 – Solution

Strategy. Follow the eigenvectors as well as the roots. Distinct roots can merge into one direction at the transition.

Step 1: Classify the roots. The polynomial is $\lambda^2 - 2\lambda + \tau$. For $\tau < 1$, set $d = \sqrt{1-\tau} > 0$; the roots are $1 \pm d$. At $\tau = 1$ the root is 1 twice. For $\tau > 1$, set $b = \sqrt{\tau - 1} > 0$; the roots are $1 \pm ib$. Their sum is $2 = \operatorname{tr} A_\tau$ and product is $\tau = \det A_\tau$ in all cases.

Step 2: Find the eigenspaces and field restrictions. The first row of $A_\tau x = \lambda x$ forces $x_2 = \lambda x_1$. The second row is then exactly the characteristic equation. Thus every root has complex eigenspace spanned by $(1, \lambda)^T$; for real roots, the real eigenspace is the real span of the same vector. At $\tau = 1$ this is only one-dimensional, despite algebraic multiplicity 2. Consequently

	$\mathbb{R}$-diagonalizable	$\mathbb{C}$-diagonalizable
$\tau < 1$	yes	yes
$\tau = 1$	no	no
$\tau > 1$	no	yes

The real failure above 1 comes from nonreal eigenvalues, not a shortage of complex eigenvectors.

Step 3: Construct a real representation for the complex pair. For $\lambda = 1 + ib$, write $(1, 1+ib)^T = p + iq$ with $p = (1,1)^T$, $q = (0,b)^T$. Equating parts gives $A_\tau p = p - bq$ and $A_\tau q = bp + q$. Since $\det(p\ q) = b \neq 0$, this is a real basis, with representation

$$\boxed{(p\ q)^{-1} A_\tau (p\ q) = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}.}$$

It is a real block, not a real diagonalization.

Step 4: Inspect the collapsing eigenvector basis. Below 1, use $P = \begin{pmatrix} 1 & 1 \\ 1+d & 1-d \end{pmatrix}$. Its determinant is $-2d$, and

$$P^{-1} = \frac{1}{2d} \begin{pmatrix} d-1 & 1 \\ 1+d & -1 \end{pmatrix}.$$

In particular, the second column is unbounded as $d \to 0^+$. Both columns tend to $(1,1)^T$ as $d \to 0^+$. The limiting basis is singular and cannot diagonalize the limiting matrix. Diagonalizability need not persist when distinct eigendirections coalesce.

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Question 9

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = A^2.$$

For a polynomial p, define $p(A)$ by replacing each scalar power with the corresponding matrix power and each constant with that constant times I_3.

Tasks

1. Find the eigenvalues and eigenspaces of A and B, and explain why squaring merges some eigenspaces.
2. Give a nonzero eigenvector of B that is not an eigenvector of A. Prove the general forward implication from an eigenpair of A to an eigenpair of $p(A)$, and explain why its converse can fail.
3. Find the monic polynomial of least degree satisfying $p(A) = 0$. Prove both the identity and the minimality of its degree.
4. Use that polynomial identity to express A^{-1} as a polynomial in A. Compute the inverse explicitly and verify it.

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Question 9 – Solution

Strategy. A polynomial changes eigenvalues without changing each original eigenvector, but different eigenvalues can acquire the same image.

Step 1: Compare the two spectra. For A, the eigenspaces are

$$E_1 = \text{span}\{(1,1,0)^T\}, \quad E_{-1} = \text{span}\{(1,-1,0)^T\}, \quad E_2 = \text{span}\{(0,0,1)^T\}.$$

These vectors form a basis. Since $B = \text{diag}(1,1,4)$, its eigenspace for 1 is the entire plane $x_3 = 0$, and its eigenspace for 4 is the third coordinate axis. Squaring sends both 1 and -1 to 1.

Step 2: Prove the forward rule and refute the converse. The vector $e_1 = (1,0,0)^T$ satisfies $Be_1 = e_1$ but $Ae_1 = (0,1,0)^T$, not a multiple of e_1. In general, $Av = \lambda v$ implies $A^k v = \lambda^k v$ by induction; linearity then gives $\boxed{p(A)v = p(\lambda)v}$. The merged eigenspace permits new combinations that are eigenvectors of $p(A)$ without being eigenvectors of A.

Step 3: Find the least-degree annihilating polynomial. Any polynomial with $p(A) = 0$ must vanish at $1, -1, 2$, by applying the identity to the three nonzero eigenvectors. A nonzero polynomial therefore needs degree at least three. The monic candidate is

$$\boxed{p(z) = (z-1)(z+1)(z-2) = z^3 - 2z^2 - z + 2.}$$

It annihilates all three basis eigenvectors, hence every vector, proving $p(A) = 0$ and minimality. It is the unique monic cubic with those roots.

Step 4: Recover the inverse from the identity. Rearrange $A^3 - 2A^2 - A + 2I = 0$ to get $A(-A^2 + 2A + I)/2 = I$. The factors commute, so the reverse product is also I. Thus

$$\boxed{A^{-1} = \frac{-A^2 + 2A + I}{2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.}$$

Direct multiplication confirms the inverse without division by a matrix expression.

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Question 10

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

The symmetric matrix

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

measures differences along the path 1–2–3. Let $\mathbf{1} = (1, 1, 1)^T$. A vector has zero mean when $x_1 + x_2 + x_3 = 0$.

Tasks

1. Find all eigenvalues and an orthogonal eigenvector basis. Verify each pair and show that the list is complete.
2. Express $x^T Lx$ as a sum of squares. Determine all vectors for which this expression is zero, and explain the zero eigenvalue.
3. Give a necessary and sufficient condition for $Lx = b$ to have a real solution. For $b = (2, -1, -1)^T$, find all solutions and the unique one with zero mean.
4. On the zero-mean subspace, prove the sharp bounds $\|x\|^2 \leq x^T Lx \leq 3\|x\|^2$ and give every equality case. Explain why the lower bound fails without the zero-mean restriction.

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Question 10 – Solution

Strategy. Separate the constant direction, which the matrix cannot detect, from the two directions of variation.

Step 1: Verify a complete orthogonal basis. Direct multiplication gives

$$L\mathbf{1} = 0, \qquad Lv = v, \quad v = (1, 0, -1)^T, \qquad Lw = 3w, \quad w = (1, -2, 1)^T.$$

The three vectors are nonzero and mutually perpendicular, so they form a basis of $\mathbb{R}^3$. Thus the complete eigenvalue list is $\boxed{0, 1, 3}$, with the corresponding one-dimensional eigenspaces.

Step 2: Interpret the nonnegative quadratic form. Expanding gives

$$\boxed{x^T Lx = (x_1 - x_2)^2 + (x_2 - x_3)^2 \geq 0.}$$

It vanishes exactly when $x_1 = x_2 = x_3$, the constant direction. The zero eigenvalue reflects the fact that adding the same constant to all three entries changes no adjacent difference.

Step 3: Solve subject to the compatibility condition. Since $\mathbf{1}^T L = 0$, solvability requires $\mathbf{1}^T b = 0$. Conversely, every such b can be written $b = \alpha v + \beta w$, and $x = \alpha v + (\beta/3)w + c\mathbf{1}$ solves the system for every real c. For the stated b, dot products give $\alpha = 3/2$ and $\beta = 1/2$. Therefore all solutions are

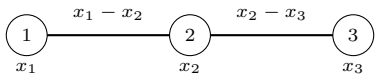
$$\boxed{x = (5/3, -1/3, -4/3)^T + c\mathbf{1}, \qquad c \in \mathbb{R}.}$$

The displayed particular vector has zero mean; the total sum is $3c$, so zero mean uniquely selects $c = 0$.

Step 4: Prove the sharp restricted bounds. A zero-mean vector has form $x = av + bw$. Orthogonality and $\|v\|^2 = 2$, $\|w\|^2 = 6$ give

$$\|x\|^2 = 2a^2 + 6b^2, \qquad x^T Lx = 2a^2 + 18b^2.$$

The lower equality holds exactly on the line through v; the upper exactly on the line through w, with zero included in both. This proves both sharp bounds. A nonzero constant vector has positive length but zero quadratic form, so the lower bound cannot hold on all of $\mathbb{R}^3$.



Two edges of weight 1; energy is the sum of squared differences.

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Question 1

Consider the scalar initial-value problem, for $t > 0$,

$$y^{(3)} + \frac{1}{t}y'' - 2y' + y = e^t, \qquad y(1) = 2, \quad y'(1) = -1, \quad y''(1) = 0.$$

A first-order state must contain enough information to determine all of its first derivatives. A system is autonomous when its right-hand side has no explicit dependence on the independent variable.

Tasks

1. Use $X = (y, y\prime, y\prime\prime)^T$ to write $X\prime = A(t)X + g(t)$ and the complete initial state.
2. Classify the system as linear or nonlinear, homogeneous or nonhomogeneous, and autonomous or nonautonomous. State its dimension and the largest coefficient-continuity interval containing $t = 1$.
3. Use instead $Z = (y\prime\prime, y, y\prime)^T$. Find the transformed system and initial state, explaining why reordering state variables changes neither the underlying problem nor the number of initial values.
4. Prove equivalence in both directions between the scalar equation and your first state system. Explain what the coefficient singularity does, and does not, establish about continuation through $t = 0$.

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Question 1 – Solution

Strategy. Store successive derivatives as independent state coordinates, then recover their required relations from the first rows.

Step 1: Build the derivative chain. Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Solving for the highest derivative gives

$$\boxed{X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -1/t \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}, \quad X(1) = (2, -1, 0)^T.}$$

In particular the last row is $x_3' = -x_1 + 2x_2 - x_3/t + e^t$.

Step 2: Separate the classifications. The three-dimensional system is linear: coefficients depend on t, not on the unknown state. It is nonhomogeneous because the forcing is nonzero, and nonautonomous because $1/t$ and e^t depend explicitly on time. The largest open coefficient-continuity interval containing 1 is $(0, \infty)$.

Step 3: Reorder the coordinates consistently. Here $z_1 = x_3$, $z_2 = x_1$, $z_3 = x_2$, so

$$\boxed{Z' = \begin{pmatrix} -1/t & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Z + \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad Z(1) = (0, 2, -1)^T.}$$

A fixed permutation is invertible. It relabels coordinates and initial data without adding or removing information; there are still three initial values.

Step 4: Check equivalence and the domain limitation. Any scalar solution supplies a state satisfying all three rows. Conversely, for a continuously differentiable state solution, the first two rows imply $x_2 = x_1'$ and $x_3 = x_1''$; the third then makes x_1 three times differentiable and gives the original equation and initial data. The standard linear-system theorem gives a unique solution on $(0, \infty)$. It does not apply through 0, where $1/t$ is undefined. This alone does not prove that a particular solution cannot have a smooth extension; any extension would require a separately specified equation at the singular point.

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Question 2

Two masses move horizontally without friction. Their displacements u, v are measured to the right from equilibrium. Mass 1 has mass 1, is attached to a fixed left wall by a spring of stiffness 2, and is coupled to mass 2 by a spring of stiffness 3. Mass 2 has mass 2 and no wall spring. A force $f(t)$ acts to the right on mass 2. Use consistent mechanical units. Initially $(u, u', v, v') = (1, 0, 0, 0)$.

Tasks

1. Derive the two second-order force equations, checking the opposite signs of the coupling forces.
2. For $X = (u, u\prime, v, v\prime)^T$, give the first-order matrix system and initial state. Explain why (u, v) alone is insufficient.
3. When $f = 0$, differentiate $E = \frac{1}{2}(u\prime)^2 + (v\prime)^2 + u^2 + \frac{3}{2}(v - u)^2$ and prove conservation. For general f, find the exact power balance.
4. Find all equilibrium states for a constant force $f(t) = F$. Explain why equilibrium requires conditions on velocities as well as displacements.

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Question 2 – Solution

Strategy. Apply Newton's law to each mass and use energy to check the coupling signs independently.

Step 1: Account for the spring forces. The wall spring exerts $-2u$ on mass 1. The coupling spring exerts $3(v-u)$ on mass 1 and its opposite on mass 2. Therefore

$$\boxed{u'' = -5u + 3v, \qquad 2v'' = 3u - 3v + f(t).}$$

Dividing the second equation by its mass is essential.

Step 2: Include positions and velocities. Writing $p = u'$, $q = v'$ gives

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 3/2 & 0 & -3/2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t)/2 \end{pmatrix}, \quad X(0) = (1,0,0,0)^T.$$

Equal positions can have different velocities, so positions alone do not determine the first derivatives. The four coordinates supply the complete state.

Step 3: Verify the work-energy balance. Differentiate the stated energy before substituting the equations:

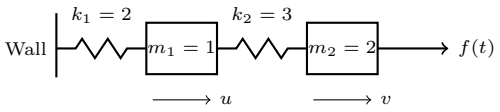
$$E' = p(-5u + 3v) + q(3u - 3v + f) + 2up + 3(v-u)(q-p).$$

Every spring term cancels, leaving $\boxed{E' = f(t)v'}$. Thus $f = 0$ conserves E; for the stated initial data, $E(0) = 5/2$. The two kinetic terms reflect the different masses. The remaining terms are the nonnegative spring potential energies.

Step 4: Balance a constant force. An equilibrium has $p = q = 0$, $-5u + 3v = 0$, and $3u - 3v + F = 0$. Consequently $u = F/2$, $v = 5F/6$, and

$$\boxed{X_* = (F/2, 0, 5F/6, 0)^T.}$$

This is the unique equilibrium. Zero acceleration at one instant would not make a state constant if either velocity were nonzero.



Displacements from equilibrium; rightward positive.

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Question 3

Two perfectly mixed tanks have constant volumes 100 L and 200 L. Fresh water enters tank 1 at 3 L/min. Tank 1 sends 5 L/min to tank 2; tank 2 returns 2 L/min to tank 1 and drains 3 L/min outside. Let x, y be the salt amounts in grams, with $x(0) = 100$, $y(0) = 0$. You may use the inward-boundary criterion: for a locally Lipschitz vector field, the nonnegative quadrant is forward invariant if its boundary derivatives point inward or are tangent.

Tasks

1. Check both volume balances and derive the first-order system for salt amounts. Identify the concentration used in each outflow term.
2. Write the amount system in matrix form. Then derive the system for concentrations $c_1 = x/100$, $c_2 = y/200$, including initial data.
3. Find the derivative of total salt and explain why the internal transfer rates cancel even though the volumes differ. Find every equilibrium.
4. Prove that nonnegative initial amounts remain nonnegative and that total salt cannot increase. Give explicit bounds for this initial state without solving the coupled system.

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Question 3 – Solution

Strategy. Multiply each flow rate by the concentration in its source tank, then check the total balance.

Step 1: Balance volumes and salt separately. Tank 1 receives $3 + 2 = 5$ L/min and sends 5; tank 2 receives 5 and sends $2 + 3 = 5$. Their volumes are constant. The return concentration is $y/200$ and the forward concentration is $x/100$, giving

$$\boxed{x' = -x/20 + y/100, \qquad y' = x/20 - y/40.}$$

Fresh water supplies no salt. Each coefficient has units min^{-1}.

Step 2: Change from amounts to concentrations. For $X = (x, y)^T$,

$$X' = \begin{pmatrix} -1/20 & 1/100 \\ 1/20 & -1/40 \end{pmatrix} X, \qquad X(0) = (100, 0)^T.$$

Dividing each balance by its own tank volume gives

$$\boxed{c' = \begin{pmatrix} -1/20 & 1/50 \\ 1/40 & -1/40 \end{pmatrix} c, \qquad c(0) = (1, 0)^T.}$$

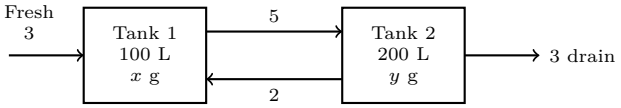
This is a constant invertible change of state; the two matrices need not match.

Step 3: Check the external balance and equilibria. Adding the amount equations gives

$$\boxed{(x + y)' = -3y/200},$$

exactly the outside drain rate times its source concentration. At an equilibrium this forces $y = 0$, and then $y' = x/20 = 0$ forces $x = 0$. The only equilibrium is the salt-free state.

Step 4: Prove positivity and bounds. The field is linear and therefore locally Lipschitz. On $x = 0$, $y \geq 0$, we have $x' = y/100 \geq 0$; on $y = 0$, $x \geq 0$, we have $y' = x/20 \geq 0$. The given inward-boundary criterion therefore proves that the nonnegative quadrant is forward invariant. The constant linear system exists for all forward times, so the conclusion holds for every $t \geq 0$. Now the total balance implies $0 \leq x + y \leq 100$. Hence $0 \leq x, y \leq 100$ g, $0 \leq c_1 \leq 1$ and $0 \leq c_2 \leq 1/2$ g/L. These are guaranteed bounds, not claims that every bound is attained.



All flow rates in L/min; arrows show actual transfer directions.

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Question 4

Consider the first-order system on the real time axis

$$x' = x + y, \qquad y' = 2x - y + t, \qquad x(0) = 1, \quad y(0) = 2.$$

An equivalent scalar equation must preserve both the dynamics and the initial information; an eliminated coordinate must be recoverable.

Tasks

1. Eliminate y to derive a second-order equation for x. Find the two corresponding initial values.
2. Prove the converse: from any solution of your scalar IVP, reconstruct y and verify both original equations and both original initial values.
3. Instead eliminate x to obtain a scalar IVP for y and a reconstruction formula for x. Verify the forcing signs.
4. A calculation uses the correct scalar equation for x but imposes $x(0) = 1$, $x\prime(0) = 2$. Identify exactly which original initial condition this changes. Explain why prescribing both $x\prime(0) = 2$ and the original $y(0) = 2$ is inconsistent.

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Question 4 – Solution

Strategy. Differentiate one row, substitute the other, and retain the algebraic formula that recovers the missing coordinate.

Step 1: Eliminate y while retaining initial information. The first row gives $y = x' - x$. Differentiating it and using the second gives $x'' = x' + 2x - y + t = 3x + t$. Thus

$$\boxed{x'' - 3x = t, \qquad x(0) = 1, \quad x'(0) = 3.}$$

The initial slope is $x(0) + y(0)$, not the initial value of y alone.

Step 2: Reconstruct and verify. Given a twice differentiable solution of this scalar IVP, define $y = x' - x$. Then $x' = x + y$ identically, and

$$y' = x'' - x' = 3x + t - x' = 2x - (x' - x) + t = 2x - y + t.$$

Also $y(0) = 3 - 1 = 2$. This proves the converse as well as the forward implication: no extra scalar solutions remain after reconstruction and data matching.

Step 3: Eliminate in the other direction. The second row gives $x = (y' + y - t)/2$. Differentiating that row yields

$$y'' = 2x' - y' + 1 = 2x + 2y - y' + 1 = 3y + 1 - t.$$

Therefore the equivalent alternative is

$$\boxed{y'' - 3y = 1 - t, \quad y(0) = 2, \quad y'(0) = 0, \qquad x = \frac{y' + y - t}{2}.}$$

Indeed $x' = (y'' + y' - 1)/2 = (3y + y' - t)/2 = x + y$; the reconstruction also gives $x(0) = 1$ and the second row directly.

Step 4: Diagnose the mismatched data. If $x(0) = 1$ and $x'(0) = 2$, reconstruction forces $y(0) = 2 - 1 = 1$. The calculation solves a different original IVP, with initial state $(1, 1)$. Insisting simultaneously on $y(0) = 2$ would require $x'(0) = 1 + 2 = 3$, contradicting the imposed slope. Correct differential equations alone do not make two initial-value problems equivalent.

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Question 5

Let

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X(0) = (0,1)^T,$$

and introduce a time-dependent coordinate system $X = P(t)Z$, where

$$P(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The physical state X and its coordinate column Z describe the same vector using different bases at different times.

Tasks

1. Derive the differential equation for Z by differentiating $X = PZ$. Explain why the constant-basis formula $P^{-1}AP$ is insufficient.
2. Compute the coefficient matrix and initial state for Z. Classify the transformed system as linear, homogeneous, and autonomous or nonautonomous.
3. Verify the supplied physical trajectory $X(t) = (\sin t, \cos t)^T$. Obtain $Z(t)$ and check both transformed equations directly.
4. Compare $X\prime(0)$ with $Z\prime(0)$. Explain how a zero instantaneous coordinate derivative can coexist with a changing physical vector.

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Question 5 – Solution

Strategy. A moving basis contributes its own derivative; keep that contribution before multiplying by the inverse.

Step 1: Differentiate the whole change of coordinates. Since $X' = P'Z + PZ' = APZ$, invertibility of P gives

$$\boxed{Z' = P^{-1}(AP - P')Z.}$$

Dropping P' would treat a moving basis as fixed and generally produce incorrect coordinate derivatives. Here $\det P = 1$ for all real t.

Step 2: Compute the transformed system. The matrices are

$$P^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \qquad AP - P' = \begin{pmatrix} 0 & 0 \\ -1 & -t \end{pmatrix}.$$

Consequently

$$\boxed{Z' = \begin{pmatrix} t & t^2 \\ -1 & -t \end{pmatrix} Z, \qquad Z(0) = (0,1)^T.}$$

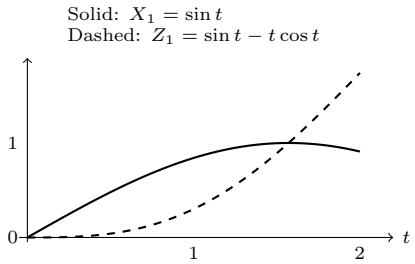
This system is linear and homogeneous, but nonautonomous. Explicit time dependence can arise entirely from a time-dependent choice of coordinates.

Step 3: Check the supplied trajectory in both descriptions. For $X = (\sin t, \cos t)^T$, $X' = (\cos t, -\sin t)^T = AX$, with the stated data. Multiplication by P^{-1} gives

$$\boxed{Z = (\sin t - t\cos t,\, \cos t)^T.}$$

Its derivative is $(t\sin t, -\sin t)^T$. The transformed right-hand side is also $(t(\sin t - t\cos t) + t^2\cos t, -(\sin t - t\cos t) - t\cos t)^T$, which simplifies to the same vector.

Step 4: Interpret the instantaneous difference. At 0, $X' = (1,0)^T$ but $Z' = (0,0)^T$. The identity $X' = P'Z + PZ'$ explains this: $P'(0)Z(0) = (1,0)^T$. The basis motion accounts for the entire instantaneous change. The figure compares the first components; the coordinate curve starts with a horizontal tangent while the physical component has slope 1.



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Question 6

Consider the nonautonomous system

$$x' = tx, \qquad y' = -y + \sin t, \qquad x(2) = 3, \quad y(2) = -1.$$

A common way to remove explicit time is to add a clock as another state. In an autonomous system, the right-hand side depends only on the state.

Tasks

1. Write this as a two-dimensional linear system $X\prime = A(t)X + g(t)$ and classify homogeneity and autonomy.
2. Introduce a new independent variable τ and a clock $s(\tau)$ satisfying $s\prime = 1$. Give an autonomous three-dimensional system with initial data at $\tau = 0$ that reproduces the stated IVP.
3. Prove the correspondence between solutions of the two systems, specifying exactly how t and τ are related. Explain why the clock initial value is required.
4. Classify the augmented system as linear or nonlinear and find its equilibria. Explain why autonomous does not mean linear, and why adding a clock can eliminate all equilibria.

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Question 6 – Solution

Strategy. Turn the old independent variable into a dependent variable whose rate is fixed and whose initial value records the time origin.

Step 1: Identify the original structure. With $X = (x, y)^T$,

$$X' = \begin{pmatrix} t & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}, \qquad X(2) = (3, -1)^T.$$

It is linear, nonhomogeneous and nonautonomous. Time-dependent coefficients do not violate linearity in the dependent variables.

Step 2: Add and initialize the clock. Write $U(\tau) = (s(\tau), u(\tau), v(\tau))^T$. The autonomous system is

$$\boxed{s' = 1, \qquad u' = su, \qquad v' = -v + \sin s, \qquad U(0) = (2, 3, -1)^T.}$$

Here every prime denotes differentiation with respect to τ. There is no explicit τ on the right-hand side.

Step 3: Prove the change-of-clock correspondence. The first row forces $s(\tau) = 2 + \tau$. Given an original solution, define $u(\tau) = x(2 + \tau)$ and $v(\tau) = y(2 + \tau)$. The chain rule verifies the last two rows and the initial data. Conversely, define $x(t) = u(t - 2)$ and $y(t) = v(t - 2)$ from an augmented solution with $s(0) = 2$; substitution recovers the original system. Intervals shift by the same amount. Choosing $s(0) = 0$ would instead sample the coefficients at $t = \tau$, changing the stated starting time.

Step 4: Distinguish autonomy, linearity and equilibrium. The augmented system is nonlinear: su multiplies two dependent variables, and $\sin s$ is a nonlinear function of a dependent variable. It cannot be classified as a homogeneous linear system merely because the original equations were linear. An equilibrium would require all three right-hand sides to vanish, but $s' = 1$ never vanishes. Thus the augmented autonomous system has no equilibrium. A clock keeps moving even if some physical state components happen to be constant.

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Question 7

For a real constant κ, consider the implicit derivative equations

$$x' + y' = -x, \qquad x' + \kappa y' = y.$$

A classical solution is a continuously differentiable pair satisfying both relations throughout an interval. An algebraically consistent derivative at one state is not necessarily part of a solution through that state.

Tasks

1. Write $M_\kappa X\prime = BX$ and determine exactly when it can be solved for a unique derivative at every state. Find the explicit system in that case.
2. At $\kappa = 1$, determine the states at which the two equations are algebraically consistent for a derivative. Describe the possible derivatives at such a state.
3. Still at $\kappa = 1$, find every classical solution on an interval of positive length. Explain why most instantaneously consistent states cannot be used as initial data.
4. For $\kappa \neq 1$ and initial state $(x, y) = (1, -1)$, compute the initial derivative and the derivative of $x + y$. Does a finite, parameter-independent initial velocity imply that a solution exists through the same state at $\kappa = 1$?

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Question 7 – Solution

Strategy. Separate invertibility of the derivative matrix from the extra consistency conditions imposed when its rank drops.

Step 1: Solve when the derivative matrix is invertible. Here

$$M_\kappa = \begin{pmatrix} 1 & 1 \\ 1 & \kappa \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det M_\kappa = \kappa - 1.$$

For $\kappa \neq 1$, inversion yields

$$\boxed{X' = \frac{1}{\kappa - 1} \begin{pmatrix} -\kappa & -1 \\ 1 & 1 \end{pmatrix} X.}$$

This constant linear system has a unique solution for every initial state.

Step 2: Test instantaneous algebraic consistency. At $\kappa = 1$, the left sides coincide, so the right sides must satisfy $y = -x$. At a state $(a, -a)$, the derivative pairs (p, q) satisfy $p + q = -a$, an entire line of possibilities. Outside $y = -x$, no derivative can satisfy both equations even at that instant.

Step 3: Enforce the constraint along a solution. A classical solution must have $y(t) = -x(t)$ at every time. Differentiating this identity gives $x' + y' = 0$. The first equation then forces $x = 0$, and the constraint forces $y = 0$. Therefore

$$\boxed{X(t) \equiv (0, 0)^T \text{ is the only classical solution at } \kappa = 1.}$$

It indeed satisfies both equations. Nonzero points on the consistency line admit instantaneous derivatives but no solution interval through them. Even at the origin, most algebraic derivative choices cannot persist.

Step 4: Check the limiting initial velocity. Substituting $(1, -1)$ into the explicit system gives $x' = (-\kappa + 1)/(\kappa - 1) = -1$ and $y' = 0$ for every $\kappa \neq 1$. Thus $(x + y)' = -1$, although initially $x + y = 0$. This velocity does not keep the state on the constraint line required at $\kappa = 1$. Its finiteness does not imply a limiting solution through $(1, -1)$; Step 3 proves that no such solution exists at that parameter.

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Question 8

For $t \geq 0$, consider

$$x' = 2\sqrt{|x|}, \qquad y' = -y, \qquad x(0) = 0, \quad y(0) = 1.$$

You may use the local uniqueness theorem: continuity of a vector field and local Lipschitz continuity in the state guarantee local uniqueness. Failure of its hypotheses alone does not prove nonuniqueness.

Tasks

1. Check continuity and test the local Lipschitz condition at states with $x = 0$. Explain exactly which uniqueness guarantee is unavailable.
2. For any waiting time $a \geq 0$, construct a solution that has $x = 0$ until time a and then becomes positive. Verify the derivative at the joining time as well as both differential equations.
3. Find y and compare the initial state and initial derivative of the waiting-time solutions. Include the solution that never leaves $x = 0$.
4. If an additional observation says $x(1) = 0$, does that restore uniqueness for later times? Exhibit two distinct solutions satisfying all the data and explain what this example says about a complete initial state.

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Question 8 – Solution

Strategy. Prove nonuniqueness by constructing different differentiable solutions; do not infer it solely from a theorem's failure.

Step 1: Test the regularity hypothesis. The field $F(x, y) = (2\sqrt{|x|}, -y)^T$ is continuous everywhere. For $h > 0$, the change in its first component between (h, y) and $(0, y)$, divided by the state distance, is $2\sqrt{h}/h = 2/\sqrt{h}$. This is unbounded as $h \downarrow 0$, so no local Lipschitz constant exists near $x = 0$. The quoted uniqueness theorem cannot guarantee uniqueness there.

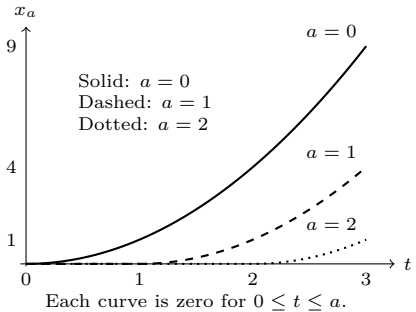
Step 2: Construct and check the waiting family. For each $a \geq 0$, define

$$\boxed{x_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ (t-a)^2, & t > a. \end{cases}}$$

Before a, both x_a' and $2\sqrt{|x_a|}$ are zero. After a, both are $2(t-a)$. At a, the left and right derivatives are zero, so the function is continuously differentiable and satisfies the equation there too. At $a = 0$, the same check uses the right derivative at the endpoint.

Step 3: Check the common data. The second scalar equation uniquely gives $\boxed{y(t) = e^{-t}}$. Every pair (x_a, e^{-t}) has initial state $(0, 1)$ and initial derivative $(0, -1)$. The pair $(0, e^{-t})$ is another solution, corresponding to waiting forever. Different finite waiting times produce different first components, despite identical initial states and slopes.

Step 4: Test whether one extra observation selects a solution. Every $a \geq 1$ gives $x_a(1) = 0$, as does the never-departing solution. For example, $a = 1$ and $a = 2$ satisfy all the data, but at $t = 3/2$ their first components are $1/4$ and 0. Uniqueness is still absent. A complete state specifies all initial coordinates; unique evolution also requires appropriate properties of the differential equations. The figure shows three members of this family, with time increasing rightward.



Each curve is zero for $0 \leq t \leq a$.

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Question 9

Three nonnegative dimensionless amounts undergo internal transfers: amount moves from compartment 1 to 2 at rate xy, and from 2 to 3 at rate $2y$. There are no external gains or losses. Their initial amounts are $(x, y, z) = (1, 1, 0)$. The rate laws are valid for nonnegative states.

Tasks

1. Derive the vector differential equation and classify it as autonomous and linear or nonlinear. Explain why a state-dependent coefficient does not make this a linear system.
2. Find the total-amount balance. Show that nonnegative initial states remain nonnegative and use this to bound every component for the stated data.
3. For the stated initial data, prove $x(t) > 0$ at every finite forward time. Derive a second conserved expression involving $x + y$ and $\ln x$.
4. Use the two conserved expressions to reduce the dynamics to one scalar equation for x, with formulas recovering y, z. State the restrictions that make the reduction meaningful; no explicit solution is required.

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Question 9 – Solution

Strategy. Each internal transfer leaves one compartment and enters another at the same rate; this exposes conserved quantities before any solution is sought.

Step 1: Translate the transfers. The balances are

$$\boxed{x' = -xy, \qquad y' = xy - 2y, \qquad z' = 2y, \qquad (x, y, z)(0) = (1, 1, 0).}$$

The system is autonomous, but nonlinear because of xy. Writing $x' = -yx$ does not make $-y$ a prescribed coefficient: y is another unknown state coordinate, not a known function of time.

Step 2: Prove conservation and nonnegativity. Adding the rows gives $(x + y + z)' = 0$, hence $x + y + z = 2$. Along any solution,

$$x(t) = x(0)e^{-\int_0^t y(s)\,ds}, \qquad y(t) = y(0)e^{\int_0^t (x(s)-2)\,ds}, \quad z(t) = z(0) + 2\int_0^t y(s)\,ds.$$

These formulas prove nonnegativity for nonnegative data on every forward existence interval. Thus $0 \le x, y, z \le 2$. The polynomial field is locally Lipschitz; boundedness prevents finite-time escape, so the solution continues for all $t \ge 0$.

Step 3: Obtain a second conserved quantity. With $x(0) = 1$ and $0 \le y \le 2$, the exponential formula gives $x(t) \ge e^{-2t} > 0$ at finite times. Therefore $\ln x$ is legitimate, and

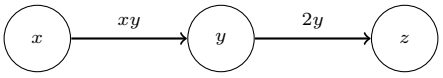
$$(x + y - 2\ln x)' = -2y - 2(-xy)/x = 0.$$

Its initial value is 2, so $\boxed{x + y - 2\ln x = 2}$.

Step 4: Reduce and recover the state. The two conservation laws give

$$\boxed{y = 2 - x + 2\ln x, \quad z = -2\ln x, \quad x' = -x(2 - x + 2\ln x), \quad x(0) = 1.}$$

The physical branch has $x > 0$, $y \ge 0$, $z \ge 0$, in particular $x \le 1$. Conversely, on this branch differentiation gives $y' = (x - 2)y$ and $z' = 2y$, proving recovery of the original system. Taking logarithms at $x = 0$ would be invalid; that boundary is excluded from this reduction.



Internal transfers only: $x + y + z = 2$ for the given data.

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Question 10

An unknown autonomous affine system has the form $X' = AX+b$ in $\mathbb{R}^2$, where A, b are constant. Independent measurements of its velocity field F give

$$F(0,0) = (1,-1)^T, \quad F(1,0) = (-1,2)^T, \quad F(0,1) = (2,-3)^T.$$

An affine system is written as a nonhomogeneous linear differential system; its constant forcing need not vanish.

Tasks

1. Recover A and b, and prove uniqueness within the class of affine systems. Explain why the measured inputs are states rather than initial derivatives of a single trajectory.
2. Find every equilibrium and translate it to the origin. Write the differential equation in the translated state and explain what happens to the constant forcing.
3. A fourth measurement reports $F(1,1) = (0,1)^T$. Is it compatible with the first three under the affine assumption? Quantify the discrepancy.
4. Construct a smooth nonlinear vector field fitting all four measurements. Explain why the first three determine an affine model uniquely but do not determine an unrestricted vector field.

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Question 10 – Solution

Strategy. Subtract the velocity at the origin to isolate the matrix columns, then test the model assumption with new data.

Step 1: Recover the affine coefficients. At the origin, $b = F(0,0) = (1,-1)^T$. The differences $F(1,0) - b = (-2,3)^T$ and $F(0,1) - b = (1,-2)^T$ are the first and second columns of A. Therefore

$$\boxed{A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \qquad b = (1,-1)^T.}$$

These columns and b are forced, proving uniqueness within the affine class. The measurements prescribe the field at three different states; they are not three independently assignable derivatives at one initial state.

Step 2: Translate the unique equilibrium. The equation $AX_* + b = 0$ reads $-2x + y + 1 = 0$ and $3x - 2y - 1 = 0$. It gives $X_* = (1,1)^T$; uniqueness also follows from $\det A = 1$. For $Z = X - X_*$, the translation is constant, and

$$Z' = A(Z + X_*) + b = AZ, \qquad \boxed{Z' = AZ.}$$

The forcing vanishes because the translation is by an equilibrium, not because every translation removes every forcing term.

Step 3: Check the fourth report. The recovered model predicts $F(1,1) = (0,0)^T$, while the report is $(0,1)^T$. The reported-minus-predicted discrepancy is $(0,1)^T$. No affine system fits all four measurements, since the first three already fix its coefficients. This conclusion tests the affine model assumption.

Step 4: Exhibit a nonlinear alternative. Define

$$\boxed{\widetilde{F}(x,y) = (-2x + y + 1,\ 3x - 2y - 1 + xy)^T.}$$

The extra term xy vanishes at the first three states and equals 1 at $(1,1)$, so all four measurements are met. This polynomial field is smooth and nonlinear. In it, $(1,1)$ is not an equilibrium. More generally, fields obtained by adding $(0, cxy)^T$ all fit the first three data for any real c. Finite measurements uniquely identify this affine class, but leave unrestricted nonlinear models undetermined.

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Question 1

For $X' = AX$, with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, consider

$$U(t) = e^t(1,0)^T, \quad V(t) = e^t(t,1)^T, \quad W(t) = e^t(1,t)^T.$$

A vector solution must satisfy both rows at every time in its interval. A fundamental matrix has solution columns that form a basis at each time.

Tasks

1. Compute the residual $F\prime - AF$ for each candidate $F = U, V, W$. Decide which are solutions on a nonempty open interval.
2. Use the valid candidates to form a fundamental matrix. Prove that every solution is their constant linear combination, rather than merely checking that combinations work.
3. Solve the IVP $X(0) = (2, -1)^T$ and verify both the equations and initial data.
4. For this IVP, find the first zero of its first component for $t \geq 0$. Determine the full state and derivative there, and explain why a zero component does not mean a zero solution.

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Question 1 – Solution

Strategy. Check vector residuals first; then solve the triangular equations to establish completeness.

Step 1: Test all candidates. Direct differentiation gives $U' - AU = 0$ and $V' - AV = 0$. For the third candidate,

$$W' = e^t(1, t+1)^T, \quad AW = e^t(1+t, t)^T, \quad \boxed{W' - AW = e^t(-t, 1)^T.}$$

Its second residual component never vanishes, so W is not a solution on any nonempty open interval.

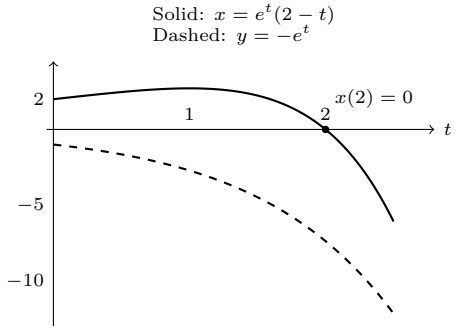
Step 2: Prove that the valid family is complete. The matrix $\Phi = (U\ V) = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has determinant $e^{2t} > 0$. More explicitly, the second equation gives $y = c_2 e^t$. Then $(e^{-t}x)' = c_2$, so

$$\boxed{X = e^t(c_1 + c_2 t, c_2)^T = c_1 U + c_2 V.}$$

This derivation reaches every solution and introduces exactly two arbitrary constants. The family is valid on the whole real axis.

Step 3: Select and verify the initial-value solution. At 0, the constants are $c_1 = 2$, $c_2 = -1$, hence $\boxed{X = e^t(2 - t, -1)^T}$. Its derivative is $e^t(1 - t, -1)^T$, exactly AX, and its initial state is $(2, -1)^T$ as required.

Step 4: Interpret a component zero. Since $e^t \neq 0$, the first component first vanishes at $t = 2$. There $X(2) = (0, -e^2)^T$ and $X'(2) = (-e^2, -e^2)^T$. The state is nonzero and continues moving; the first component crosses zero with negative slope. For a homogeneous linear system with continuous coefficients, only a full zero state at a time would force the identically zero solution by uniqueness. The plot displays both components.



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Question 2

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}.$$

A principal fundamental matrix at 0 is a fundamental matrix Y satisfying $Y(0) = I$. For real a, also define $R_a = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $\Psi_a = \Phi R_a$.

Tasks

1. Verify $\Phi\prime = A\Phi$ and calculate its determinant. Explain why its columns describe a complete homogeneous solution family.
2. Construct the principal fundamental matrix $Y(t)$ and verify $Y(0) = I$. Express its entries using hyperbolic functions or exponentials.
3. For arbitrary initial data $X(0) = (p, q)^T$, find the constants in $X = \Phi c$ and the same solution in the form $X = YX(0)$.
4. Classify when Ψ_a is a fundamental matrix. At $a = 1$ and $a = -1$, determine exactly which initial states can still be represented by $X = \Psi_a d$, even though every column remains a solution.

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Question 2 – Solution

Strategy. Distinguish solution columns from a complete basis, and normalize by the matrix at the initial time.

Step 1: Verify the columns and independence. Differentiating each exponential column gives $\Phi' = A\Phi$. Its determinant is -2, never zero. Every initial vector can therefore be represented by $\Phi(0)c$; uniqueness of the linear IVP proves that the resulting constant combination supplies every solution.

Step 2: Normalize at the initial time. Since $\Phi(0)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$$\boxed{Y(t) = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.}$$

Here $\cosh t = (e^t + e^{-t})/2$, $\sinh t = (e^t - e^{-t})/2$. Thus $Y(0) = I$ and $Y' = AY$ by differentiation or constant right multiplication.

Step 3: Recover arbitrary data. Solving $\Phi(0)c = (p, q)^T$ gives

$$\boxed{c_1 = (p+q)/2, \quad c_2 = (p-q)/2, \quad X = (p\cosh t + q\sinh t,\ p\sinh t + q\cosh t)^T.}$$

This equals $Y(t)(p, q)^T$. The constants in a nonnormalized basis are not generally the initial state coordinates themselves.

Step 4: Detect a collapsed family. Constant combinations remain solutions, but $\det \Psi_a = -2(1 - a^2)$, so the family is fundamental exactly when $\boxed{a \neq \pm 1}$. At $a = 1$, both columns are $U + V$, whose value at 0 is $(2, 0)^T$; exactly the initial states $(p, 0)^T$ are representable. At $a = -1$, the columns are $U - V$ and its negative, with initial value $(0, 2)^T$; exactly $(0, q)^T$ are representable. Here U, V denote the columns of Φ. Neither singular case supplies all initial states, despite having two displayed solution columns.

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Question 3

Consider the continuous time-dependent linear system

$$X' = A(t)X, \qquad A(t) = \begin{pmatrix} 2t & 1 \\ 0 & -t \end{pmatrix}.$$

For two solution columns $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, write $W = u_1v_2 - u_2v_1$. Do not assume a determinant evolution formula without deriving it.

Tasks

1. Differentiate W and use the original system to derive its scalar differential equation.
2. Find $W(t)$ when the matrix of columns equals I at $t = 0$. Is the claim $W = e^{t^2}$ consistent with the system?
3. Construct the principal fundamental matrix at 0. A definite integral is an acceptable exact entry; verify the initial matrix and both column equations.
4. Find the solution with initial state $(0, 1)^T$. Explain why growth of the fundamental determinant for $t > 0$ does not imply growth of every solution component.

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Question 3 – Solution

Strategy. Derive the determinant law by the product rule, and construct the columns by solving the triangular equations.

Step 1: Differentiate the determinant. Substitution gives

$$W' = (2tu_1 + u_2)v_2 + u_1(-tv_2) - (-tu_2)v_1 - u_2(2tv_1 + v_2) = tW.$$

The off-diagonal terms cancel. This is the trace law here, since $\operatorname{tr} A = t$, obtained directly rather than assumed.

Step 2: Apply the determinant's initial value. At 0, $W(0) = 1$, and solving $W' = tW$ gives $\boxed{W(t) = e^{t^2/2}}$. The proposed e^{t^2} has derivative $2te^{t^2}$ and fails the required equation except at the isolated time 0; matching the initial value is insufficient.

Step 3: Construct and verify the matrix. Put $J(t) = \int_0^t e^{-3s^2/2}\,ds$. The second equation gives $y = c_2 e^{-t^2/2}$, and the first gives $(e^{-t^2}x)' = c_2 e^{-3t^2/2}$. Therefore

$$\boxed{\Phi(t) = \begin{pmatrix} e^{t^2} & e^{t^2}J(t) \\ 0 & e^{-t^2/2} \end{pmatrix}.}$$

Since $J(0) = 0$, $\Phi(0) = I$. The derivative of its upper-right entry is $2te^{t^2}J + e^{-t^2/2}$, precisely the required first-row expression. The remaining entries satisfy their rows directly. Its determinant agrees with Step 2 and never vanishes, so this is a principal fundamental matrix.

Step 4: Interpret the selected solution. The initial vector $(0, 1)^T$ selects the second column: $X = (e^{t^2}J(t), e^{-t^2/2})^T$. Its second component strictly decreases for $t > 0$, even while W grows. A determinant concerns a pair of independent columns, not the separate monotonicity of each component. The nonzero determinant guarantees a complete independent family throughout the real axis.

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Question 4

For the nonautonomous system

$$x' = 2ty, \qquad y' = 0,$$

define the transition matrix $T(t, s)$ by $X(t) = T(t, s)X(s)$ for arbitrary real starting time s. It must satisfy $T(s, s) = I$.

Tasks

1. Derive $T(t, s)$ from the component equations and verify its differential equation and normalization.
2. Prove $T(t, r)T(r, s) = T(t, s)$ and determine the inverse of $T(t, s)$. Explain the role of the intermediate time.
3. Let $Y(t) = T(t, 0)$. Determine whether $Y(t + s) = Y(t)Y(s)$ holds for all real t, s, and give a concrete counterexample if it fails.
4. Solve the IVP $X(-1) = (2, 3)^T$. Compare $X(1)$ with $X(-1)$ and compute $X\prime(0)$. Does either equality imply that the solution is constant?

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Question 4 – Solution

Strategy. Keep both endpoint times: a time-dependent equation need not evolve according only to elapsed time.

Step 1: Integrate from the actual starting time. The second component is $y(t) = y(s)$, and $x(t) = x(s) + y(s)\int_s^t 2u\,du$. Hence

$$\boxed{T(t,s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.}$$

Its derivative with respect to t is $\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}$, which equals $A(t)T(t,s)$, and $T(s,s) = I$.

Step 2: Check composition and inversion. Multiplying two such shears adds their upper-right entries: $(t^2 - r^2) + (r^2 - s^2) = t^2 - s^2$. Thus

$$\boxed{T(t,r)T(r,s) = T(t,s), \qquad T(t,s)^{-1} = T(s,t).}$$

The intermediate time cancels only because it is the end of the first journey and the start of the second, in the correct order.

Step 3: Test the one-parameter shortcut. The upper-right entry of $Y(t)Y(s)$ is $t^2 + s^2$, whereas that of $Y(t+s)$ is $(t+s)^2 = t^2 + s^2 + 2ts$. They agree only when $ts = 0$, not for all times. For example, $Y(1)Y(1)$ has upper-right entry 2, while $Y(2)$ has entry 4. This does not contradict the two-time composition identity.

Step 4: Apply the transition to the IVP. The solution is

$$\boxed{X(t) = T(t,-1)(2,3)^T = (3t^2 - 1, 3)^T.}$$

It returns to $(2,3)^T$ at $t = 1$, and $X'(0) = (0,0)^T$. Nevertheless $X' = (6t, 0)^T$ is nonzero away from 0, so the solution is not constant. In a nonautonomous system, a repeated state at different times or a zero derivative at one instant does not force stationary evolution.

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Question 5

Consider

$$X' = AX + f(t), \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad f(t) = (1, 2e^{-t})^T.$$

The proposed particular solution is $P(t) = (1, 2e^{-t})^T$. Distinguish combinations of homogeneous solutions from combinations of solutions of this fixed nonhomogeneous system.

Tasks

1. Verify P, and prove that the full solution family is $P + c_1(e^{-t}, 0)^T + c_2(0, e^{-2t})^T$.
2. If U, V both solve the fixed forced system, determine exactly when $\alpha U + \beta V$ also solves it. Prove the condition for real constants α, β.
3. Solve the IVP $X(0) = (0, 0)^T$ and check its initial derivative against the original right-hand side.
4. For $t \geq 0$, find the maximum of the second component of this IVP and its time. Find the limiting state and decide whether that limiting vector, held constant, is itself a solution of the forced system.

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Question 5 – Solution

Strategy. Subtract one particular solution to expose the homogeneous family; keep the forcing when testing superposition.

Step 1: Verify and complete the affine family. Here $P' = (0, -2e^{-t})^T = AP + f$. For any other solution X, the difference $H = X - P$ satisfies $H' = AH$, so $H = (c_1 e^{-t}, c_2 e^{-2t})^T$. Conversely each such sum solves the forced equation. Thus the proposed family is complete.

Step 2: Test a weighted combination. For $Z = \alpha U + \beta V$,

$$Z' - AZ = (\alpha + \beta) f(t).$$

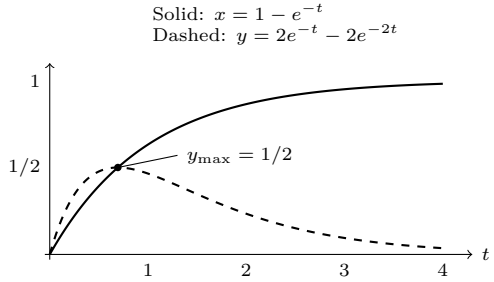
Because the first component of f is always 1, equality to f holds exactly when $\boxed{\alpha + \beta = 1}$. In particular an average is a solution, while a sum generally is not. The difference of two forced solutions solves the homogeneous equation.

Step 3: Impose the zero initial state. The constants are $c_1 = -1$, $c_2 = -2$, yielding

$$\boxed{X(t) = (1 - e^{-t},\, 2e^{-t} - 2e^{-2t})^T.}$$

At 0 this is zero, but $X'(0) = (1, 2)^T = f(0)$. A zero initial state therefore does not imply a zero solution in this nonhomogeneous problem.

Step 4: Locate the peak and interpret the limit. For $y = 2e^{-t}(1 - e^{-t})$, $y' = 2e^{-t}(2e^{-t} - 1)$. It is positive before $\ln 2$ and negative after, so the unique maximum on $t \geq 0$ is $\boxed{y(\ln 2) = 1/2}$. The state tends to $(1, 0)^T$. However the constant vector $(1, 0)^T$ has zero derivative while $A(1, 0)^T + f(t) = (0, 2e^{-t})^T \neq 0$ at finite times. A limiting state need not itself be a constant solution of a time-dependent forced equation. The plot marks the actual second-component peak.



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Question 6

For $X' = AX$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, consider

$$U(t) = (\cos t, -\sin t)^T, \qquad V(t) = (\sin t, \cos t)^T.$$

You may use uniqueness of linear initial-value problems with continuous coefficients. The independent variable is real time.

Tasks

1. Verify both solutions and state their initial vectors.
2. Find all real times when their first components agree. At those times compare their second components and decide whether the full states coincide.
3. Prove that two solutions of the same continuous linear system agreeing in their full state at one time must agree throughout their common interval. Apply this to the zero solution of a homogeneous system.
4. Find a constant time shift relating V to U. Explain why identical states at different times are compatible with your uniqueness statement.

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Question 6 – Solution

Strategy. Keep equality of one coordinate, equality of full states at the same time, and a time-shift relation separate.

Step 1: Verify the two vector equations. Differentiation gives $U' = (-\sin t, -\cos t)^T = AU$ and $V' = (\cos t, -\sin t)^T = AV$. Their initial vectors are $U(0) = (1, 0)^T$ and $V(0) = (0, 1)^T$.

Step 2: Find the coordinate coincidences. The equation $\cos t = \sin t$ holds exactly at

$$\boxed{t = \pi/4 + k\pi,\ k \in \mathbb{Z}}.$$

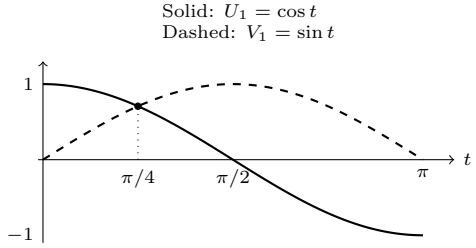
At such a time the common first component is $(-1)^k/\sqrt{2}$. The second components are its negative for U and its positive for V, so they differ. More generally,

$$\|U(t) - V(t)\|^2 = (\cos t - \sin t)^2 + (-\sin t - \cos t)^2 = 2.$$

The full states never agree at the same time. The figure displays first components only, and marks one coordinate coincidence.

Step 3: State what uniqueness actually implies. Suppose two solutions share the full vector $X(t_0) = x_0$. They solve the same IVP at t_0, so uniqueness makes them identical on their common interval. Equivalently their difference solves the homogeneous equation with zero initial data and must vanish throughout that interval. Taking one solution to be zero proves that a homogeneous solution reaching the full zero state cannot later depart from it.

Step 4: Check the shifted identity. Trigonometric identities give $\boxed{V(t) = U(t - \pi/2)}$. This compares states at different time arguments; it does not assert $U(t) = V(t)$. An autonomous system permits translating a solution in time, and the shifted function normally has different initial data at time 0. Thus the shift creates no conflict with same-time IVP uniqueness.



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Question 7

For a real parameter a, solve the nonlinear IVP

$$x' = x^2, \qquad y' = -y, \qquad x(0) = a, \quad y(0) = 1.$$

A maximal interval means the largest connected open time interval containing 0 on which the classical solution exists. The vector field is smooth.

Tasks

1. Derive the solution for $a \neq 0$, and handle $a = 0$ without dividing by the zero solution.
2. Find the maximal interval for each sign of a. Explain why smoothness of the field guarantees local uniqueness but not necessarily existence for all real times.
3. At $a = 1$, the rational formula is finite at $t = 2$. Does that value belong to a continuation of the stated IVP through $t = 1$? Justify your answer.
4. Compare $a = 0$ with $a = \varepsilon > 0$. Prove a small-data bound on any fixed $[0, T]$ when $\varepsilon T \leq 1/2$, and evaluate the difference at $t = 1/\varepsilon - 1$ for $0 < \varepsilon < 1$. Explain why the two conclusions are consistent.

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Question 7 – Solution

Strategy. Derive explicit solutions, then distinguish the algebraic formula's domain from the connected interval of an initial-value solution.

Step 1: Solve and verify, retaining the zero case. For $a \neq 0$, separation gives $1/x = 1/a - t$, hence

$$\boxed{X_a(t) = \left(\frac{a}{1-at}, e^{-t}\right)^T.}$$

The first derivative is $a^2/(1-at)^2 = x^2$, and the data match at 0. For $a = 0$, the solution is $\boxed{X_0 = (0, e^{-t})^T}$; local uniqueness of the smooth field excludes other departures from $x = 0$.

Step 2: Find the maximal connected intervals. The first component blows up at $1/a$ when $a \neq 0$. Thus

$$\boxed{a > 0 : \ (-\infty, 1/a); \qquad a = 0 : \ \mathbb{R}; \qquad a < 0 : \ (1/a, \infty).}$$

The exponential component introduces no further restriction. A finite-valued continuous extension across the pole is impossible. Smoothness is a local regularity condition and does not prevent unbounded growth in finite time.

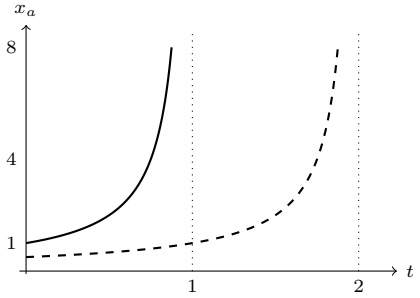
Step 3: Reject a disconnected continuation. At $a = 1$, the formula gives $x(2) = -1$, but the IVP through 0 exists only for $t < 1$. Its first component tends to $+\infty$ as $t \uparrow 1$. The branch for $t > 1$ solves the differential equation separately; it cannot be joined into a classical solution through the intervening pole.

Step 4: Compare finite-time and long-time sensitivity. For $0 \leq t \leq T$ with $\varepsilon T \leq 1/2$,

$$\|X_\varepsilon(t) - X_0(t)\| = \frac{\varepsilon}{1-\varepsilon t} \leq 2\varepsilon.$$

But at $t = 1/\varepsilon - 1$, still before the pole, the difference equals 1. That comparison time tends to infinity as $\varepsilon \downarrow 0$. There is no contradiction: closeness on each fixed finite interval does not give a uniform bound over arbitrarily long, parameter-dependent times. The plot shows only the initial-value branches before their poles.

Solid: $a = 1$; dashed: $a = 1/2$
Dotted verticals: poles; $a = 0$ stays on the time axis.



Positive branches shown up to $x = 8$; no continuation across a pole.

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Question 8

Let $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e_1 = (1, 0)^T$. Consider the periodically forced system

$$X' = AX + R(t)e_1.$$

You may use $R' = AR$, $R^{-1} = R(-t)$ and $R(t+2\pi) = R(t)$. A 2π-periodic solution satisfies $X(t+2\pi) = X(t)$ for all real t.

Tasks

1. Verify that $X_c(t) = R(t)(c + te_1)$ is a solution for every constant vector c. Determine its initial state.
2. Prove that this family contains every solution, by differentiating $R(-t)X(t)$.
3. Compute the change in state over one forcing period and decide whether any choice of initial data produces a 2π-periodic solution.
4. Replace e_1 in the forcing by an arbitrary constant vector $b = (\alpha, \beta)^T$. Find the full solution family and classify exactly when a 2π-periodic solution exists.

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Question 8 – Solution

Strategy. Factor out the supplied rotating matrix and test periodicity on the full solution, not just on its coefficients and forcing.

Step 1: Verify the proposed family directly. The product rule gives

$$X_c' = AR(t)(c + te_1) + R(t)e_1 = AX_c + R(t)e_1.$$

Also $X_c(0) = c$. Thus every initial state has a member of this family.

Step 2: Prove completeness independently. Set $Z = R(-t)X$. Differentiating the inverse identity gives $(R^{-1})' = -R^{-1}A$. Consequently

$$Z' = -R^{-1}AX + R^{-1}(AX + Re_1) = e_1.$$

It follows that $Z = c + te_1$, so $\boxed{X = R(t)(c + te_1)}$ is the entire family, valid on the real axis.

Step 3: Examine one full forcing period. Periodicity of R gives

$$\boxed{X(t + 2\pi) - X(t) = 2\pi R(t)e_1.}$$

This difference has length 2π and never vanishes, independently of c. In particular $X(2\pi) = X(0) + (2\pi, 0)^T$. No initial state gives a 2π-periodic solution. A periodic forcing can produce a response with an accumulating change rather than a periodic response.

Step 4: Classify the general constant-amplitude forcing. The same calculation gives

$$\boxed{X = R(t)(c + tb), \qquad X(t + 2\pi) - X(t) = 2\pi R(t)b.}$$

Since $R(t)$ is invertible, a periodic solution exists exactly when $b = 0$, that is, $\alpha = \beta = 0$. In this case every member $X = R(t)c$ is 2π-periodic, including the zero solution. If $b \neq 0$, no choice of c can cancel the per-period difference. The classification concerns this specified forcing family; it does not claim that every nonzero periodic forcing precludes a periodic response.

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Question 9

For

$$x' = y, \qquad y' = -x,$$

only the measurements $x(0) = 1$ and $x(T) = b$ are given, where $T > 0$. The second initial component is unknown. You may verify solutions using sine and cosine identities; no phase-plane analysis is required.

Tasks

1. Find every solution consistent with $x(0) = 1$, using one arbitrary constant, and identify what that constant means.
2. Classify existence and uniqueness for all $T > 0$ and $b \in \mathbb{R}$, including the exceptional times $T = k\pi$.
3. Apply the classification to $(T, b) = (\pi, -1)$ and $(\pi/2, 2)$. Give the full solution set in each case.
4. When $\sin T \neq 0$, suppose the reported value of b has an error δ. Find the exact resulting error in the recovered second initial component. Explain what happens as $T \to \pi$ and why this is not failure of uniqueness for a fully specified IVP.

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Question 9 – Solution

Strategy. Treat the unknown initial velocity as a parameter and ask whether the observations actually determine it.

Step 1: Parametrize all compatible initial states. Since $x'' = -x$ and $x(0) = 1$, the complete family is

$$\boxed{x = \cos t + c \sin t, \qquad y = -\sin t + c \cos t.}$$

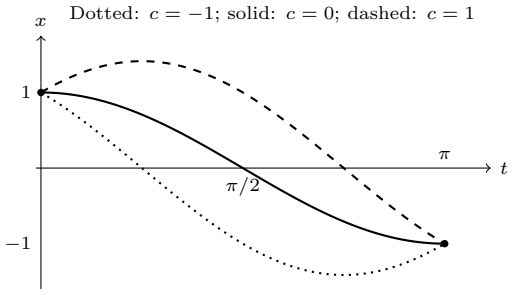
Direct differentiation verifies both rows. The arbitrary constant is $c = y(0) = x'(0)$, the missing initial component.

Step 2: Test the observation equation. The second measurement requires $\cos T + c \sin T = b$. If $\sin T \neq 0$, there is exactly one solution, with $\boxed{c = (b - \cos T)/\sin T}$. If $T = k\pi$ for a positive integer k, the measurement is independent of c: there are infinitely many solutions when $b = (-1)^k$, and none when $b \neq (-1)^k$. These cases exhaust all positive times.

Step 3: Evaluate the two specified experiments. At $(\pi, -1)$ every real c is allowed, giving the full family in Step 1. The figure shows three of these indistinguishable endpoint measurements. At $(\pi/2, 2)$, $c = 2$, so the unique solution is

$$\boxed{X = (\cos t + 2 \sin t,\ -\sin t + 2 \cos t)^T.}$$

Step 4: Quantify sensitivity and interpret it. Replacing b by $b+\delta$ changes c by exactly $\boxed{\Delta c = \delta/\sin T}$. Thus the absolute amplification factor is $1/|\sin T|$, which is unbounded as $T \to \pi$ through nonexceptional times. A fixed nonzero measurement error can then create a large recovered initial-velocity error. At $T = \pi$, that component is unobservable from these two measurements. Each fully specified initial vector still has a unique solution; the difficulty lies in recovering it from partial observations.



All share $x(0) = 1$ and $x(\pi) = -1$; initial velocities differ.

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Question 10

An unknown continuous homogeneous linear system has the two specified solution columns

$$\Phi(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.$$

We ask whether these columns determine a unique system $X' = A(t)X$, rather than assuming that a displayed pair automatically comes from a chosen matrix.

Tasks

1. Find $A(t)$ and verify $\Phi\prime = A\Phi$. Prove uniqueness of the recovered coefficient matrix at every real time.
2. Write the complete solution family and solve the IVP $X(1) = (2, 3)^T$. Verify the recovered initial state.
3. Could both columns instead solve an uncoupled system $x\prime = a(t)x$, $y\prime = b(t)y$ on any nonempty open interval? Give a direct contradiction using their first components.
4. Compute the determinant and trace relevant to this fundamental matrix. Does its constant determinant imply that every solution has constant Euclidean length? Test the second column and explain the conclusion.

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Question 10 – Solution

Strategy. An invertible matrix of solution columns determines its coefficient matrix through its derivative, while its determinant alone does not determine lengths.

Step 1: Recover and verify the matrix. The determinant of Φ is 1, so

$$\Phi^{-1} = \begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}, \qquad \boxed{A = \Phi'\Phi^{-1} = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}.}$$

Multiplication gives $A\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix} = \Phi'$. Any coefficient matrix satisfying the column equations must equal $\Phi'\Phi^{-1}$, proving uniqueness at every real time.

Step 2: Select the constants for the IVP. The full solution is $\boxed{X = \Phi(t)c}$ with constant $c \in \mathbb{R}^2$. Indeed $(\Phi^{-1}X)' = 0$ for any solution, proving completeness. At time 1,

$$c = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} (2,3)^T = (1,1)^T.$$

Thus $\boxed{X = (1+t, 1+t+t^2)^T}$, which equals $(2,3)^T$ at 1. Its derivative follows either directly or from the verified column identity.

Step 3: Rule out an uncoupled alternative. The first component of the first column is the constant 1. Its equation would force $a(t) = 0$ throughout the interval. The first component of the second column is t, whose derivative is 1; with $a = 0$ its proposed equation would instead give derivative 0. This is impossible on any nonempty interval. The coupling is essential.

Step 4: Separate independence from length preservation. Here $\det\Phi = 1$ and $\operatorname{tr} A = -t + t = 0$, consistent with the constant determinant. But the second solution column has squared length

$$\boxed{t^2 + (1+t^2)^2 = 1 + 3t^2 + t^4,}$$

which is not constant. A nonzero determinant ensures independence of the columns; even a constant determinant does not force individual solution lengths to stay fixed. The explicit column supplies a counterexample.

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Question 1

Consider $x' = y$, $y' = -4x$, with initial state $(x, y) = (1, 0)$ at $t = 0$. A phase trajectory is the curve traced by $(x(t), y(t))$, with its direction as time increases. Nullclines are sets where one coordinate derivative is zero.

Tasks

1. Find the equilibrium and both nullclines. Determine the directions of the vector field on each coordinate axis away from the origin.
2. Find a conserved quadratic expression and the phase curve through the initial state. Explain why the curve must be drawn with equal coordinate scales.
3. Find a parametrization solving the IVP, its least positive period, and the direction in which the phase curve is traversed.
4. Locate every horizontal and vertical tangent on this phase curve. Explain why writing it globally as a single-valued function $y(x)$ would lose information.

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Question 1 – Solution

Strategy. Use a conserved quantity to identify the geometric curve, then use the original equations to recover direction and timing.

Step 1: Locate zero derivatives. The x-nullcline is $y = 0$, and the y-nullcline is $x = 0$. Their only common point, $(0, 0)$, is the equilibrium. At $(x, 0)$ the field is $(0, -4x)$: downward for $x > 0$, upward for $x < 0$. At $(0, y)$ it is $(y, 0)$: rightward for $y > 0$, leftward for $y < 0$.

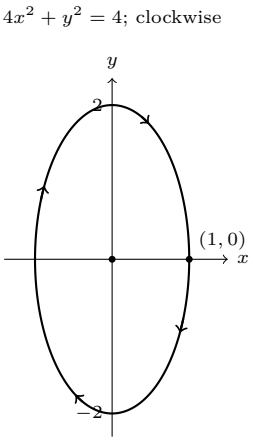
Step 2: Determine the conserved curve. Differentiating gives $(4x^2 + y^2)' = 8xy - 8xy = 0$. The initial state therefore lies on

$$\boxed{4x^2 + y^2 = 4,}$$

an ellipse with horizontal semiaxis 1 and vertical semiaxis 2. Equal coordinate scales preserve this actual geometric aspect ratio.

Step 3: Restore orientation and time. Direct substitution verifies $\boxed{x = \cos(2t), \quad y = -2\sin(2t)}$ and the initial data. The least positive period is π. At the initial point $(1, 0)$ the velocity is $(0, -4)$, so the ellipse is traversed clockwise. The conserved equation alone does not specify this orientation or period.

Step 4: Identify all tangent directions. Horizontal tangents occur where $y' = 0$, namely $(0, 2)$ and $(0, -2)$, where $x' \neq 0$. Vertical tangents occur where $x' = 0$, namely $(1, 0)$ and $(-1, 0)$, where $y' \neq 0$. For $-1 < x < 1$, the same x has two possible values $y = \pm 2\sqrt{1 - x^2}$ on this orbit. A single branch cannot represent the full closed trajectory. The origin is a separate equilibrium, not part of this ellipse. Arrows in the figure indicate increasing time.



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Question 2

For $x' = x$, $y' = -y$, a student divides the equations to obtain $dy/dx = -y/x$ and concludes that every trajectory is a hyperbola $xy = C$. Investigate both the usefulness and the limitations of this calculation.

Tasks

1. Derive the conserved quantity without dividing by x or y. Find all equilibria.
2. Classify the trajectories lying on the coordinate axes, including their time directions. Explain what division can omit.
3. For $xy \neq 0$, identify the separate trajectory branches and their directions in all four quadrants. Explain why an entire two-branch hyperbola is not one trajectory.
4. Solve the IVP $(x(0), y(0)) = (2, 1/2)$ and describe both time limits. Determine all initial states whose solutions tend to the origin as $t \to \infty$.

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Question 2 – Solution

Strategy. Keep zero-coordinate cases before separating variables, and use explicit signs to orient each connected branch.

Step 1: Find a division-free invariant. The product rule gives $(xy)' = x'y + xy' = xy - xy = 0$. Thus $xy = C$ along every solution, including axis solutions. The only equilibrium is $(0, 0)$.

Step 2: Recover the axis trajectories. If $y = 0$, then $x = x_0 e^t$; each nonzero horizontal half-axis is a trajectory directed away from the origin. If $x = 0$, then $y = y_0 e^{-t}$; each nonzero vertical half-axis is directed toward the origin. The origin itself is a constant trajectory. Division by x loses the vertical axis; subsequent division by y can also lose the horizontal axis.

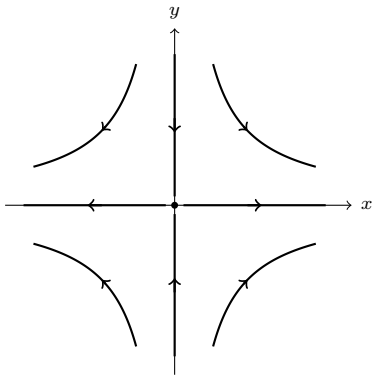
Step 3: Separate the hyperbola branches. For nonzero coordinates, $x = x_0 e^t$ and $y = y_0 e^{-t}$ preserve both signs. In quadrants I, II, III, IV the directions are respectively right/down, left/down, left/up, right/up. Each fixed nonzero C has two disconnected branches; no continuous solution switches between them. The equation $xy = 0$ likewise combines several distinct trajectories rather than describing one orbit through the origin.

Step 4: Apply the initial data and characterize attraction. For the stated IVP,

$$\boxed{x = 2e^t, \qquad y = \tfrac{1}{2}e^{-t}, \qquad xy = 1.}$$

As $t \to \infty$, $x \to \infty$, $y \to 0$; as $t \to -\infty$, $x \to 0$, $y \to \infty$. Neither limit is the origin. For general initial data, convergence to the origin in forward time occurs exactly when $\boxed{x_0 = 0}$, including the zero state. The figure distinguishes the axis directions from representative hyperbola branches.

$xy = \pm 1$ and the four axis half-orbits



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Question 3

Consider the autonomous nonlinear system

$$x' = y - x^2, \qquad y' = 1 - x.$$

A nullcline describes a zero coordinate velocity. It need not be a trajectory. No explicit solution of the full system is required.

Tasks

1. Find the equilibrium and both nullclines.
2. Use inequalities relative to the nullclines to determine the signs of $x\prime, y\prime$ in every region. Give the actual velocity at $(0,0)$, $(0,1)$ and $(1,0)$.
3. Determine tangent directions along each nullcline away from the equilibrium. Prove that neither whole nullcline is an invariant trajectory.
4. For a solution starting at $(0,0)$, calculate $x\prime(0), y\prime(0), x\prime\prime(0), y\prime\prime(0)$. Explain how it can have a vertical tangent while immediately leaving the x-nullcline.

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Question 3 – Solution

Strategy. Read component signs from the two defining expressions, then distinguish instantaneous tangency from remaining on a curve.

Step 1: Locate the nullclines and their intersection. The x-nullcline is $y = x^2$, and the y-nullcline is $x = 1$. Their sole intersection is the equilibrium $\boxed{(1, 1)}$.

Step 2: Determine the regional directions. Above the parabola $x' > 0$; below it $x' < 0$. To the left of $x = 1$, $y' > 0$; to its right, $y' < 0$. Combining these independent signs gives right/up, left/up, right/down or left/down wherever the corresponding inequalities hold. The requested velocities are $F(0, 0) = (0, 1)$, $F(0, 1) = (1, 1)$ and $F(1, 0) = (-1, 0)$.

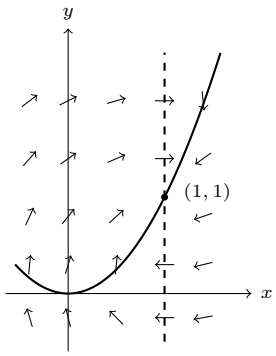
Step 3: Check nullcline tangency correctly. On $y = x^2$, away from $(1, 1)$, the nonzero velocity is vertical. The parabola has finite-slope tangent $(1, 2x)$, so this velocity is not tangent to that parabola. On $x = 1$, away from the equilibrium, the velocity is horizontal and not tangent to the vertical line. Thus neither complete nullcline is invariant. Each is crossed by typical solutions; the equilibrium is the exception where both velocities vanish.

Step 4: Examine departure from the origin. Differentiating the equations gives $x'' = y' - 2xx'$ and $y'' = -x'$. At the initial state,

$$\boxed{x' = 0, \quad y' = 1, \quad x'' = 1, \quad y'' = 0.}$$

Hence $x(t) = t^2/2 + o(t^2)$ and $y(t) = t + o(t^2)$ as $t \to 0$. For small positive t, $y - x^2 = t + o(t) > 0$, so the solution leaves the nullcline and acquires positive horizontal velocity. A vertical tangent at one instant does not mean that x stays constant. The figure shows nullclines and local velocity arrows, not guessed solution curves.

Solid: $y = x^2$
Dashed: $x = 1$; small arrows: velocities



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Question 4

Compare the autonomous vector fields

$$F(x,y) = (-y, x), \quad G(x,y) = (1 + x^2 + y^2)(-y, x), \quad H(x,y) = (y, -x).$$

All three have the origin as an equilibrium. A geometric orbit and the time needed to traverse it are different pieces of information.

Tasks

1. Prove that every nonzero trajectory of each field lies on a circle centered at the origin.
2. On a circle of radius $R > 0$, compute the angular velocity for each field and determine its time direction.
3. Find the least positive period on that circle for each field. Give explicit solutions starting at $(2, 0)$ for F and G.
4. Explain precisely which information is shared by the three phase pictures and which requires arrows or time parametrization. Does a positive scalar multiple of a field reverse its directions?

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Question 4 – Solution

Strategy. Compute radial and angular rates separately; multiplication of a field can change timing without changing its curves.

Step 1: Verify the common invariant circles. For F, $(x^2+y^2)' = 2x(-y) + 2yx = 0$. Multiplication by $1 + x^2 + y^2$ preserves this cancellation for G, and changing the sign preserves it for H. A nonzero initial radius stays fixed and each field is nonzero on that circle.

Step 2: Compute orientation from angular velocity. For $R^2 = x^2 + y^2 > 0$,

$$\theta' = \frac{xy' - yx'}{R^2}.$$

Substitution gives $\boxed{\theta'_F = 1, \quad \theta'_G = 1 + R^2, \quad \theta'_H = -1}$. Thus F, G traverse circles counterclockwise, while H traverses them clockwise.

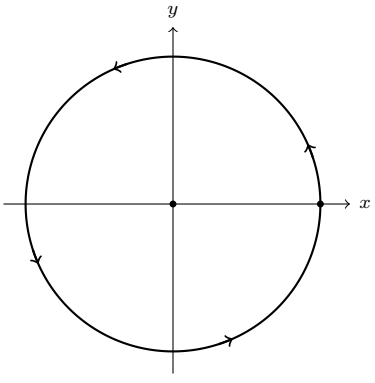
Step 3: Calculate periods and the given trajectories. A full revolution requires angular change of magnitude 2π, so

$$\boxed{T_F = T_H = 2\pi, \qquad T_G = \frac{2\pi}{1+R^2}.}$$

At radius 2, the requested solutions are $X_F = (2\cos t, 2\sin t)$ and $X_G = (2\cos 5t, 2\sin 5t)$. They start at the same state and cover the same circle at different rates. The zero equilibrium is not assigned a least positive revolution period.

Step 4: Distinguish geometry, direction and speed. All three fields have the same nonzero orbit sets and the same equilibrium. Arrows separate H from F, G; the time parametrization separates F from G. A strictly positive multiplier preserves velocity direction wherever the field is nonzero, so it cannot reverse an orbit. A negative multiplier does reverse it. The plotted circle is shared by F, G; its arrows show their common orientation, and the caption records their different periods at radius 2.

Same radius-2 orbit for F, G: counterclockwise



$T_F = 2\pi$, $T_G = 2\pi/5$; H reverses the arrows.

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Question 5

The time-dependent system $x' = 1$, $y' = 2t$ has two proposed solutions

$$U(t) = (t, t^2), \qquad V(t) = (t + 1, t^2), \qquad t \in \mathbb{R}.$$

Investigate whether their projected curves in the (x, y) plane can cross without contradicting uniqueness of the original initial-value problem.

Tasks

1. Verify the candidates, derive the full solution family, and find the equations of their two projected curves.
2. Find the intersection of those curves, the time at which each solution visits it, and the velocity of each at its visit.
3. Decide whether $U(t)$ and $V(t)$ ever agree at the same time. Explain why the usual no-crossing conclusion for a locally Lipschitz autonomous plane field does not apply to these projections.
4. Add time as a state to obtain an autonomous three-dimensional system. Show that the two lifted trajectories do not intersect, including above their projected intersection.

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Question 5 – Solution

Strategy. Record the time attached to each projected point; the vector field here is not determined by the plane coordinates alone.

Step 1: Verify and eliminate time. Both derivatives are $(1, 2t)$, so both candidates solve the system. Integration gives the full family $(x, y) = (t + a, t^2 + b)$. The candidate plane curves are respectively $\boxed{y = x^2}$ and $\boxed{y = (x-1)^2}$, each directed toward increasing x.

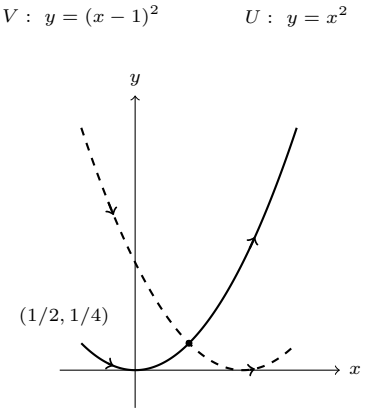
Step 2: Locate the projected crossing and its times. Solving $x^2 = (x-1)^2$ gives $(x, y) = (1/2, 1/4)$. The first solution reaches it at $t = 1/2$, with velocity $(1, 1)$; the second reaches it at $t = -1/2$, with velocity $(1, -1)$. The different tangent directions reflect different evaluation times.

Step 3: Check what uniqueness prohibits. At any common time, $V(t) - U(t) = (1, 0)$, so the full states never agree. The IVP is unique, but the projected point (x, y) alone does not fix $y' = 2t$. There is no single autonomous plane velocity at the projected crossing. Thus the autonomous no-crossing rule is inapplicable. Geometric intersection at different times is not an intersection of these solutions at the same specified time and state.

Step 4: Restore the clock in the state. With new independent variable τ, use

$$\boxed{s' = 1, \quad x' = 1, \quad y' = 2s.}$$

The lifted candidates are $(s, x, y) = (t, t, t^2)$ and $(t, t+1, t^2)$. An intersection of lifted curves would first require equal s, hence equal t, and would then contradict the fixed horizontal difference 1. Above the plane crossing their clock coordinates are $1/2$ and $-1/2$, so the lifted states are distinct. The figure deliberately labels only projected plane curves and shows their separate rightward directions.



Projected curves; intersection occurs at different times.

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Question 6

For $r^2 = x^2 + y^2$, consider

$$x' = x(1 - r^2) - y, \qquad y' = y(1 - r^2) + x.$$

For $r > 0$, use polar coordinates with a continuously followed angle along each trajectory. A periodic orbit is a nonconstant closed trajectory.

Tasks

1. Find every equilibrium. Derive differential equations for r and the angular coordinate.
2. Identify the invariant unit circle, its orientation and least positive period. Determine whether nearby trajectories move toward it from each side.
3. For any initial radius $r_0 > 0$, solve the equation for $u = r^2$ and prove convergence to the unit circle in forward time. State separately what happens at $r_0 = 0$.
4. Does a nonzero solution converge to one particular point on the unit circle? Prove your answer and distinguish approaching an orbit from approaching an equilibrium.

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Question 6 – Solution

Strategy. Separate the radial attraction from the angular motion; approaching a circle does not require stopping on it.

Step 1: Find the radial and angular equations. For a nonzero state, the equilibrium coefficient matrix has determinant $(1 - r^2)^2 + 1 > 0$, so it cannot annihilate that state. The only equilibrium is the origin. Direct differentiation gives

$$\boxed{r' = r(1 - r^2), \qquad \theta' = 1 \quad (r > 0).}$$

The formulas follow from $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.

Step 2: Identify the periodic orbit and radial directions. At $r = 1$, the radius is constant and the angle increases at unit rate. This circle is a counterclockwise periodic orbit of least period 2π. For $0 < r < 1$, $r' > 0$; for $r > 1$, $r' < 0$. Both sides move radially toward the circle rather than away from it.

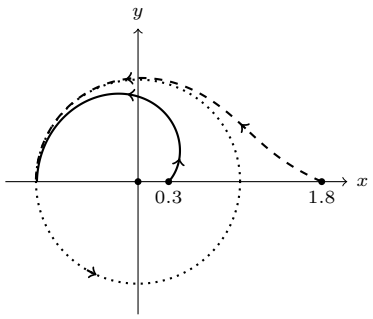
Step 3: Prove the forward radial limit. The squared radius satisfies $u' = 2u(1 - u)$, with solution

$$\boxed{u(t) = \frac{1}{1 + (r_0^{-2} - 1)e^{-2t}} \quad (t \geq 0).}$$

The denominator is positive for every $t \geq 0$, and direct differentiation and evaluation at zero verify the formula. It tends to 1, hence $r \to 1$. With $\theta = \theta_0 + t$, this also supplies a forward-global solution. At $r_0 = 0$ the solution remains at the equilibrium; the reciprocal formula is not used there.

Step 4: Rule out convergence to a single point. Along times $t = 2\pi n$ the states tend to $(\cos\theta_0, \sin\theta_0)$. Along $t = 2\pi n + \pi$ they tend to its negative. These are distinct, so no nonzero solution has a single limiting point. Its distance to the unit circle, $|r - 1|$, tends to zero while it continues rotating. The figure shows the unit orbit and sample inward/outward spirals; all arrows represent increasing time.

Solid: $r_0 = 0.3$; dashed: $r_0 = 1.8$
Dotted: invariant unit orbit



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Question 7

Consider the damped nonlinear system

$$x' = y, \qquad y' = -x - x^3 - cy, \qquad c > 0,$$

with initial state $(1, 0)$. Define $E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$. A level curve of E need not itself be a solution curve.

Tasks

1. Find the equilibrium and derive the exact change of E along a solution.
2. Use the initial energy to prove bounds on both coordinates for every forward time. Explain why the smooth system then has a global forward solution.
3. At the initial state, $E\prime = 0$. Does this mean that the solution is stationary or remains on its original energy level? Use the original equations to decide.
4. Prove that no nonconstant periodic orbit exists, by integrating the energy balance over a hypothetical period. Explain what the enclosing energy curve represents in the phase picture.

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Question 7 – Solution

Strategy. A decreasing energy supplies a containing region and rules out a closed return without requiring an explicit trajectory.

Step 1: Compute the equilibrium and dissipation. Equilibrium requires $y = 0$ and $x + x^3 = 0$, hence only $(0, 0)$. Along solutions,

$$E' = y(-x - x^3 - cy) + (x + x^3)y = \boxed{-cy^2 \le 0}.$$

Thus energy decreases except at instants when $y = 0$.

Step 2: Bound the forward solution. Initially $E = 3/4$. Since all energy terms are nonnegative, $y^2/2 \le 3/4$ gives $|y| \le \sqrt{3/2}$. Also $x^4 + 2x^2 \le 3$, so $(x^2 + 1)^2 \le 4$ and $|x| \le 1$. Therefore

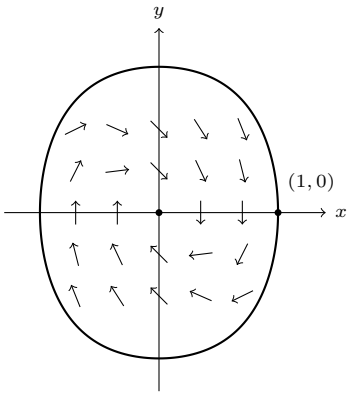
$$\boxed{|x(t)| \le 1, \qquad |y(t)| \le \sqrt{3/2} \quad (t \ge 0).}$$

The polynomial field is locally Lipschitz. These bounds keep the state in a compact region, preventing finite-time escape and allowing continuation for all forward times.

Step 3: Interpret zero instantaneous energy loss. At $(1, 0)$, the velocity is $(0, -2)$, not zero. Moreover $y(t) = -2t + o(t)$ for small positive t, so y becomes nonzero and $E' = -cy^2 < 0$. The solution enters lower energy levels. A horizontal zero of E' at one instant does not make energy constant on an interval or make its level curve a trajectory.

Step 4: Exclude nonconstant periodic motion. If a solution had period $T > 0$, integrating gives $0 = E(T) - E(0) = -c \int_0^T y(t)^2 \, dt$. Continuity and $c > 0$ force $y \equiv 0$, then $x' = 0$ and $x + x^3 = 0$, so the solution is the zero equilibrium. Thus no nonconstant periodic orbit exists. The drawn $E = 3/4$ curve bounds a forward-invariant region; it is not a closed orbit. Interior arrows show velocities for the illustrative choice $c = 1$.

$E = 3/4$ is a containing boundary, not an orbit
Interior velocity arrows use $c = 1$



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Question 8

Start with $x' = -y$, $y' = x$, and introduce the nonlinear change of coordinates $u = x$, $v = y + x^2$. The original unit-circle trajectory is $(x, y) = (\cos t, \sin t)$. Both coordinate planes should be drawn with equal scales when interpreting their geometry.

Tasks

1. Find the inverse coordinate change and derive the autonomous system for (u, v).
2. Find the image of the unit circle as an implicit equation and a time parametrization. Verify the parametrization in the transformed differential equations.
3. Determine the equilibrium, the least positive period of the transformed unit orbit, and its velocity at the image of $(1, 0)$.
4. Can this invertible coordinate change turn the unit circle into a self-intersecting trajectory? Prove your conclusion and explain which geometric features can change even though the time parametrization is preserved.

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Question 8 – Solution

Strategy. Transform both the curve and its velocity by the chain rule; invertibility preserves distinct states even when shapes change.

Step 1: Derive the inverse and new field. The inverse is $x = u$, $y = v - u^2$, valid on the whole plane. Therefore

$$\boxed{u' = u^2 - v, \qquad v' = u + 2u(u^2 - v).}$$

The term $2xx'$ in $v' = y' + 2xx'$ is essential. This is a nonlinear autonomous system, despite the linear original field.

Step 2: Transform and verify the orbit. Substituting into $x^2 + y^2 = 1$ gives

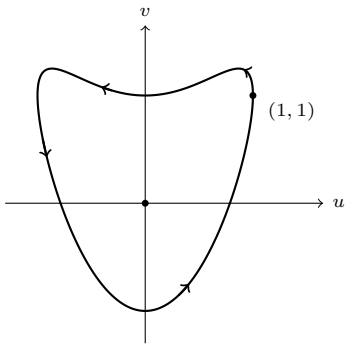
$$\boxed{u^2 + (v - u^2)^2 = 1, \quad (u, v) = (\cos t, \sin t + \cos^2 t).}$$

Here $u' = -\sin t = u^2 - v$, and $v' = \cos t - 2\cos t \sin t = u + 2u(u^2 - v)$. Thus this is the actual transformed trajectory, not merely a guessed level curve.

Step 3: Check equilibrium, period and orientation locally. From $u' = 0$ we have $v = u^2$; then $v' = u$ forces $u = v = 0$. The only equilibrium is the image of the original origin. The transformed orbit has period 2π. A smaller period would, through the inverse map, give a smaller period of $(\cos t, \sin t)$, which is impossible. The image of $(1, 0)$ is $(1, 1)$, with velocity $(0, 1)$: upward.

Step 4: Exclude a self-intersection. If two transformed states agree, the inverse gives equal original states. On the unit circle this occurs only for times differing by an integer multiple of 2π. No new self-intersection can appear within one traversal. The circle becomes a distorted closed curve, so Euclidean lengths, angles and curvature can change. The same time variable and invertible state map preserve the period and prevent distinct states from being identified. The figure plots the transformed orbit in equal u, v scales.

$u^2 + (v - u^2)^2 = 1$; equal coordinate scales



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Question 9

For $x' = 1-x^2$, $y' = -y$, focus on the strip $-1 < x < 1$ and its boundaries. You may use $(\tanh t)' = 1-\tanh^2 t$ and $\operatorname{artanh} x = \frac{1}{2}\ln((1+x)/(1-x))$ on this strip.

Tasks

1. Find the equilibria and verify the trajectory $(x, y) = (\tanh t, 0)$, including its direction and its limits as $t \to \pm\infty$.
2. Explain why the closed segment joining the two equilibria is not a single trajectory, even though it is the closure of the connecting orbit.
3. For initial state $(0, b)$, find a parametrization and eliminate time on the strip. Describe the forward and backward limits, treating $b = 0$ separately.
4. Determine trajectories on $x = -1$ and $x = 1$. Can a nonconstant one reach its equilibrium in finite time? Distinguish approaching an endpoint from attaining it.

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Question 9 – Solution

Strategy. Keep equilibria separate from nearby nonconstant orbits, and retain the open domain when eliminating time.

Step 1: Identify the connecting trajectory. The equilibria are $\boxed{(-1,0),(1,0)}$. The function $(\tanh t, 0)$ solves both equations, has $x' > 0$, and tends to $(-1,0)$ backward in time and $(1,0)$ forward in time. Its orbit is the open horizontal segment $-1 < x < 1$, directed rightward.

Step 2: Separate the orbit from its closure. For every finite t, $-1 < \tanh t < 1$, so neither endpoint is reached. Each endpoint is its own constant trajectory. Uniqueness for the smooth field also prevents a solution from reaching an equilibrium and then leaving it. The closed segment is the union of three trajectories, not one.

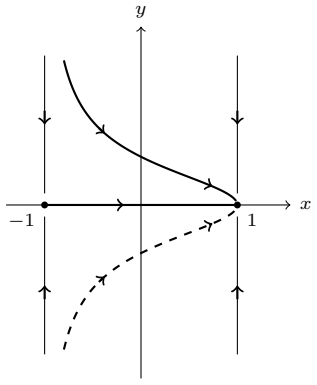
Step 3: Recover the whole one-parameter family. For initial $(0, b)$,

$$\boxed{x = \tanh t, \quad y = be^{-t}, \qquad y = b\sqrt{\frac{1-x}{1+x}}, \quad -1 < x < 1.}$$

The inverse hyperbolic tangent formula proves the eliminated equation. Every member tends to $(1,0)$ as $t \to \infty$. If $b = 0$, the backward limit is $(-1,0)$. If $b \neq 0$, then $x \to -1$ while $y \to +\infty$ for $b > 0$ or $-\infty$ for $b < 0$; there is no finite backward limiting state.

Step 4: Check the invariant boundary lines. On either $x = \pm 1$, the first coordinate stays fixed and $y = y_0 e^{-(t-t_0)}$. For $y_0 \neq 0$, each vertical half-line is an orbit directed toward $(\pm 1, 0)$, reached only as $t \to \infty$. At $y_0 = 0$ the state is already the equilibrium. The figure marks equilibria as separate filled points; connecting arrows and nearby trajectories approach them without asserting finite-time arrival.

$b = 1/2$ (solid), $b = -1/2$ (dashed), $b = 0$
Filled equilibria are not part of the connecting orbit.



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Question 10

On the positive quadrant $x > 0$, $y > 0$, consider

$$x' = x(1 - y), \qquad y' = y(x - 1), \qquad (x(0), y(0)) = (2, 1).$$

You may use the elementary geometric fact that a compact regular level of a smooth strictly convex function in the plane, enclosing its unique minimum, is a simple closed curve.

Tasks

1. Find the positive equilibrium and nullclines. Determine the velocity and direction at the stated initial point.
2. Prove that $H = x - \ln x + y - \ln y$ is conserved, and find its level for the initial data.
3. Use convexity, boundary behavior and the vector field to show that this nonconstant orbit is closed and traversed in a finite period. No period formula is required.
4. Find exact equations for the minimum and maximum of each population on this orbit, identify where those extrema occur, and explain why their maxima are not simultaneous.

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Question 10 – Solution

Strategy. A proper conserved function confines the positive orbit, and nonzero tangential velocity turns its level curve into a periodic trajectory.

Step 1: Read the positive nullclines. In the positive quadrant, $x' = 0$ exactly at $y = 1$, and $y' = 0$ exactly at $x = 1$. The only positive equilibrium is $(1, 1)$. At $(2, 1)$ the velocity is $(0, 1)$, so the orbit initially moves upward. The positive axes are not included in the stated domain.

Step 2: Derive the conserved level. Differentiating yields

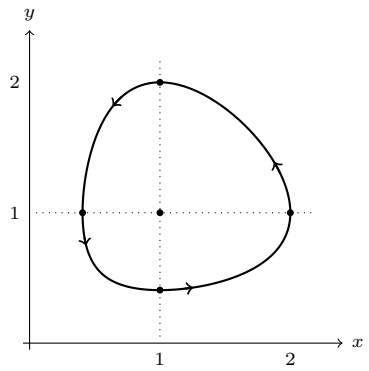
$$H' = (1 - 1/x)x(1 - y) + (1 - 1/y)y(x - 1) = 0.$$

For the stated data, $\boxed{H = 3 - \ln 2}$. The Hessian is diagonal with entries $1/x^2, 1/y^2 > 0$, so H is strictly convex with unique minimum $H(1, 1) = 2$.

Step 3: Justify a closed periodic orbit. The function $z - \ln z$ tends to infinity as $z \downarrow 0$ or $z \to \infty$. Thus this level is compact within the positive quadrant. Its gradient vanishes only at $(1, 1)$, below the stated level; it is regular. The given convex-level fact makes it a simple closed curve. The vector field is tangent by conservation and never zero on this curve. Its continuous speed therefore has a positive minimum, so traversal of the finite-length smooth closed curve takes finite time. Uniqueness then gives a periodic orbit. It is counterclockwise, as its velocity at the rightmost point $(2, 1)$ is upward.

Step 4: Locate and compare the extrema. Extrema of x occur at $y = 1$, hence $x - \ln x = 2 - \ln 2$. This has roots 2 and a unique $a \in (0, 1)$, since $z - \ln z$ decreases on $(0, 1)$ and increases on $(1, \infty)$. By symmetry, $\boxed{x_{\min} = y_{\min} = a, \quad x_{\max} = y_{\max} = 2}$. The four extreme points are $(a, 1), (2, 1), (1, a), (1, 2)$. In particular the two maxima occur at different states, so they are not simultaneous. The plotted level uses the exact conserved equation.

$x - \ln x + y - \ln y = 3 - \ln 2$



Counterclockwise; dotted nullclines $x = 1$, $y = 1$.

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Question 1

Let $X = (x, y)^T$ solve

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

For geometric interpretation use modal coordinates $u = (x + y)/2$ and $v = (x - y)/2$. A stable node means that all nearby states tend to the origin forward in time, with two real negative eigenvalues.

Tasks

1. Find both eigenpairs and solve the IVP. Explain why the two modes give every solution.
2. Eliminate time for this IVP in the (u, v) coordinates, retaining the domain. Translate the result to the original phase plane.
3. Determine the time direction and the limiting tangent line at the origin. Find $\lim_{t\to\infty} y(t)/x(t)$.
4. For arbitrary $X(0) = (p, q)^T$, identify precisely the initial states that approach along the faster eigendirection instead. Classify the origin.

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Question 1 – Solution

Strategy. Decompose into independent eigendirections, then distinguish the dominant slow mode from initial data that cancel it.

Step 1: Resolve the modes. The characteristic polynomial is $(\lambda+1)(\lambda+3)$. Eigenvectors are $v_1 = (1,1)^T$ for -1 and $v_2 = (1,-1)^T$ for -3. They form a basis, so modal coordinates satisfy $u' = -u$, $v' = -3v$. Initially $u = v = 1$, giving

$$\boxed{X = e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix}.}$$

This decoupling also proves completeness for arbitrary initial data.

Step 2: Retain the orbit domain. Here $u = e^{-t} > 0$ and $v = e^{-3t} = u^3$ for all real t. Thus the orbit is

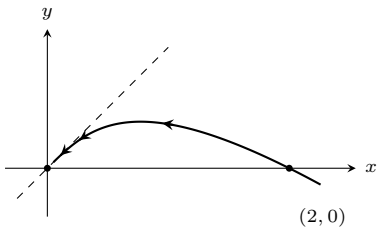
$$\frac{x-y}{2} = \left(\frac{x+y}{2}\right)^3, \qquad x+y>0.$$

The origin is a limiting point, not a point attained at finite time.

Step 3: Read the approach direction. As time increases, u decreases toward zero. Moreover $y/x = (1-e^{-2t})/(1+e^{-2t}) \to 1$. Both the state direction and the velocity direction approach the line $y = x$; indeed $dy/dx = (1-3e^{-2t})/(1+3e^{-2t}) \to 1$. At $(2,0)$ the velocity is $(-4,2)$, consistent with the arrows.

Step 4: Keep the exceptional line. For initial (p,q), the slow and fast coefficients are $(p+q)/2$ and $(p-q)/2$. Every solution tends to zero. A nonzero solution approaches along the fast line $y = -x$ exactly when $p+q = 0$; otherwise its limiting line is $y = x$. The zero state remains fixed and has no approach direction. Both eigenvalues are negative, so the origin is a stable node.

Stable node: slow line $y = x$ (dashed)



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Question 2

Consider

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X.$$

Use the Euclidean distance to the line $y = -x$. In the perturbed initial data below assume $0 < \varepsilon < 1/\sqrt{2}$.

Tasks

1. Find the eigenpairs and general solution. Identify every initial state producing a bounded solution for $t \geq 0$.
2. Explain the saddle geometry and the forward directions on its two eigenlines. Can a nonconstant trajectory cross either eigenline?
3. For $X(0) = (1 + \varepsilon, -1 + \varepsilon)^T$, find when the magnitudes of its stable and unstable modal coefficients first become equal.
4. Find the first $t \geq 0$ at which this solution has distance 1 from $y = -x$. Explain why small initial error can eventually cause a large departure.

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Question 2 – Solution

Strategy. Track stable and unstable coefficients separately; perpendicular distance measures departure from the stable line.

Step 1: Separate decay from growth. Eigenpairs are $3, (1,1)^T$ and $-1, (1,-1)^T$. For data (p, q),

$$X = \frac{p+q}{2}e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{p-q}{2}e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

These independent vectors give the full family. A forward solution is bounded exactly when $p + q = 0$, and then it tends to zero.

Step 2: Interpret the invariant lines. The unstable line $y = x$ points away from the origin; the stable line $y = -x$ points toward it. The opposite signs make the origin a saddle. Each modal coefficient preserves its sign unless identically zero. Thus off-line trajectories cannot cross either eigenline at finite time. Each equilibrium or nonzero half-line is a separate trajectory.

Step 3: Compute modal balance. The perturbed solution is $X = \varepsilon e^{3t}(1,1)^T + e^{-t}(1,-1)^T$. Equality of modal magnitudes requires $\varepsilon e^{3t} = e^{-t}$, hence

$$\boxed{t_{\text{balance}} = \tfrac{1}{4}\ln(1/\varepsilon).}$$

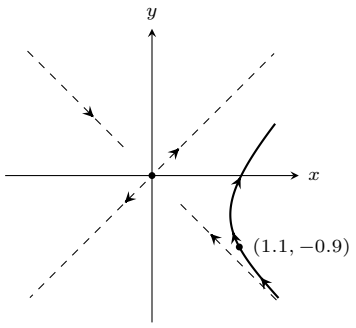
The eigenvectors have equal length, so the same time balances their Euclidean contributions. It is nonnegative under the stated assumption.

Step 4: Measure the delayed departure. Distance to $y = -x$ is $|x+y|/\sqrt{2} = \sqrt{2}\varepsilon e^{3t}$, strictly increasing from a value below one. Its first value one occurs at

$$\boxed{t_{\text{exit}} = \tfrac{1}{3}\ln\left(\frac{1}{\sqrt{2}\varepsilon}\right).}$$

The delay diverges as $\varepsilon \downarrow 0$, but every positive error has a growing unstable component. Only exact cancellation stays on the stable line. The figure illustrates $\varepsilon = 1/10$.

Saddle; solid curve uses $\varepsilon = 1/10$



Dashed eigenline orbits: arrows show increasing time

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Question 3

For

$$X' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

a student claims that positive eigenvalues force both coordinates to increase for all real time. Investigate the entire trajectory, including $t < 0$.

Tasks

1. Find the eigenpairs and the solution, and classify the equilibrium.
2. Eliminate time to find the exact phase curve and its domain. Determine its time direction.
3. Find every stationary point of $x(t)$ and classify it. Determine when $x(t)$ is negative and when it vanishes.
4. Identify the limiting eigenline as $t \to -\infty$ and the limiting direction as $t \to \infty$. Explain what is wrong with the student's claim.

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Question 3 – Solution

Strategy. Growing modal amplitudes can combine with opposite signs; compute coordinate derivatives before making a monotonicity claim.

Step 1: Solve in an eigenbasis. Eigenpairs are $2, (1,0)^T$ and $1, (-1,1)^T$. The initial vector is their sum, so

$$\boxed{x = e^{2t} - e^t, \qquad y = e^t.}$$

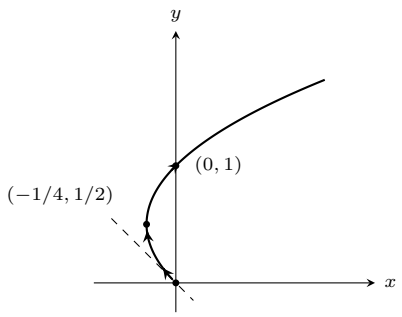
Both eigenvalues are positive, making the origin an unstable node (a source). Every solution tends to zero backward in time, not forward.

Step 2: Recover the geometric branch. Since $y = e^t > 0$, the phase curve is $\boxed{x = y^2 - y,\ y > 0}$. It is traversed toward increasing y. The limiting origin is excluded; the finite-time point with $x = 0$ is $(0, 1)$. The curve's leftmost point has a vertical tangent, not an equilibrium.

Step 3: Locate coordinate reversal. We have $x' = e^t(2e^t - 1)$, zero only at $t = -\ln 2$. It is negative before this time and positive after it. Therefore the unique global minimum is $\boxed{x = -1/4}$, at $y = 1/2$. For $t < 0$, $x = e^t(e^t - 1) < 0$; it vanishes only at $t = 0$ and is positive for $t > 0$. The limit $x \to 0$ as $t \to -\infty$ is not another zero.

Step 4: Distinguish modal and coordinate behavior. Backward, $X/e^t \to (-1,1)^T$, so the approach line is $y = -x$. Forward, $X/e^{2t} \to (1,0)^T$, giving the positive horizontal direction. Positive eigenvalues describe growth of modes with fixed coefficients; they do not force each original coordinate to increase. Here the two contributions to x have opposite signs, producing a genuine reversal. The faster eigenmode eventually dominates because its coefficient is nonzero.

$x = y^2 - y$, $y > 0$; upward time direction



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Question 4

Let

$$X' = \begin{pmatrix} -1 & 6 \\ 0 & -2 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Write $\|X\| = \sqrt{x^2 + y^2}$. Investigate whether negative eigenvalues alone force this ordinary distance from the origin to decrease at every time.

Tasks

1. Find both eigenpairs and solve the IVP. Show that all initial states decay as $t \to \infty$.
2. Find the maximum of $x(t)$ on $t \geq 0$. Evaluate $\|X\|$ there and compare it with $\|X(0)\|$.
3. Compute $(\|X\|^2)'$ for a general state. Give a state at which it is positive, and determine its sign at the stated initial point.
4. Construct a positive quadratic quantity using $u = x + 6y$, $v = y$ that strictly decreases along every nonzero solution. Explain how this reconciles decay with transient growth.

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Question 4 – Solution

Strategy. Euclidean distance mixes nonorthogonal modes; a quadratic quantity in modal coordinates separates their decay.

Step 1: Decouple the eigenmodes. The eigenpairs are $-1, (1,0)^T$ and $-2, (-6,1)^T$. Since $(0,1)^T = 6(1,0)^T + (-6,1)^T$,

$$\boxed{x = 6(e^{-t} - e^{-2t}), \qquad y = e^{-2t}.}$$

Every initial vector has this two-mode form with suitable coefficients, so all solutions tend to zero.

Step 2: Exhibit transient amplification. The derivative $x' = 6(-e^{-t} + 2e^{-2t})$ changes from positive to negative at $t = \ln 2$. Thus $x_{\max} = 3/2$, while $y = 1/4$ there. Consequently

$$\boxed{\|X(\ln 2)\| = \sqrt{37}/4 > 1 = \|X(0)\|}.$$

This is the maximum of x, not a claim about the time of maximum norm.

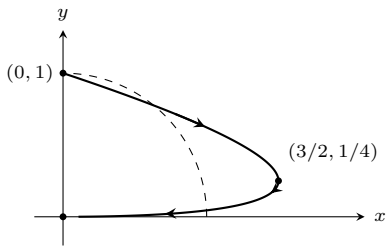
Step 3: Test instantaneous norm growth. Direct differentiation gives

$$(\|X\|^2)' = -2x^2 + 12xy - 4y^2.$$

At $(1,1)$ it equals $6 > 0$, so the norm can increase. At the given initial state $(0,1)$ it equals -4: this trajectory first moves closer, but later exceeds its initial distance. Negative eigenvalues do not make this quadratic expression negative at every state.

Step 4: Use a norm adapted to the modes. The invertible coordinates satisfy $u' = -u$, $v' = -2v$. Thus $E = (x + 6y)^2 + y^2 = u^2 + v^2$ is positive except at zero and $\boxed{E' = -2u^2 - 4v^2 < 0}$ for every nonzero state. The Euclidean circles and the level ellipses of E measure different sizes. Decrease of the latter is compatible with temporary Euclidean growth, and the explicit modes still establish eventual decay.

Solid: IVP; dashed: unit-circle arc



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Question 5

For a real parameter $-1 < a < 1$, consider

$$X' = A_a X, \qquad A_a = \frac{1}{2}\begin{pmatrix} a-2 & a+2 \\ a+2 & a-2 \end{pmatrix}.$$

Treat the zero-eigenvalue case explicitly; do not assume that every equilibrium is isolated.

Tasks

1. Find the eigenvalues, an eigenbasis independent of a, and the solution with arbitrary initial state $(p, q)^T$.
2. Classify the origin for $a < 0$ and $a > 0$, and identify the initial states that converge to it in each case.
3. At $a = 0$, find every equilibrium and the limiting state of every solution. Is the origin attracting all nearby states?
4. For initial state $(2, 0)^T$, compare $\lim_{a\uparrow 0} \lim_{t\to\infty} X_a(t)$ with $\lim_{t\to\infty} \lim_{a\uparrow 0} X_a(t)$. Explain the difference.

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Question 5 – Solution

Strategy. Keep the parameter-dependent mode visible; its decay disappears at the zero eigenvalue.

Step 1: Use fixed eigendirections. The vectors $(1,1)^T$ and $(1,-1)^T$ have eigenvalues a and -2. They remain independent, and their eigenvalues are distinct throughout the stated parameter interval. Setting $m=(p+q)/2$, $d=(p-q)/2$ gives

$$\boxed{X_a(t)=me^{at}\begin{pmatrix}1\\1\end{pmatrix}+de^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}.}$$

This formula satisfies the system and the initial data for every allowed a.

Step 2: Classify the nonzero-parameter cases. For $a<0$, both modes decay, so the origin is a stable node and all initial states converge to it. For $a>0$, the origin is a saddle; convergence occurs exactly when $m=0$, that is, on $y=-x$. The $y=x$ mode then grows. There are no other equilibria in these two cases because $\det A_a=-2a\neq 0$.

Step 3: Resolve the equilibrium line. At $a=0$, every point on $y=x$ is an equilibrium and $X_0(t)\to(m,m)^T$. This is the orthogonal projection of the initial state onto that line. The origin does not attract all nearby states: nearby nonzero points of the equilibrium line remain fixed. Nevertheless it is stable in the distance sense, since the orthogonal constant and decaying modes never increase the Euclidean norm.

Step 4: Compare the orders of limiting. For the specified data, $m=d=1$. At each fixed $a<0$ the time limit is zero; hence the first iterated limit is $\boxed{(0,0)^T}$. At each fixed finite t, taking $a\uparrow 0$ instead gives $(1,1)^T+e^{-2t}(1,-1)^T$, whose time limit is $\boxed{(1,1)^T}$. The decay e^{at} becomes arbitrarily slow as a approaches zero from below. It is not uniform over such parameters on an unbounded time interval, so these two operations need not commute.

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Question 6

An unknown constant real 2×2 matrix A is known to satisfy

$$A\begin{pmatrix}1\\2\end{pmatrix} = -\begin{pmatrix}1\\2\end{pmatrix}, \qquad A\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix}.$$

These are exact eigenvector observations, not merely measured slopes.

Tasks

1. Reconstruct A and justify its uniqueness.
2. Solve $X\prime = AX$ with $X(0) = (3, 3)^T$. Verify the initial state and derivative directly.
3. Find the entire set of initial states that decay forward in time. Determine the limiting direction for the stated IVP.
4. Would knowing only $\operatorname{tr} A = 1$ and $\det A = -2$ determine that decaying line? Give a concrete comparison matrix and explain the effect of rescaling either supplied eigenvector.

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Question 6 – Solution

Strategy. An eigenbasis fixes the linear map, while eigenvalues alone do not fix its directions.

Step 1: Reconstruct the operator. Let $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \operatorname{diag}(-1, 2)$. Since $\det P = -3 \neq 0$,

$$\boxed{A = PDP^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.}$$

A linear map is uniquely determined by its values on a basis, so there is no other matrix satisfying both given observations.

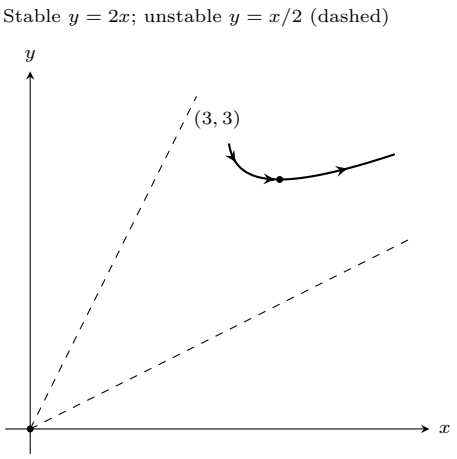
Step 2: Solve and check the IVP. The initial vector is $(1, 2)^T + (2, 1)^T$, giving

$$\boxed{X = e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.}$$

At zero this is $(3, 3)^T$, and its derivative is $(-1, -2)^T + (4, 2)^T = (3, 0)^T = A(3, 3)^T$. More generally each mode differentiates to its eigenvalue times itself, verifying the ODE for every time.

Step 3: Separate stable data from generic growth. For initial (p, q), the unstable coefficient is $(2p - q)/3$. Decay therefore occurs exactly on $q = 2p$, the line spanned by $(1, 2)^T$. For the stated IVP the unstable coefficient is one, so $X/e^{2t} \to (2, 1)^T$ and $X/\|X\| \to (2, 1)^T/\sqrt{5}$. The drawn eigenlines are invariant; the off-line curve cannot cross them.

Step 4: Distinguish rates from directions. The matrix $B = \operatorname{diag}(-1, 2)$ has the same trace and determinant, hence the same eigenvalues, but its decaying line is the horizontal axis. Trace and determinant alone do not determine the stable line. Multiplying either supplied eigenvector by any nonzero scalar changes its coordinate coefficient but leaves PDP^{-1} unchanged: the diagonal rescaling commutes with D. Zero rescaling is not allowed.



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Question 7

Consider the three-dimensional homogeneous system

$$X' = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A solution is called bounded forward (backward) if its Euclidean norm is bounded on $[0, \infty)$ (respectively $(-\infty, 0]$).

Tasks

1. Find an eigenbasis and use it to write the complete solution for arbitrary (p, q, r).
2. Find exactly the initial states giving bounded forward solutions. Within that set, distinguish the slow and fast nonzero decay directions.
3. Find exactly the initial states giving bounded backward solutions, and describe their forward behavior.
4. Determine every solution bounded on the entire real line. Explain why cancellation between distinct growing modes cannot produce additional cases.

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Question 7 – Solution

Strategy. Express the initial state in a basis of three real modes, then read boundedness from individual modal coordinates.

Step 1: Resolve all three components. The triangular matrix has distinct eigenvalues $-2, -1, 1$. Corresponding vectors are $v_1 = (1, 0, 0)^T$, $v_2 = (1, 1, 0)^T$, $v_3 = (1, 1, 1)^T$, forming an invertible triangular basis. Solving for coefficients yields

$$\boxed{X = (p - q)e^{-2t}v_1 + (q - r)e^{-t}v_2 + re^t v_3.}$$

Each term satisfies the ODE; their sum at zero is $(p, q, r)^T$. Equivalently the independent coordinates $x - y, y - z, z$ solve scalar equations with rates $-2, -1, 1$, proving completeness.

Step 2: Find the stable plane and its exception. Forward boundedness requires and is implied by $r = 0$; this is the plane $z = 0$ of initial states. If $q \neq 0$, then $X/e^{-t} \to qv_2$, so the slow limiting line is $\text{span}(v_2)$. If $q = 0$ and $p \neq 0$, only the faster mode $pe^{-2t}v_1$ remains. The zero state is separate. All states in this plane tend to zero.

Step 3: Find the backward-bounded line. As $t \to -\infty$, the negative-rate exponentials grow. Their coefficients must vanish: $p - q = 0$ and $q - r = 0$. Thus the exact set is the line $\boxed{p = q = r}$. On it $X = re^t v_3$, tending to zero backward and, for $r \neq 0$, growing without bound forward. At $r = 0$ it is the equilibrium.

Step 4: Intersect the two conditions. A solution bounded for all real time lies both in $r = 0$ and in $p = q = r$, so $\boxed{X \equiv 0}$ is the only one. This is not merely an inference from the norm of a sum. The inverse basis map is fixed and linear; bounded X forces all its modal coordinates $x - y, y - z, z$ to be bounded. Each is one scalar exponential, so a forbidden growing coefficient cannot be hidden by cancellation with another mode.

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Question 8

Let

$$X' = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} X, \qquad x(0) = 1, \qquad y(T) = b, \quad T > 0.$$

Only the initial value $q = y(0)$ is unknown. These conditions concern different times; they are not a complete initial state at one time.

Tasks

1. Find the real eigenpairs and write the solution in terms of q.
2. Derive the equation that q must satisfy. Identify every positive time at which its coefficient vanishes.
3. At the exceptional time, classify all values of b by whether there are zero, one, or infinitely many solutions. Explain why uniqueness for an IVP is not contradicted.
4. For other times, give q explicitly and its change under a measurement error $b \mapsto b + \delta$. Describe what happens near the exceptional time.

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Question 8 – Solution

Strategy. Two real exponential modes can cancel in an endpoint observation even when the full state remains uniquely determined by its initial data.

Step 1: Find the two modes. The characteristic polynomial is $\lambda^2 - 1$. Eigenvectors are $(1,1)^T$ for 1 and $(1,2)^T$ for -1. The data $(1,q)^T$ give coefficients $2-q$ and $q-1$, so

$$\boxed{X = (2-q)e^t\begin{pmatrix}1\\1\end{pmatrix} + (q-1)e^{-t}\begin{pmatrix}1\\2\end{pmatrix}.}$$

Step 2: Locate the singular observation time. The endpoint condition becomes

$$b = 2e^T - 2e^{-T} + q(2e^{-T} - e^T).$$

The coefficient of q vanishes exactly when $e^{2T} = 2$, hence $\boxed{T_* = \frac{1}{2}\ln 2}$, the unique positive exceptional time.

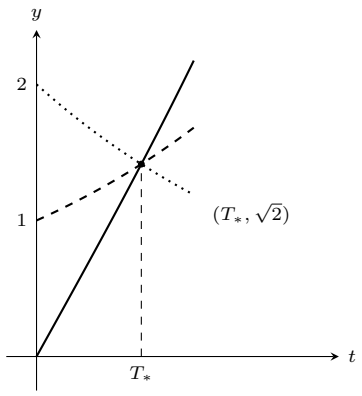
Step 3: Classify compatibility at that time. At T_*, the equation reduces to $b = \sqrt{2}$. If $b = \sqrt{2}$, every real q gives a solution; otherwise none does. There is never exactly one at this time. Distinct q specify distinct initial states. In fact $x(T_*) = (3-q)/\sqrt{2}$, so the full states remain different even though their second components agree. Each individual IVP still has its unique solution. The figure shows three such second components.

Step 4: Quantify sensitivity away from the exception. For $T \neq T_*$ there is exactly one solution, with

$$\boxed{q = \frac{b - 2e^T + 2e^{-T}}{2e^{-T} - e^T}, \qquad \Delta q = \frac{\delta}{2e^{-T} - e^T}.}$$

The denominator tends to zero as $T \to T_*$, so the absolute amplification factor diverges. This is a poorly conditioned recovery of missing initial data, not loss of uniqueness for a fully specified IVP. For a fixed nonzero δ, the recovered initial error can become arbitrarily large.

Solid $q = 0$; dashed $q = 1$; dotted $q = 2$



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Question 9

Let

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}, \qquad P = A + 2I, \qquad Q = -(A + I).$$

Here a projection means a matrix R with $R^2 = R$; it need not be an orthogonal projection. Develop a solution formula without choosing normalized eigenvectors.

Tasks

1. Find the eigenvalues and verify $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$.
2. Identify the range and kernel of each projection. Are these projections orthogonal in the Euclidean plane?
3. Prove that $F(t) = e^{-t}P + e^{-2t}Q$ satisfies $F\prime = AF$ and $F(0) = I$. Use it to solve the IVP $X(0) = (1, 0)^T$.
4. Prove $F(t + s) = F(t)F(s)$ and find its inverse. Explain why these identities hold for every pair of real times here.

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Question 9 – Solution

Strategy. Polynomial projections isolate the two real eigenspaces and make the evolution law a scalar calculation on each one.

Step 1: Verify complementary projections. The characteristic polynomial is $(\lambda+1)(\lambda+2)$, and direct multiplication gives $A^2+3A+2I=0$. Here $P=\begin{pmatrix}-2 & 2\\ -3 & 3\end{pmatrix}$ and $Q=\begin{pmatrix}3 & -2\\ 3 & -2\end{pmatrix}$. Thus $P+Q=I$, $P^2=P$, $Q^2=Q$, and $PQ=QP=0$. For example $PQ=-(A+2I)(A+I)=0$; the other identities follow by multiplication or from $Q=I-P$.

Step 2: Identify the oblique directions. The range of P is $\operatorname{span}(2,3)^T$, the -1 eigenspace; its kernel is $\operatorname{span}(1,1)^T$, the -2 eigenspace. For Q the range and kernel are interchanged. These two directions have nonzero dot product 5, so neither projection is orthogonal. The matrices are also visibly nonsymmetric.

Step 3: Construct and verify evolution. The same polynomial identity gives $AP=-P$, $AQ=-2Q$. Consequently $F'=AF$ and $F(0)=P+Q=I$. For arbitrary initial vector X_0, $F(t)X_0$ solves the IVP; constant linear-system uniqueness gives the complete solution. In particular

$$\boxed{X=e^{-t}\begin{pmatrix}-2\\ -3\end{pmatrix}+e^{-2t}\begin{pmatrix}3\\ 3\end{pmatrix}.}$$

Its initial state is $(1,0)^T$ and its initial derivative $(-4,-3)^T=A(1,0)^T$.

Step 4: Multiply by modes. All mixed products vanish, so

$$F(t)F(s)=e^{-(t+s)}P+e^{-2(t+s)}Q=F(t+s).$$

Taking $s=-t$ yields $\boxed{F(t)^{-1}=F(-t)=e^{t}P+e^{2t}Q}$. No finite real time makes either scalar exponential vanish. This is the normalized evolution of one constant matrix; the fixed complementary projections justify the multiplication rule for every $s,t\in\mathbb{R}$. No orthogonality or eigenvector normalization was used.

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Question 10

An unknown real matrix has eigenvectors $(1,1)^T$ and $(1,-1)^T$ with distinct negative eigenvalues λ and μ, respectively. For the solution with $X(0) = (2,0)^T$, define $u = (x+y)/2$, $v = (x-y)/2$. The observed entire phase curve is $v = u^2$, $u > 0$, with direction toward the origin. A clock measurement gives $u(\ln 2) = 1/2$.

Tasks

1. Determine what the phase curve alone implies about λ and μ. Explain why its shape and arrows do not determine their absolute values.
2. Use the clock measurement to recover both eigenvalues and the matrix.
3. Write and verify the full IVP solution, including the observed curve, its domain, and the clock measurement.
4. Find the limiting tangent at the origin and the maximum of $y(t)$ on $t \geq 0$. Explain how uniformly speeding up this system would affect the curve, maximum value, and time of that maximum.

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Question 10 – Solution

Strategy. Geometry determines a ratio of modal rates; one elapsed-time observation fixes their common scale.

Step 1: Extract a rate ratio. Initially $u = v = 1$, so $u = e^{\lambda t}$ and $v = e^{\mu t}$. The identity $v = u^2$ for every time implies $\mu = 2\lambda$. Any $\lambda < 0$ with this ratio gives the same oriented orbit as t ranges over the real line. Thus the curve and its arrows alone leave one positive speed factor undetermined.

Step 2: Calibrate the clock. The measurement yields $e^{\lambda \ln 2} = 1/2$, so $\lambda = -1$ and $\mu = -2$. Using the specified eigenbasis reconstructs

$$\boxed{A = \frac{1}{2}\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.}$$

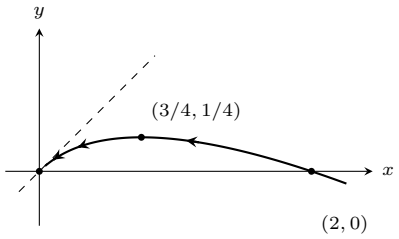
Step 3: Verify the reconstructed trajectory. The solution is

$$\boxed{x = e^{-t} + e^{-2t}, \qquad y = e^{-t} - e^{-2t}.}$$

Differentiating and multiplying by the displayed matrix agree component by component, and the initial state is $(2, 0)^T$. Moreover $u = e^{-t} > 0$, $v = e^{-2t} = u^2$, and $u(\ln 2) = 1/2$. Every $u > 0$ occurs at a finite real time; the limiting origin is excluded from the orbit.

Step 4: Separate geometric and temporal conclusions. As $t \to \infty$, $dy/dx = (1-2e^{-t})/(1+2e^{-t}) \to 1$, giving the tangent line $y = x$. On $t \geq 0$, $y' = e^{-t}(2e^{-t} - 1)$ changes from positive to negative at $t = \ln 2$. Hence $\boxed{y_{\max} = 1/4}$, attained at $x = 3/4$. Replacing A by kA, $k > 0$, changes the solution to $X(kt)$: the oriented curve and maximum value remain the same, but the maximum occurs at $(\ln 2)/k$. Except for $k = 1$, it fails the given clock measurement.

Recovered orbit; dashed tangent $y = x$



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Question 1

Let

$$X' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use a complex eigenvector to construct real solutions. Angles in the original (x, y) plane are measured counterclockwise from the positive x-axis.

Tasks

1. Find the eigenvalues and an eigenvector for the eigenvalue with positive imaginary part. Separate its exponential solution into real and imaginary parts.
2. Prove that the resulting two real solutions are independent and solve the IVP.
3. Find the radius, an unwrapped angle, and the first positive time at which this trajectory reaches the positive x-axis.
4. Find the radius multiplier after one full revolution. Is the nonzero solution periodic? Explain why the conjugate eigenpair does not supply two additional independent real solutions.

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Question 1 – Solution

Strategy. Take real and imaginary parts with their signs intact, then use polar coordinates to separate decay from rotation.

Step 1: Extract real solutions. The eigenvalues are $-1 \pm 2i$. For $\lambda = -1 + 2i$, choose $v = (1, -i)^T$, which satisfies $Av = \lambda v$. Writing $e^{\lambda t}v = U + iV$ gives

$$U = e^{-t}\begin{pmatrix}\cos 2t \\ \sin 2t\end{pmatrix}, \qquad V = e^{-t}\begin{pmatrix}\sin 2t \\ -\cos 2t\end{pmatrix}.$$

Because A is real, both parts solve the original real system.

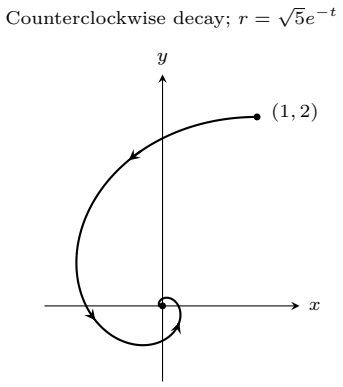
Step 2: Resolve the initial state. The determinant of the columns (U, V) is $-e^{-2t} \neq 0$. At zero they are $(1, 0)^T$ and $(0, -1)^T$, so $X = U - 2V$:

$$\boxed{X = e^{-t}\begin{pmatrix}\cos 2t - 2\sin 2t \\ \sin 2t + 2\cos 2t\end{pmatrix}.}$$

The nonzero determinant and linear-system uniqueness give the complete real family as arbitrary real combinations of U, V.

Step 3: Locate the first positive-axis crossing. Let $\theta_0 = \arctan 2$, which lies in $(0, \pi/2)$. Here $r = \sqrt{5}e^{-t}$ and a continuous angle is $\theta = \theta_0 + 2t$. Rotation is counterclockwise. The first positive time on the positive x-axis requires $\theta = 2\pi$, hence $\boxed{t_* = (2\pi - \arctan 2)/2}$. The earlier negative-axis crossing is not the requested event.

Step 4: Separate revolution from repetition. One revolution takes π, reducing the radius by $e^{-\pi}$. Since the radius strictly decreases, no nonzero solution is periodic; the origin is a stable spiral. Conjugating the exponential solution gives $U - iV$, whose parts are $U, -V$, already in the same real span. Thus a conjugate pair supplies two real degrees of freedom, not four. The figure shows a finite forward segment of the stated IVP.



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Question 2

Consider

$$x' = 4y, \qquad y' = -x, \qquad (x(0), y(0)) = (2, 0).$$

A student sees eigenvalues $\pm 2i$ and claims that the trajectory must be a circle rotating with constant Euclidean angular speed 2.

Tasks

1. Find the eigenvalues and solve the IVP in real form.
2. Find a conserved positive quadratic expression and determine the complete phase curve. Locate its intercepts.
3. Find the least positive period and the direction of traversal. Identify coordinates in which the motion is a uniform circular rotation.
4. Compute the original Euclidean angular velocity $\theta\prime = (xy\prime - yx\prime)/(x^2 + y^2)$. Find its range on the orbit and evaluate the student's claim.

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Question 2 – Solution

Strategy. A linear coordinate change can turn a circle into an ellipse while preserving elapsed time and period.

Step 1: Solve the oscillatory IVP. The characteristic polynomial is λ^2+4, giving $\pm 2i$. From $x''=-4x$ and $x'(0)=0$,

$$\boxed{x=2\cos 2t, \qquad y=-\sin 2t.}$$

Differentiation verifies both first-order equations and the initial data.

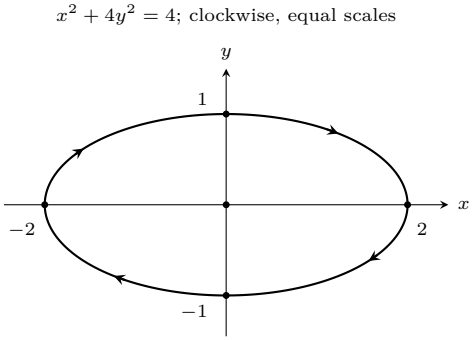
Step 2: Recover the invariant ellipse. The derivative of $H=x^2+4y^2$ is $8xy-8xy=0$. Thus the orbit is $\boxed{x^2+4y^2=4}$, with intercepts $(\pm 2,0)$ and $(0,\pm 1)$. The parametrization covers the whole ellipse. Its unequal semiaxes must remain unequal in an equal-scale drawing; it is not a circle in the original Euclidean coordinates.

Step 3: Determine period and orientation. The least positive period is π. At $(2,0)$ the velocity is $(0,-2)$, so traversal is clockwise. The coordinates $u=x$, $v=2y$ satisfy $u'=2v$, $v'=-2u$ and $u^2+v^2=4$. In that transformed plane the angular velocity is constantly -2. The invertible transformation preserves the least period.

Step 4: Compute the actual angular speed. In the original plane,

$$\theta'=-\frac{x^2+4y^2}{x^2+y^2}=-\frac{4}{x^2+y^2}.$$

On the ellipse, $1\le x^2+y^2\le 4$, so $\boxed{-4\le\theta'\le -1}$. It equals -1 at the horizontal intercepts and -4 at the vertical ones. Thus imaginary eigenvalues determine a temporal frequency, but do not force a Euclidean circle or constant Euclidean angular speed. The average signed angular velocity over a complete traversal is nevertheless -2.



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Question 3

Let

$$X' = \begin{pmatrix} -1 & 8 \\ -1/2 & -1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Use $\|X\| = \sqrt{x^2 + y^2}$. A stable spiral means that all nearby states approach the origin with continuing rotation, not necessarily that their ordinary distance decreases at every instant.

Tasks

1. Find the eigenvalues and solve the IVP. Determine its rotation direction.
2. Evaluate the norm at $t = \pi/4$ and compare it with its initial value. Compute $(\|X\|^2)\prime$ at a general state.
3. Find a positive quadratic expression $H = x^2 + cy^2$ with $H\prime = -2H$. Use it to establish decay of every solution.
4. For the given IVP and $0 < \rho < 1$, give an explicit time after which the state is guaranteed to remain in $\|X\| \le \rho$. Explain why this guarantee need not be the first entry time.

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Question 3 – Solution

Strategy. Use coordinates that balance the oscillatory coupling, then compare the adapted quadratic size with Euclidean distance.

Step 1: Solve the damped rotation. The characteristic polynomial is $(\lambda+1)^2+4$, so the rates are $-1\pm 2i$. Direct substitution verifies

$$\boxed{x = 4e^{-t}\sin 2t, \qquad y = e^{-t}\cos 2t.}$$

For any nonzero state, $xy' - yx' = -x^2/2 - 8y^2 < 0$; rotation is clockwise. The transformed coordinates $(x, 4y)$ rotate uniformly with decay.

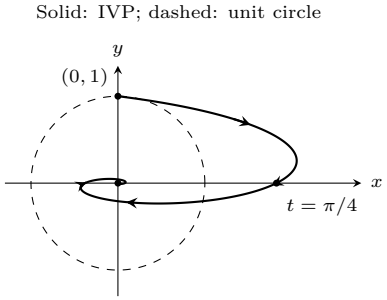
Step 2: Exhibit temporary growth. Initially the norm is one, whereas $\boxed{\|X(\pi/4)\| = 4e^{-\pi/4} > 1}$. More generally,

$$(\|X\|^2)' = -2x^2 + 15xy - 2y^2,$$

which equals $11 > 0$ at $(1, 1)$. Its value at the given initial state is -2. Hence even this trajectory can first shrink and subsequently exceed its initial distance. The negative real parts guarantee eventual decay, not a sign for this derivative at every state.

Step 3: Find a decreasing quadratic size. Differentiating $x^2 + cy^2$ cancels the mixed term when $c = 16$. Then $\boxed{H = x^2 + 16y^2, \quad H' = -2H}$, so $H(t) = H(0)e^{-2t}$. Since $\|X\|^2 \le H$, every initial state tends to zero. Together with the strict rotation sign this identifies a stable spiral, even though the drawn orbit can move outside the unit circle.

Step 4: Give a permanent-entry bound. For the specified data, $H(0) = 16$ and $\|X(t)\| \le 4e^{-t}$. Thus every $t \ge \boxed{\ln(4/\rho)}$ satisfies $\|X(t)\| \le \rho$. This bound controls all later times. It need not be the first entry: H overestimates the ordinary squared norm except where $y = 0$. Transient contraction and expansion can occur before the guaranteed time; no claim of a sharp first-entry time is made.



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Question 4

For $a \in \mathbb{R}$, consider

$$X' = \begin{pmatrix} a & -3 \\ 3 & a \end{pmatrix} X.$$

For $r > 0$, define the positive-axis return map $P_a(r)$ as the radius the solution starting at $(r, 0)$ has on its first later return to the positive x-axis. A return to this ray need not be a return to the same point.

Tasks

1. Find the eigenvalues and the solution from an arbitrary nonzero initial state in polar form.
2. Classify the origin for $a < 0$, $a = 0$, and $a > 0$. Determine exactly when a nonzero solution is periodic.
3. Find the first return time, the map P_a, and all its fixed radii in $r > 0$. Explain their relation to periodic trajectories.
4. An experiment finds $P_a(r) = r/2$ for one nonzero radius. Recover a, and find the radius after n complete turns for every integer $n \geq 0$.

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Question 4 – Solution

Strategy. The real part sets radial growth and the imaginary part sets turning time; a ray return becomes periodic only when the radius is restored.

Step 1: Read radial and angular evolution. The eigenvalues are $a \pm 3i$. For initial radius $r_0 > 0$ and angle θ_0, the solution has

$$\boxed{r(t) = r_0 e^{at}, \qquad \theta(t) = \theta_0 + 3t.}$$

Equivalently $X(t) = e^{at} R(3t) X(0)$, where $R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. The only equilibrium is zero because the determinant is $a^2 + 9 > 0$.

Step 2: Classify all parameter cases. For $a < 0$ the origin is a stable spiral; for $a > 0$ it is an unstable spiral. At $a = 0$ it is a center: every nonzero state follows a circle with least period $2\pi/3$. For $a \neq 0$, the radius is strictly monotone, so no nonzero solution is periodic. Zero remains constant and has no least positive period.

Step 3: Compute the return map. The first return to the positive ray takes $T = 2\pi/3$, independent of r and a. Consequently

$$\boxed{P_a(r) = e^{2\pi a/3} r.}$$

For $r > 0$, the fixed-point equation holds for every radius if $a = 0$ and for none if $a \neq 0$. These fixed radii are exactly the nonzero periodic trajectories. Returning to an angle alone is insufficient.

Step 4: Recover damping from one revolution. The measurement gives $e^{2\pi a/3} = 1/2$, hence

$$\boxed{a = -\frac{3 \ln 2}{2\pi}, \qquad r_n = r_0\, 2^{-n}.}$$

The coefficient a is unique because the real exponential is one-to-one. The same multiplier applies to every radius by linearity; a single nonzero measurement therefore determines the damping rate in this specified family.

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Question 5

An unknown real matrix A has eigenvalues $-1 \pm 3i$. Every solution of $X' = AX$ satisfies

$$\frac{d}{dt}(x^2 + 4y^2) = -2(x^2 + 4y^2).$$

At the state $(2, 0)$, the velocity has a negative second component. Use all three observations to reconstruct the system.

Tasks

1. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and extract the restrictions imposed by the quadratic identity at arbitrary states.
2. Use the eigenvalues to find every matrix consistent with that identity before applying the direction observation.
3. Select the correct matrix and solve the IVP starting at $(2, 0)$. Verify its quadratic size and rotation direction.
4. Find the time and contraction factor for one complete turn. Explain precisely which ambiguity would remain if the direction observation were omitted.

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Question 5 – Solution

Strategy. An identity valid for all initial states determines quadratic coefficients; the remaining sign is geometric orientation.

Step 1: Compare polynomial coefficients. Differentiation gives $2ax^2+2(b+4c)xy+8dy^2 = -2x^2-8y^2$. Every point can be an initial state, so coefficients must agree:

$$\boxed{a = d = -1, \qquad b = -4c.}$$

Step 2: Use the spectral determinant. The eigenvalues require trace -2 and determinant 10. The trace is already correct, while $\det A = 1 - bc = 1 + 4c^2 = 10$ gives $c = \pm 3/2$. Thus precisely two matrices remain:

$$\begin{pmatrix} -1 & 6 \\ -3/2 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & -6 \\ 3/2 & -1 \end{pmatrix}.$$

Both have the required quadratic decay and the required eigenvalues.

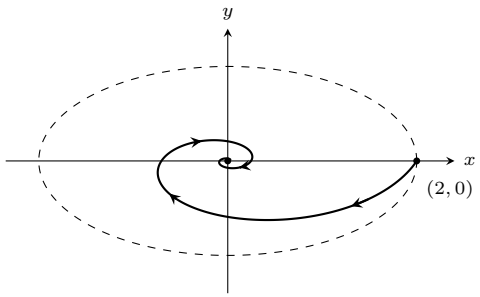
Step 3: Use orientation and solve. At $(2, 0)$, $y' = 2c < 0$, so select $c = -3/2$, $b = 6$. The IVP is

$$\boxed{x = 2e^{-t}\cos 3t, \qquad y = -e^{-t}\sin 3t.}$$

It satisfies the selected equations, initial data and $x^2 + 4y^2 = 4e^{-2t}$. Moreover $xy' - yx' = -3x^2/2 - 6y^2 < 0$ off zero, confirming clockwise rotation. The shrinking ellipses are size contours, not individual orbits.

Step 4: Separate timing, contraction and sign. The coordinates $(x, 2y)$ make one clockwise revolution in $2\pi/3$. An invertible positive diagonal rescaling preserves a full turn, so this is also the time to return to the initial positive ray. The whole state then equals $e^{-2\pi/3}X(0)$; this is not a periodic return. Without the direction observation, the second matrix would remain equally possible, with counterclockwise rotation and the same contraction factor. The unordered conjugate eigenvalues do not determine orientation.

Clockwise spiral; dashed initial-size contour



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Question 6

Let

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 0 \end{pmatrix}, \qquad B = A - I.$$

Find the normalized evolution matrix for $X' = AX$ without selecting or normalizing complex eigenvectors. A normalized evolution matrix F satisfies $F' = AF$ and $F(0) = I$.

Tasks

1. Find the eigenvalues and prove a simple identity for B^2.
2. Construct a real formula for $F(t)$ using that identity, and verify both defining conditions.
3. Solve the IVP $X(0) = (1, 1)^T$. Compute $F(\pi)$ and explain why a complete oscillation does not produce a periodic nonzero solution.
4. Prove the multiplication law $F(t + s) = F(t)F(s)$, find $F(t)^{-1}$, and compute its determinant. Does the determinant alone determine how every vector length changes?

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Question 6 – Solution

Strategy. Remove the real part of the spectrum; the remaining matrix acts algebraically like multiplication by an imaginary number.

Step 1: Isolate the oscillatory matrix. The characteristic polynomial is $\lambda^2 - 2\lambda + 5$, with roots $1 \pm 2i$. Here $B = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$ and direct multiplication gives $\boxed{B^2 = -4I}$.

Step 2: Construct a real evolution formula. Set

$$\boxed{F(t) = e^t \left(I \cos 2t + \tfrac{1}{2} B \sin 2t \right).}$$

Its value at zero is I. Differentiating the bracket gives $-2I \sin 2t + B \cos 2t$, which equals B times that bracket because $B^2 = -4I$. The product rule therefore yields $F' = (I + B)F = AF$. Uniqueness verifies that this formula evolves every initial state.

Step 3: Solve and inspect one oscillation. Since $B(1,1)^T = (-4,0)^T$,

$$\boxed{X = e^t \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix}.}$$

The initial state is $(1,1)^T$ and initial derivative $(-3,1)^T = A(1,1)^T$. Also $F(\pi) = e^\pi I$: a full trigonometric cycle multiplies the state by e^π. More generally the periodic bracket and its inverse are bounded, so every nonzero solution grows without bound forward in time. Thus there is no nonzero periodic solution.

Step 4: Verify the evolution algebra. Multiplying two brackets and using $B^2 = -4I$ gives the sine and cosine addition formulas, hence $F(t+s) = F(t)F(s)$. Therefore $\boxed{F(t)^{-1} = F(-t)}$. For scalars c, s, $\det(cI + sB/2) = c^2 + s^2$, so $\boxed{\det F(t) = e^{2t}}$. This controls area, not every vector length: at the state $(1,1)$ the squared-length derivative is $2(1,1) \cdot (-3,1) = -4 < 0$, despite increasing area. Shape distortion and eventual growth can coexist.

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Question 7

A damped oscillator is written as a state system

$$x' = y, \qquad y' = -5x - 2y, \qquad (x(0), y(0)) = (1, 0).$$

Here x is displacement and y is velocity. A turning point means a time when the displacement has a local maximum or minimum.

Tasks

1. Find the complex eigenvalues and the real IVP solution.
2. Find the first positive zero of the displacement and the velocity there, retaining its sign.
3. Find the first positive turning time and classify that turning point. Determine the ratio of the magnitudes of consecutive displacement extrema.
4. Compute the derivative of the mechanical energy $E = (y^2 + 5x^2)/2$. Explain why $E\prime(0) = 0$ does not make the motion constant or periodic.

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Question 7 – Solution

Strategy. The state components have different phases: displacement zeros and velocity zeros mark different events on the spiral.

Step 1: Solve the state IVP. The characteristic polynomial is $\lambda^2 + 2\lambda + 5$, giving $-1 \pm 2i$. The initial conditions yield

$$\boxed{x = e^{-t}(\cos 2t + \tfrac{1}{2}\sin 2t), \qquad y = -\tfrac{5}{2}e^{-t}\sin 2t.}$$

Differentiation gives $x' = y$ and $y' = -5x - 2y$, including $y'(0) = -5$.

Step 2: Locate the first displacement crossing. Write $\alpha = \arctan 2 \in (0, \pi/2)$. The first positive root of $\cos 2t + \frac{1}{2}\sin 2t = 0$ occurs at

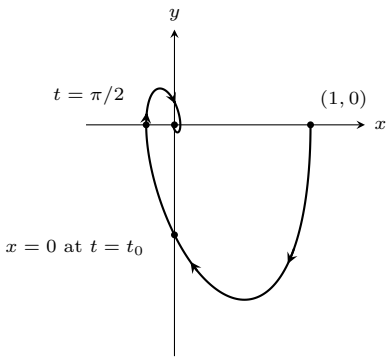
$$\boxed{t_0 = (\pi - \alpha)/2, \qquad y(t_0) = -\sqrt{5}e^{-t_0}.}$$

Indeed $\sin(\pi - \alpha) = 2/\sqrt{5}$ and $\cos(\pi - \alpha) = -1/\sqrt{5}$. The negative velocity shows a crossing from positive to negative displacement, not a turning point.

Step 3: Separate the turning events. Velocity vanishes exactly at $t = n\pi/2$, for integers n. The first positive turning time is $\pi/2$, with displacement $-e^{-\pi/2}$. There $x'' = y' = 5e^{-\pi/2} > 0$, so it is a minimum. Successive extrema have $x(n\pi/2) = (-1)^n e^{-n\pi/2}$; their absolute values have ratio $\boxed{e^{-\pi/2}}$ for consecutive forward extrema. Successive positive maxima are one full oscillation apart and have ratio $e^{-\pi}$.

Step 4: Interpret dissipation at a zero derivative. The energy derivative is $E' = yy' + 5xx' = -2y^2 \le 0$. At zero, $y = 0$ so $E' = 0$, but the acceleration is -5 and the state immediately changes. On any nontrivial time interval this solution has $y \neq 0$ somewhere, so its energy strictly drops over that interval. A nonconstant periodic solution would instead require zero total energy loss over a period, forcing $y \equiv 0$ and then $x \equiv 0$, a contradiction. The phase arrows distinguish the crossing and turning points.

Displacement–velocity phase curve



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Question 8

For a nonzero real frequency ω, consider

$$X' = \begin{pmatrix} -1 & -\omega \\ \omega & -1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An instrument records only selected times. At every integer $n \geq 0$ it reports $X(n) = e^{-n}(1, 0)^T$. Assume the measurements are exact.

Tasks

1. Solve the IVP and determine every nonzero frequency consistent with all the integer-time measurements.
2. Can these measurements determine rotation direction or the number of revolutions per unit time? Explain why the observed points do not imply a straight-line continuous trajectory.
3. An additional measurement is $X(1/4) = e^{-1/4}(0, 1)^T$. Find every frequency still allowed, including negative frequencies.
4. If one also knows $0 < \omega < 4\pi$, recover the frequency uniquely. Exhibit two different frequencies that fit both sets of measurements when that extra bound is absent.

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Question 8 – Solution

Strategy. Sampling observes angles only modulo complete revolutions; additional information is needed to recover the continuous frequency.

Step 1: Solve and impose integer samples. The eigenvalues are $-1 \pm i\omega$, and

$$\boxed{X(t) = e^{-t}\begin{pmatrix}\cos \omega t \\ \sin \omega t\end{pmatrix}.}$$

The measurement at $n = 1$ requires $\omega = 2\pi k$, $k \in \mathbb{Z}$. Every such value satisfies all integer samples; the nonzero-frequency assumption excludes $k = 0$.

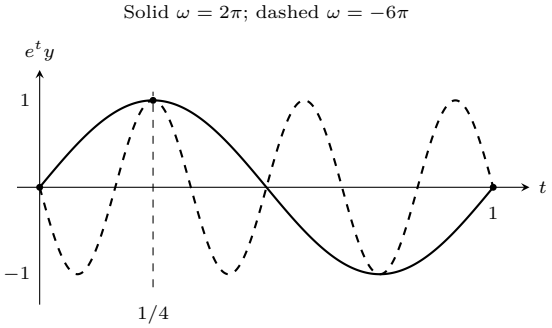
Step 2: Identify what the samples conceal. The sign of k gives counterclockwise or clockwise rotation, and $|k|$ gives revolutions per unit time. Neither is fixed by the integer samples. For every permitted nonzero frequency the continuous angle changes and the radius decays, tracing a spiral. The sampled points all lie on one ray because whole turns occur between them, not because that ray is an invariant trajectory.

Step 3: Use the quarter-time observation. The additional angle condition is $\omega/4 = \pi/2 + 2\pi m$. Thus

$$\boxed{\omega = 2\pi(1 + 4m), \qquad m \in \mathbb{Z}.}$$

These values already satisfy the integer constraints. Negative possibilities remain, for example $m = -1$ gives $\omega = -6\pi$. No zero frequency occurs in this family.

Step 4: Use the external frequency bound. Under $0 < \omega < 4\pi$, only $m = 0$ remains, giving $\boxed{\omega = 2\pi}$. Without that bound, 2π and -6π both fit all stated observations, despite opposite directions and different speeds. The figure plots $e^t y(t)$ for these two solutions: they agree at the recorded times but differ between them. The vertical variable is the rescaled component, not the radius or a phase-plane coordinate.



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Question 9

Consider

$$X' = \begin{pmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use Euclidean boundedness on $[0, \infty)$ or $(-\infty, 0]$ as appropriate. For a nonzero state its direction means the normalized vector $X/\|X\|$.

Tasks

1. Find the eigenvalues and the complete real solution for arbitrary initial data.
2. Classify the initial states giving bounded solutions forward in time, backward in time, or for all real time.
3. When $r \neq 0$, determine the limiting direction as $t \to \infty$ and justify why the rotating components do not affect it.
4. When $r = 0$ but $(p, q) \neq (0, 0)$, determine the state limit and decide whether its direction has a limit. Give a sequence-based justification.

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Question 9 – Solution

Strategy. Decouple the rotating plane from the real exponential axis, and distinguish convergence of a state from convergence of its direction.

Step 1: Separate the blocks. The eigenvalues are $-1 \pm 2i$ and 1. The complete solution is

$$\boxed{\begin{aligned} x &= e^{-t}(p\cos 2t - q\sin 2t), \\ y &= e^{-t}(p\sin 2t + q\cos 2t), \\ z &= re^{t}. \end{aligned}}$$

The planar rotation is invertible at every time, so the arbitrary three initial components are represented without loss.

Step 2: Classify boundedness in both time directions. Exactly $r = 0$ gives forward boundedness. For backward boundedness, $x^2 + y^2 = (p^2 + q^2)e^{-2t}$ forces $p = q = 0$; the remaining z component then tends to zero backward. The intersection of this axis with the forward-bounded plane is zero, so $\boxed{X \equiv 0}$ is the only solution bounded on the entire real line. Rotation cannot cancel the planar radius.

Step 3: Find the direction with an unstable component. If $r \neq 0$, the ratio of planar radius to $|z|$ is $\sqrt{p^2 + q^2}\, e^{-2t}/|r| \to 0$. Consequently $\boxed{X/\|X\| \to (0, 0, \operatorname{sgn} r)^T}$. The state itself grows without bound; its direction can still converge because the real growing mode dominates both decaying components.

Step 4: Test direction on the stable plane. If $r = 0$ and $s = \sqrt{p^2 + q^2} > 0$, then $X \to 0$, but its normalized planar state is $R(2t)(p, q)^T/s$. At $t_n = n\pi$ it equals $(p, q)^T/s$; at $u_n = n\pi + \pi/2$ it equals its negative. Both sequences tend to infinity and the two unit vectors are distinct, so no directional limit exists. At $n\pi + \pi/4$ the direction is perpendicular to its value at $n\pi$, so even the unoriented line of approach has no limit. A convergent state need not behave like a nonzero real dominant eigenmode.

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Question 10

For $\gamma > 0$, consider the four-dimensional system

$$x_1' = -x_2, \quad x_2' = x_1, \qquad x_3' = -\gamma x_4, \quad x_4' = \gamma x_3.$$

A state is periodic if there is $T > 0$ with $X(t+T) = X(t)$ for all real t. A constant solution has no least positive period.

Tasks

1. Find the eigenvalues and solve the IVP $X(0) = (1, 0, 1, 0)^T$. Describe its projections onto the two coordinate planes.
2. For $\gamma = \sqrt{2}$, prove that this full state is not periodic, even though both planar projections are periodic.
3. Still with $\gamma = \sqrt{2}$, classify exactly which arbitrary initial states give periodic solutions, and give their least periods when nonconstant.
4. Replace γ by $3/2$. Find the least period for the stated IVP and explain why purely imaginary eigenvalues alone do not guarantee a common period in higher dimensions.

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Question 10 – Solution

Strategy. A full state repeats only when every active rotating block repeats at the same time.

Step 1: Solve each oscillatory block. The eigenvalues are $\pm i$ and $\pm i\gamma$. For the stated initial data,

$$\boxed{X(t) = (\cos t, \sin t, \cos \gamma t, \sin \gamma t)^T.}$$

The two projections are unit circles with respective least periods 2π and $2\pi/\gamma$. Their closure as plane curves alone does not settle repetition of the combined state.

Step 2: Exclude a common period for irrational frequency ratio. At $\gamma = \sqrt{2}$, any period must satisfy $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$ for positive integers m, n. This would imply $\sqrt{2} = n/m$, impossible. Hence the four-dimensional state is not periodic. Its norm nevertheless remains $\sqrt{2}$, so the obstruction is incompatible timing, not growth or damping.

Step 3: Classify all initial states in the irrational case. Each initially zero two-component block remains zero. A nonzero first block repeats exactly at integer multiples of 2π; a nonzero second block repeats exactly at integer multiples of $2\pi/\sqrt{2}$. Thus precisely the states with at least one block zero give periodic solutions. If only the first block is nonzero, the least period is 2π; if only the second is nonzero, it is $\sqrt{2}\pi$. If both vanish, the solution is constant. If both are nonzero, the same irrationality argument rules out every positive period, regardless of their initial phases.

Step 4: Recover a common period for rational frequency ratio. With $\gamma = 3/2$ and both blocks active, $T = 2\pi m$ must also satisfy $3m/2 \in \mathbb{Z}$. The least positive choice is $m = 2$, giving $\boxed{T = 4\pi}$. In these independent rotation systems all states are bounded, but common periodicity requires commensurate active frequencies. Two planar centers therefore need not combine into a periodic full state; purely imaginary eigenvalues alone do not provide the missing frequency ratio.

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Question 1

Compare the systems $X' = AX$ and $X' = BX$, where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Both have the characteristic polynomial $(\lambda + 2)^2$.

Tasks

1. Find the eigenspace of each matrix and explain why the repeated characteristic root does not determine the number of independent eigenvectors.
2. Write the complete real solution of each system for initial state $(p, q)^T$. Verify any term containing t.
3. For the B system starting at $(0, 1)$, eliminate time with the correct domain. Find the maximum of $x(t)$ on $t \geq 0$.
4. Compare the two systems' phase trajectories and forward limits. Does every repeated eigenvalue require a polynomial factor in every solution?

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Question 1 – Solution

Strategy. The eigenspace dimension decides whether a repeated root already supplies a full basis or needs another solution.

Step 1: Distinguish the eigenspaces. For A, every nonzero vector is a -2 eigenvector, so the eigenspace has dimension two. For B, $(B+2I)(x,y)^T = (y,0)^T$, so its eigenspace is $\operatorname{span}(1,0)^T$, of dimension one. Algebraic multiplicity is two in both cases; the geometric multiplicities differ.

Step 2: Recover both complete families. For initial (p,q),

$$\boxed{X_A = e^{-2t}\begin{pmatrix} p \\ q \end{pmatrix}, \qquad X_B = e^{-2t}\begin{pmatrix} p+qt \\ q \end{pmatrix}.}$$

For B, first solve $y' = -2y$, then $(e^{2t}x)' = q$. This derives every solution, and direct differentiation verifies the qt term. In particular $te^{-2t}(1,0)^T$ alone is not the missing solution; the accompanying $e^{-2t}(0,1)^T$ is essential.

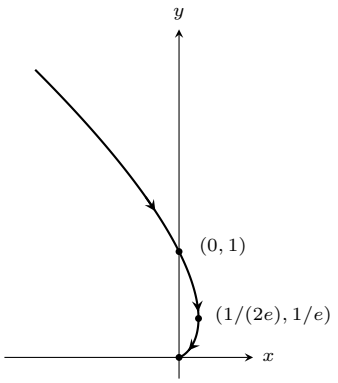
Step 3: Analyze the selected orbit. For B with $(p,q) = (0,1)$, $x = te^{-2t}$ and $y = e^{-2t} > 0$. Eliminating t gives

$$\boxed{x = -\tfrac{1}{2}y\ln y, \quad y > 0.}$$

Forward time decreases y. On $t \geq 0$, $x' = e^{-2t}(1-2t)$ changes sign at $t = 1/2$, giving $x_{\max} = 1/(2e)$ at $y = e^{-1}$.

Step 4: Interpret multiplicity and geometry. Every nonzero A trajectory is a straight ray toward zero. For B, only the eigenline $y = 0$ gives straight rays; otherwise $x/y = p/q + t$ changes, giving curved trajectories. All solutions of both systems decay. A repeated root needs no polynomial factor when there is a full eigenbasis; even in the defective system, eigenline initial data have $q = 0$ and no t term. The figure shows the selected B orbit, not a generic ray.

$x = -\frac{1}{2}y\ln y$; arrows show increasing time



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Question 2

Let

$$X' = NX, \qquad N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Both eigenvalues are zero. A matrix is nilpotent if some positive power of it is zero; investigate the effect on the actual motion.

Tasks

1. Compute N^2, find the entire equilibrium set, and determine its dimension.
2. Find the normalized evolution matrix and the solution from arbitrary $(p, q)^T$. Verify the evolution law directly.
3. For initial data outside the equilibrium line, find the phase trajectory and its time direction. Can it reach that line at a finite or infinite forward time?
4. Classify all forward-bounded solutions. Decide whether the origin is stable in the sense that sufficiently small initial states remain small for all $t \geq 0$.

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Question 2 – Solution

Strategy. A zero eigenvalue can preserve an equilibrium line while a nilpotent part creates unbounded drift parallel to it.

Step 1: Identify the nilpotent part and equilibria. Direct multiplication gives $N^2 = 0$ but $N \neq 0$. The equation $NX = 0$ requires $x = y$, so the equilibria form the one-dimensional line $L = \{(s, s) : s \in \mathbb{R}\}$. The double zero eigenvalue has only one independent eigendirection.

Step 2: Construct the complete flow. The matrix $F(t) = I + tN$ has $F(0) = I$ and $F' = N = NF$, since $N^2 = 0$. Thus

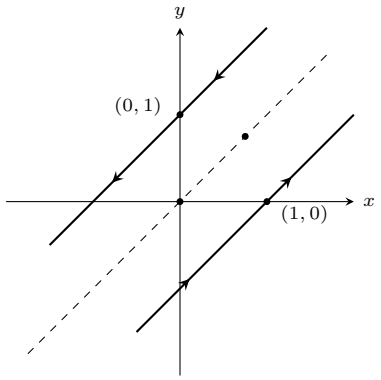
$$\boxed{X(t) = \begin{pmatrix} p \\ q \end{pmatrix} + t(p - q)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$$

Also $(I + tN)(I + sN) = I + (t + s)N$, verifying the evolution law and $F(t)^{-1} = F(-t)$. All real initial states and all real times are allowed.

Step 3: Find the drifting straight lines. Put $d = p - q$. If $d \neq 0$, then $x - y = d$ for every time and the velocity is the constant vector $d(1, 1)^T$. Thus the full orbit is the entire line $x - y = d$, traversed up/right for $d > 0$ and down/left for $d < 0$. Its distance to L stays $|d|/\sqrt{2} > 0$, so it neither reaches nor approaches the equilibrium line. If $d = 0$, the orbit is one fixed point, not the entire line L.

Step 4: Test boundedness and stability. Exactly $p = q$ gives a forward-bounded solution; all other solutions drift without bound. The origin is unstable: initial $(\varepsilon, 0)$ can be arbitrarily small, yet evolves to $(\varepsilon(1 + t), \varepsilon t)$, eventually leaving any fixed neighborhood. Zero real parts alone do not ensure boundedness or stability when a nonzero nilpotent part is present. The dashed line in the figure consists of separate equilibria.

Solid: drifting orbits; dashed: equilibrium line



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Question 3

Consider

$$X' = AX, \qquad A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For a repeated eigenvalue λ, a length-two chain means vectors v, w with $(A-\lambda I)v = 0$ and $(A-\lambda I)w = v \neq 0$.

Tasks

1. Find the eigenvalue, its eigenspace, and a chain with $v = (1, 1)^T$.
2. Derive and verify two independent real solutions using this chain. Explain why $te^{\lambda t}v$ alone fails.
3. Solve the IVP and verify the initial state and derivative.
4. Find every generalized vector w compatible with the fixed v. Explain how changing w changes the coefficients without changing the IVP solution.

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Question 3 – Solution

Strategy. The generalized vector cancels the extra derivative introduced by the polynomial factor.

Step 1: Build a chain explicitly. The characteristic polynomial is $(\lambda+2)^2$. Let $N = A+2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; then $N^2 = 0$. Its kernel is $\text{span}(1,1)^T$. With $v = (1,1)^T$, choose $w = (1,0)^T$, since $Nw = v$. The two vectors are independent.

Step 2: Verify the missing solution. Set $U = e^{-2t}v$ and $V = e^{-2t}(w + tv)$. Using $Nv = 0$, $Nw = v$, differentiation gives $U' = AU$ and $V' = AV$. Their determinant is $e^{-4t}\det(v, w) = -e^{-4t} \neq 0$, so they form a complete real basis. For $Z = te^{-2t}v$, however, $Z' - AZ = e^{-2t}v \neq 0$: the polynomial term by itself is not a solution.

Step 3: Determine the IVP coefficients. Since $(2,1)^T = v + w$, the required solution is $U + V$:

$$\boxed{X = e^{-2t}\begin{pmatrix} 2+t \\ 1+t \end{pmatrix}.}$$

At zero it is $(2,1)^T$, and its derivative is $(-3,-1)^T$, agreeing with $A(2,1)^T$. The mode calculation verifies the equation for every time, not just at the initial point.

Step 4: Account for chain freedom. The equation $Nw = v$ means $w_1 - w_2 = 1$, so every choice is $w_c = (1+c, c)^T = w + cv$, $c \in \mathbb{R}$. The new second solution is $V_c = V + cU$. Hence the same IVP becomes $\boxed{X = (1-c)U + V_c}$. In general a combination $aU + bV$ becomes $(a - bc)U + bV_c$. The basis changes, but its span and every physical solution remain the same. Scaling or shifting a chain must be accompanied by the corresponding coefficient change.

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Question 4

For $\varepsilon > 0$, let

$$X_\varepsilon' = \begin{pmatrix} -1 & 1 \\ 0 & -1-\varepsilon \end{pmatrix} X_\varepsilon, \qquad X_\varepsilon(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Study the limit as the two distinct eigenvalues merge. Let X_0 denote the solution of the same IVP with $\varepsilon = 0$.

Tasks

1. Solve the IVP for $\varepsilon > 0$ by using the distinct real modes.
2. Find the limit at each fixed time as $\varepsilon \downarrow 0$ and verify the resulting repeated-root solution.
3. For $0 \le t \le T$, prove bounds on $|x_\varepsilon - x_0|$ and $|y_\varepsilon - y_0|$ that tend uniformly to zero as $\varepsilon \downarrow 0$. You may use $0 \le 1 - e^{-u} \le u$ for $u \ge 0$.
4. At $t = 1/\varepsilon$, compare x_ε / x_0 with 1. Explain how coefficients that diverge and a relative error that persists can coexist with the fixed-time limit.

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Question 4 – Solution

Strategy. Keep the difference of nearby exponentials together; an integral representation exposes the limit without subtracting divergent pieces.

Step 1: Solve before the roots merge. Eigenvectors are $(1,0)^T$ for -1 and $(1,-\varepsilon)^T$ for $-1-\varepsilon$. Decomposition of $(0,1)^T$ gives coefficients $1/\varepsilon, -1/\varepsilon$, hence

$$\boxed{x_\varepsilon = e^{-t}\frac{1-e^{-\varepsilon t}}{\varepsilon}, \qquad y_\varepsilon = e^{-(1+\varepsilon)t}.}$$

Step 2: Pass to the repeated-root solution. Since $(1-e^{-\varepsilon t})/\varepsilon = \int_0^t e^{-\varepsilon s}\,ds$, its limit at each fixed real t is t. Thus $\boxed{X_0 = e^{-t}(t,1)^T}$. Direct differentiation gives $x_0' = -x_0 + y_0$, $y_0' = -y_0$ and the stated initial data. The polynomial factor is the limit of the combined modes.

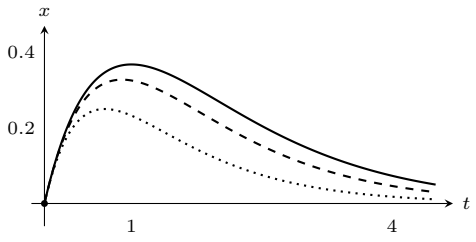
Step 3: Prove uniform finite-interval control. For $0 \le t \le T$, the supplied inequality yields

$$0 \le x_0 - x_\varepsilon = e^{-t}\int_0^t (1-e^{-\varepsilon s})\,ds \le \tfrac{1}{2}\varepsilon t^2 e^{-t} \le \tfrac{1}{2}\varepsilon T^2.$$

Similarly $0 \le y_0 - y_\varepsilon = e^{-t}(1-e^{-\varepsilon t}) \le \varepsilon t e^{-t} \le \varepsilon T$. Both bounds tend uniformly to zero on every fixed finite interval.

Step 4: Separate relative and absolute effects. At $t = 1/\varepsilon$, $\boxed{x_\varepsilon/x_0 = 1-e^{-1}}$, so the relative discrepancy is e^{-1}, independent of ε. This time moves to infinity, and $x_0 = \varepsilon^{-1}e^{-1/\varepsilon} \to 0$, so the absolute discrepancy still vanishes there. The separate modal coefficients diverge, but their exponential contributions cancel to a finite limit. Neither divergent coefficients nor this moving-time relative error contradicts the proved finite-interval convergence.

Solid $\varepsilon = 0$; dashed $1/4$; dotted 1



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Question 5

Let

$$X' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X, \qquad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Examine the whole trajectory for $-\infty < t < \infty$, including its approach to the origin backward in time.

Tasks

1. Find the repeated eigenvalue and eigenspace, and solve the IVP.
2. Eliminate time and retain the full orbit domain. Find the unique minimum of $x(t)$ and all its finite zeros.
3. Find the limiting normalized directions as $t \to -\infty$ and $t \to \infty$. Does approaching an eigenline imply that the trajectory lies on it?
4. Determine whether this orbit ever reaches the origin or repeats a state. Explain why its coordinate reversal is not an oscillation.

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Question 5 – Solution

Strategy. The exponential controls overall growth, while the polynomial factor can reverse one coordinate and dominate the limiting direction.

Step 1: Recover the defective solution. The double eigenvalue is 1, with eigenspace $y = 0$. Solving $y' = y$ and then $(e^{-t}x)' = 3$ gives

$$\boxed{x = 3te^t, \qquad y = e^t.}$$

The factor 3 is part of the coupling and cannot be omitted from the generalized mode. Both components satisfy the stated initial data.

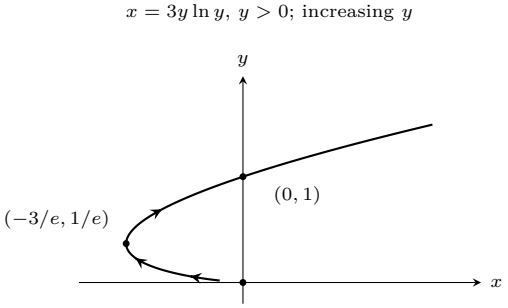
Step 2: Find the curve and minimum. Since $y > 0$, the orbit is $\boxed{x = 3y \ln y, \quad y > 0}$, traversed toward increasing y. Its derivative $x' = 3e^t(t + 1)$ vanishes only at $t = -1$, changing from negative to positive. The global minimum is $x = -3/e$ at $y = 1/e$. The only finite zero of x is $t = 0$; its backward limit zero is not another attained zero.

Step 3: Find the two limiting directions. The normalized state is

$$\frac{X}{\|X\|} = \frac{(3t, 1)^T}{\sqrt{9t^2 + 1}}.$$

It tends to $(-1, 0)^T$ backward and $(1, 0)^T$ forward. Although the limiting line is the eigenline $y = 0$, every finite-time state has $y > 0$ and lies off that line. The origin is an unstable defective node; this IVP tends to it only as $t \to -\infty$.

Step 4: Distinguish reversal from repetition. The origin is never reached at finite time because $e^t > 0$. Also $y = e^t$ is strictly increasing, so two different times cannot give the same state. The single minimum of x comes from the factor te^t, not recurring oscillations. There is no periodic orbit or repeated cycle: a coordinate can reverse even with a positive repeated real eigenvalue. The equal-scale figure marks the minimum and the initial state.



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Question 6

Let

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad X' = AX, \qquad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use the Euclidean vector norm. For $N = A + I$, the operator bounds $\|NX\| \le \|X\|$ and $\|N^2X\| \le \|X\|$ may be used.

Tasks

1. Compute the powers of N needed to find the normalized evolution and the complete solution.
2. For initial $(0, 0, 1)^T$, find the maximum of each of the first two components on $t \ge 0$ and its time.
3. For arbitrary nonzero initial data, identify when the largest polynomial factor in the solution is quadratic, linear, or constant. Do all solutions decay?
4. Can one constant C satisfy $\|X(t)\| \le Ce^{-t}\|X(0)\|$ for all $t \ge 0$ and all initial data? Prove instead a bound with decay $e^{-t/2}$ and an explicit constant.

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Question 6 – Solution

Strategy. A longer nilpotent chain truncates the exponential after a quadratic term; its degree matters for sharp decay estimates.

Step 1: Truncate the nilpotent exponential. Here N^2 has only entry $(1,3)$ equal to one; $N^3 = 0$. Direct differentiation verifies $F(t) = e^{-t}(I + tN + t^2N^2/2)$ and $F(0) = I$. Consequently

$$\boxed{X = e^{-t}\begin{pmatrix} p + qt + rt^2/2 \\ q + rt \\ r \end{pmatrix}.}$$

This gives all initial states and shows why a length-three chain needs $t^2/2$, rather than another unrelated eigenvector.

Step 2: Locate the component peaks. For $(p,q,r) = (0,0,1)$, the first components are $t^2e^{-t}/2$ and te^{-t}. Their derivatives are $e^{-t}t(2-t)/2$ and $e^{-t}(1-t)$. Thus their maxima are $\boxed{2e^{-2} \text{ at } t = 2}$ and $\boxed{e^{-1} \text{ at } t = 1}$, respectively. These peaks are not simultaneous.

Step 3: Classify polynomial degrees. If $r \neq 0$, the leading vector factor is $(r/2)t^2(1,0,0)^T$. If $r = 0$ but $q \neq 0$, it is $qt(1,0,0)^T$. If $q = r = 0$ and $p \neq 0$, it is the constant $p(1,0,0)^T$. Every polynomial factor times e^{-t} tends to zero, so all solutions decay; algebraic multiplicity three does not force degree two for every state.

Step 4: Prove an honest exponential estimate. For initial $(0,0,1)$, $e^t\|X(t)\| \geq t^2/2$, ruling out the proposed constant C. The given operator bounds instead imply $\|X(t)\| \leq e^{-t}(1 + t + t^2/2)\|X(0)\|$. Using $e^{-t/2} \leq 1$, $te^{-t/2} \leq 2/e$ and $(t^2/2)e^{-t/2} \leq 8/e^2$ for $t \geq 0$ gives

$$\boxed{\|X(t)\| \leq (1 + 2/e + 8/e^2)e^{-t/2}\|X(0)\|.}$$

A slightly slower exponential absorbs the polynomial transient.

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Question 7

An unknown constant real 2×2 matrix A has characteristic polynomial $(\lambda + 1)^2$. Its normalized evolution F is measured exactly at one time:

$$F(1) = e^{-1} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

You may use the Cayley–Hamilton identity $(A + I)^2 = 0$.

Tasks

1. Recover A from the measured matrix and prove uniqueness under the stated spectral assumption.
2. Find $F(t)$ for every real time and verify the evolution and inverse laws.
3. Solve the IVP $X(0) = (1, 0)^T$. Does its first component cross zero for $t \geq 0$?
4. Would measuring only $\det F(1) = e^{-2}$ recover the same matrix? Give a different admissible matrix and explain what information the full observation supplies.

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Question 7 – Solution

Strategy. Under the repeated-root assumption the nilpotent exponential is linear in time, making the full one-step observation invertible algebraically.

Step 1: Recover the nilpotent part. Let $N = A + I$. Since $N^2 = 0$, direct verification gives $F(t) = e^{-t}(I + tN)$. Therefore

$$N = eF(1) - I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \boxed{A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.}$$

This N indeed squares to zero and is nonzero. Every matrix with the stated characteristic polynomial has the same formula, so the full measurement determines it uniquely within the specified class.

Step 2: Verify all-time evolution. The normalized evolution is

$$\boxed{F(t) = e^{-t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.}$$

Its derivative equals AF and $F(0) = I$. Since $N^2 = 0$, multiplying $(I + tN)(I + sN)$ gives $I + (t + s)N$. Thus $F(t + s) = F(t)F(s)$ and $\boxed{F(t)^{-1} = F(-t)}$ for all real times.

Step 3: Inspect the recovered IVP. Selecting the first column gives $\boxed{X = e^{-t}(1 + t, -t)^T}$. At zero the state is $(1, 0)^T$ and the derivative is $(0, -1)^T = A(1, 0)^T$. For $t \geq 0$, the first component stays strictly positive because $1 + t > 0$, though it tends to zero. Its only finite zero on the entire real line is $t = -1$, outside the requested forward interval.

Step 4: Separate area data from the full map. The different matrix $\widetilde{A} = -I$ has the same repeated eigenvalue and $\det \widetilde{F}(1) = e^{-2}$, but $\widetilde{F}(1) = e^{-1}I$. Indeed $\det(I + tN) = 1$, so the nilpotent shear is invisible to this area multiplier. The full observed matrix supplies directional and coupling information, which the determinant alone discards. Uniqueness above uses both the spectral assumption and the complete matrix observation.

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Question 8

For $k \geq 0$, consider

$$X' = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix} X.$$

Investigate Euclidean monotonicity and construct an alternative quadratic size when the standard squared norm $x^2 + y^2$ is unsuitable.

Tasks

1. Find the complete solution and classify the eigenspace dimension as k varies. Prove forward decay for all initial data.
2. Classify all k for which $(x^2 + y^2)\prime$ is negative at every nonzero state, nonpositive with nonzero equality states, or positive at some state.
3. At the boundary parameter between those cases, decide whether the norm strictly decreases between any two distinct times along every nonzero solution. Address the instantaneous equality line.
4. Choose a positive coefficient $c = c(k)$ so that $H = x^2 + cy^2$ has strictly negative derivative at every nonzero state for every $k \geq 0$. Give an explicit choice and proof.

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Question 8 – Solution

Strategy. The nilpotent coupling does not change the eigenvalues, but it can change which quadratic size decreases at every state.

Step 1: Solve for every coupling strength. The repeated eigenvalue is -1. If $k = 0$, the eigenspace is the whole plane; if $k > 0$, it is $y = 0$. For initial (p, q),

$$\boxed{X = e^{-t}(p + kqt, q)^T.}$$

The component equations directly verify completeness. Constant and linear factors times e^{-t} both vanish forward, so every initial state decays.

Step 2: Locate the exact norm threshold. The squared-norm derivative is

$$(x^2 + y^2)' = -2x^2 + 2kxy - 2y^2.$$

For $0 \le k < 2$, $2|xy| \le x^2 + y^2$ makes this strictly negative off zero. At $k = 2$ it equals $-2(x - y)^2$, with equality on $x = y$. For $k > 2$, its value at $(1, 1)$ is $2k - 4 > 0$. Thus the threshold is $\boxed{k = 2}$, despite identical eigenvalues throughout the family.

Step 3: Distinguish instantaneous equality from an interval. At $k = 2$, a trajectory cannot stay on $x = y$ over an interval unless it is zero. On that line the velocity is $(x, -x)$, which is tangent to the line only when $x = 0$. Integrating $-2(x - y)^2$ over any interval therefore gives strictly negative change for every nonzero solution. The norm strictly decreases between distinct times even if its derivative vanishes at an isolated time.

Step 4: Construct a uniformly valid quadratic choice. Choose $c = 1 + k^2/2 > 0$. Completing the square yields

$$\boxed{H' = -2(x - ky/2)^2 - (2 + k^2/2)y^2 < 0 \quad (X \neq 0).}$$

The quadratic $H = x^2 + (1 + k^2/2)y^2$ is positive definite for every k, including zero. Its decrease is compatible with temporary Euclidean growth when $k > 2$; the two quantities measure different families of ellipses. The spectral decay is unchanged by this choice of measurement.

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Question 9

The critically damped equation

$$x'' + 2x' + x = 0, \qquad x(0) = 1, \qquad x'(0) = -a, \quad a \geq 0$$

has a repeated negative characteristic root. A student claims that critical damping prevents the displacement from crossing its equilibrium for every choice of initial velocity.

Tasks

1. Write the equation as a state system with $y = x\prime$, find its eigenspace, and solve the stated IVP.
2. Classify exactly the values of a for which $x(t)$ crosses zero at a positive time. Give that time when it exists.
3. For the crossing case, find the subsequent minimum displacement and its time. For the other cases, describe forward monotonicity.
4. If the first displacement crossing is observed at a prescribed $T > 0$, recover the initial velocity. Explain why a single crossing does not imply oscillatory motion.

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Question 9 – Solution

Strategy. The linear polynomial multiplying the repeated-root exponential determines whether the initial velocity produces an overshoot.

Step 1: Find the defective state solution. The state matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, with double eigenvalue -1 and eigenspace $\operatorname{span}(1,-1)^T$. The initial data give

$$\boxed{x = e^{-t}[1 + (1-a)t], \qquad y = e^{-t}[-a - (1-a)t].}$$

These formulas verify $x' = y$, $y' = -x - 2y$, and $(x(0), y(0)) = (1, -a)$.

Step 2: Classify positive-time crossings. Because $e^{-t} > 0$, a positive-time zero occurs exactly when $a > 1$. It is unique and equals $\boxed{t_0 = 1/(a-1)}$. The polynomial has negative slope, so this is a crossing from positive to negative displacement. For $0 \le a \le 1$, the displacement remains positive at every finite forward time. Thus critical damping alone does not prevent overshoot.

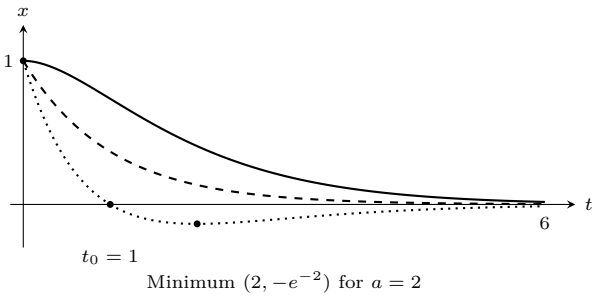
Step 3: Locate the minimum or prove monotonicity. If $a > 1$, $y = e^{-t}[(a-1)t - a]$ changes from negative to positive at

$$\boxed{t_{\min} = \frac{a}{a-1} = t_0 + 1, \qquad x_{\min} = -(a-1)e^{-a/(a-1)}.}$$

The displacement then increases toward zero from below. For $0 \le a \le 1$, $y < 0$ for all $t > 0$, so the positive displacement strictly decreases to zero; when $a = 0$, only the initial derivative vanishes.

Step 4: Infer the velocity from the crossing. Solving $T = 1/(a-1)$ gives $\boxed{x'(0) = -a = -(1 + 1/T)}$. The solution has only one zero because its exponential factor never vanishes and its polynomial factor is linear. After the single minimum it approaches zero from below without further crossings. This is an initial-data-dependent overshoot, not a repeating oscillation. The time plot compares $a = 0, 1, 2$; only the $a = 2$ curve crosses in those examples.

Solid $a = 0$; dashed $a = 1$; dotted $a = 2$



Minimum $(2, -e^{-2})$ for $a = 2$

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Question 10

For $\varepsilon \geq 0$, consider

$$A_\varepsilon = \begin{pmatrix} -1 & \varepsilon & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad X' = A_\varepsilon X, \qquad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A matrix is diagonalizable if it has a basis of eigenvectors. For nonzero solutions, compare their normalized directions $X/\|X\|$.

Tasks

1. Find the eigenvalues and eigenspaces. For which ε is the matrix diagonalizable?
2. Find the complete solution for arbitrary initial data, including the polynomial term when present.
3. Identify all forward-bounded initial states. Within that set, classify the limiting forward directions for every nonzero initial state.
4. For initial $(0, 1, 0)^T$, compare the iterated limits of $X_\varepsilon/\|X_\varepsilon\|$ as $t \to \infty$ and $\varepsilon \downarrow 0$, in both orders. Explain what changes even though the eigenvalues do not.

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Question 10 – Solution

Strategy. A repeated stable eigenvalue can have a two-dimensional eigenspace or a defective chain; the limiting direction detects that difference.

Step 1: Determine the eigenspaces. The eigenvalues are -1 (multiplicity two) and 2. For -1, $z = 0$ and $\varepsilon y = 0$. Thus its eigenspace is the whole plane $z = 0$ when $\varepsilon = 0$, but only the x-axis when $\varepsilon > 0$. The 2 eigenspace is always $\operatorname{span}(1, 0, 1)^T$. There is a full eigenbasis exactly when $\boxed{\varepsilon = 0}$; this diagonalizable three-dimensional matrix is not a scalar matrix.

Step 2: Solve the coupled components. First $z = re^{2t}$ and $y = qe^{-t}$. Then $(e^t x)' = \varepsilon q + 3re^{3t}$, giving

$$\boxed{X = e^{-t}\begin{pmatrix} p - r + \varepsilon q t \\ q \\ 0 \end{pmatrix} + re^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.}$$

This satisfies every component and recovers $(p, q, r)^T$ at zero.

Step 3: Classify the stable-plane directions. Forward boundedness holds exactly when $r = 0$. On that plane all states decay. For $\varepsilon = 0$, the nonzero direction stays constantly $(p, q, 0)^T/\sqrt{p^2 + q^2}$. For $\varepsilon > 0$ and $q \neq 0$, the normalized direction tends to $(\operatorname{sgn} q, 0, 0)^T$. If $q = 0$, $p \neq 0$, it is constantly $(\operatorname{sgn} p, 0, 0)^T$. The zero solution has no normalized direction.

Step 4: Compare the two orders of limits. For the specified initial state, the normalized vector is $(\varepsilon t, 1, 0)^T/\sqrt{\varepsilon^2 t^2 + 1}$. Taking $t \to \infty$ first with $\varepsilon > 0$, then $\varepsilon \downarrow 0$, gives $\boxed{(1, 0, 0)^T}$. Reversing the operations gives $\boxed{(0, 1, 0)^T}$. At every fixed finite time the solution depends continuously on ε, but long-time direction is not uniform near zero. The eigenvalues stay fixed while the eigenspace dimension changes.

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Question 1

Consider the constant-input system

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Separate the equilibrium response from the transient response.

Tasks

1. Find the equilibrium and transform the equation into a homogeneous system by translating the state.
2. Use the homogeneous eigenmodes to find the complete family, then solve the stated IVP.
3. Determine whether either component overshoots its equilibrium value for $t \geq 0$. Give the limiting state and justify the signs of the derivatives.
4. Explain why the nonhomogeneous solution family is not a vector space. Which linear combinations of two solutions necessarily solve the same forced equation?

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Question 1 – Solution

Strategy. A constant particular solution shifts the equilibrium; the translated state evolves through the familiar homogeneous modes.

Step 1: Translate to the forced equilibrium. Solving $-2x+y+3=0$, $x-2y=0$ gives $\boxed{X_* = (2,1)^T}$. With $Z = X - X_*$, the equation becomes $Z' = AZ$. The origin is not an equilibrium of the forced system.

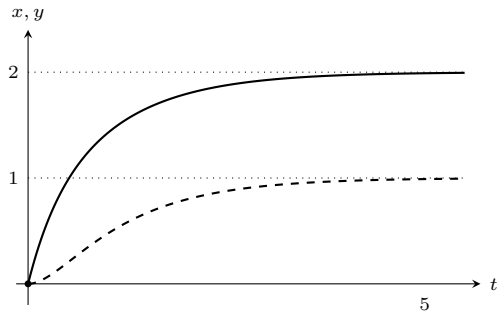
Step 2: Resolve the transient coefficients. The eigenpairs of A are $-1, (1,1)^T$ and $-3, (1,-1)^T$. Thus every solution is $X_* + c_1 e^{-t}(1,1)^T + c_2 e^{-3t}(1,-1)^T$. The initial state requires $c_1 = -3/2$, $c_2 = -1/2$, giving

$$\boxed{x = 2 - \tfrac{3}{2}e^{-t} - \tfrac{1}{2}e^{-3t}, \qquad y = 1 - \tfrac{3}{2}e^{-t} + \tfrac{1}{2}e^{-3t}.}$$

Step 3: Check monotonicity and the limit. Here $x' = \frac{3}{2}(e^{-t} + e^{-3t}) > 0$ for $t \geq 0$, while $y' = \frac{3}{2}(e^{-t} - e^{-3t}) > 0$ for $t > 0$ and $y'(0) = 0$. Both start at zero and tend to 2 and 1, respectively. Their monotone approach excludes overshoot; the initial velocity is $(3,0)^T$, as the original equation requires. The two horizontal dotted lines in the figure are component limits, not additional time-dependent solutions.

Step 4: Identify the affine superposition rule. If U, V solve the same forced equation, then $(\alpha U + \beta V)' - A(\alpha U + \beta V) = (\alpha+\beta)(3,0)^T$. It is a solution exactly when $\boxed{\alpha + \beta = 1}$. In particular the zero function is not a solution, and the sum of two solutions generally fails. Differences of solutions are homogeneous; adding any homogeneous solution to one particular solution gives the complete affine family derived above.

Solid x; dashed y; dotted component limits



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Question 2

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 1 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both homogeneous eigenvalues are negative, but the forcing has an unbounded component. Use a polynomial particular solution.

Tasks

1. Find a particular solution of the form $X_p = ut + v$, with constant vectors u, v. Explain why a constant trial cannot suffice.
2. Add the complete homogeneous family and impose the initial data.
3. Verify the result in the original equations and determine the limits of $x(t) - t$ and $y(t)$.
4. Decide which components are bounded and whether two solutions with different initial data approach one another. Explain why homogeneous stability does not force convergence to a constant here.

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Question 2 – Solution

Strategy. Match polynomial coefficients before imposing initial data; stability controls differences of responses, not the size of an unbounded input.

Step 1: Match the polynomial coefficients. Substituting $X_p = ut + v$ gives $u = A(ut + v) + (t, 1)^T$. The coefficient of t requires $Au = -(1, 0)^T$, hence $u = (1, 0)^T$. The constant equation $u = Av + (0, 1)^T$ gives $v = (0, 1)^T$. Thus $\boxed{X_p = (t, 1)^T}$. A constant trial cannot cancel the forcing's nonzero coefficient of t.

Step 2: Complete the family and solve the IVP. The homogeneous solution is $e^{-t}(c_1 + c_2 t, c_2)^T$. Initial zero gives $c_1 = 0$, $c_2 = -1$, so

$$\boxed{x = t(1 - e^{-t}), \qquad y = 1 - e^{-t}.}$$

Step 3: Check the original equation and asymptotics. We have $x' = 1 - e^{-t} + te^{-t}$ and $y' = e^{-t}$. The right sides $-x + y + t$ and $-y + 1$ give those same expressions. The initial values vanish. Moreover $x - t = -te^{-t} \to 0$ and $y \to 1$. The first component tracks a growing ramp; its tracking error decays, but its value does not tend to a finite constant.

Step 4: Separate stability from bounded response. On $t \geq 0$, y is bounded between zero and one, while $x \to \infty$. For any two solutions, their difference has the form $e^{-t}(d_1 + d_2 t, d_2)^T \to 0$. Thus initial-data effects disappear even though the common forced response grows. Negative eigenvalues of A guarantee decay of the homogeneous transient; they do not turn an unbounded polynomial forcing into a bounded steady state. A complete answer therefore describes both the particular response and the transient.

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Question 3

For real constants a, b, consider

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X + e^t \begin{pmatrix} a \\ b \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A student says that because the forcing exponent equals an eigenvalue, a factor t must occur in a particular solution for every nonzero (a, b).

Tasks

1. Determine exactly when a trial $X_p = e^t v$ with constant vector v works, and describe all such v.
2. Construct a particular solution valid for every (a, b) and then solve the zero-state IVP.
3. Explain how the component of the forcing in the resonant eigendirection decides whether a polynomial factor is unavoidable.
4. For $a = 0, b \neq 0$, determine the forward growth of the solution. Is growth by itself evidence of resonance? Address the student's claim.

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Question 3 – Solution

Strategy. Resonance is a compatibility problem for the forcing direction, not merely a comparison of two scalar exponents.

Step 1: Test the simple exponential trial. Substitution gives $(I - A)v = (a, b)^T$, or $0 = a$, $3v_2 = b$. Thus the trial works exactly when $a = 0$, with $\boxed{v = (c, b/3)^T, \quad c \in \mathbb{R}}$. The free first component is a homogeneous contribution, not a new forced effect.

Step 2: Construct the general forced response. A particular solution for all parameters is $X_p = (ate^t, (b/3)e^t)^T$. Adding $(c_1e^t, c_2e^{-2t})^T$ and imposing zero initial data gives

$$\boxed{x = ate^t, \qquad y = \tfrac{b}{3}(e^t - e^{-2t}).}$$

Differentiation gives $x' - x = ae^t$ and $y' + 2y = be^t$, verifying both forcing components and the initial values.

Step 3: Identify the resonant component. The resonant eigendirection is $(1, 0)^T$. Its forced scalar equation is $(e^{-t}x)' = a$, so a nonzero a necessarily creates ate^t. When $a = 0$, there is no forcing in that direction and no polynomial factor is needed. No choice of homogeneous constants can cancel a nonzero ate^t for all time. The condition is directional, not just spectral.

Step 4: Separate forced growth from resonance. For $a = 0, b \neq 0$, the zero-state solution has $x = 0$ and $y/e^t \to b/3$. It grows because its forcing grows, although the forced second component has homogeneous rate -2 and is not resonant. The first eigenvalue 1 is present in the matrix but receives no input. Thus growth alone does not prove resonance, and the student's claim fails for every nonzero forcing vector of the form $(0, b)^T$.

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Question 4

Consider the time-dependent system

$$X' = A(t)X + f(t), \qquad A(t) = \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use variation of parameters and check the correct two-time transition matrix.

Tasks

1. Find a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ and its inverse.
2. Derive the variation-of-parameters equation by writing $X = \Phi C$, then compute the IVP solution.
3. Find $T(t, s) = \Phi(t)\Phi(s)^{-1}$ and express the same solution as an integral of $T(t, s)f(s)$. Evaluate it.
4. A proposed shortcut replaces $T(t, s)$ by $\Phi(t - s)$. Compute the resulting candidate and its residual in the original system. Explain why the shortcut fails.

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Question 4 – Solution

Strategy. For time-dependent coefficients, propagation depends on both the departure time and the arrival time.

Step 1: Normalize the homogeneous matrix. Solving $y' = 0$, $x' = 2ty$ gives

$$\Phi(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}, \qquad \Phi(t)^{-1} = \begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix}.$$

Its determinant is one, and direct differentiation verifies $\Phi' = A(t)\Phi$ and $\Phi(0) = I$.

Step 2: Derive and integrate the varying coefficients. Substituting $X = \Phi C$ cancels $\Phi' C = A\Phi C$, leaving $C' = \Phi^{-1} f = (t - t^2, 1)^T$. Zero initial data gives $C = (t^2/2 - t^3/3, t)^T$. Therefore

$$\boxed{X = (t^2/2 + 2t^3/3, t)^T.}$$

Indeed $x' = t + 2t^2 = 2ty + t$ and $y' = 1$.

Step 3: Use the genuine two-time transition. Matrix multiplication yields

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Thus $X(t) = \int_0^t T(t, s) f(s)\, ds = \int_0^t (s + t^2 - s^2, 1)^T\, ds$, reproducing the same formula. This expression also follows directly from $\Phi(t) \int_0^t \Phi(s)^{-1} f(s)\, ds$.

Step 4: Expose the incorrect shortcut. Using $\Phi(t - s)$ instead gives $\widetilde{X} = (t^2/2 + t^3/3, t)^T$. Its residual is

$$\boxed{\widetilde{X}' - A(t)\widetilde{X} - f(t) = (-t^2, 0)^T,}$$

which is not identically zero. The two matrices differ because $t^2 - s^2 \neq (t - s)^2$ in general. A fundamental matrix normalized at zero cannot automatically be shifted in time when the coefficient matrix itself changes with time.

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Question 5

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The forcing matches a repeated homogeneous exponent and enters through the second component of a length-two chain.

Tasks

1. Use $Z = e^{t}X$ to derive a simpler forced system and solve the IVP.
2. Test the trial $X_p = te^{-t}v$ with constant vector v. Explain why it cannot supply a particular solution here.
3. Find the forward maximum of each component and its time. Do the maxima occur together?
4. Determine the long-time behavior and compare the polynomial degree of this forced response with the homogeneous family. Does resonance necessarily imply an unbounded response?

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Question 5 – Solution

Strategy. Removing the common exponential reveals successive integrations along the nilpotent chain.

Step 1: Integrate along the chain. With $N = A + I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the transformed system is $Z' = NZ + (0,1)^T$, $Z(0) = 0$. Thus $z_2' = 1$, $z_1' = z_2$, giving $z_2 = t$, $z_1 = t^2/2$ and

$$\boxed{x = \tfrac{1}{2}t^2e^{-t}, \qquad y = te^{-t}.}$$

Substitution verifies the original equation and zero initial state.

Step 2: Reject an insufficient polynomial trial. The trial would give $Z = tv$, so its equation requires $v = tNv + (0,1)^T$ for every time. Matching coefficients forces $Nv = 0$ and $v = (0,1)^T$, but $N(0,1)^T = (1,0)^T \neq 0$. These conditions contradict each other. A factor t multiplying a constant vector is insufficient; the chain requires a quadratic term in the first component.

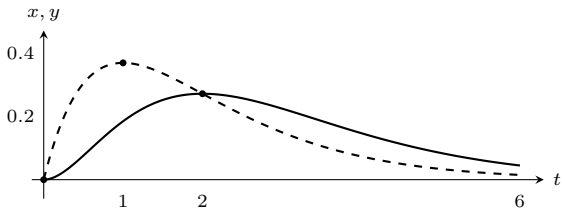
Step 3: Locate the separate response peaks. For $t \geq 0$, $x' = e^{-t}t(2-t)/2$ and $y' = e^{-t}(1-t)$. Their sign changes show

$$\boxed{x_{\max} = 2e^{-2} \text{ at } t = 2, \qquad y_{\max} = e^{-1} \text{ at } t = 1.}$$

The maxima occur at different states and different times. The figure marks these component peaks on the two time graphs.

Step 4: Interpret resonance without growth to infinity. Both components tend to zero, since every fixed polynomial times e^{-t} vanishes. The homogeneous family is $e^{-t}(c_1 + c_2t, c_2)^T$, whose largest degree is one; the forcing raises the first component's degree to two. This is a resonant polynomial response, even though the negative exponential ultimately wins. Resonance describes an enhanced polynomial factor, not a universal guarantee of an unbounded solution. Adding any homogeneous solution still leaves forward decay.

Solid x; dashed y; dots mark component maxima



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Question 6

For $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, consider

$$X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Investigate the response to a forcing that rotates at the homogeneous oscillation frequency.

Tasks

1. Use $X = R(t)Z$ to derive and solve a constant-forcing equation for Z. Give the full family.
2. Find the unique 2π-periodic solution and prove its uniqueness.
3. For $X(0) = 0$, find the state, its distance from the periodic solution at the same time, and its distance from the unit circle.
4. Does the zero-state response become exactly periodic at a finite time or tend to a single point? Explain how its state curve should be interpreted in this time-dependent forced problem.

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Question 6 – Solution

Strategy. A rotating coordinate frame removes the oscillation and leaves scalar damping toward a constant vector.

Step 1: Pass to the rotating frame. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $R' = JR$ and $A = -I + J$. Substitution cancels the rotational terms, giving $Z' = -Z + (1,0)^T$. Hence

$$\boxed{X(t) = R(t)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t}\left(c - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right],}$$

where $c = X(0)$ is an arbitrary real vector. This gives every solution.

Step 2: Identify and prove the periodic response. Taking $c = (1,0)^T$ gives $\boxed{P(t) = (\cos t, \sin t)^T}$. If another solution were 2π-periodic, its difference from P would be $e^{-t}R(t)d$ and would satisfy $d = e^{-2\pi}d$ at times zero and 2π. Thus $d = 0$, proving uniqueness. Damping prevents a secular term despite the matched rotation frequency.

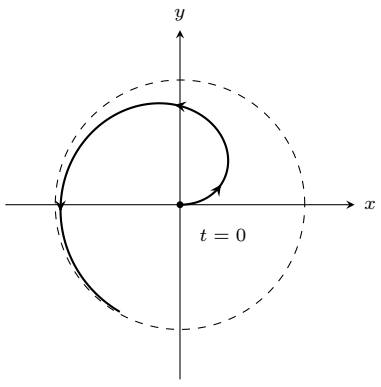
Step 3: Quantify zero-state convergence. For $c = 0$,

$$\boxed{X(t) = (1 - e^{-t})(\cos t, \sin t)^T \quad (t \geq 0).}$$

Since rotations preserve length, $\|X(t) - P(t)\| = e^{-t}$. Its radius is $1 - e^{-t}$, so its distance to the unit circle is also e^{-t}. Its initial velocity is $(1,0)^T$, agreeing with the input at zero.

Step 4: Distinguish asymptotic response from exact return. The radius is strictly increasing at every finite forward time, so the solution neither repeats a state nor becomes exactly periodic later. Along $t = 2\pi n$ and $t = 2\pi n + \pi$, its states tend to $(1,0)^T$ and $(-1,0)^T$, respectively; there is no single point limit. The drawn curve is a nonautonomous state path approaching a periodic response, not an autonomous phase portrait or a claim of a finite-time merger with the circle. Only a finite initial segment is drawn.

Solid: initial state path; dashed: periodic-response circle



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Question 7

On $t \geq 0$, consider the bounded input in

$$x' = x + e^{-t}, \qquad y' = -y + 1, \qquad (x(0), y(0)) = (p, q).$$

Investigate which initial states suppress the unstable homogeneous mode.

Tasks

1. Find the complete solution and classify every initial state yielding a bounded forward response.
2. Find the limiting state for those bounded solutions. Is that limiting vector, held constant, itself a solution of the original forced equation?
3. Express the bounded first component using an integral from the current time to infinity. Explain how this condition selects its initial value.
4. Perturb the selected initial first component by δ, with $0 < |\delta| < 1$. Find when its difference from the bounded response first reaches magnitude one. Does bounded forcing ensure bounded responses here?

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Question 7 – Solution

Strategy. Boundedness can require an exact cancellation of an unstable mode rather than follow from bounded forcing alone.

Step 1: Solve and isolate the growing coefficient. Integrating factors give

$$\boxed{x = (p + \tfrac{1}{2})e^{t} - \tfrac{1}{2}e^{-t}, \qquad y = 1 + (q-1)e^{-t}.}$$

The second component is bounded for every q. The first is bounded exactly when $\boxed{p = -1/2}$, so the full bounded family has that fixed first initial value and arbitrary q.

Step 2: Check the limit against the original equation. For bounded data, $X(t) \to (0,1)^T$. However the constant function $X = (0,1)^T$ has zero derivative, whereas the original right side is $(e^{-t}, 0)^T \neq 0$ at finite times. It is the limiting equilibrium of the limiting equation, not an exact constant solution of the stated time-dependent equation.

Step 3: Recover the cancellation from a future integral. Solving the first equation with no growing term gives

$$\boxed{x(t) = -\int_{t}^{\infty} e^{t-s}e^{-s}\,ds = -\tfrac{1}{2}e^{-t}.}$$

At zero it selects $p = -1/2$. This is a boundedness condition imposed over the whole future, used to choose an initial value. It does not replace the ordinary forward initial-value evolution once that value is fixed.

Step 4: Measure sensitivity to the unstable mode. Changing p by δ adds δe^{t} to the first component. Its absolute difference starts below one and grows strictly, first reaching one at $\boxed{t = \ln(1/|\delta|)}$. The input remains bounded throughout, but every such perturbed response is unbounded. The unstable homogeneous coefficient, not the input size alone, determines the failure. Exact cancellation selects a special family; it does not make the underlying system stable.

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Question 8

For $t \geq 0$, let

$$x' = -x + y, \qquad y' = -y + u(t), \qquad x(0) = y(0) = 0,$$

where $u(t) = 1$ for $0 \leq t < 1$ and $u(t) = 0$ for $t > 1$. Take the state to be continuous at the switch; use one-sided derivatives there. The value assigned to $u(1)$ does not change the continuous solution.

Tasks

1. Solve on $0 \leq t \leq 1$, then continue for $t \geq 1$ using $\tau = t - 1$ and the state at the switch.
2. Find the one-sided state derivatives at $t = 1$. Which component derivative is continuous, and which jumps?
3. Find the global forward maximum of x and its time. Explain why the maximum occurs after the input has switched off.
4. Find the long-time limit and the integrals $\int_0^\infty x(t)\,dt$ and $\int_0^\infty y(t)\,dt$ by integrating the differential equations.

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Question 8 – Solution

Strategy. Match the state at the switch, then track the stored second-component response as it continues to feed the first.

Step 1: Solve and match the two intervals. For $0 \le t \le 1$, integration gives $x = 1 - (1+t)e^{-t}$, $y = 1 - e^{-t}$. Set $a = 1 - 2/e$, $b = 1 - 1/e$, the switch values. For $\tau = t - 1 \ge 0$, the homogeneous continuation is

$$\boxed{x = e^{-\tau}(a + b\tau), \qquad y = be^{-\tau}.}$$

At $\tau = 0$ these match the first formulas, so both components are continuous.

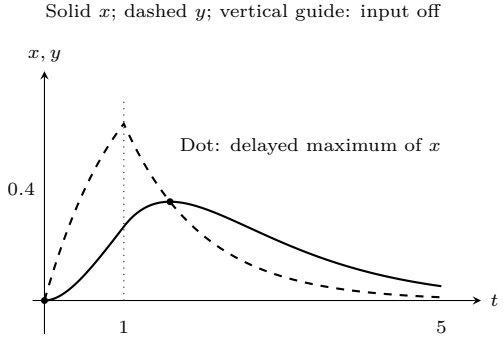
Step 2: Compare one-sided derivatives. Before the switch, $(x', y') = (te^{-t}, e^{-t})$. Thus $X'(1^-) = (1/e, 1/e)^T$. Immediately after it, $X'(1^+) = (-a+b, -b)^T = (1/e, -1+1/e)^T$. The derivative x' is continuous; y' has jump -1. A finite step input changes a derivative, not the state itself.

Step 3: Find the delayed first-component peak. Before switching, $x' > 0$ for $t > 0$. Afterward, $x' = e^{-\tau}(b - a - b\tau)$, with $b - a = 1/e$. It changes sign once, at $\tau = 1/(e-1)$, hence

$$\boxed{t_{\max} = 1 + \frac{1}{e-1}, \qquad x_{\max} = (1 - 1/e)e^{-1/(e-1)}.}$$

The still-positive y initially exceeds x after the input ends, so $x' = -x + y$ remains positive until the two components become equal.

Step 4: Integrate the total response. Both components tend to zero exponentially times a linear factor, so their integrals converge. Integrating $y' = -y + u$ over $[0, \infty)$ and using its zero endpoints gives $\int y = \int u = 1$. Integrating $x' = -x + y$ similarly gives $\boxed{\int_0^\infty x(t)\,dt = \int_0^\infty y(t)\,dt = 1}$. Equal total areas do not mean equal graphs or simultaneous peaks. The figure marks the switch and the delayed maximum of x.



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Question 9

Consider the input-driven system

$$x' = y, \qquad y' = u(t), \qquad x(0) = y(0) = 0, \quad 0 \le t \le 1.$$

Choose an input so that $(x(1), y(1)) = (1, 0)$. First restrict attention to inputs of the form $u(t) = a + bt$.

Tasks

1. Express the endpoint conditions as two integral conditions on a general continuous input u.
2. Find the unique affine input $a + bt$ satisfying both conditions. Explain why a constant input cannot work.
3. Compute the entire state path for that input. Determine whether displacement is monotone and find the maximum velocity.
4. Is the input unique among all continuous functions? Test additions of $c(6t^2 - 6t + 1)$, with arbitrary real c, and distinguish endpoint agreement from identical intermediate motion.

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Question 9 – Solution

Strategy. The endpoint conditions constrain two weighted integrals of the input, not all of its time-dependent values.

Step 1: Derive the two input moments. Integrating gives $y(t) = \int_0^t u(s)\,ds$. A second integration yields $x(t) = \int_0^t (t-s)u(s)\,ds$, verified by differentiation and the zero initial data. The endpoint requirements are

$$\boxed{\int_0^1 u(s)\,ds = 0, \qquad \int_0^1 (1-s)u(s)\,ds = 1.}$$

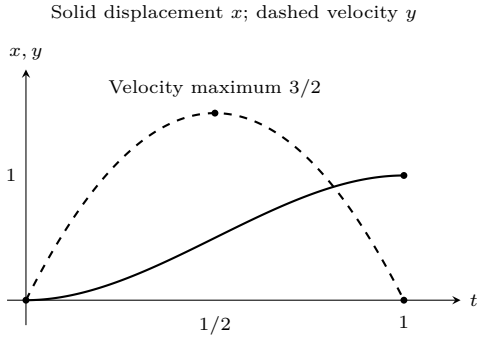
Step 2: Solve within the affine family. For $u = a + bt$, these conditions become $a + b/2 = 0$ and $a/2 + b/6 = 1$. Their unique solution is $\boxed{a = 6,\ b = -12}$, so $u = 6 - 12t$. A constant input would need $a = 0$ from the final velocity condition, then could not create displacement one.

Step 3: Recover and interpret the state path. Integration gives

$$\boxed{y = 6t(1-t), \qquad x = 3t^2 - 2t^3.}$$

The state starts at zero and ends at $(1, 0)$. Since $y > 0$ for $0 < t < 1$, displacement strictly increases through the interior. Velocity has its unique maximum $\boxed{3/2}$ at $t = 1/2$, where the input changes sign. The negative input in the second half slows the positive velocity to zero; it does not reverse the displacement.

Step 4: Expose the remaining freedom. Let $g(t) = 6t^2 - 6t + 1$. Direct integration gives $\int_0^1 g = 0$ and $\int_0^1 sg(s)\,ds = 0$, hence also $\int_0^1 (1-s)g(s)\,ds = 0$. Therefore every $u = 6 - 12t + cg(t)$ reaches the same endpoint. For $c \neq 0$, g is not identically zero, so the velocity derivative and intermediate motion differ. Uniqueness holds within the affine trial family, not among all continuous inputs. No monotonicity claim is made for the added-input family.



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Question 10

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Consider

$$X' = JX + e^{-t^2} R(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad t \geq 0.$$

You may use $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$. Exact definite-integral answers are expected; no elementary antiderivative is assumed.

Tasks

1. Use variation of parameters to find the solution for arbitrary $X(0) = c$. Specialize to $c = 0$.
2. For $c = 0$, find the limiting norm. Does the state converge to a point or become periodic? Explain why forcing that tends to zero need not produce a state that tends to zero.
3. Find exactly the initial state that makes $X(t) \to 0$. Express that solution with an integral from t to infinity.
4. Prove a quantitative decay bound for this selected solution using $\int_t^\infty e^{-s^2}\, ds \leq e^{-t^2}/(2t)$ for $t > 0$, deriving the inequality as part of the solution.

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Question 10 – Solution

Strategy. A rotating frame converts the forcing into accumulated scalar amplitude; cancelling its total integral removes the persistent free rotation.

Step 1: Integrate in rotating coordinates. Since $R' = JR$ and $R^{-1} = R(-t)$, writing $X = R(t)Z$ gives $Z' = e^{-t^2}(1,0)^T$. Define $G(t) = \int_0^t e^{-s^2}\,ds$. Then

$$\boxed{X(t) = R(t)\left[c + G(t)\begin{pmatrix}1\\0\end{pmatrix}\right].}$$

For $c = 0$ the state is $G(t)(\cos t, \sin t)^T$. Differentiation verifies the equation, including the non-elementary amplitude derivative.

Step 2: Interpret the zero-state asymptotics. Let $L = \sqrt{\pi}/2$. For $c = 0$, the norm is $G(t) \uparrow L$, strictly increasing at every finite time, so the solution is not periodic. At times $2\pi n$ and $2\pi n + \pi$ it tends to $(L,0)^T$ and $(-L,0)^T$, so no point limit exists. The input decays, but the undamped homogeneous rotation retains the accumulated amplitude; there is no dissipative mechanism to erase it.

Step 3: Select the unique cancelling initial state. Because rotations preserve length, $\|X(t)\| \to \|c + L(1,0)^T\|$. Thus $X(t) \to 0$ exactly when $\boxed{c = (-L,0)^T}$. For that state,

$$\boxed{X(t) = -R(t)\begin{pmatrix}1\\0\end{pmatrix}\int_t^\infty e^{-s^2}\,ds.}$$

Any other initial state leaves a nonzero limiting amplitude.

Step 4: Bound the remaining tail. For $s \geq t > 0$, $1 \leq s/t$. Therefore

$$0 \leq \int_t^\infty e^{-s^2}\,ds \leq \frac{1}{t}\int_t^\infty s e^{-s^2}\,ds = \frac{e^{-t^2}}{2t}.$$

The selected solution consequently obeys $\boxed{\|X(t)\| \leq e^{-t^2}/(2t) \quad (t > 0)}$. This is a tail estimate, not a formula at $t = 0$; the initial norm there is L. The definite integral and the bound avoid inventing an elementary antiderivative or replacing the exact response by numerical samples.

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Question 1

For $t \geq 0$, consider the coupled initial-value problem

$$x' = -2x + y, \qquad y' = 2x - 3y, \qquad x(0) = 1, \quad y(0) = 0.$$

Write $F(s) = \mathcal{L}\{x\}(s)$ and $G(s) = \mathcal{L}\{y\}(s)$, using one-sided Laplace transforms. Recover both components by solving the transformed algebraic system.

Tasks

1. Derive the two transformed equations, retaining the initial terms. Solve for F and G and state a common half-plane of convergence.
2. Invert both transforms and verify the initial values and original differential equations.
3. Find the transform and time response of $w = x - y$. Explain why one pole of the full system is absent from this observation.
4. Find the exact time and value of the largest $y(t)$ on $t \geq 0$. Explain how y can initially increase while both system modes decay.

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Question 1 – Solution

Strategy. Keep the initial data on the right side of the transformed system, then distinguish component responses from observed combinations.

Step 1: Solve the algebraic system. The derivative rule gives $(s+2)F-G=1$ and $-2F+(s+3)G=0$. Their determinant is $(s+1)(s+4)$, so

$$F=\frac{s+3}{(s+1)(s+4)}, \qquad G=\frac{2}{(s+1)(s+4)}.$$

Both transforms converge for $\operatorname{Re} s>-1$. Dropping the initial term 1 would incorrectly give the zero solution.

Step 2: Invert and check. Partial fractions give

$$\boxed{x=\tfrac{2}{3}e^{-t}+\tfrac{1}{3}e^{-4t}, \qquad y=\tfrac{2}{3}(e^{-t}-e^{-4t}).}$$

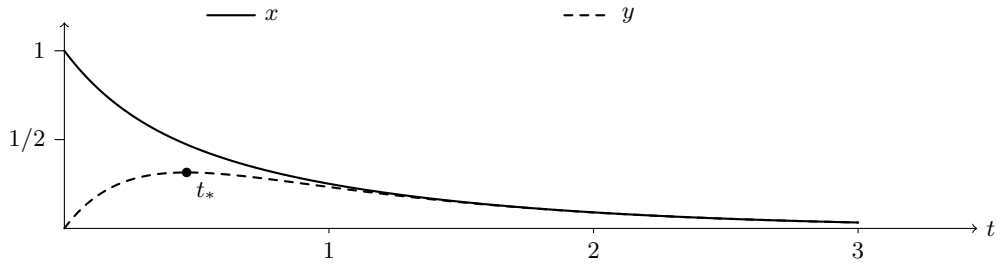
At zero these are $(1,0)$. Differentiation gives $x'=-\frac{2}{3}e^{-t}-\frac{4}{3}e^{-4t}$ and $y'=-\frac{2}{3}e^{-t}+\frac{8}{3}e^{-4t}$, which equal $-2x+y$ and $2x-3y$.

Step 3: Explain the cancelled pole. Subtracting the transforms gives $F-G=1/(s+4)$, hence $\boxed{w=e^{-4t}}$. The slow mode is a multiple of $(1,1)^T$; subtracting its components annihilates it. The state still contains this mode. Cancellation in an observation does not remove a mode from the system.

Step 4: Locate the component peak. Since $y'=\frac{2}{3}e^{-t}(4e^{-3t}-1)$, it changes sign once, from positive to negative, at $t_*=\log(4)/3$. Thus

$$\boxed{\max_{t\ge 0} y(t)=\tfrac{1}{2}\,4^{-1/3}, \qquad t_*=\log(4)/3.}$$

Here $y'(0)=2$ because the first component feeds the second. Decay of each mode does not imply monotonic decay of every component: their signed combination initially grows. The curve tends back to zero.



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Question 2

Let $Z = (x, y, z)^T$ satisfy $Z' = AZ$ for $t \geq 0$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \qquad Z(0) = (a, b, c)^T.$$

The characteristic polynomial has a triple root. Investigate how the initial vector determines the poles actually present in each transformed component.

Tasks

1. Compute $(sI - A)^{-1}$ directly, and verify its product with $sI - A$ is I. State where the inverse matrix exists.
2. Use the resolvent to find all three Laplace transforms and invert them for arbitrary real a, b, c.
3. For the initial state $(0, 0, 1)^T$, verify the original equations and explain the different polynomial degrees in the three components.
4. Classify the pole order of $\mathcal{L}\{x\}$ at $s = -2$ for every (a, b, c), including the case of no pole. Does a triple characteristic root force a triple pole in every component?

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Question 2 – Solution

Strategy. Invert the triangular matrix before substituting data; its powers of $(s+2)^{-1}$ display how a chain transmits the initial state.

Step 1: Compute the resolvent. Put $q = s+2$. For $q \neq 0$,

$$\boxed{(sI - A)^{-1} = \begin{pmatrix} q^{-1} & q^{-2} & q^{-3} \\ 0 & q^{-1} & q^{-2} \\ 0 & 0 & q^{-1} \end{pmatrix}.}$$

Multiplication by the upper triangular matrix with diagonal q and superdiagonal -1 gives diagonal entries 1; each off-diagonal entry cancels. The inverse exists exactly for $s \neq -2$. This algebraic domain is larger than the convergence half-plane $\operatorname{Re} s > -2$ for the nonzero responses.

Step 2: Multiply by the initial vector. The transforms are $a/q + b/q^2 + c/q^3$, $b/q + c/q^2$, and c/q. Since $\mathcal{L}\{t^n e^{-2t}\} = n!/(s+2)^{n+1}$,

$$\boxed{x = e^{-2t}(a + bt + ct^2/2), \quad y = e^{-2t}(b + ct), \quad z = ce^{-2t}.}$$

The factorial is essential: the inverse of q^{-3} is $t^2 e^{-2t}/2$.

Step 3: Check transmission along the chain. For the specified initial state,

$$Z = (t^2 e^{-2t}/2, te^{-2t}, e^{-2t})^T.$$

Directly, $z' = -2z$, $y' = -2y + z$ and $x' = -2x + y$, with the required initial state. After multiplication by e^{2t}, the equations become $(e^{2t}z)' = 0$, $(e^{2t}y)' = e^{2t}z$ and $(e^{2t}x)' = e^{2t}y$. Two successive integrations produce the quadratic factor.

Step 4: Classify actual pole orders. The numerator of the first transform over q^3 is $aq^2 + bq + c$. Its order at $q = 0$ gives the exhaustive classification:

initial-data condition	pole order of $\mathcal{L}\{x\}$
$c \neq 0$	3
$c = 0,\ b \neq 0$	2
$c = b = 0,\ a \neq 0$	1
$a = b = c = 0$	no pole

Thus the characteristic multiplicity supplies a possible order, not an order every response must attain. Even for $c \neq 0$, the third component has only a simple pole, whereas the first has a triple pole.

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Question 3

A delayed input drives the zero-state system

$$x' = -x + H(t-2), \qquad y' = x - 2y, \qquad x(0) = y(0) = 0, \quad t \geq 0.$$

Here $H(t-a)$ is zero for $t < a$ and one for $t > a$; its value at the switch does not affect the continuous state. Interpret derivatives there one-sidedly.

Tasks

1. Find the transforms of both components and identify the factor that encodes the input delay.
2. Invert them to obtain a piecewise solution. Explain why multiplying an undelayed response by e^{-2t} does not implement a delay.
3. Determine continuity of x, y, x', y' at $t = 2$, and find the one-sided values of y'' there.
4. Prove both components increase after the switch and approach their limits without overshoot. Sketch their time responses, marking the switch and limiting levels.

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Question 3 – Solution

Strategy. First invert the undelayed rational factors, then translate the entire response using the second shifting theorem.

Step 1: Transform both equations. With $F = \mathcal{L}\{x\}$ and $G = \mathcal{L}\{y\}$, $(s{+}1)F = e^{-2s}/s$ and $(s{+}2)G = F$. Hence

$$F = \frac{e^{-2s}}{s(s+1)}, \qquad G = \frac{e^{-2s}}{s(s+1)(s+2)}, \qquad \operatorname{Re} s > 0.$$

It is e^{-2s}, rather than an exponential in time, that encodes the delay.

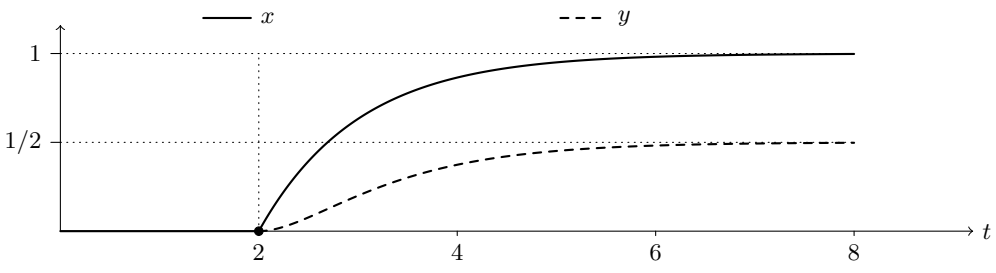
Step 2: Shift the inverse transforms. The undelayed inverses are $1 - e^{-t}$ and $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2}(1 - e^{-t})^2$. Writing $\tau = t - 2$, the solution is zero for $t < 2$, and for $t \geq 2$,

$$\boxed{x = 1 - e^{-\tau}, \qquad y = \tfrac{1}{2}(1 - e^{-\tau})^2.}$$

These formulas give zero at the joining point. Multiplication by e^{-2t} would change decay rates while generally leaving a nonzero response before $t = 2$; it is a different transform operation.

Step 3: Check the switch regularity. Both states are continuous. Before the switch their derivatives vanish; after it $x' = e^{-\tau}$ and $y' = e^{-\tau} - e^{-2\tau}$. Thus $x'(2^-) = 0$, $x'(2^+) = 1$, whereas $y'(2^-) = y'(2^+) = 0$. Afterward $y'' = -e^{-\tau} + 2e^{-2\tau}$, so $\boxed{y''(2^-) = 0,\ y''(2^+) = 1}$. In particular y is continuously differentiable at the switch but is not twice differentiable there.

Step 4: Establish shape and limits. For $\tau > 0$, $x' > 0$ and $y' = e^{-\tau}(1 - e^{-\tau}) > 0$. Also $0 < x < 1$ and $0 < y < 1/2$, with limits 1 and 1/2. These inequalities prove the absence of overshoot for all later times. The dotted horizontal guides in the sketch are limits; the solid and dashed curves are the components. Both remain zero throughout the waiting interval.



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Question 4

For $a > 0$, a single impulse acts on an undamped system:

$$x' = y + \alpha\delta(t-a), \qquad y' = -x + \beta\delta(t-a), \qquad (x(0), y(0)) = (1, 0).$$

Here $\delta(t-a)$ has unit mass, α, β are real, and states are piecewise smooth with jumps determined by integrating the equations. Use right-hand values at the impulse when writing the response.

Tasks

1. Transform the system and find $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ for general a, α, β.
2. Invert the transforms and verify both jump conditions directly. Explain why this impulse may create jumps in the states themselves.
3. Find the unique pair (α, β) that leaves the state identically zero for all $t > a$.
4. If only the second equation can receive an impulse ($\alpha = 0$), find all admissible stopping times and amplitudes. Identify the earliest time and sketch its two component responses, showing one-sided values.

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Question 4 – Solution

Strategy. The shifted impulse produces a shifted homogeneous response; stopping requires cancellation of the complete pre-impulse state.

Step 1: Solve in the transform domain. The equations are $sF - G = 1 + \alpha e^{-as}$ and $F + sG = \beta e^{-as}$. Thus, for $\operatorname{Re} s > 0$,

$$F = \frac{s}{s^2+1} + e^{-as}\frac{\alpha s + \beta}{s^2+1}, \qquad G = \frac{-1}{s^2+1} + e^{-as}\frac{-\alpha + \beta s}{s^2+1}.$$

This half-plane suffices for all parameter choices, even when cancellation makes the actual response have compact support.

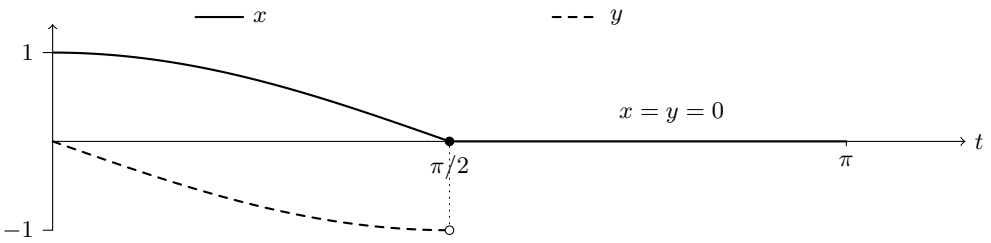
Step 2: Invert and check jumps. For $t < a$, $(x, y) = (\cos t, -\sin t)$. For $t \geq a$, with $\tau = t - a$,

$$x = \cos t + \alpha \cos \tau + \beta \sin \tau, \qquad y = -\sin t - \alpha \sin \tau + \beta \cos \tau.$$

Hence $\boxed{[x]_a = \alpha,\ [y]_a = \beta}$, where $[f]_a = f(a^+) - f(a^-)$. Integrating the original equations across a gives these same jumps: the ordinary bounded terms contribute zero in the shrinking interval, while the impulses contribute their masses. Away from a the ordinary equations hold.

Step 3: Cancel the full state. The state just before a is $(\cos a, -\sin a)$. The only kick making the right-hand state zero is $\boxed{(\alpha, \beta) = (-\cos a, \sin a)}$. The ensuing homogeneous IVP then stays zero by uniqueness. Conversely, an identically zero future requires that same right-hand state, so the kick is unique.

Step 4: Restrict the available impulse. With $\alpha = 0$, cancellation requires $\cos a = 0$. Thus $\boxed{a = \pi/2 + k\pi,\ \beta = (-1)^k,\ k = 0, 1, 2, \ldots}$. The earliest is $a = \pi/2$, $\beta = 1$. There x is continuous at zero, while y jumps from -1 to 0. Both vanish afterward. The vertical dotted segment marks a jump, not values traversed continuously by the solution.



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Question 5

Consider, for $t \geq 0$,

$$Z' = AZ + B, \qquad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad Z(0) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

A student cancels a factor $s - 2$ in the zero-initial-state transform and concludes that every solution of this forced system is bounded.

Tasks

1. Compute $(sI - A)^{-1}B$ and find the response when $p = q = 0$ using Laplace transforms.
2. Explain the cancellation using the direction of B, and identify the system mode that the constant input does not excite from zero.
3. Retain the initial-state term in the transformed system and determine exactly which (p, q) produce bounded solutions on $[0, \infty)$.
4. Perturb the zero initial state to $(\varepsilon, 2\varepsilon)^T$, with $\varepsilon \neq 0$. Find the exact difference from the zero-state response and assess the student's conclusion.

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Question 5 – Solution

Strategy. Separate the input contribution from the initial-state contribution before cancelling rational factors.

Step 1: Calculate the zero-state response. The determinant is $s^2 - s - 2 = (s-2)(s+1)$, and

$$(sI - A)^{-1}B = \frac{1}{(s-2)(s+1)}\begin{pmatrix} s-1 & 1 \\ 2 & s \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{s+1}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The constant input contributes an additional $1/s$. Inverting yields $\boxed{Z_0(t) = (1 - e^{-t})(1,-1)^T}$, which is bounded. Its transform converges for $\operatorname{Re} s > 0$; the cancelled expression extends through $s = 2$, where the full matrix resolvent itself is undefined.

Step 2: Identify why the pole disappears. We have $AB = -B$, so the input lies entirely in the stable eigendirection. The other eigenpair is $2, (1,2)^T$. From zero, this input never excites that direction. The pole cancellation describes the chosen input and data; it does not change either eigenvalue of A.

Step 3: Include all initial conditions. Write $(p,q)^T = c_+(1,2)^T + c_-(1,-1)^T$, where $c_+ = (p+q)/3$ and $c_- = (2p-q)/3$. The transform contains

$$\frac{c_+}{s-2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{c_-}{s+1}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{s(s+1)}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$Z = c_+ e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left[1 + (c_- - 1)e^{-t}\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The growing vector cannot be cancelled by a bounded term, so boundedness holds exactly when $\boxed{p + q = 0}$. Then the limit is $(1,-1)^T$. A common convergence half-plane for the general family is $\operatorname{Re} s > 2$.

Step 4: Test sensitivity to the omitted term. For the specified perturbation, $c_+ = \varepsilon$ and $c_- = 0$. The exact difference is $\boxed{Z - Z_0 = \varepsilon e^{2t}(1,2)^T}$, whose Euclidean norm is $\sqrt{5}|\varepsilon|e^{2t} \to \infty$. Thus arbitrarily small initial perturbations in that direction destroy boundedness. The student's inference confuses a bounded zero-state response with boundedness for all initial states.

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Question 6

Two inputs with the same frequency drive the same system from rest:

$$x' = -y + \cos t, \qquad y' = x + \sigma \sin t, \qquad x(0) = y(0) = 0, \qquad \sigma \in \{1, -1\}.$$

Use Laplace transforms to investigate why a frequency match alone does not settle whether this vector input generates growing oscillations.

Tasks

1. Derive both transformed components for general σ, showing the denominator before any cancellation.
2. For $\sigma = 1$, invert the transforms and verify the original equations. Determine the Euclidean norm for $t \geq 0$.
3. For $\sigma = -1$, simplify before inversion, solve the IVP and prove boundedness.
4. Compare the pole orders and the input directions in the two cases. Is every sinusoidal vector forcing at the natural frequency resonant in the sense of an unbounded zero-state response?

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Question 6 – Solution

Strategy. Solve the coupled algebra first; vector numerators can either retain or cancel the repeated oscillator poles.

Step 1: Derive the rational responses. The transformed equations are $sF + G = s/(s^2+1)$ and $-F + sG = \sigma/(s^2+1)$. Solving gives, on the sufficient half-plane $\operatorname{Re} s > 0$,

$$\boxed{F = \frac{s^2 - \sigma}{(s^2+1)^2}, \qquad G = \frac{(1+\sigma)s}{(s^2+1)^2}.}$$

The numerator, not the unsimplified denominator alone, decides actual pole order.

Step 2: Invert the co-rotating input. For $\sigma = 1$, differentiating the standard transforms with respect to s gives $\mathcal{L}\{t\cos t\} = (s^2-1)/(s^2+1)^2$ and $\mathcal{L}\{t\sin t\} = 2s/(s^2+1)^2$. Hence

$$\boxed{x = t\cos t, \qquad y = t\sin t, \qquad \|Z(t)\| = t.}$$

Indeed $x' = \cos t - t\sin t = -y + \cos t$ and $y' = \sin t + t\cos t = x + \sin t$, with zero initial state. The response is unbounded even though each forcing component is bounded.

Step 3: Invert the opposite rotation. For $\sigma = -1$, cancellation gives $F = 1/(s^2+1)$ and $G = 0$. Consequently $\boxed{x = \sin t,\ y = 0}$, with norm $|\sin t| \le 1$. The first equation reads $\cos t = 0 + \cos t$; the second reads $0 = \sin t - \sin t$. Both initial conditions hold. No polynomial factor survives in this response.

Step 4: Interpret the distinction. For $\sigma = 1$, both transforms have double poles at $s = \pm i$; for $\sigma = -1$, the first has simple poles and the second is zero. The free system rotates counterclockwise. The input $(\cos t, \sin t)$ rotates with it, whereas $(\cos t, -\sin t)$ rotates oppositely. More explicitly, write $Z = R(t)V$, where R is the counterclockwise rotation matrix. Then $V' = (1,0)^T$ in the first case and $V' = (\cos 2t, -\sin 2t)^T$ in the second. Their integrals respectively grow linearly and remain bounded. A frequency match alone therefore does not force an unbounded response; the vector direction and resulting pole cancellations matter.

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Question 7

A scalar input drives a two-stage system:

$$x' = -x + u(t), \qquad y' = x - 3y, \qquad x(0) = y(0) = 0, \quad t \geq 0.$$

Assume u is continuous and of exponential order. In the inverse task, assume a prescribed target y and its first two derivatives are continuous and of exponential order. All claims about input bounds apply for every $t \geq 0$.

Tasks

1. Derive both transfer expressions in the Laplace domain and invert their kernels to express x and y as convolutions with u.
2. If $0 \leq u \leq 1$, prove a sharp constant upper bound on $y(t)$. Explain in what sense the bound is sharp.
3. Recover x and u from a prescribed y. State the initial compatibility conditions and justify uniqueness of the recovered input.
4. Apply the inverse formula to $y(t) = t^2 e^{-t}$. Find x, u, decide whether $0 \leq u \leq 1$ is possible, and give a second obstruction using the target's maximum.

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Question 7 – Solution

Strategy. Positive inverse kernels give all-time bounds; elimination then turns the same system into an exact input-recovery formula.

Step 1: Invert the transfer kernels. If $U = \mathcal{L}\{u\}$, the zero initial values give $F = U/(s+1)$ and $G = U/[(s+1)(s+3)]$. Partial fractions yield

$$x(t) = \int_0^t e^{-(t-r)}u(r)\,dr, \qquad \boxed{y(t) = \tfrac{1}{2}\int_0^t \big(e^{-(t-r)} - e^{-3(t-r)}\big)u(r)\,dr.}$$

These identities hold on any common right half-plane of convergence and, after inversion, for all $t \geq 0$.

Step 2: Use positivity to obtain a sharp bound. The kernel $k(v) = (e^{-v} - e^{-3v})/2$ is nonnegative for $v \geq 0$. For $0 \leq u \leq 1$,

$$0 \leq y(t) \leq \int_0^t k(v)\,dv = \tfrac{1}{3} - \tfrac{1}{2}e^{-t} + \tfrac{1}{6}e^{-3t} < \tfrac{1}{3} \quad (t < \infty).$$

The strict upper inequality follows from the positive remaining tail of k; at $t = 0$ it also holds. With $u \equiv 1$, $y(t) \to 1/3$. Thus $1/3$ is the least uniform constant upper bound, although it is not attained at finite time from zero.

Step 3: Recover the input and its compatibility conditions. The second equation forces $x = y' + 3y$. Substitution into the first gives

$$\boxed{u = y'' + 4y' + 3y, \qquad y(0) = 0, \quad y'(0) = 0.}$$

The conditions are necessary for both initial states to vanish. Conversely, under the stated regularity, these formulas satisfy both equations and the zero initial values whenever the conditions hold. They prove uniqueness: any input producing that y must give exactly this x and u.

Step 4: Test the proposed target. For $y = t^2e^{-t}$, the compatibility conditions hold, and direct calculation gives $\boxed{x = (2t + 2t^2)e^{-t},\ u = (2 + 4t)e^{-t}}$. The unique input is nonnegative but already has $u(0) = 2 > 1$; its maximum is $4e^{-1/2} > 1$ at $t = 1/2$. Independently, the target peaks at $t = 2$ with value $4e^{-2} > 1/3$, contradicting the kernel bound. It is therefore achievable by a continuous input, but not by an input confined to $[0, 1]$.

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Question 8

For $\varepsilon \geq 0$, let

$$x' = -\varepsilon x - y + 1, \qquad y' = x - \varepsilon y, \qquad x(0) = y(0) = 0, \quad t \geq 0.$$

A student evaluates $\lim_{s\to 0^+} s\mathcal{L}\{x\}$ and $\lim_{s\to 0^+} s\mathcal{L}\{y\}$ and declares these the limiting state for every ε. For rational transforms, the final-value theorem requires all poles of the simplified $s\mathcal{L}\{f\}$ to have negative real part.

Tasks

1. Find both transformed components for general ε and calculate the student's two proposed limits.
2. Solve the case $\varepsilon = 0$. Check the theorem's hypothesis and determine whether either component has a limit as $t \to \infty$.
3. For $\varepsilon > 0$, invert the transforms and establish the true limiting state. Explain why the theorem now applies.
4. Show that $y_\varepsilon \to y_0$ at each fixed time as $\varepsilon \to 0^+$ but not uniformly on $[0, \infty)$. Use times $2\pi n$ to obtain a lower bound on the uniform error. Sketch y_0 and $y_{1/2}$.

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Question 8 – Solution

Strategy. Check pole locations before applying the final-value theorem, and distinguish fixed-time convergence from an infinite-time assertion.

Step 1: Solve the transformed system. With $D = (s+\varepsilon)^2 + 1$, the equations give

$$F = \frac{s+\varepsilon}{sD}, \qquad G = \frac{1}{sD}.$$

A sufficient convergence half-plane is $\operatorname{Re} s > 0$. The proposed final values are $\varepsilon/(1+\varepsilon^2)$ and $1/(1+\varepsilon^2)$.

Step 2: Examine the undamped case. At $\varepsilon = 0$, inversion yields $\boxed{x = \sin t,\ y = 1 - \cos t}$. Neither has a limit: the values along successive maxima and minima differ. Both sF and sG have poles at $\pm i$, so the stated theorem's hypothesis fails. The finite algebraic values 0 and 1 do not prove convergence in time.

Step 3: Resolve the damped response. For $\varepsilon > 0$, decompose, for example,

$$G = \frac{1}{1+\varepsilon^2}\left(\frac{1}{s} - \frac{s+\varepsilon}{D} - \frac{\varepsilon}{D}\right).$$

Inverting this and the analogous decomposition of F gives

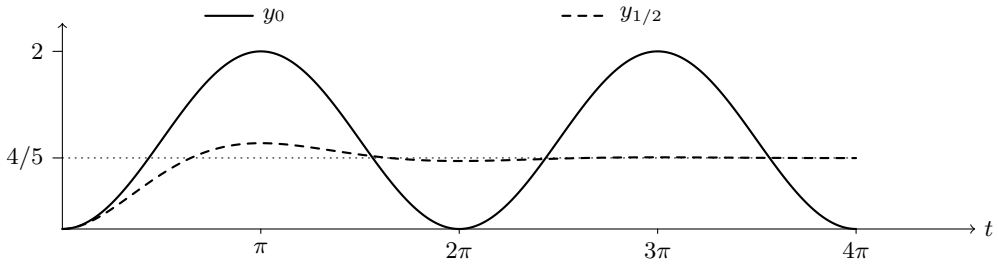
$$x_\varepsilon = \frac{\varepsilon - e^{-\varepsilon t}(\varepsilon\cos t - \sin t)}{1+\varepsilon^2}, \qquad y_\varepsilon = \frac{1 - e^{-\varepsilon t}(\cos t + \varepsilon \sin t)}{1+\varepsilon^2}.$$

Both start at zero. The exponentially decaying terms leave the limit $\boxed{(\varepsilon, 1)^T/(1+\varepsilon^2)}$. Now the poles of sF and sG are $-\varepsilon \pm i$, strictly in the left half-plane, as required. The limit also solves the equilibrium equations.

Step 4: Test uniformity on the whole half-line. At fixed t, the formula gives $y_\varepsilon(t) \to 1 - \cos t = y_0(t)$. But $y_0(2\pi n) = 0$, whereas $y_\varepsilon(2\pi n) = (1 - e^{-2\pi n\varepsilon})/(1+\varepsilon^2)$. Consequently

$$\boxed{\sup_{t \geq 0} |y_\varepsilon(t) - y_0(t)| \geq \frac{1}{1+\varepsilon^2}.}$$

This lower bound tends to 1, not 0. Fixed-time convergence therefore cannot be used to pass a limiting-state conclusion to the undamped system. The dotted guide is the damped limit 4/5 for $\varepsilon = 1/2$.



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Question 9

A rectangular input drives an initially resting oscillator:

$$x' = y, \qquad y' = -4x + u_T(t), \qquad x(0) = y(0) = 0, \qquad u_T(t) = H(t) - H(t-T), \quad T > 0.$$

Use the causal step convention for $t \geq 0$. States are continuous at T, and equations at the switch are understood one-sidedly.

Tasks

1. Find both Laplace transforms and express the response as an undelayed step response minus its delayed copy.
2. Calculate $(x(T), y(T))$ and determine exactly which positive pulse durations leave the state at rest forever after the pulse ends.
3. For arbitrary T, calculate the constant value of $y^2 + 4x^2$ after switch-off. Relate complete cancellation to zeros of the input transform at the oscillator poles.
4. Compare $T = \pi$ and $T = \pi/2$: give their post-pulse responses and sketch $x(t)$ on $0 \leq t \leq 2\pi$. Explain why a nonzero, nonnegative pulse can leave no residual oscillation.

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Question 9 – Solution

Strategy. Shift a single step response, then check both state components at switch-off; displacement alone is not a stopping condition.

Step 1: Transform and invert by shifting. Since $sF = G$ and $sG = -4F + (1 - e^{-Ts})/s$,

$$F = \frac{1 - e^{-Ts}}{s(s^2 + 4)}, \qquad G = \frac{1 - e^{-Ts}}{s^2 + 4}, \qquad \operatorname{Re} s > 0.$$

Let $h(t) = (1 - \cos 2t)/4$ and $j(t) = \sin 2t/2$. Then

$$\boxed{x = h(t) - H(t - T)h(t - T), \quad y = j(t) - H(t - T)j(t - T)}.$$

Both delayed terms begin at zero, proving state continuity.

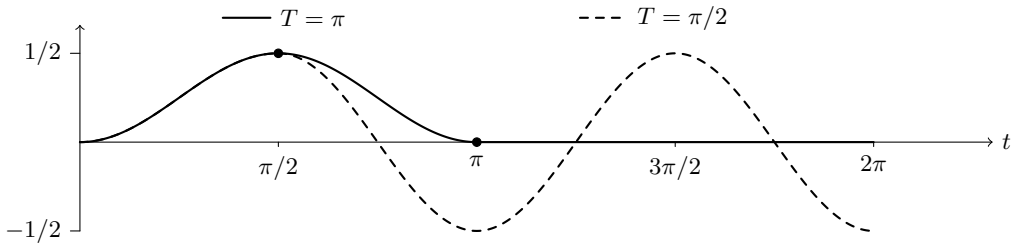
Step 2: Enforce complete rest at the endpoint. At T, the state is $((1 - \cos 2T)/4, \sin 2T/2)^T$. Both components vanish exactly when $\cos 2T = 1$, giving $\boxed{T = k\pi,\ k = 1, 2, \dots}$; the earliest is π. Zero endpoint data then yield the identically zero unforced continuation. Conversely, rest for all later times requires these endpoint values by continuity.

Step 3: Compute the residual and its spectral cancellation. After the pulse, differentiation gives $(y^2 + 4x^2)' = 2y(-4x) + 8xy = 0$. Evaluation at T yields

$$\boxed{y^2 + 4x^2 = \tfrac{1}{4}\sin^2 2T + \tfrac{1}{4}(1 - \cos 2T)^2 = \sin^2 T.}$$

The finite-pulse transform $U_T = (1 - e^{-Ts})/s$ has zeros at $s = \pm 2i$ exactly when $e^{2iT} = 1$. These are precisely the stopping durations; they cancel the oscillator poles. At $s = 0$ its singularity is removable, with value T, so a nonzero total input area does not prevent this cancellation.

Step 4: Compare two durations. For $T = \pi$, $x = y = 0$ after π. For $T = \pi/2$, writing $\tau = t - \pi/2 \geq 0$, the endpoint state is $(1/2, 0)$ and $\boxed{x = \frac{1}{2}\cos 2\tau,\ y = -\sin 2\tau}$ afterward. During either pulse, $x = h(t)$ and $y = j(t)$; the plotted curves coincide up to $\pi/2$. A full-period pulse lets the forced trajectory return to rest before switch-off; accumulated state contributions have different oscillatory phases and can cancel even when the input never becomes negative. Equal input sign is not equal state phase.



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Question 10

A known system has an unknown continuous input u and unknown initial state:

$$x' = -x + y, \qquad y' = -2x - 4y + u(t), \qquad t \geq 0.$$

The input and states are of exponential order. A proposed pair of measured one-sided transforms is

$$F(s) = \frac{s+3}{(s+1)(s+2)}, \qquad G(s) = \frac{1}{s+2}.$$

Assess compatibility before trying to infer the input.

Tasks

1. Recover both initial values from the transforms using the initial-value rule, then invert F and G.
2. Check the first transformed equation, including its initial term, and verify the corresponding time-domain equation.
3. Recover $\mathcal{L}\{u\}$ and $u(t)$ from the second equation. Prove whether knowledge of $x(t)$ alone would determine both $y(t)$ and $u(t)$ under the stated assumptions.
4. Suppose F is unchanged but the proposed second transform is instead $\widetilde{G}(s) = 1/(s+3)$. Can any continuous input in the second equation repair the discrepancy? Quantify the violated transformed identity.

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Question 10 – Solution

Strategy. Initial terms and the equation without the unknown input provide independent consistency checks before input reconstruction.

Step 1: Recover the initial state and signals. The initial-value rule gives $x(0) = \lim_{s\to\infty} sF(s) = 1$ and $y(0) = \lim_{s\to\infty} sG(s) = 1$. Partial fractions give

$$F = \frac{2}{s+1} - \frac{1}{s+2}, \qquad \boxed{x = 2e^{-t} - e^{-2t}, \quad y = e^{-2t}.}$$

Both recovered signals indeed start at one; their transforms share the convergence half-plane $\operatorname{Re} s > -1$.

Step 2: Check the input-free constraint. The first transformed equation is $(s+1)F - G = x(0) = 1$. Using the measured expressions, $(s+1)F - G = (s+3)/(s+2) - 1/(s+2) = 1$. In time, $x' = -2e^{-t} + 2e^{-2t} = -x + y$. Thus the two measurements pass a constraint that adjusting the second-equation input cannot alter.

Step 3: Reconstruct and establish uniqueness. The second equation gives $U = sG - y(0) + 2F + 4G$. Therefore

$$U = \frac{s+4}{s+2} - 1 + \frac{2(s+3)}{(s+1)(s+2)} = \frac{4}{s+1}, \qquad \boxed{u = 4e^{-t}.}$$

Directly, $y' = -2e^{-2t} = -2x - 4y + 4e^{-t}$. Even x alone determines $y = x' + x$ and then $\boxed{u = x'' + 5x' + 6x}$. The continuous input makes these derivatives well defined through the system. Thus two inputs giving the same exact x must give the same y and the same u; the initial state is recovered as well. This is an exact-data uniqueness claim, not a claim about noise sensitivity.

Step 4: Reject an incompatible second signal. The replacement still has initial value one, but

$$(s+1)F - \widetilde{G} - 1 = \frac{1}{s+2} - \frac{1}{s+3} = \boxed{\frac{1}{(s+2)(s+3)} \neq 0}.$$

Equivalently, $x' + x - \widetilde{y} = e^{-2t} - e^{-3t}$ is nonzero for $t > 0$. No choice of input confined to the second equation can repair this failure of the first equation. Matching initial values is necessary but does not guarantee compatibility of the complete measured signals.

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Question 1

Two perfectly mixed tanks contain 100 L and 200 L of solution. Two pumps transfer 10 L/min in each direction; there is no external inflow, outflow, or reaction. Initially the first tank contains 10 kg of salt and the second contains none. Let $x(t), y(t)$ be their salt amounts in kg, with t in minutes.

Tasks

1. Derive the amount equations from salt fluxes, explaining why the volumes stay constant and why the two transfer coefficients differ.
2. Find a conserved quantity and the equilibrium amounts. Does equilibrium mean equal amounts or equal concentrations?
3. Solve the initial-value problem and verify nonnegativity and conservation for all $t \geq 0$.
4. Find the earliest time at which the concentration difference is at most 0.005 kg/L, and show it stays below this tolerance thereafter. Sketch the two concentrations and their common limiting level.

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Question 1 – Solution

Strategy. Use volume flow times concentration for each salt flux, then use conservation to reduce the coupled balances.

Step 1: Construct the two balances. Each tank gains and loses 10 L/min, so its volume is fixed. Its outgoing salt rate is 10 times its own concentration. Thus

$$\boxed{x' = -\tfrac{1}{10}x + \tfrac{1}{20}y, \qquad y' = \tfrac{1}{10}x - \tfrac{1}{20}y.}$$

The coefficients have units min^{-1}: equal volume flows remove different fractions of the unequal tank volumes per minute.

Step 2: Use the conserved total. Adding gives $(x+y)' = 0$, so $x + y = 10$ kg. At equilibrium $x/100 = y/200$. Consequently $\boxed{(x_*, y_*) = (10/3, 20/3) \text{ kg}}$ and the common concentration is 1/30 kg/L. Equal amounts would leave unequal concentrations and hence a nonzero net salt transfer.

Step 3: Solve and check the physical range. Substituting $y = 10 - x$ gives $x' = 1/2 - 3x/20$. The initial data yield

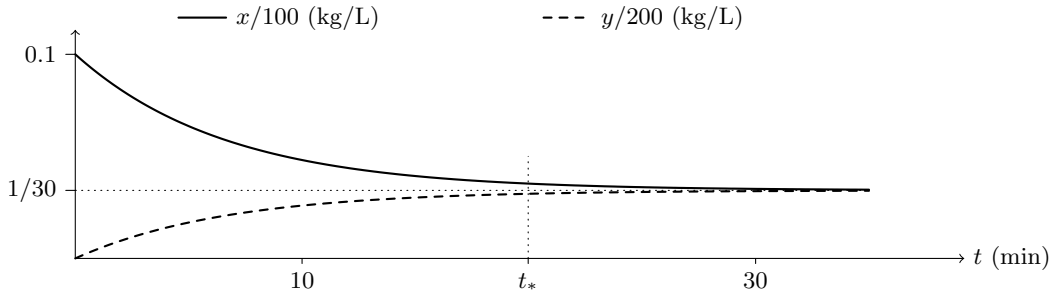
$$\boxed{x = \tfrac{10}{3} + \tfrac{20}{3}e^{-3t/20}, \qquad y = \tfrac{20}{3}(1 - e^{-3t/20}).}$$

Their sum is 10, both are nonnegative, and their initial values are $(10, 0)$. Differentiation gives $x' = -e^{-3t/20}$ and $y' = e^{-3t/20}$ kg/min; substitution into the original flux balances gives the same rates.

Step 4: Compute a lasting tolerance time. The concentration difference is

$$\frac{x}{100} - \frac{y}{200} = \frac{1}{10}e^{-3t/20} > 0.$$

It decreases strictly, so its first value 0.005 occurs at $\boxed{t = \tfrac{20}{3}\log 20 \text{ min} \approx 19.97 \text{ min}}$. The inequality holds for this time and every later time. The first concentration approaches the dotted limit from above; the second approaches it from below.



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Question 2

An idealized drug model has a central compartment and a peripheral compartment. Let x, y be drug amounts in mg, with time in hours. Each hour, transfer from central to peripheral is 1 times the current central amount; reverse transfer is 1 times the peripheral amount. Elimination occurs only from the central compartment, at rate 1 times its amount per hour. Immediately after a bolus, $(x(0), y(0)) = (D, 0)$, where $D > 0$ is the numerical dose in mg; no further drug enters.

Tasks

1. Derive the two amount equations and the rate of change of total drug. State which modeling assumptions justify linear transfer laws.
2. Explain why nonnegative initial amounts remain nonnegative. Add an eliminated-drug state $e(t)$ and find the conserved total, with $e(0) = 0$.
3. Solve the two-compartment IVP exactly in terms of D. Verify the initial data and determine the limiting retained and eliminated amounts.
4. Find $\int_0^\infty x(t)\,dt$ and $\int_0^\infty y(t)\,dt$ directly from the balance equations. Explain their units and why adding them does not count an eliminated mass twice.

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Question 2 – Solution

Strategy. Distinguish retained mass from cumulative elimination; integrated compartment balances recover exposure without integrating long formulas.

Step 1: Write transfer and loss rates. Well-mixed compartments and constant, unsaturated first-order rate constants give $\boxed{x' = -2x + y,\ y' = x - y}$. The numerical coefficients have units h^{-1}. Adding gives $(x + y)' = -x$: internal transfers cancel, but central elimination removes drug. Amounts, rather than concentrations, are the states; no unspecified compartment volume is needed for these laws.

Step 2: Establish the physical state constraints. On $x = 0, y \geq 0$, $x' = y \geq 0$; on $y = 0, x \geq 0$, $y' = x \geq 0$. These inward boundary directions and uniqueness preserve the nonnegative quadrant. With $e' = x$, $e(0) = 0$, $\boxed{x + y + e = D}$. Thus retained amounts lie between 0 and D, and e is nondecreasing. The third state records removal, not return flow.

Step 3: Resolve the two decay modes. The roots are $\lambda_{\pm} = (-3 \pm \sqrt{5})/2$, both negative. Solving for the constants gives, with $r = \sqrt{5}$,

$$\boxed{x = De^{-3t/2}\big(\cosh(rt/2) - r^{-1}\sinh(rt/2)\big), \quad y = \frac{2D}{r}e^{-3t/2}\sinh(rt/2).}$$

These give $x(0) = D, y(0) = 0, x'(0) = -2D, y'(0) = D$. Differentiating the formulas gives $x' = -2x + y$ and $y' = x - y$ for all t. Since $r < 3$, both modal exponentials decay: $x, y \to 0$ and $e \to D$.

Step 4: Integrate the balances. Decay makes both integrals finite. Integrating $(x + y)' = -x$ gives $\int_0^\infty x\,dt = D$ numerically, and integrating $y' = x - y$ then gives $\int_0^\infty y\,dt = D$. Restoring the rate units,

$$\boxed{\int_0^\infty x(t)\,dt = \int_0^\infty y(t)\,dt = D\ \mathrm{mg \cdot h}.}$$

These are amount–time exposures in mg·h, not eliminated masses. Their sum is $2D$ mg·h of total residence exposure, while the eliminated mass is only D mg. A molecule can contribute residence time in both compartments.

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Question 3

Two uniform bodies each have heat capacity 1 kJ/°C. Their mutual thermal conductance is 1 kJ/(min·°C), and each has that same conductance to a room maintained at 20°C. Heat flow equals conductance times temperature difference; there are no internal heat sources. Initially $T_1 = 80$°C and $T_2 = 20$°C. Time is in minutes.

Tasks

1. Derive the two energy balances. Introduce excess temperatures $\theta_i = T_i - 20$ and write the homogeneous system.
2. Find the total excess thermal energy and its decay law. Explain which heat transfers cancel in this sum.
3. Solve for both temperatures. Find the exact maximum temperature of the initially cooler body and when it occurs.
4. Explain why one body can warm while total excess energy decreases. Prove neither body leaves the interval $[20, 80]$°C, and sketch the temperatures.

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Question 3 – Solution

Strategy. Write heat-capacity balances before shifting the reference temperature; sum and difference variables isolate two decay rates.

Step 1: Balance heat for each body. For body 1, capacity times T_1' equals $(T_2 - T_1) + (20 - T_1)$ in kJ/min; the analogous balance holds for body 2. Hence

$$\boxed{\theta_1' = -2\theta_1 + \theta_2, \qquad \theta_2' = \theta_1 - 2\theta_2, \qquad (\theta_1(0), \theta_2(0)) = (60, 0).}$$

The numerical decay coefficients have units min^{-1}.

Step 2: Track total excess energy. With each capacity equal to 1 kJ/°C, excess energy is $E = \theta_1 + \theta_2$ numerically in kJ. Interbody exchange cancels, leaving $E' = -E$ and $\boxed{E(t) = 60e^{-t} \text{ kJ}}$. Only heat delivered to the room decreases this total.

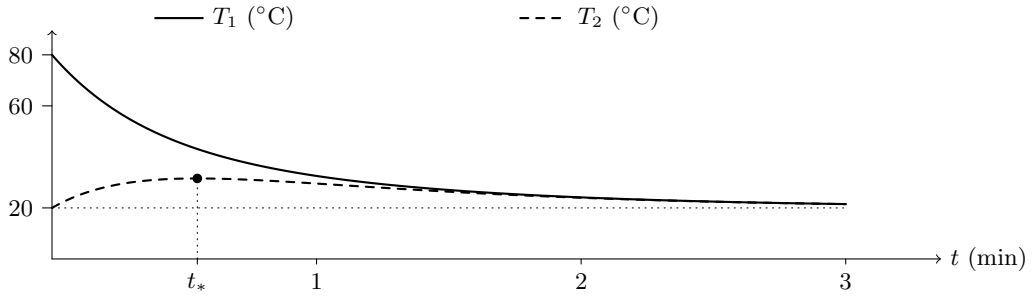
Step 3: Solve the modes and locate the warm-up peak. The difference obeys $(\theta_1 - \theta_2)' = -3(\theta_1 - \theta_2)$. Together with the sum this gives

$$\boxed{T_1 = 20 + 30(e^{-t} + e^{-3t}), \quad T_2 = 20 + 30(e^{-t} - e^{-3t}).}$$

Here $T_2' = 30(-e^{-t} + 3e^{-3t})$, which changes sign from positive to negative at $t_* = (\log 3)/2$. Therefore

$$\boxed{T_{2,\max} = (20 + 20/\sqrt{3})\ ^\circ\text{C}, \qquad t_* = (\log 3)/2 \text{ min}.}$$

Step 4: Reconcile local warming with total cooling. Initially the hotter body supplies heat to body 2 faster than body 2 loses heat to the room. Nevertheless $E' < 0$ throughout. The formulas give $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 60e^{-t} \leq 60$, so each temperature stays in $[20, 80]$°C. Both tend to room temperature; the second has a genuine temporary maximum, not a higher final equilibrium.



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Question 4

Two 1 kg masses slide without friction on a horizontal line. Each is attached to its outer fixed wall by a spring of stiffness 1 N/m, and a third spring of stiffness 1 N/m joins the masses. Let q_1, q_2 be displacements in metres from equilibrium, positive to the right. Initially $q_1 = a$, $q_2 = 0$ and both velocities vanish, where $a > 0$. Time is in seconds; Hooke's law applies.

Tasks

1. Derive the forces, paying attention to the change in length of the middle spring. Write a first-order system with four states and its initial vector.
2. Use the sum and difference of the displacements to solve for q_1, q_2 and their velocities.
3. Construct the total mechanical energy from the three springs and two masses. Show it is constant and find its initial value.
4. Determine whether the full initial state ever returns at a positive time. Justify your answer exactly, and sketch q_1, q_2 for $a = 0.1$ m on $0 \le t \le 10$ s.

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Question 4 – Solution

Strategy. The coupling spring exerts opposite forces on the masses; symmetric and antisymmetric motions then become independent oscillators.

Step 1: Derive the complete state model. The middle spring's extension is $q_2 - q_1$, giving force $q_2 - q_1$ on mass 1 and its negative on mass 2. Thus $q_1'' = -2q_1 + q_2$ and $q_2'' = q_1 - 2q_2$, with numerical acceleration coefficients in s^{-2}. For $Z = (q_1, v_1, q_2, v_2)^T$,

$$\boxed{Z' = (v_1, -2q_1 + q_2, v_2, q_1 - 2q_2)^T, \quad Z(0) = (a, 0, 0, 0)^T.}$$

Step 2: Solve the two normal modes. Set $s = q_1 + q_2$, $d = q_1 - q_2$. Then $s'' = -s$, $d'' = -3d$, with $s(0) = d(0) = a$ and zero initial derivatives. Therefore

$$\boxed{q_1 = \tfrac{a}{2}(\cos t + \cos\sqrt{3}t), \quad q_2 = \tfrac{a}{2}(\cos t - \cos\sqrt{3}t).}$$

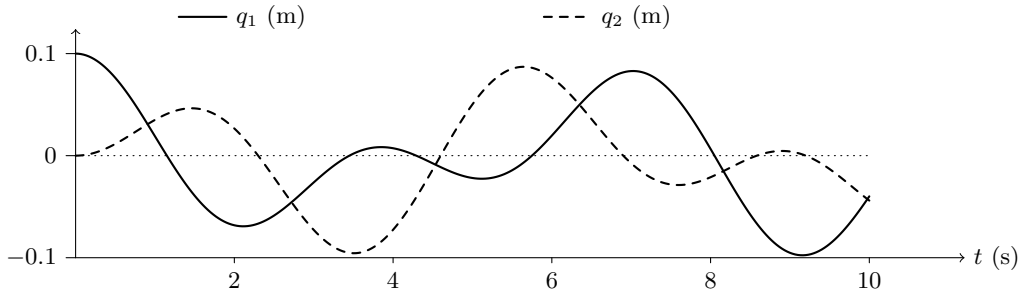
The velocities are $v_1 = -a(\sin t + \sqrt{3}\sin\sqrt{3}t)/2$ and $v_2 = -a(\sin t - \sqrt{3}\sin\sqrt{3}t)/2$. They satisfy all four initial data.

Step 3: Check the energy balance. In joules, with the stated numerical masses and stiffnesses,

$$E = \tfrac{1}{2}(v_1^2 + v_2^2) + \tfrac{1}{2}\big(q_1^2 + q_2^2 + (q_2 - q_1)^2\big).$$

Its derivative is $v_1(q_1'' + 2q_1 - q_2) + v_2(q_2'' + 2q_2 - q_1) = 0$. At the initial state $\boxed{E = a^2 \text{ J}}$ when a is the numerical initial displacement in metres. Energy includes the middle spring as well as the wall springs.

Step 4: Test an exact return. A return of the full state requires $s = d = a$ and $s' = d' = 0$. Thus $t = 2\pi m$ and $\sqrt{3}t = 2\pi n$ for integers m, n. For $t > 0$ this would imply $\sqrt{3} = n/m$, impossible. No exact positive-time return occurs. A near return visible in a plot cannot prove periodicity.



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Question 5

Two magnetically coupled loops have inductances $L_1 = L_2 = 2$ H, mutual inductance $M = 1$ H, and resistances $R_1 = R_2 = 1\ \Omega$. Reference current directions and coil orientations are chosen so the mutual derivative terms have positive signs in both voltage-drop balances. At $t = 0$ a constant 3 V source is connected to loop 1; loop 2 has no source. Initially both currents are zero. Time is in seconds; the ideal linear circuit model applies.

Tasks

1. Use Kirchhoff's voltage law to derive the coupled derivative equations, then solve them for i_1', i_2'. Explain why the inductance matrix is invertible.
2. Solve the initial-value problem using current sum and difference, and find the limiting currents.
3. Determine the sign of $i_2(t)$ for $t > 0$ and its most negative value and time. Explain what a negative current means here, and sketch both currents.
4. With magnetic energy $W = (2i_1^2 + 2i_1 i_2 + 2i_2^2)/2$, derive the power balance. Explain what changes when both voltage sources are zero.

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Question 5 – Solution

Strategy. Invert the inductance matrix, not just its diagonal; the sum and difference currents see different inductances.

Step 1: Form the coupled voltage balances. Voltage drops give $2i_1' + i_2' = 3 - i_1$ and $i_1' + 2i_2' = -i_2$. The inductance determinant is $4 - 1 = 3\ \mathrm{H}^2 > 0$, so

$$\boxed{i_1' = 2 - \tfrac{2}{3}i_1 + \tfrac{1}{3}i_2, \qquad i_2' = -1 + \tfrac{1}{3}i_1 - \tfrac{2}{3}i_2.}$$

Each right side has units A/s. Ignoring the mutual terms would predict $i_2 \equiv 0$ and would fail the second voltage balance during the transient.

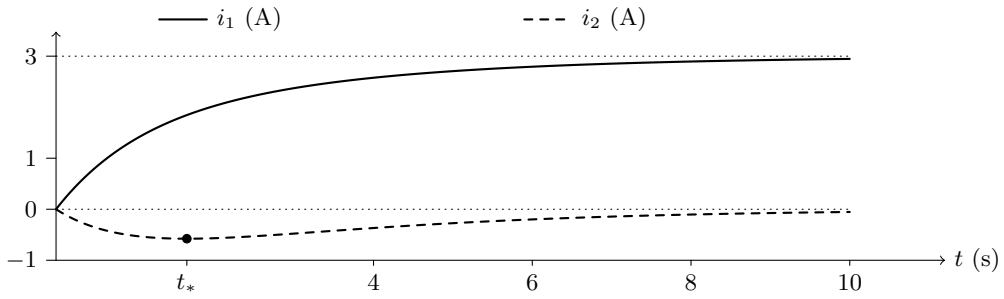
Step 2: Solve the sum and difference. Let $s = i_1 + i_2$, $d = i_1 - i_2$. Adding and subtracting the voltage equations gives $3s' = 3 - s$ and $d' = 3 - d$, with $s(0) = d(0) = 0$. Thus $s = 3(1 - e^{-t/3})$, $d = 3(1 - e^{-t})$, and

$$\boxed{i_1 = 3 - \tfrac{3}{2}(e^{-t/3} + e^{-t}), \qquad i_2 = \tfrac{3}{2}(e^{-t} - e^{-t/3}).}$$

They tend to $(3, 0)$ A, the resistive steady state. Their initial slopes are $(2, -1)$ A/s, agreeing with the inverted model.

Step 3: Locate the induced-current extremum. For $t > 0$, $e^{-t} < e^{-t/3}$, so $i_2 < 0$. Its derivative vanishes when $e^{-t/3}/3 = e^{-t}$, at $t_* = (3/2)\log 3$ s. The derivative changes from negative to positive, giving $\boxed{i_{2,\min} = -1/\sqrt{3}\ \mathrm{A}}$. The sign means actual current flows opposite its chosen reference direction; it is not a negative stored energy or a violation of the circuit model.

Step 4: Verify the power identity. Writing $L = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, $W' = i^T L i' = i^T(V - i)$ gives $\boxed{W' = 3i_1 - i_1^2 - i_2^2}$ in watts. With $V = 0$ instead, $W' = -i_1^2 - i_2^2 \le 0$. Since L has positive eigenvalues $1, 3$, $W > 0$ for nonzero currents. The driven circuit can gain magnetic energy from its source; the source-free circuit can only dissipate it.



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Question 6

In an idealized closed-population epidemic model, s, i, r are susceptible, infectious and removed fractions. There are no births, deaths or reinfections. Homogeneous mixing gives new infections at rate $2si$ per day and removal at rate i per day. Initially $(s, i, r) = (0.99, 0.01, 0)$, and t is in days. Assume the stated rates remain constant throughout the modeled outbreak.

Tasks

1. Derive the three balance equations and explain conservation and nonnegativity. Reduce to two independent states.
2. Eliminate time to derive a relation between i and s. Find the susceptible fraction and infectious fraction at the unique infectious peak; justify that this peak is reached.
3. Show $i(t) \to 0$ and $s(t) \to s_\infty > 0$. Derive an equation uniquely selecting s_∞ in $(0, 1/2)$ and estimate it to three decimals.
4. Sketch the physical path in the (s, i) plane with equal scales, its initial point, peak and limiting endpoint. Explain why the peak condition does not mean infections have ended.

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Question 6 – Solution

Strategy. Conservation confines the dynamics; eliminating time gives an exact curve and identifies both the peak and final-size constraint.

Step 1: Balance the three populations. The equations are $\boxed{s' = -2si,\ i' = 2si - i,\ r' = i}$, with $s + i + r = 1$. Boundary rates cannot create negative fractions. More explicitly, $s = 0.99e^{-2\int_0^t i}$ and $i = 0.01e^{\int_0^t (2s-1)}$ are positive at finite times; $r' = i \geq 0$. Use $r = 1 - s - i$ to reduce the system.

Step 2: Eliminate time and locate the peak. Since $s, i > 0$, $di/ds = -1 + 1/(2s)$. Integrating through the initial point gives

$$\boxed{i = 1 - s + \tfrac{1}{2}\log(s/0.99).}$$

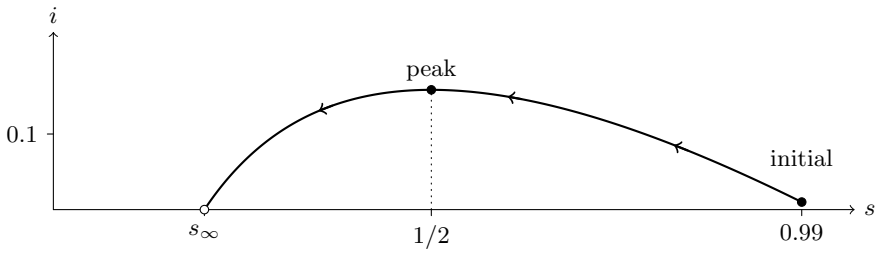
Here $i' = i(2s - 1)$ changes sign at $s = 1/2$. This value must be reached: if $s \geq 1/2$ forever, then $i \geq 0.01$ and $s' \leq -0.02s$, a contradiction. Strict decrease of s permits only one crossing. Hence $\boxed{i_{\max} = \tfrac{1}{2} + \tfrac{1}{2}\log(0.5/0.99) \approx 0.15845}$.

Step 3: Identify the physical final size. Because $\int_0^\infty i = r_\infty \leq 1$, $s \geq 0.99e^{-2} > 0$. Thus decreasing s has a positive limit. Also i is integrable and has a bounded derivative on the invariant simplex, so $i \to 0$: nonvanishing peaks would have widths bounded below and contradict integrability. The peak crossing implies $s_\infty < 1/2$. Substitution gives

$$\boxed{1 - s_\infty + \tfrac{1}{2}\log(s_\infty/0.99) = 0, \quad s_\infty \approx 0.200.}$$

On $(0, 1/2)$ the left side is strictly increasing, tends to $-\infty$ at zero and is positive at $1/2$, so this root is unique.

Step 4: Interpret the physical trajectory. Time moves toward smaller s, up to the peak and then down toward zero infectious fraction. The limiting endpoint is open because $i(t) > 0$ at finite times. At the peak, incidence and removal balance; both are positive. The peak marks zero net infectious growth, not the end of infection.



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Question 7

A continuous two-stage population model uses juvenile and adult abundances J, A, measured in thousands, with time in years. Juveniles mature at per-capita rate 1 yr^{-1} and die at rate 1 yr^{-1}. Adults die at rate 1 yr^{-1} and produce juveniles at per-capita rate b yr^{-1}, where $b \geq 0$. Assume no crowding and strictly positive initial J_0, A_0.

Tasks

1. Derive the stage equations and explain why maturation must appear with opposite signs in the two balances.
2. Find the eigenvalues and classify long-time extinction, a critical threshold, and unbounded growth as b varies. Justify that the growing mode is actually present for the stated initial data.
3. At the critical value, find a conserved weighted total, the exact solution and its limiting stage abundances.
4. For every $b \geq 0$, find the limiting ratio J/A and interpret it. Explain why this linear model cannot determine a realistic carrying capacity.

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Question 7 – Solution

Strategy. Separate transfers between stages from births and deaths, then use the dominant mode and the exceptional zero eigenvalue.

Step 1: Construct the stage balances. Maturation removes juveniles and adds adults, whereas juvenile death removes individuals without adding to the adult stage. Therefore $\boxed{J' = bA - 2J,\ A' = J - A}$. All coefficients have units yr^{-1}. Nonnegative off-diagonal rates preserve the nonnegative quadrant; the strictly positive initial populations remain positive.

Step 2: Determine the growth threshold. The characteristic equation is $\lambda^2 + 3\lambda + 2 - b = 0$, so $\lambda_\pm = (-3 \pm \sqrt{1+4b})/2$. Put $d = \sqrt{1+4b}$. The adult solution has dominant coefficient

$$A(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}, \qquad c_+ = \frac{J_0 + (1+d)A_0/2}{d} > 0.$$

Thus both populations tend to zero for $0 \le b < 2$, approach a nonzero critical equilibrium for $b = 2$, and grow without bound for $b > 2$. For $b > 2$, the dominant juvenile coefficient $(\lambda_+ + 1)c_+$ is also positive.

Step 3: Solve the critical case exactly. When $b = 2$, $K = J + 2A = J_0 + 2A_0$ is conserved. Since $A' = K - 3A$,

$$\boxed{A = K/3 + (A_0 - K/3)e^{-3t}, \qquad J = K/3 - 2(A_0 - K/3)e^{-3t}.}$$

Both tend to $K/3$ thousand. The critical equilibria form the line $J = A$; there is no single equilibrium amount independent of initial data.

Step 4: Interpret the asymptotic stage ratio. Since $J = A' + A$ and $c_+ > 0$, division by the dominant adult mode gives

$$\boxed{\lim_{t\to\infty} \frac{J}{A} = \lambda_+ + 1 = \frac{\sqrt{1+4b} - 1}{2}.}$$

For $b = 0$, this limit is zero: juveniles decay at rate 2, adults at rate 1. The ratio describes relative stage composition even when the entire population tends to zero; it does not imply constant abundance. For $b > 2$ there is no crowding term to halt growth. A carrying capacity requires additional density-dependent assumptions absent from this model.

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Question 8

An ideal temperature controller is modeled in dimensionless variables. The room temperature is 0, the target is 1, and the scaled heater output is $P(t)$. The plant obeys $T' = -T + P$. A controller with memory sets

$$P(t) = k \int_0^t [1 - T(\tau)]\, d\tau, \qquad T(0) = 0, \qquad k > 0.$$

There is no heater saturation, delay or heat-source noise in this ideal model.

Tasks

1. Convert the plant and controller into a two-state autonomous IVP. Find its equilibrium and explain why P is a state rather than a fixed parameter.
2. Shift to the equilibrium and show it attracts every initial state for every $k > 0$. Classify the eigenvalues at the critical gain.
3. For $k = 1/4$, solve the specified IVP. Prove both T and P stay in $[0, 1)$ at finite times and increase toward 1. Sketch them.
4. Replace the memory controller by the proportional rule $P = k(1 - T)$, with the same $k > 0$. Find its limiting temperature and explain the difference in steady error and the practical scope of the ideal memory model.

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Question 8 – Solution

Strategy. The accumulated error supplies an additional state; its equilibrium condition eliminates the temperature offset of proportional feedback.

Step 1: Introduce the controller state. Differentiation gives

$$\boxed{T' = -T + P, \quad P' = k(1 - T), \quad (T(0), P(0)) = (0, 0).}$$

At equilibrium, $T = 1$ and $P = 1$. Heater output depends on accumulated past error, so two states with the same temperature can have different future motion if their current P values differ.

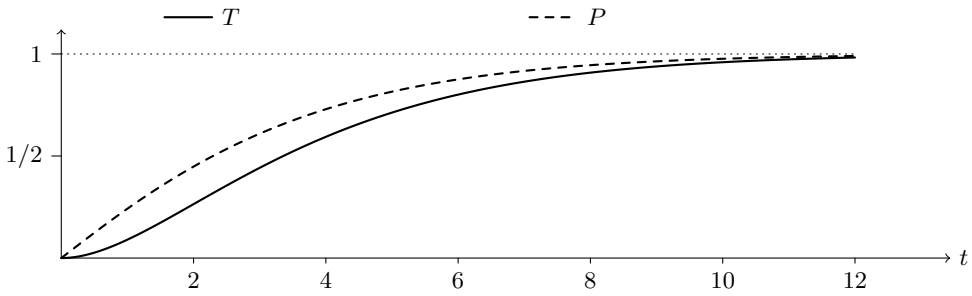
Step 2: Check stability and the critical gain. For $x = T - 1$, $u = P - 1$, the matrix is $\left(\begin{smallmatrix} -1 & 1 \\ -k & 0 \end{smallmatrix}\right)$. Its roots are $(-1 \pm \sqrt{1 - 4k})/2$. For $0 < k < 1/4$ both are real and negative; at $k = 1/4$ there is a repeated root $-1/2$; for $k > 1/4$ they are complex with real part $-1/2$. Every homogeneous mode decays, including the polynomial times exponential at the repeated root. Thus the equilibrium attracts every initial state within this ideal linear model.

Step 3: Solve the critical-gain response. Eliminating P gives $T'' + T' + T/4 = 1/4$, with $T(0) = T'(0) = 0$. The repeated-root solution and $P = T' + T$ yield

$$\boxed{T = 1 - (1 + t/2)e^{-t/2}, \qquad P = 1 - (1 + t/4)e^{-t/2}.}$$

The initial values vanish and $T' = te^{-t/2}/4 \geq 0$, $P' = (1/4 + t/8)e^{-t/2} > 0$. The latter equals $(1 - T)/4$, checking the controller equation. The positive subtracted terms and monotonicity imply $0 \leq T, P < 1$ at finite t, and both tend to 1. Here the heater output stays nonnegative without needing a saturation rule.

Step 4: Compare steady errors. Proportional feedback instead gives $T' = k - (1+k)T$, so $T \to k/(1+k)$ and the steady error is $\boxed{1/(1+k) > 0}$. The memory controller can retain $P = 1$ even when the current error is zero; a proportional controller would then set $P = 0$. Real actuator limits and delays can change stability or feasibility, so the ideal conclusions apply under the stated assumptions, not to every physical controller.



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Question 9

In a closed, well-mixed vessel of constant volume, the reversible reaction $A \rightleftharpoons 2B$ has forward reaction rate $k_f a$ and reverse reaction rate $k_r b^2$ in mol/(L·min), where a, b are concentrations in mol/L. Here $k_f = 1$ min^{-1} and $k_r = 1$ L/(mol·min). Initially $a = 1$, $b = 0$ mol/L. There are no other reactions. Use numerical concentration values in these units.

Tasks

1. Derive the concentration equations from the stoichiometric coefficients. Identify a conserved weighted concentration and explain why $a + b$ is not conserved.
2. Reduce to a scalar equation for b and identify its physically allowed interval. Show this interval is invariant.
3. Find the unique physical equilibrium and prove the stated initial response approaches it monotonically in b.
4. Solve explicitly for $b(t)$ by separation, then recover $a(t)$. Explain why the second algebraic root of the equilibrium equation is inadmissible and why the physical equilibrium is not reached at finite time.

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Question 9 – Solution

Strategy. Each forward reaction consumes one A and creates two B; the resulting weighted conservation law reduces the nonlinear system.

Step 1: Respect the reaction stoichiometry. A net forward reaction rate $a - b^2$ gives

$$\boxed{a' = -a + b^2, \qquad b' = 2a - 2b^2}$$

in the chosen numerical units. Hence $(2a + b)' = 0$ and $\boxed{2a + b = 2}$ mol/L. The sum $a + b$ has derivative $a - b^2$ and is not conserved: the reaction changes the number of molecules, while preserving the weighted building-block total.

Step 2: Reduce within the physical domain. With $a = 1 - b/2$, the scalar equation is $\boxed{b' = 2 - b - 2b^2}$, with $0 \le b \le 2$ required by $a, b \ge 0$. At $b = 0$ the derivative is $2 > 0$, and at $b = 2$ it is $-8 < 0$. Uniqueness and inward boundary directions make the interval invariant. The concentrations cannot be extended to negative values merely because an algebraic equation has another root.

Step 3: Identify the attracting physical equilibrium. Let $d = \sqrt{17}$ and $r_\pm = (-1 \pm d)/4$. Then $b' = -2(b - r_+)(b - r_-)$, with $r_- < 0 < r_+ < 2$. The only physical equilibrium is

$$\boxed{b_* = r_+, \qquad a_* = 1 - r_+/2.}$$

For $0 \le b < r_+$, $b' > 0$. Uniqueness prevents crossing the equilibrium, so the solution from zero increases and stays below r_+. Its limit must be a zero of the scalar right side; otherwise its positive derivative would remain bounded away from zero. Thus it tends to r_+.

Step 4: Separate and impose the initial concentration. Since $2(r_+ - r_-) = d$, integration gives

$$\log\left|\frac{b - r_+}{b - r_-}\right| = -dt + \log\left|\frac{r_+}{r_-}\right|.$$

The ratio is negative on the physical branch starting at zero. Solving gives

$$\boxed{b(t) = \frac{r_+(1 - e^{-dt})}{1 - (r_+/r_-)e^{-dt}}, \qquad a(t) = 1 - \frac{b(t)}{2}.}$$

The denominator exceeds 1 at finite times because $r_+/r_- < 0$. The formulas give $(a, b) = (1, 0)$ initially and the stated equilibrium as $t \to \infty$. For every finite t, $b < r_+$, so arrival is asymptotic.

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Question 10

A conserved tracer moves among three well-mixed compartments. Transfers occur only along $1 \rightleftharpoons 2 \rightleftharpoons 3$, with constant first-order rates, no external input and no loss. Let x_j be fractions of the total tracer, and use time in hours. Three experiments start with all tracer in compartment j; their measured initial derivative vectors are

$$v_1 = (-2, 2, 0)^T, \qquad v_2 = (1, -4, 3)^T, \qquad v_3 = (0, 2, -2)^T.$$

Treat these measurements as exact within the stated model class.

Tasks

1. Recover all four transfer rates and the matrix in $X' = AX$. Explain why the experiment vectors give columns rather than rows.
2. Verify conservation and nonnegativity, and explain why the unit simplex is the physical state space.
3. Find the unique equilibrium with total fraction 1. Check balance of each pair of opposing transfer fluxes.
4. Solve the first experiment exactly using the three modes. Determine whether equilibrium is reached at finite time, and state what the identification result does and does not establish about alternative model classes.

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Question 10 – Solution

Strategy. Initial states equal to coordinate vectors reveal matrix columns; conservation and nonnegative transfer rates then provide physical checks.

Step 1: Identify the directed rates. Since $X'(0) = Ae_j$, the jth experimental vector is column j. Thus

$$\boxed{A = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 3 & -2 \end{pmatrix}.}$$

The rates are $k_{1\to2} = 2$, $k_{2\to1} = 1$, $k_{2\to3} = 3$, $k_{3\to2} = 2\ \mathrm{h}^{-1}$. Each diagonal entry is minus the total outgoing rate from its compartment. Transposing the measurements would incorrectly replace these column balances with row balances.

Step 2: Check the physical state space. Every column sums to zero, so $(x_1 + x_2 + x_3)' = 0$. At a boundary $x_j = 0$ with the other fractions nonnegative, the corresponding derivative is a sum of nonnegative incoming fluxes. The simplex $\boxed{x_j \geq 0,\ x_1 + x_2 + x_3 = 1}$ is therefore invariant. Conservation bounds every fraction by 1.

Step 3: Balance the equilibrium fluxes. The end-compartment equations require $x_2 = 2x_1$ and $2x_3 = 3x_2$. Normalization then gives $\boxed{X_* = (1, 2, 3)^T/6}$. Indeed the opposing fluxes on the first edge are $2(1/6) = 1(2/6)$, and on the second are $3(2/6) = 2(3/6)$. These two relations and the total also prove uniqueness of the normalized equilibrium.

Step 4: Resolve the first experiment and its interpretation. The eigenpairs are $0, (1, 2, 3)^T$, $-2, (1, 0, -1)^T$ and $-6, (1, -4, 3)^T$. Resolving $(1, 0, 0)^T$ gives

$$X = X_* + \tfrac{3}{4}e^{-2t}(1, 0, -1)^T + \tfrac{1}{12}e^{-6t}(1, -4, 3)^T.$$

Equivalently,

$$\boxed{x_1 = \tfrac{1}{6} + \tfrac{3}{4}e^{-2t} + \tfrac{1}{12}e^{-6t}, \quad x_2 = \tfrac{1}{3}(1 - e^{-6t}), \quad x_3 = \tfrac{1}{2} - \tfrac{3}{4}e^{-2t} + \tfrac{1}{4}e^{-6t}.}$$

The coefficients reproduce the initial state, and each mode satisfies the matrix equation. Since $x_2 < 1/3$ at finite times, equilibrium is approached but never reached then. The experiments uniquely identify this constant linear transfer matrix; they do not rule out nonlinear or time-dependent models sharing the same three measured initial slopes.

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Question 1

Consider the real power series

$$F(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

A student reports a radius of convergence and assumes that this alone specifies where the series represents a finite function.

Tasks

1. Identify the center and radius of convergence. Explain what the ratio test says inside and outside the radius, and what it leaves undecided.
2. Test both boundary points separately. Give the exact interval of convergence and distinguish absolute from conditional convergence.
3. Derive a closed formula for the sum inside the open interval by integrating a geometric series. Include the constant fixed by the value at the center.
4. Determine the sum at the included boundary point without substituting into an unjustified endpoint identity. Sketch the convergence interval and explain why the excluded endpoint behaves differently.

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Question 1 – Solution

Strategy. Normalize the displacement from the center, test the endpoints as numerical series, then justify any boundary value independently.

Step 1: Apply the ratio test away from the center. Write $z = (x-2)/3$. For $z \neq 0$, the ratio of successive term magnitudes is $|z|n/(n+1) \to |z|$. Thus the center is 2 and $\boxed{R = 3}$. The series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$. At the center all terms vanish. The ratio test is inconclusive at $z = \pm 1$.

Step 2: Resolve the two endpoints. At $x = -1$, the series is $\sum_{n\geq 1}(-1)^n/n$, which converges by alternation but not absolutely. At $x = 5$, it is the divergent harmonic series. Therefore

$$\boxed{I = [-1, 5), \quad \text{absolute on } (-1, 5), \quad \text{conditional only at } -1.}$$

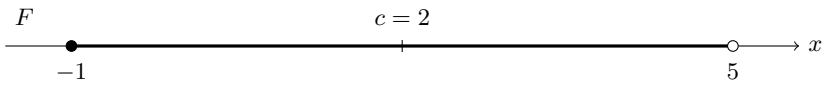
A radius specifies distances, not whether either boundary point is included.

Step 3: Integrate the geometric series. On any compact subinterval of $(-1, 1)$, the geometric series converges uniformly, so integration from 0 to z gives

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \frac{du}{1-u} = -\log(1-z).$$

This also holds for negative z with the usual oriented integral. Both sides vanish at $z = 0$, fixing the constant. Hence $\boxed{F(x) = -\log(1-(x-2)/3)}$ for $-1 < x < 5$.

Step 4: Establish the boundary value. The finite geometric identity gives $\sum_{k=0}^{N-1}(-u)^k = (1-(-u)^N)/(1+u)$. Integrating over $[0, 1]$, its difference from $\log 2$ has magnitude at most $\int_0^1 u^N\, du = 1/(N+1) \to 0$. Negating the resulting alternating harmonic sum proves $\boxed{F(-1) = -\log 2}$. At $x = 5$ there is no alternation and harmonic divergence persists; the interior logarithm also tends to $+\infty$ there.



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Question 2

A power series with large gaps between nonzero coefficients is written as

$$L(x) = \sum_{k=0}^{\infty} 2^k x^{2^k}.$$

The first terms are $x + 2x^2 + 4x^4 + 8x^8 + \cdots$. Use the ordinary definition of a power series even though many coefficients vanish.

Tasks

1. Write its coefficient rule in the form $\sum_{n=0}^{\infty} a_n x^n$. Explain why applying $|a_{n+1}/a_n|$ to all consecutive coefficients is inappropriate.
2. Prove the exact radius using comparison when $|x| < 1$ and the necessary term test when $|x| > 1$.
3. Test both endpoints, paying attention to the parity of 2^k. Does the endpoint $x = -1$ produce an alternating tail?
4. Let $x_m = e^{-1/2^m}$ for integers $m \geq 0$. Use a single term to bound $L(x_m)$ from below, and decide whether the sparse exponents make L bounded as $x \to 1^-$.

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Question 2 – Solution

Strategy. Treat the missing coefficients explicitly. A convergent positive comparison series controls the interior, while selected large terms control boundary growth.

Step 1: Identify the coefficients. We have $a_n = n$ when $n = 2^k$ for an integer $k \geq 0$, and $a_n = 0$ otherwise, including $n = 0$. Consecutive coefficient ratios repeatedly have zero denominators, so the usual coefficient-ratio shortcut is unavailable. This does not prevent other convergence tests from determining the radius.

Step 2: Establish the radius directly. For $r = |x| < 1$,

$$\sum_{k=0}^{\infty} 2^k r^{2^k} \leq \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} < \infty.$$

Thus the series converges absolutely throughout $|x| < 1$. If $|x| > 1$, the magnitudes $2^k |x|^{2^k}$ do not tend to zero; indeed they grow without bound. Therefore $\boxed{R = 1}$. The comparison sum follows by differentiating the geometric series inside its radius.

Step 3: Test parity at the endpoints. At $x = 1$, the terms are 2^k and fail the term test. At $x = -1$, the first term is -1, but every term with $k \geq 1$ is $+2^k$, since 2^k is even. There is no alternating tail. Both endpoint series diverge, so $\boxed{I = (-1, 1)}$, with absolute convergence at every point of this interval.

Step 4: Prove unbounded growth despite the gaps. For $0 < x_m < 1$, every term is positive. Keeping just the term indexed by $k = m$ gives

$$\boxed{L(x_m) \geq 2^m x_m^{2^m} = 2^m / e \longrightarrow \infty.}$$

Also $x_m \uparrow 1$. Thus L is not bounded near 1. In fact the sum is increasing on $[0, 1)$ because each term is increasing, so this sequence of lower bounds proves $L(x) \to +\infty$ as $x \to 1^-$. Sparse powers do not offset the growth of the coefficients and their near-boundary terms.

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Question 3

Let

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad G(x) = \int_0^x F(t)\,dt.$$

Use termwise differentiation and integration only where their hypotheses hold. No closed formula for F itself is required.

Tasks

1. Determine the radius and exact convergence interval of the series for F, including the type of convergence at each endpoint.
2. Obtain power series for F' and G. Determine each radius and interval, testing the endpoints afresh rather than copying them from F.
3. Sum the derivative series for $|x| < 1$ using the logarithmic series, treating $x = 0$ separately. Find its limit as $x \to -1^+$ and as $x \to 1^-$.
4. Prove F is continuous on $[-1, 1]$. Determine its one-sided endpoint slopes and explain why a finite endpoint value need not have a finite slope. Compare the three convergence intervals in a diagram.

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Question 3 – Solution

Strategy. Differentiation and integration preserve the radius but can alter endpoints; continuity and the mean value theorem settle endpoint slopes.

Step 1: Check the original series. The ratio test gives $R = 1$. At $x = 1$ and $x = -1$, the absolute values sum to $\sum 1/n^2 < \infty$. Thus $\boxed{I_F = [-1, 1]}$, absolutely throughout. For $|x| > 1$ the terms do not tend to zero.

Step 2: Differentiate and integrate inside the radius. For $|x| < 1$,

$$\boxed{F'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \qquad G(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)}.}$$

The constant in G is zero because $G(0) = 0$. Both new radii are 1. For the derivative series, $x = 1$ gives harmonic divergence; $x = -1$ gives conditional alternating convergence. Hence $I_{F'} = [-1, 1)$. The integrated series converges absolutely at both endpoints, so $I_G = [-1, 1]$. Here these interval labels refer to the displayed series, not an assumed endpoint differentiation rule.

Step 3: Sum the derivative in the open interval. Using $\sum_{n\geq 1} x^n/n = -\log(1-x)$ gives

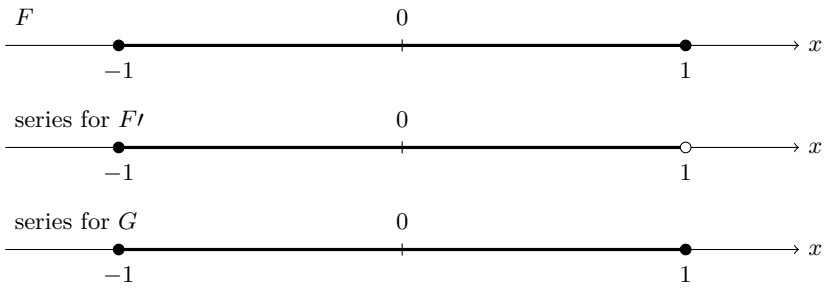
$$F'(x) = \frac{-\log(1-x)}{x} \quad (0 < |x| < 1), \qquad F'(0) = 1.$$

The value at zero follows from the constant term and also from the limit. Consequently $\lim_{x\to -1^+} F'(x) = \log 2$, whereas $\lim_{x\to 1^-} F'(x) = +\infty$.

Step 4: Justify what happens at the boundary. Since $|x^n/n^2| \leq 1/n^2$ on $[-1, 1]$, the uniform majorant test gives uniform convergence there and continuity of F. Apply the mean value theorem between an interior point and either endpoint. The intervening derivative then tends to the corresponding limit found above. Therefore

$$\boxed{F'_+(-1) = \log 2, \qquad \lim_{x\to 1^-} \frac{F(1) - F(x)}{1 - x} = +\infty.}$$

So the finite value $F(1)$ has no finite left derivative. Endpoint behavior must be proved separately even when the power-series radius is unchanged.



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Question 4

For an integer $N \geq 0$, define

$$S_N(x) = \sum_{n=0}^{N} x^n, \qquad f(x) = \frac{1}{1-x}, \qquad |x| < 1.$$

A student says that choosing a sufficiently large N makes this approximation uniformly accurate throughout the entire convergence interval.

Tasks

1. Derive the exact remainder $f - S_N$ from a finite geometric identity. State its sign for positive and negative x.
2. Find the least N guaranteeing absolute error at most 10^{-3} for every $x \in [-1/2, 1/2]$. State both the largest exponent and the number of retained terms.
3. Express the relative error $|f - S_N|/|f|$. For any fixed N, find a point in $(0, 1)$ where this error is exactly $1/2$.
4. Decide whether convergence is uniform on $(-1, 1)$, for either absolute or relative error, and justify the distinction from convergence at each fixed point. Plot S_2, S_5, S_{10} and f on $[0, 0.85]$.

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Question 4 – Solution

Strategy. An exact remainder supplies a sharp interval bound and exposes what fails when points approach the boundary as the degree changes.

Step 1: Retain the exact geometric remainder. Multiplication gives $(1-x)S_N = 1-x^{N+1}$. Therefore

$$\boxed{f(x) - S_N(x) = \frac{x^{N+1}}{1-x}.}$$

For $0 < x < 1$ it is positive. For $-1 < x < 0$, its sign is $(-1)^{N+1}$; at $x = 0$ it is zero. The denominator is positive throughout the interval.

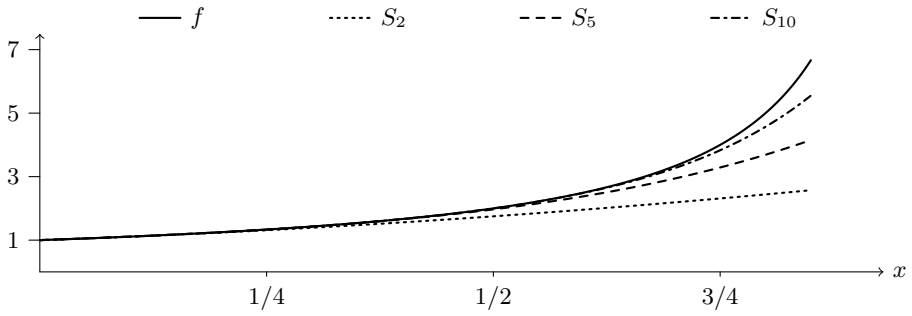
Step 2: Find the least degree on the compact interval. For $|x| \le 1/2$,

$$|f - S_N| \le \frac{(1/2)^{N+1}}{1-1/2} = 2^{-N}.$$

Equality holds at $x = 1/2$, so this is the exact maximum, not merely a convenient estimate. Since $2^{-9} > 10^{-3}$ and $2^{-10} < 10^{-3}$, $\boxed{N = 10}$ is least. This retains $\boxed{11 \text{ terms}}$, including the constant.

Step 3: Select a moving point with fixed relative error. The relative error simplifies to $\boxed{|x|^{N+1}}$. Taking $x_N = 2^{-1/(N+1)}$ gives a point in $(0, 1)$ with relative error exactly $1/2$ for every $N \ge 0$. These points approach 1 as N increases.

Step 4: Distinguish fixed-point from uniform accuracy. At each fixed $|x| < 1$, both errors tend to zero. Yet for every fixed N, $x^{N+1}/(1-x) \to +\infty$ as $x \to 1^-$, so the supremum of the absolute error on $(-1, 1)$ is infinite. The supremum of relative error is 1. Neither error converges uniformly to zero on that whole interval. The plot shows slower approximation near the right boundary; on any fixed $[-r, r]$, $r < 1$, the bound $r^{N+1}/(1-r) \to 0$ does give uniform convergence.



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Question 5

Consider two series centered at zero:

$$A(x) = \sum_{n=0}^{\infty} x^n, \qquad B(x) = 1 - \sum_{n=1}^{\infty} \frac{x^n}{2^n}.$$

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the series obtained by their Cauchy product, using finite coefficient convolutions.

Tasks

1. Find the sums and exact convergence intervals of A and B separately.
2. Compute c_n for every $n \geq 0$, including the constant coefficient. Determine the radius and interval of the resulting series C.
3. Identify the interval where the Cauchy-product theorem directly proves $C = AB$. Explain the rational cancellation that permits the new series to converge on a larger interval.
4. Evaluate the series C at $x = 1$ and $x = 3/2$. At each point, decide whether this value can legitimately be called the product of the two original series sums.

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Question 5 – Solution

Strategy. Compute the product coefficients before interpreting the enlarged domain; a new convergent series does not make a divergent factor converge.

Step 1: Sum and test the two factors. Geometric summation gives

$$A(x) = \frac{1}{1-x} \quad (|x| < 1), \qquad B(x) = 1 - \frac{x/2}{1-x/2} = \frac{1-x}{1-x/2} \quad (|x| < 2).$$

At each respective endpoint the terms fail to tend to zero. Thus the exact intervals are $(-1, 1)$ and $(-2, 2)$, both absolutely convergent inside.

Step 2: Compute the finite coefficient sums. The coefficients of A are all 1; those of B are $b_0 = 1$ and $b_n = -2^{-n}$ for $n \geq 1$. Hence

$$c_n = \sum_{k=0}^{n} b_k = 1 - \sum_{k=1}^{n} 2^{-k} = 2^{-n} \qquad (n \geq 0).$$

In particular $c_0 = 1$. Therefore $\boxed{C(x) = \sum_{n\geq 0} (x/2)^n = 1/(1-x/2)}$ for $|x| < 2$. Its radius is 2 and both endpoints diverge.

Step 3: State where multiplication is justified. Both factors converge absolutely on $(-1, 1)$, so the Cauchy-product theorem proves equality there. Their rational formulas give

$$\frac{1}{1-x}\frac{1-x}{1-x/2} = \frac{1}{1-x/2}.$$

Cancellation removes the factor responsible for A's boundary at 1. The coefficient series C consequently has radius 2, exceeding the smaller input radius. The theorem guarantees convergence on the common interior; it does not require the resulting radius to equal the smaller one.

Step 4: Interpret the additional points carefully. The new series gives $\boxed{C(1) = 2,\ C(3/2) = 4}$. At $x = 1$, A diverges and B sums to zero; an undefined series sum cannot be multiplied by zero to produce 2. At $x = 3/2$, A again diverges, although B converges to -2. Thus neither value is a product of the two original series sums. They are valid values of the independently convergent product-coefficient series and its rational formula.

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Question 6

Let $f(x) = 1/(4 - x)$, defined for real $x \neq 4$. Represent this same function by geometric power series with two different centers, $c_1 = 1$ and $c_2 = -1$. Do not use a Taylor-coefficient formula.

Tasks

1. Rewrite the denominator about each center and obtain the two power series explicitly.
2. Find both radii and exact real convergence intervals, testing their endpoints. Mark centers and intervals in a diagram.
3. At $x = -4$, decide which series converges and find its sum. At $x = 5$, decide whether either series converges even though $f(5)$ exists.
4. Explain why the two series agree wherever both converge, why changing the center changes the radius here, and why the domain of the rational function is not the convergence interval of either series.

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Question 6 – Solution

Strategy. Factor out the distance from each center to the denominator's zero, then apply the geometric-series condition to the normalized displacement.

Step 1: Expand about each center. Using $4 - x = 3 - (x - 1)$ and $4 - x = 5 - (x + 1)$ gives

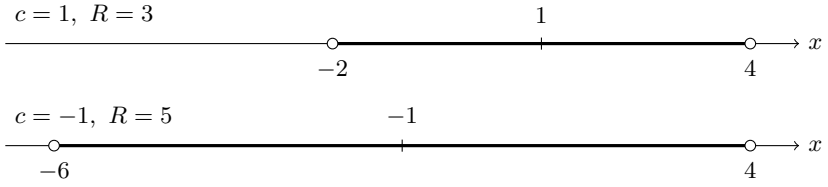
$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^{n+1}}, \qquad f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{5^{n+1}}.}$$

Each equality is initially asserted only when its geometric ratio has magnitude less than one.

Step 2: Determine the two intervals. The first ratio is $(x - 1)/3$, so $R_1 = 3$ and $I_1 = (-2, 4)$. The second is $(x + 1)/5$, so $R_2 = 5$ and $I_2 = (-6, 4)$. At each endpoint the terms have constant nonzero magnitude ($1/3$ or $1/5$), so all four endpoint tests give divergence. Both series converge absolutely inside their respective intervals.

Step 3: Test the specified points. At $x = -4$, the first ratio is $-5/3$, so its terms do not tend to zero. The second ratio is $-3/5$, so it converges and sums to $\boxed{(1/5)/(1 + 3/5) = 1/8 = f(-4)}$. At $x = 5$, the ratios are $4/3$ and $6/5$, so both series diverge even though $\boxed{f(5) = -1}$ is a valid rational-function value.

Step 4: Compare representations and domains. On the common interval $(-2, 4)$, each convergent geometric identity gives the same function $1/(4 - x)$, so their sums agree. Their right endpoints coincide with the denominator's zero, but this point is distance 3 from one center and distance 5 from the other. The left boundaries follow from the same absolute-ratio conditions, even though f is finite there. The rational function has domain $\mathbb{R} \setminus \{4\}$, while each series represents it on only its own convergence interval. Evaluation of the rational formula outside that interval is not evaluation of the series.



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Question 7

Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$ has positive radius R. Consider $H(x) = xy'(x) + y(x)$ only for $|x-2| < R$. A student claims that $H(x) = \sum_{n=0}^{\infty} (n+1) a_n (x-2)^n$.

Tasks

1. Derive the correct coefficient of $(x-2)^n$ in H for every $n \geq 0$, reindexing the derivative carefully.
2. Write the constant, linear, quadratic and cubic coefficients explicitly in terms of the a_n.
3. Identify the student's missing contribution and characterize exactly which series y make the claimed identity valid throughout the open interval.
4. Apply your formula to $a_n = 3^{-n-1}$. Sum y geometrically, compute H directly from this closed form, and verify the coefficients agree. State the resulting series' radius.

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Question 7 – Solution

Strategy. Work in $h = x - 2$ and remember that multiplication by x means multiplication by $h + 2$, not merely by h.

Step 1: Reindex before multiplying. Inside the radius, $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}h^n$. Now $hy' = \sum_{n=1}^{\infty} na_n h^n$; its constant coefficient is zero. Adding $2y'$ and y yields

$$\boxed{H = \sum_{n=0}^{\infty}\big((n+1)a_n + 2(n+1)a_{n+1}\big)h^n.}$$

The formula also works for $n = 0$, where the constant term of xy' comes entirely from $2y'$.

Step 2: Display the first four coefficients. The requested coefficients are

$$\boxed{a_0 + 2a_1, \quad 2a_1 + 4a_2, \quad 3a_2 + 6a_3, \quad 4a_3 + 8a_4.}$$

As a direct constant-term check, $H(2) = 2y'(2) + y(2) = 2a_1 + a_0$. Keeping the same summation index through a differentiation would miss the shift from a_n to a_{n+1}.

Step 3: Determine exactly when the shortcut works. The claimed series actually represents $hy' + y$. Its difference from H is $2y'$. The identity therefore holds throughout the interval exactly when $y' \equiv 0$, or equivalently $\boxed{a_n = 0 \text{ for every } n \geq 1}$. One justification of the equivalence is uniqueness of power-series coefficients, obtained by successive differentiation at the center. Any constant a_0 is allowed. Equality at a single isolated point would not imply this identity.

Step 4: Check against a geometric closed form. For $a_n = 3^{-n-1}$, $y = (1/3)\sum(h/3)^n = 1/(3-h) = 1/(5-x)$ when $|h| < 3$. Hence

$$\boxed{H = \frac{x}{(5-x)^2} + \frac{1}{5-x} = \frac{5}{(5-x)^2}.}$$

The coefficient formula gives $(n+1)/3^{n+1} + 2(n+1)/3^{n+2} = 5(n+1)/3^{n+2}$. This agrees with the differentiated geometric identity $5/(3-h)^2 = \sum_{n\geq 0} 5(n+1)h^n/3^{n+2}$. The radius is $\boxed{3}$; the factors growing linearly in n do not change it.

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Question 8

Consider a series of polynomials,

$$P(x) = \sum_{n=0}^{\infty} \left(\frac{x^2 - 4}{2} \right)^n .$$

It is geometric in $(x^2 - 4)/2$, but its displayed terms are not single monomials in x. Investigate how its convergence set compares with that of an ordinary power series representing the same sum.

Tasks

1. Determine the exact real convergence set of the displayed series, including all boundary tests. Explain why it need not be one interval centered at zero.
2. Sum the series wherever it converges. Compare the domain of that rational formula with the convergence set just found.
3. Starting afresh from the rational formula, derive an ordinary power series centered at zero. Find its radius and endpoint behavior.
4. Compare the two series at $x = 0$ and $x = 1$, and state exactly where their sums are known to agree. Diagram both convergence sets and explain why one should not silently substitute the new domain for the original one.

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Question 8 – Solution

Strategy. Apply the geometric test to the substituted variable first; then derive and test a separate series in powers of x.

Step 1: Pull back the geometric convergence condition. The displayed series converges exactly when $|(x^2-4)/2|<1$, or $2<x^2<6$. Thus its convergence set is

$$\boxed{(-\sqrt{6},-\sqrt{2})\ \cup\ (\sqrt{2},\sqrt{6}).}$$

At $x=\pm\sqrt{2}$ the ratio is -1; at $x=\pm\sqrt{6}$ it is 1. In each case the terms fail to approach zero. Elsewhere outside the two intervals the ratio has magnitude greater than one. These polynomial terms do not have the ordinary form $a_n(x-c)^n$ for a fixed center.

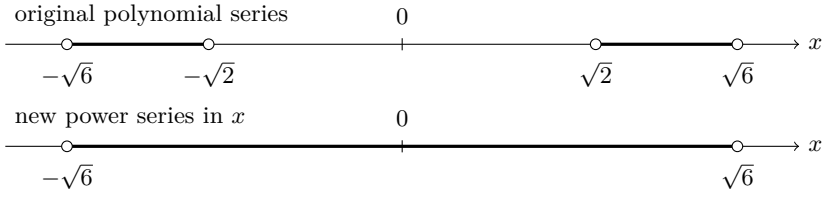
Step 2: Sum on the justified set. Geometric summation gives $\boxed{P(x)=2/(6-x^2)}$ on the two intervals above. This rational formula is defined for every real $x\neq\pm\sqrt{6}$, which is a much larger set. Its being finite at other points says nothing by itself about convergence of the displayed polynomial series there.

Step 3: Construct a distinct centered power series. Factoring the denominator differently gives

$$\boxed{\frac{2}{6-x^2}=\frac{1/3}{1-x^2/6}=\sum_{n=0}^{\infty}\frac{x^{2n}}{3\cdot 6^n}},\qquad |x|<\sqrt{6}.$$

Its odd coefficients vanish. Its radius is $\sqrt{6}$ and both endpoints diverge because the terms there equal $1/3$. The new convergence interval is $(-\sqrt{6},\sqrt{6})$.

Step 4: Compare actual sums, not just formulas. At $x=0$, the original terms are $(-2)^n$, so the original series diverges; the new series sums to $\boxed{1/3}$. At $x=1$, the original ratio is $-3/2$ and its series diverges, while the new series sums to $\boxed{2/5}$. On $\sqrt{2}<|x|<\sqrt{6}$, both converge and both equal $2/(6-x^2)$. The new representation adds valid points; it does not retroactively change the convergence set of the original sequence of partial sums.



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Question 9

For a real parameter p, consider

$$F_p(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^p}.$$

Use this family to investigate which endpoint patterns can be designed by changing coefficient decay, and which require changing coefficient signs.

Tasks

1. Find the center and radius for every real p, including $p \leq 0$.
2. Give a complete classification of convergence and absolute convergence at both endpoints as p varies. Include the threshold cases $p = 0$ and $p = 1$.
3. Determine all p yielding each of the intervals $(-2, 4)$, $[-2, 4)$ and $[-2, 4]$. Supply one concrete value for each pattern.
4. Can any member of this family have convergence interval $(-2, 4]$? Prove your answer, then construct another power series centered at 1 with radius 3 and exactly that interval.

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Question 9 – Solution

Strategy. The exponential coefficient scale fixes the radius, while power-law decay and signs decide the endpoints.

Step 1: Determine the radius independently of p. For $x \neq 1$, the successive absolute-term ratio is

$$\frac{|x-1|}{3}\left(\frac{n}{n+1}\right)^p \longrightarrow \frac{|x-1|}{3}.$$

Thus the center is 1 and $\boxed{R=3}$ for every finite real p. At the center the sum is zero. The open interval $(-2,4)$ is always a region of absolute convergence, and outside the closed interval the series diverges.

Step 2: Classify the endpoints. At $x=4$, the series is $\sum 1/n^p$, convergent exactly when $p>1$. At $x=-2$, it is $\sum(-1)^n/n^p$: it converges for $p>0$, since the magnitudes then decrease to zero, and fails the term test for $p\leq 0$. Absolute convergence at either endpoint occurs exactly for $p>1$. Thus the left endpoint is conditional for $0<p\leq 1$, including $p=1$; at $p=0$ neither endpoint's terms tend to zero.

Step 3: List all parameter regimes. Combining the tests gives

parameter range	convergence interval	example
$p\leq 0$	$(-2,4)$	$p=0$
$0<p\leq 1$	$[-2,4)$	$p=1$
$p>1$	$[-2,4]$	$p=2$

No other parameter cases remain. Changing p affects the boundary tests, not the radius or the open interval of absolute convergence.

Step 4: Reverse the endpoint pattern by signs. Within the given family, convergence at the right endpoint implies $p>1$, which forces absolute convergence at the left endpoint too. Hence $(-2,4]$ is impossible in this family. A different series that works is

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n3^n}.}$$

Its radius is 3. At $x=4$ it is alternating harmonic and converges conditionally; at $x=-2$ it is the divergent positive harmonic series. This explicitly separates a restriction of the chosen family from a restriction on power series in general.

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Question 10

An unknown real power series $\sum_{n=0}^{\infty} a_n(x-2)^n$ converges at $x=5$ and diverges at $x=-2$. No coefficient formula is supplied. Use only what these two observations imply; boundary convergence may be conditional.

Tasks

1. Find the strongest possible bounds on its radius of convergence R. Explain why either inequality may be an equality.
2. Classify the behavior forced at $x = 0, -1, 11/2, 6, 7$: absolute convergence, divergence, or not determined by the information. Give reasons using distances from the center.
3. Construct two explicit series satisfying both observations that establish all the undetermined cases in your table. Verify the observations and the disputed points for each example.
4. Show every radius in your proposed closed range can occur. Use explicit examples for the endpoints and a general construction for the intermediate radii.

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Question 10 – Solution

Strategy. Convergence at a point controls smaller distances absolutely; divergence controls larger distances. Boundary tests cannot be transferred automatically to the opposite side.

Step 1: Bound the radius sharply. The convergent point is distance 3 from the center, so $R \geq 3$. The divergent point is distance 4, so $R \leq 4$: if $R > 4$ that point would be inside the region of absolute convergence. Thus $\boxed{3 \leq R \leq 4}$. Convergence at distance 3 could be boundary convergence, and divergence at distance 4 could be boundary divergence.

Step 2: Record only forced conclusions.

x	$\lvert x-2\rvert$	conclusion
0	2	absolute convergence
-1	3	not determined
11/2	7/2	not determined
6	4	not determined
7	5	divergence

The absolute convergence at distance 2 follows from the convergent point at distance 3. The opposite point at distance 3 need not share an endpoint test, and distances between 3 and 4 are not settled by the bounds.

Step 3: Construct examples proving the uncertainty. Take, starting at $n = 1$ and setting $a_0 = 0$,

$$U(x) = \sum_{n\geq 1} \frac{(-1)^n (x-2)^n}{n3^n}, \qquad V(x) = \sum_{n\geq 1} \frac{(-1)^n (x-2)^n}{n4^n}.$$

For U, $x = 5$ gives alternating harmonic convergence and $x = -2$ fails the term test; $R_U = 3$. At $x = -1$ it is positive harmonic and diverges, and at 11/2 and 6 its terms do not tend to zero. For V, $x = 5$ converges absolutely, while $x = -2$ is positive harmonic and diverges; $R_V = 4$. At -1 and 11/2 it converges absolutely, and at 6 it converges conditionally. These two examples resolve every "not determined" entry without changing the given observations.

Step 4: Realize the entire radius range. The examples above realize $R = 3$ and $R = 4$. For any $3 < \rho < 4$, use $\boxed{\sum_{n=0}^{\infty} ((x-2)/\rho)^n}$. Its radius is ρ; at 5 its ratio has magnitude $3/\rho < 1$, and at -2 it has magnitude $4/\rho > 1$. Thus every radius in $[3, 4]$ occurs, and no narrower range follows from the two observations alone.

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Question 1

Let $f(x) = \ln x$, and expand about $x = 2$. A student proposes

$$P_4(x) = \ln 2 + \sum_{n=1}^{4} \frac{(-1)^{n-1}(n-1)!}{2^n}(x-2)^n.$$

A Taylor coefficient and a derivative value are different kinds of data.

Tasks

1. Derive $f^{(n)}(2)$ for $n \geq 1$, correct the proposed polynomial, and identify its first incorrect coefficient.
2. Find the full Taylor series and its interval of convergence, including both endpoints. Justify that its sum equals $\ln x$ on the open interval.
3. Use the Lagrange remainder to give a uniform absolute error bound for the corrected fourth-degree polynomial on $[1.8, 2.2]$.
4. Decide whether the corrected polynomial lies above or below $\ln x$ on each side of 2 in that interval. Support the conclusion with a signed remainder.

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Question 1 – Solution

Strategy. Divide derivatives by factorials, then distinguish series convergence from finite-polynomial accuracy.

Step 1: Normalize the coefficients. For $n \geq 1$, $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$. Thus

$$\boxed{T_4(2+h) = \ln 2 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{24} - \frac{h^4}{64}.}$$

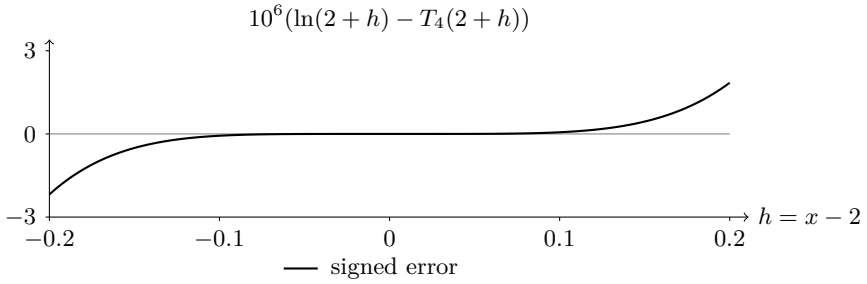
The student omitted division by $n!$; the first error is the h^2 coefficient, $-1/4$ instead of $-1/8$.

Step 2: Establish the represented function. Integrating the geometric series for $1/(2+h)$ from 0 to h gives $\ln(2+h) = \ln 2 + \sum_{n\geq 1}(-1)^{n-1}h^n/(n2^n)$ for $|h| < 2$. Uniform convergence on the integration segment justifies this step. The term ratio tends to $|h|/2$, so the radius is 2; $h = -2$ gives the divergent negative harmonic series and $h = 2$ gives the alternating harmonic series. The convergence interval is $\boxed{(0, 4]}$ in x.

Step 3: Certify a uniform error. Since $f^{(5)}(x) = 24/x^5$, on $[1.8, 2.2]$,

$$|f(x) - T_4(x)| \leq \frac{24}{1.8^5}\frac{0.2^5}{120} = \boxed{\frac{1}{5 \cdot 9^5} < 3.39 \times 10^{-6}}.$$

Step 4: Determine the error sign. For $x \neq 2$, $f(x) - T_4(x) = h^5/(5\xi^5)$ for some positive ξ between 2 and x. Thus T_4 is above f on $[1.8, 2)$ and below it on $(2, 2.2]$; equality holds at 2. The plot magnifies this signed error.



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Question 2

For $0 \leq x \leq 1$, let $T_n(x) = \sum_{k=0}^{n} (-x)^k/k!$ approximate e^{-x}. The number of terms in T_n is $n+1$.

Tasks

1. Prove that the Maclaurin series represents e^{-x} on $[0,1]$, using a remainder estimate that is uniform in x.
2. Find the smallest degree certified by the bound $|e^{-x} - T_n(x)| \leq 1/(n+1)!$ to achieve error at most 10^{-6} throughout $[0,1]$.
3. A sufficient bound need not identify the true smallest degree. Prove in this case that the degree you found really is minimal, using the alternating tail at $x = 1$ to exclude every smaller degree.
4. Produce a rational enclosure for e^{-1} from consecutive partial sums at that degree, and explain which endpoint is the upper bound.

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Question 2 – Solution

Strategy. Use Taylor's theorem for convergence and alternating tails for both upper and lower error bounds.

Step 1: Control all points simultaneously. Every derivative of e^{-x} has absolute value $e^{-x} \leq 1$ on $[0, 1]$. Taylor's theorem gives

$$|e^{-x} - T_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!} \longrightarrow 0.$$

This proves equality to the series and uniform convergence on the stated interval.

Step 2: Find a certified degree. Because $9! = 362880 < 10^6 < 3628800 = 10!$, the smallest degree certified by this bound is $\boxed{n = 9}$, using ten terms. Degree 8 is not certified by the upper bound alone.

Step 3: Prove actual minimality. At $x = 1$, the alternating tail has the sign of its first term, and its magnitude is strictly greater than the difference of its first two magnitudes:

$$|e^{-1} - T_n(1)| > \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{n+1}{(n+2)!} =: b_n.$$

Indeed, the remaining tail starts positive after factoring out its first sign and grouping consecutive pairs. Also $b_{n+1}/b_n = (n+2)/[(n+1)(n+3)] < 1$ for $n \geq 0$. Hence for every $0 \leq n \leq 8$, the error exceeds $b_8 = 9/10! > 10^{-6}$. No smaller degree works on all of $[0, 1]$.

Step 4: Enclose the value rationally. Odd partial sums lie below the limit and even ones above. Therefore

$$\boxed{\frac{16687}{45360} = T_9(1) < e^{-1} < T_{10}(1) = \frac{16481}{44800}.}$$

The interval width is exactly $1/10!$; the upper endpoint comes from the positive first omitted term after T_9.

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Question 3

For $x \neq 0$, define

$$H(x) = \frac{\sin x - x\cos x}{x^3}.$$

Near the origin the numerator suffers substantial cancellation.

Tasks

1. Derive the full Maclaurin series of H by combining the sine and cosine series before dividing. State its convergence domain.
2. Determine the unique continuous extension at 0, and find $H''(0)$ and $H^{(4)}(0)$ for that extension.
3. Construct its fourth-degree Taylor polynomial and certify its absolute error on $[-1, 1]$ by an alternating-series estimate.
4. Prove that the extended function is even and strictly positive on $[-1, 1]$. Explain why the series is useful for evaluating the original quotient near 0.

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Question 3 – Solution

Strategy. Expose the first nonzero numerator term before removing the apparent singularity.

Step 1: Cancel and reindex. The coefficient of x^{2k+1} in the numerator is $(-1)^k[1/(2k+1)! - 1/(2k)!]$. It is zero when $k = 0$. Setting $j = k - 1$ for the surviving terms gives

$$\boxed{H(x) = \sum_{j=0}^{\infty} \frac{(-1)^j\, 2(j+1)}{(2j+3)!} x^{2j}.}$$

Sine and cosine converge absolutely for every real x; the new series also has infinite radius by the ratio test. It agrees with the quotient when $x \neq 0$.

Step 2: Fill the removable singularity. The new series defines an analytic extension with

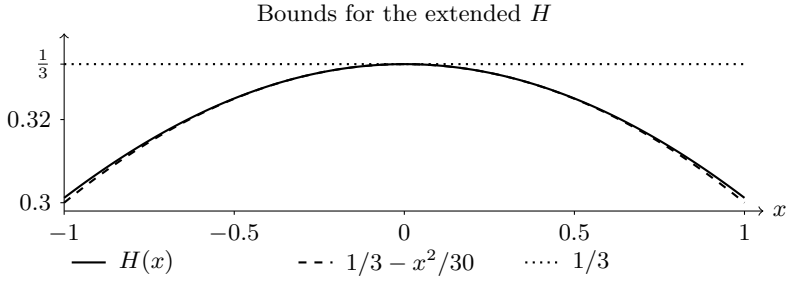
$$H(0) = \frac{1}{3}, \qquad H''(0) = 2!\left(-\frac{1}{30}\right) = -\frac{1}{15}, \qquad H^{(4)}(0) = 4!\frac{1}{840} = \frac{1}{35}.$$

Continuity forces the value $1/3$, so no other continuous extension is possible.

Step 3: Bound the truncated tail. The polynomial is $T_4(x) = 1/3 - x^2/30 + x^4/840$. For $|x| \leq 1$, successive term magnitudes have ratio $[(j+2)/(j+1)]x^2/[(2j+5)(2j+4)] \leq 1/10$. They decrease to zero, so

$$\boxed{|H(x) - T_4(x)| \leq \frac{|x|^6}{45360} \leq \frac{1}{45360}.}$$

Step 4: Interpret symmetry and stability. Only even powers occur. Alternating bounds give $H(x) \geq 1/3 - x^2/30 \geq 3/10 > 0$ on $[-1, 1]$. The numerator subtracts two quantities of order x to obtain one of order x^3; the series evaluates the finite limiting value without that subtraction. This is a cancellation concern, not a failure of the exact quotient.



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Question 4

Let $F(x) = e^{x^2}\cos x$. A product of truncated series contains both useful terms and terms beyond the intended degree.

Tasks

1. Obtain the sixth-degree Maclaurin polynomial P_6 of F by explicit coefficient convolution.
2. Find $F^{(6)}(0)$ and explain why the remainder has order x^8, rather than merely order x^7.
3. Write $A = 1 + x^2 + x^4/2 + x^6/6$ and $B = 1 - x^2/2 + x^4/24 - x^6/720$. Compute $AB - P_6$ exactly.
4. On $|x| \le 1/2$, prove the concrete bound $|F(x) - P_6(x)| < 0.0005$. Account separately for the exponential tail, cosine tail, and high-degree terms in AB.

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Question 4 – Solution

Strategy. Convolve by degree, then bound the two truncation errors and the discarded product terms separately.

Step 1: Collect only the requested powers. Using the entire exponential and cosine series,

$$[x^2]F = 1 - \tfrac{1}{2} = \tfrac{1}{2}, \quad [x^4]F = \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{24} = \tfrac{1}{24},$$

$$[x^6]F = \tfrac{1}{6} - \tfrac{1}{4} + \tfrac{1}{24} - \tfrac{1}{720} = -\tfrac{31}{720}.$$

Thus $\boxed{P_6(x) = 1 + x^2/2 + x^4/24 - 31x^6/720}$.

Step 2: Recover a derivative and its parity. We have $F^{(6)}(0) = 6!(-31/720) = -31$. Both factors are even and analytic everywhere, so all odd coefficients vanish. The next possible power after degree six is eight, proving $F - P_6 = O(x^8)$ near 0.

Step 3: Retain the discarded product explicitly. Direct multiplication gives

$$AB - P_6 = -\frac{23}{360}x^8 + \frac{1}{160}x^{10} - \frac{1}{4320}x^{12}.$$

These terms belong to the product of polynomials, not to P_6.

Step 4: Certify the error on the whole interval. Write $E = e^{x^2}$ and $C = \cos x$. Then $EC - P_6 = (E - A)C + A(C - B) + (AB - P_6)$. For $|x| \le 1/2$, $|C| \le 1$, $0 < A \le e^{1/4} < 4/3$ (compare the exponential series with the geometric series). Taylor's theorem and the alternating cosine tail give

$$|E - A| \le \frac{4}{3}\frac{|x|^8}{24}, \qquad |C - B| \le \frac{|x|^8}{8!}.$$

Consequently,

$$|F - P_6| \le \frac{4/3}{24 \cdot 256} + \frac{4/3}{40320 \cdot 256} + \frac{23}{360 \cdot 256} + \frac{1}{160 \cdot 1024} + \frac{1}{4320 \cdot 4096} < 0.000473 < 0.0005.$$

The bound is conservative but rigorous; no unbounded O-term is used as a numerical certificate.

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Question 5

Consider the nonelementary integral

$$I = \int_0^{1/2} e^{-t^2}\,dt, \qquad S_m = \sum_{k=0}^{m} \frac{(-1)^k}{2^{2k+1}k!(2k+1)}.$$

An accuracy certificate is required; a decimal from a calculator is insufficient.

Tasks

1. Derive the series for I and justify integrating the Maclaurin series term by term on the whole integration interval.
2. Find the smallest m for which the first-omitted-term bound certifies $|I - S_m| \le 10^{-6}$, and state the number of terms used.
3. Give an exact rational enclosure for I using that partial sum and the next one. Report an approximation with its certified absolute error.
4. Explain the sign of the error geometrically by comparing e^{-t^2} with the degree-$2m$ polynomial under the integral on $[0, 1/2]$.

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Question 5 – Solution

Strategy. Integrate a uniformly convergent series, then use the integrated alternating tail as an error certificate.

Step 1: Justify the interchange. On $[0, 1/2]$, $|(-t^2)^k/k!| \le (1/4)^k/k!$, whose sum is finite. The Weierstrass test gives uniform absolute convergence. Integrating yields

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}k!(2k+1)}.$$

The positive term magnitudes decrease strictly to zero.

Step 2: Choose a certified truncation. The first omitted magnitude is $b_m = 1/[2^{2m+3}(m+1)!(2m+3)]$. It decreases with m. For $m = 3$, $b_3 = 1/110592 > 10^{-6}$, whereas

$$b_4 = \frac{1}{2703360} < 3.70 \times 10^{-7} < 10^{-6}.$$

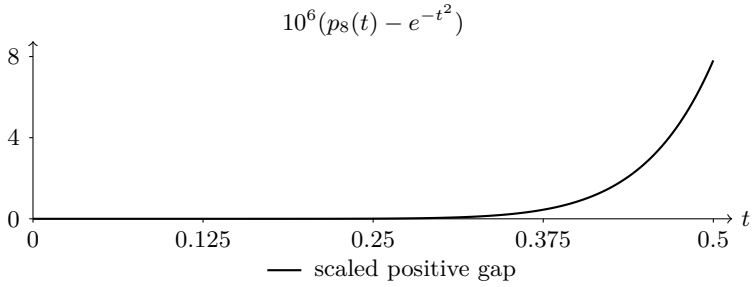
Thus $\boxed{m = 4}$ is the smallest index certified by this bound, using five terms.

Step 3: Supply an exact enclosure. The even partial sum is an upper bound and the next odd sum a lower bound:

$$\boxed{S_4 - \frac{1}{2703360} = S_5 < I < S_4 = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \frac{1}{110592}.}$$

Numerically, $S_4 \approx 0.4612813637$. Reporting $I \approx 0.46128136$ introduces less than 4×10^{-9} additional rounding error, so its total absolute error is less than 3.74×10^{-7}.

Step 4: Interpret the signed area. Set $p_8(t) = 1 - t^2 + t^4/2 - t^6/6 + t^8/24$. The alternating tail is strictly negative for $t > 0$ in this interval, so $p_8(t) > e^{-t^2}$. Hence $S_4 - I$ is the positive area between the curves. The scaled gap below makes the very small difference visible; it vanishes at $t = 0$.



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Question 6

Define a function on the real line by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

A student claims: "Every infinitely differentiable function agrees with its Taylor series near the center."

Tasks

1. Prove $|x|^{-m} e^{-1/x^2} \to 0$ as $x \to 0$ for every nonnegative integer m.
2. Prove that f is infinitely differentiable at 0 and that every derivative there vanishes. Use a polynomial-in-$1/x$ description away from 0 and difference quotients at 0.
3. Find the Maclaurin series, its radius of convergence, and the set of points where it equals f.
4. Explain precisely why Taylor's theorem is still valid but fails to prove equality to the infinite series here. Identify the remainder $R_n(x)$ at any fixed nonzero x.

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Question 6 – Solution

Strategy. Separate the existence of every derivative from the limiting behavior of the Taylor remainders.

Step 1: Exponential decay beats every power. Put $u = 1/x^2 \to \infty$. Choose an integer $k > m/2$. Since $e^u \geq u^k/k!$,

$$|x|^{-m} e^{-1/x^2} = u^{m/2} e^{-u} \leq k! u^{m/2-k} \longrightarrow 0.$$

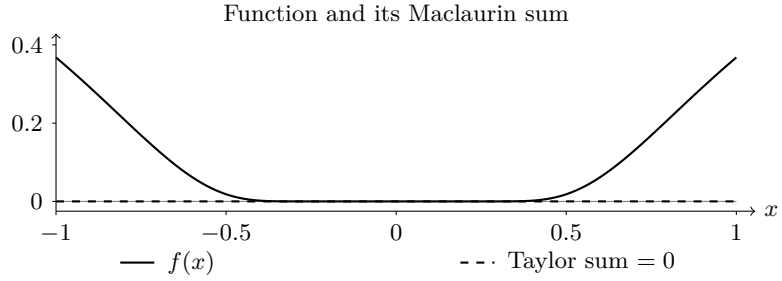
Step 2: Prove smoothness, including the center. For $x \neq 0$, induction gives $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where $P_0(z) = 1$ and

$$P_{n+1}(z) = -z^2 P_n'(z) + 2z^3 P_n(z).$$

Define each candidate derivative to be zero at 0. Step 1 proves its continuity there. Its difference quotient at 0 is $P_n(1/h)e^{-1/h^2}/h \to 0$, again by Step 1. Inductively each candidate is the derivative of the preceding one, so $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Step 3: Compare the series with the function. Every Taylor polynomial is zero. Thus the Maclaurin series is $\boxed{\sum_{n=0}^{\infty} 0x^n = 0}$, with infinite radius. It equals f only at $x = 0$, since $f(x) > 0$ for every nonzero x. Infinite radius does not by itself identify the represented function.

Step 4: Locate the missing hypothesis. Taylor's finite theorem remains valid on the segment from 0 to any fixed x. But $\boxed{R_n(x) = f(x) - T_n(x) = e^{-1/x^2} > 0}$ for $x \neq 0$, independently of n. Thus $R_n(x) \not\to 0$. Smoothness provides each finite remainder formula, not a derivative-growth bound forcing remainders to vanish. The graph and the zero Taylor sum touch to every order at 0 without agreeing nearby.



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Question 7

The direct Maclaurin expansion of $\ln(1+x)$ converges slowly at $x = 1$. Seek a faster Taylor-series evaluation of $\ln 2$ using

$$L(z) = \ln\left(\frac{1+z}{1-z}\right), \qquad |z| < 1.$$

Count terms by the number of nonzero summands, not by polynomial degree.

Tasks

1. Derive the Maclaurin series for L by differentiating and integrating a geometric series, with a justification on $|z| < 1$.
2. Select a positive z for which $L(z) = \ln 2$. For the first N terms, prove the tail bound

$$\frac{2z^{2N+1}}{(2N+1)(1-z^2)}$$

   and determine the error sign.
3. Find the smallest N for which this bound is strictly below 10^{-6} at the selected z. Give the resulting approximation and a rigorous enclosure.
4. For the direct alternating harmonic sum with N terms, how many terms does the next-term bound require to certify error strictly below 10^{-6}? Compare the two certificates and explain the limits of this comparison.

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Question 7 – Solution

Strategy. Change the evaluation point to shrink the geometric factor controlling the Taylor tail.

Step 1: Construct the odd-power series. Since $L'(z) = 2/(1-z^2) = 2\sum_{k\geq 0} z^{2k}$ and $L(0) = 0$,

$$\boxed{L(z) = 2\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.}$$

The geometric series converges uniformly on any closed segment inside $(-1, 1)$, so termwise integration is justified.

Step 2: Bound a positive tail. The equation $(1+z)/(1-z) = 2$ gives $z = 1/3$. With $A_N = 2\sum_{k=0}^{N-1} z^{2k+1}/(2k+1)$ and $0 < z < 1$,

$$\begin{aligned} 0 < L(z) - A_N &= 2\sum_{k=N}^{\infty} \frac{z^{2k+1}}{2k+1} \\ &< \frac{2z^{2N+1}}{(2N+1)(1-z^2)} =: B_N. \end{aligned}$$

Strictness follows because later denominators exceed $2N+1$.

Step 3: Count the accelerated terms. At $z = 1/3$, $B_N = 9/[4(2N+1)3^{2N+1}]$, strictly decreasing in N. Here $B_5 = 9/7794468 > 10^{-6}$ and $B_6 = 9/82904796 < 1.086 \times 10^{-7}$. Thus six terms are the smallest number certified by this bound, and

$$\boxed{A_6 < \ln 2 < A_6 + \frac{9}{82904796}}, \qquad A_6 \approx 0.6931470738.$$

Step 4: Make a fair certificate comparison. For the direct alternating sum with N terms, the stated bound is $1/(N+1)$. To make that bound strictly less than 10^{-6} requires $N \geq 1{,}000{,}000$. The comparison is six versus one million terms under these particular certificates. It does not assert that a million terms are necessary for the true direct-series error, nor that a loose bound is an optimal algorithm. The transformed series profits from the factor $z^2 = 1/9$.

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Question 8

An unknown function belongs to the family $f(x) = e^{ax}\cos(bx)$, where a, b are real. Exact local measurements give

$$f'(0) = 1, \qquad f''(0) = -3.$$

A separate report claims $f'''(0) = -10$. No measurement uncertainty is assumed.

Tasks

1. Recover every admissible parameter pair (a, b) from the first two measurements. Explain the remaining ambiguity and whether it changes f.
2. Find the fourth-degree Maclaurin polynomial of every compatible function.
3. Test the separate third-derivative report and decide whether all three measurements can arise from one member of the stated family.
4. Prove a uniform remainder bound below 5.2×10^{-6} for your polynomial on $[-0.1, 0.1]$. You may use $f(x) = \operatorname{Re} e^{(a+ib)x}$ and $|\operatorname{Re} z| \le |z|$.

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Question 8 – Solution

Strategy. Turn measured derivatives into algebraic constraints before predicting higher coefficients.

Step 1: Recover the parameters. Differentiation at 0 gives $f'(0) = a$ and $f''(0) = a^2 - b^2$. Hence $a = 1$ and $b^2 = 4$, so

$$\boxed{(a, b) = (1, 2) \text{ or } (1, -2).}$$

The sign cannot be recovered because cosine is even; both pairs produce the same function $e^x \cos 2x$.

Step 2: Predict the Taylor polynomial. Using real parts of $(1 + 2i)^n$, the derivative values for $n = 0, \ldots, 4$ are $1, 1, -3, -11, -7$. Dividing by $n!$ yields

$$\boxed{T_4(x) = 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3 - \frac{7}{24}x^4.}$$

For example, $(1 + 2i)^2 = -3 + 4i$ and $(1 + 2i)^3 = -11 - 2i$.

Step 3: Reject inconsistent exact data. Every pair compatible with the first two measurements forces $f'''(0) = -11$. The reported value -10 is incompatible. Changing the sign of b conjugates the complex powers and leaves their real parts unchanged, so it cannot repair the discrepancy.

Step 4: Bound the fifth derivative uniformly. For real t,

$$|f^{(5)}(t)| = \left|\operatorname{Re}\left((1 + 2i)^5 e^{(1+2i)t}\right)\right| \le 5^{5/2} e^t.$$

Taylor's theorem on each segment from 0 to $x \in [-0.1, 0.1]$ therefore gives

$$|f(x) - T_4(x)| \le \frac{5^{5/2} e^{0.1}}{120}\, 10^{-5} < 5.2 \times 10^{-6}.$$

For a purely rational check of the last inequality, use $\sqrt{5} < 9/4$ and $e^{0.1} < 1.106$: the latter follows by bounding the tail after $1 + 0.1 + 0.1^2/2$ by $(0.1^3/6)/(1 - 0.025)$. These bounds give less than 5.185×10^{-6}.

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Question 9

Let $T_4(x) = 1 - x^2/2 + x^4/24$, the fourth-degree Maclaurin polynomial of $\cos x$. Consider a rational competitor

$$Q(x) = \frac{1 + ax^2}{1 + bx^2}, \qquad a, b \in \mathbb{R}.$$

A student believes that replacing a Taylor polynomial by a rational expression with the same local coefficients must improve accuracy.

Tasks

1. Determine a, b so that Q and $\cos x$ agree through degree four at 0, and identify any real poles of Q.
2. Find the leading nonzero terms of $\cos x - T_4(x)$ and $\cos x - Q(x)$. Compute the limiting ratio of their absolute errors as $x \to 0$, $x \neq 0$.
3. Use an alternating Taylor enclosure for $\cos 1$ to prove which approximation is closer at $x = 1$, without relying on a calculator value of $\cos 1$.
4. Evaluate the student's claim and explain why matching coefficients alone cannot rank all approximations on a whole interval.

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Question 9 – Solution

Strategy. Match coefficients, then examine the first coefficient on which the competing approximations disagree.

Step 1: Solve the matching equations. Near 0, $Q = 1+(a-b)x^2+(b^2-ab)x^4+\cdots$. Thus $a-b=-1/2$ and $-b(a-b)=1/24$, giving

$$\boxed{b=\frac{1}{12},\quad a=-\frac{5}{12},\quad Q(x)=\frac{12-5x^2}{12+x^2}.}$$

Since $12+x^2>0$, there are no real poles.

Step 2: Compare the leading errors. The rational geometric expansion has x^6 coefficient $-1/288$. Hence

$$\cos x - T_4(x) = -\frac{x^6}{720}+O(x^8), \qquad \cos x - Q(x) = \frac{x^6}{480}+O(x^8).$$

Therefore $\boxed{\lim_{x\to 0}\frac{|\cos x - Q(x)|}{|\cos x - T_4(x)|}=3/2}$. In particular, the rational approximation is worse for all sufficiently small nonzero x.

Step 3: Make an exact comparison at one. Alternating bounds give

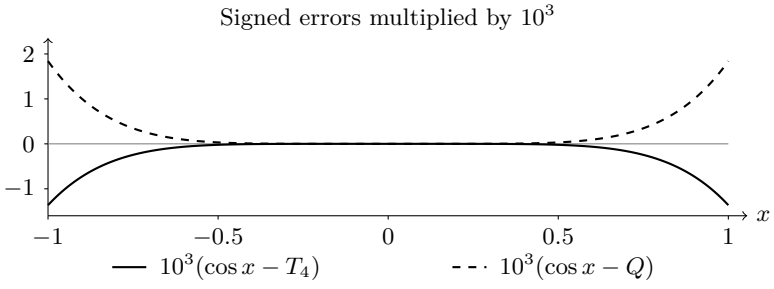
$$T_6(1)=\frac{389}{720}<\cos 1<T_4(1)=\frac{13}{24}, \qquad Q(1)=\frac{7}{13}.$$

Here $Q(1)<389/720$, and the midpoint of the two approximations is $\frac{1}{2}(7/13+13/24)=337/624<389/720$. Thus $\cos 1$ lies above their midpoint and between them: $T_4(1)$ is strictly closer than $Q(1)$.

Step 4: State the proper conclusion. The claim fails both locally and at $x=1$. Matching four derivatives fixes contact at the center, not the next error coefficient or behavior far away. For $0<|x|\le 1$, alternating bounds give $\cos x - T_4(x)<0$ and

$$\cos x - Q(x) \ge x^6\left(\frac{1}{312}-\frac{1}{720}\right)>0,$$

since $T_4-Q = x^6/[288(1+x^2/12)]$. These signs explain the plotted errors.



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Question 10

Let $f \in C^\infty([-r, r])$, with $r > 0$, and suppose constants $C, A > 0$ satisfy

$$|f^{(n)}(t)| \le CA^n n! \qquad (n \ge 0,\ |t| \le r).$$

Write $T_N(x) = \sum_{n=0}^{N} f^{(n)}(0)x^n/n!$. For a boundary example you may use $g(x) = 1/(1+x^2)$ and the identity

$$g^{(n)}(x) = \frac{(-1)^n n!}{2i}\left((x-i)^{-n-1} - (x+i)^{-n-1}\right), \qquad i^2 = -1.$$

Tasks

1. Prove that $T_N(x) \to f(x)$ for $|x| \le r$ with $A|x| < 1$. Give a uniform error estimate on $[-\rho, \rho]$ when $0 < \rho \le r$ and $A\rho < 1$.
2. Explain why this argument gives no conclusion at $A|x| = 1$. Show that both convergence and divergence there are possible under the stated derivative hypothesis, using $f = 1$ and g on $[-1, 1]$ with $C = A = 1$.
3. A function has all derivatives zero at 0 but is positive at every nearby nonzero point. Prove it cannot obey such a derivative bound on any neighborhood of 0.
4. Suppose the displayed bound is known only at $t = 0$. What can you conclude about convergence of the Taylor series, and what additional conclusion can you no longer justify? Supply a counterexample.

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Question 10 – Solution

Strategy. A bound on derivatives throughout a segment controls Taylor's remainder; center-only data control coefficients.

Step 1: Force the remainders to vanish. Taylor's theorem uses some ξ between 0 and x. The hypothesis gives

$$|f(x) - T_N(x)| \le \frac{CA^{N+1}(N+1)!}{(N+1)!}|x|^{N+1} = \boxed{C(A|x|)^{N+1}}.$$

This tends to zero whenever $|x| \le r$ and $A|x| < 1$. On the specified smaller interval the uniform bound is $C(A\rho)^{N+1} \to 0$.

Step 2: Test the boundary honestly. At $A|x| = 1$ the bound is only C, which does not tend to zero. The constant function 1 satisfies the bound with $C = A = 1$ and its Taylor series converges at ± 1. For real x, $|x \pm i| \ge 1$, so the given identity yields $|g^{(n)}(x)| \le n!$. Yet the Maclaurin series of g is $\sum_{k\ge 0}(-1)^k x^{2k}$; at $x = \pm 1$ its terms fail to tend to zero. Thus neither boundary outcome is forced.

Step 3: Rule out the bound for a flat function. If such constants existed on some $[-r, r]$, choose $0 < |x| < \min(r, 1/A)$. All Taylor polynomials would be zero, but Step 1 would force $f(x) = 0$, contradicting positivity. Thus no finite positive C, A can work on any such neighborhood.

Step 4: Separate coefficients from function values. A bound only at 0 gives $|f^{(n)}(0)/n!| \le CA^n$. Geometric comparison proves absolute convergence of the Taylor series for $|x| < 1/A$, but does not prove that its sum is $f(x)$. The smooth function e^{-1/x^2} extended by zero at 0 has every center derivative zero, so it satisfies every such center-only bound; its Taylor series is zero while the function is positive elsewhere.

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Question 1

Consider the differential equation $y'' - xy = 0$ near $x = 0$. Write $y = \sum_{n=0}^{\infty} a_n x^n$; the coefficients are not the derivatives themselves.

Tasks

1. Derive the coefficient recurrence, stating the exceptional equation at degree zero. Explain how the indices split into three chains.
2. Construct the normalized solutions $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$. Give the first four nonzero terms of each.
3. Prove that both series converge for every real x and form a fundamental pair. State the solution with arbitrary initial values $y(0) = A$, $y'(0) = B$.
4. For the degree-nine truncation U_9 of u, prove a uniform error bound below 5.88×10^{-7} on $[-1, 1]$. A numerical sample of the error is not a proof of this bound.

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Question 1 – Solution

Strategy. Keep the lowest index separate, then use the recurrence both to construct solutions and to bound their tails.

Step 1: Align the powers. The constant coefficient gives $a_2 = 0$. For $n \geq 1$, $(n+2)(n+1)a_{n+2} = a_{n-1}$, equivalently

$$\boxed{a_{m+3} = \frac{a_m}{(m+3)(m+2)} \quad (m \geq 0).}$$

The chains start at a_0, a_1, a_2; the last is identically zero.

Step 2: Build the normalized series. With empty products interpreted as 1,

$$u(x) = \sum_{k\geq 0} \frac{x^{3k}}{\prod_{j=1}^{k}(3j)(3j-1)} = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots,$$

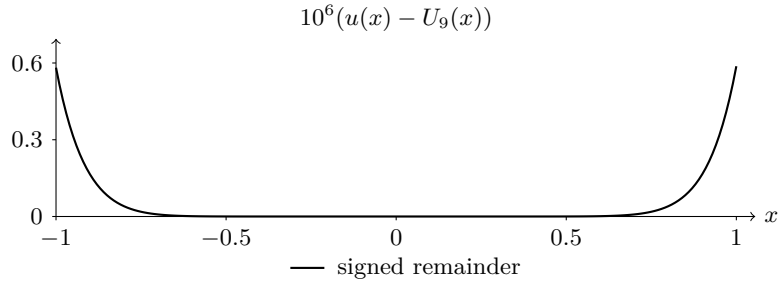
$$v(x) = \sum_{k\geq 0} \frac{x^{3k+1}}{\prod_{j=1}^{k}(3j+1)(3j)} = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots.$$

Step 3: Establish actual solutions and independence. For either series, the ratio of consecutive nonzero term magnitudes tends to zero at every fixed x. Thus both have infinite radius, may be differentiated termwise, and satisfy the equation and stated initial data. Their Wronskian satisfies $W' = uv'' - u''v = 0$ and $W(0) = 1$. Hence $\boxed{y = Au + Bv}$ is the unique solution for those initial values, on $\mathbb{R}$.

Step 4: Bound all omitted terms. The first omitted coefficient of u is $1/1710720$. On $|x| \leq 1$, ratios within the remaining tail are at most $1/(15 \cdot 14) = 1/210$. Therefore

$$\boxed{|u(x) - U_9(x)| \leq \frac{|x|^{12}}{1710720}\frac{1}{1 - 1/210} < 5.88 \times 10^{-7}.}$$

The plot magnifies the actual signed remainder; the geometric estimate covers both signs of x.



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Question 2

The equation

$$y'' + 2xy' + 2y = 0$$

admits a useful relation between its power-series construction and a first-order identity. Let $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$.

Tasks

1. Derive the recurrence for a Maclaurin solution. Compute the first four nonzero terms of u and v, and identify their parity.
2. Find a quantity involving y, y' and x whose derivative is the left-hand side. Use it to express both normalized solutions through exponentials and a definite integral.
3. Prove convergence of both series on the whole real line directly from their coefficients. Explain why the series and integral constructions agree.
4. Compute the Wronskian of the normalized pair and determine whether any nontrivial linear combination can have both its value and derivative zero at a finite point.

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Question 2 – Solution

Strategy. Compare the even and odd coefficient chains with an integrated form of the equation.

Step 1: Separate the two chains. Coefficient matching gives $(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$, so

$$\boxed{a_{n+2} = -\frac{2}{n+2}a_n \quad (n \geq 0).}$$

Thus $u = 1 - x^2 + x^4/2 - x^6/6 + \cdots$ is even, while $v = x - 2x^3/3 + 4x^5/15 - 8x^7/105 + \cdots$ is odd.

Step 2: Integrate a conserved quantity. Since $(y' + 2xy)' = 0$, $y' + 2xy = C$. At 0, $C = y'(0)$. Multiplying by e^{x^2} and integrating gives

$$y(x) = e^{-x^2}\left[y(0) + y'(0)\int_0^x e^{t^2}\,dt\right].$$

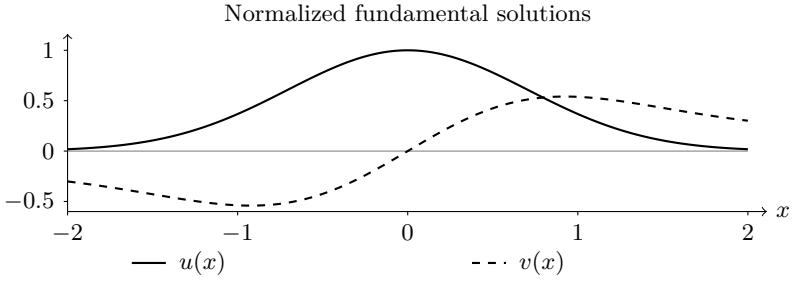
Consequently $\boxed{u = e^{-x^2}, \quad v = e^{-x^2}\int_0^x e^{t^2}\,dt}$.

Step 3: Justify convergence and equality. The even coefficients are $(-1)^k/k!$, and the odd ones are $(-2)^k/(2k+1)!!$, where $(2k+1)!! = 1 \cdot 3 \cdots (2k+1)$. Successive nonzero term ratios are $x^2/(k+1)$ and $2x^2/(2k+3)$, respectively; both tend to zero. Thus the series define solutions for all real x. The integral formulas solve the same equation with the same initial data, so uniqueness identifies the constructions.

Step 4: Test independence at every point. Differentiating v gives $v' = -2xv + 1$, while $u' = -2xu$. Hence

$$\boxed{W(u,v) = uv' - u'v = u = e^{-x^2} > 0.}$$

At any finite point, the two-by-two system for the coefficients of a combination with zero value and derivative has nonzero determinant. Both coefficients must vanish. The plotted pair remains independent even where one curve crosses zero.



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Question 3

Solve the initial-value problem

$$(1-x)y'' - y' = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

by a series about 0, then investigate what happens when that series is used far from its center.

Tasks

1. Derive the recurrence and determine every Maclaurin coefficient.
2. Identify the represented function and its maximal real interval containing 0. Independently determine the power series' radius and both endpoint outcomes.
3. Explain why the initial series fails at $x = -2$ even though the solution exists there. Is this a singularity of the solution at -2?
4. Reexpand about $x = -1$, deriving the new coefficient recurrence from the differential equation. Give the first four nonconstant terms, the new convergence interval including endpoints, and the value at $x = -2$.

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Question 3 – Solution

Strategy. Distinguish a solution's real domain from the disk reached by a particular Taylor center.

Step 1: Determine the Maclaurin coefficients. For $n \geq 0$, coefficient matching gives $(n+2)(n+1)a_{n+2} - (n+1)^2 a_{n+1} = 0$. With $a_0 = 0, a_1 = 1$, this yields

$$a_{n+2} = \frac{n+1}{n+2} a_{n+1}, \qquad \boxed{y(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}}.$$

Step 2: Identify two different domains. The differential equation says $[(1-x)y']' = 0$. Initial data give $y' = 1/(1-x)$ and $\boxed{y = -\ln(1-x)}$ on $(-\infty, 1)$, its maximal real interval through 0. The series has radius 1. It diverges at 1 (harmonic series) and converges at -1 (alternating series), so its interval is $[-1, 1)$. Integrating the geometric series identifies its sum on $(-1, 1)$; at -1, the alternating harmonic value is $-\ln 2$.

Step 3: Diagnose the failed evaluation. At -2, the terms $(-2)^n/n$ do not tend to zero. The series therefore diverges although $y(-2) = -\ln 3$ is finite. The obstruction is the center's distance to the singularity at 1, not a singularity at -2.

Step 4: Move the expansion center. Let $h = x+1$ and $y = \sum b_n h^n$. The equation is $(2-h)y'' - y' = 0$, giving $b_{n+2} = (n+1)b_{n+1}/[2(n+2)]$. Since $b_0 = -\ln 2$, $b_1 = 1/2$,

$$\boxed{y = -\ln 2 + \sum_{n \geq 1} \frac{h^n}{n 2^n} = -\ln 2 + \frac{h}{2} + \frac{h^2}{8} + \frac{h^3}{24} + \frac{h^4}{64} + \cdots}.$$

The radius in h is 2, with convergence at $h = -2$ and divergence at $h = 2$; in x the interval is $[-3, 1)$. Now $x = -2$ is interior, and the sum is $-\ln 2 - \ln(3/2) = \boxed{-\ln 3}$.

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Question 4

Consider a nonhomogeneous equation centered away from the origin:

$$y'' + (x-2)y' = e^{x-2}, \qquad y(2) = 0, \quad y'(2) = 1.$$

You may use the analytic ordinary-point theorem: a linear equation with nonzero leading coefficient and analytic normalized coefficients and forcing in a complex disk has a convergent Taylor solution throughout that disk.

Tasks

1. Set $h = x - 2$ and derive the recurrence for $y = \sum_{n\geq 0} a_n h^n$, explicitly accounting for the constant coefficient of hy'.
2. Compute a_0 through a_6 and the resulting polynomial $P_6(h)$. Identify any coefficient that vanishes despite nonzero neighboring coefficients.
3. Define the residual $\mathcal{R} = P_6'' + hP_6' - e^h$. Find its leading nonzero term. Explain why a degree-six solution truncation need not satisfy the differential equation through degree six.
4. Produce a definite-integral representation of the exact solution and justify infinite radius of convergence for its series. Check both initial values in your representation.

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Question 4 – Solution

Strategy. Expand every term in the shifted variable and keep the forcing coefficients in the same index convention.

Step 1: Match the forcing coefficient. Derivatives with respect to x and h agree. The coefficient of h^n in hy' is na_n for $n \geq 0$; it is zero when $n = 0$. Thus

$$\boxed{a_{n+2} = \frac{1/n! - na_n}{(n+2)(n+1)} \quad (n \geq 0), \qquad a_0 = 0,\ a_1 = 1.}$$

Step 2: Compute the initial segment. Successive applications give $a_2 = 1/2$, $a_3 = 0$, $a_4 = -1/24$, $a_5 = 1/120$, $a_6 = 1/144$. Therefore

$$\boxed{P_6(h) = h + \frac{h^2}{2} - \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{144}.}$$

The cubic coefficient vanishes because $1/1! - a_1 = 0$, not because the solution has a fixed parity.

Step 3: Interpret the residual correctly. Direct differentiation cancels the coefficients through h^4. At degree five, hP_6' contributes $1/24$ while e^h contributes $1/120$; hence

$$\boxed{\mathcal{R}(h) = \frac{h^5}{30} + O(h^6).}$$

The omitted term $a_7 h^7$ would contribute $42a_7 h^5$ to y''. Indeed, the recurrence gives $a_7 = -1/1260$, cancelling that residual term. Two differentiations lower the degree of an omitted term by two.

Step 4: Recover and validate the exact solution. Writing $v = y'$ gives $v' + hv = e^h$. An integrating factor yields

$$\boxed{y(2+h) = \int_0^h e^{-s^2/2} \left(1 + \int_0^s e^{t+t^2/2}\, dt\right) ds.}$$

At $h = 0$ the outer integral is zero and its integrand is 1, verifying the initial values. Differentiating gives the first-order equation for v, hence the original equation. Its leading coefficient is 1 and all coefficients and forcing are entire, so the stated theorem applies to disks of every radius. The Taylor series is therefore entire.

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Question 5

For real λ, consider

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

and seek analytic solutions near $x = 0$. A polynomial solution would remain meaningful even where the leading coefficient vanishes.

Tasks

1. Derive the recurrence and explain how it separates even and odd coefficients.
2. Classify all real λ for which a nonzero polynomial solution exists. Prove necessity from the highest-degree term and sufficiency from the recurrence; include degree zero.
3. For $\lambda = 20$, $y(0) = 3/8$ and $y'(0) = 0$, find the exact solution and verify the differential equation by substitution.
4. Find all its real zeros exactly, determine its values at $x = \pm 1$, and explain why this particular solution extends through those points although the normalized equation is singular there.

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Question 5 – Solution

Strategy. Use coefficient termination to distinguish polynomial solutions from general local series.

Step 1: Derive the parity recurrence. Matching x^n gives

$$\boxed{a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n \quad (n \geq 0).}$$

Even and odd indices evolve independently from a_0 and a_1.

Step 2: Characterize termination completely. If a nonzero polynomial has degree m, its highest coefficient obeys $[\lambda - m(m+1)]a_m = 0$, so $\lambda = m(m+1)$ for an integer $m \geq 0$. Conversely, choose the starting coefficient of parity m nonzero and the other zero. For indices of that parity below m, no numerator vanishes; at $n = m$ it does, terminating the chain. Thus these and only these parameter values work, including $m = 0$, $\lambda = 0$ (constants).

Step 3: Construct and verify the requested polynomial. For $\lambda = 20$, the recurrence yields $a_2 = -15/4$, $a_4 = 35/8$, and $a_6 = 0$. Hence

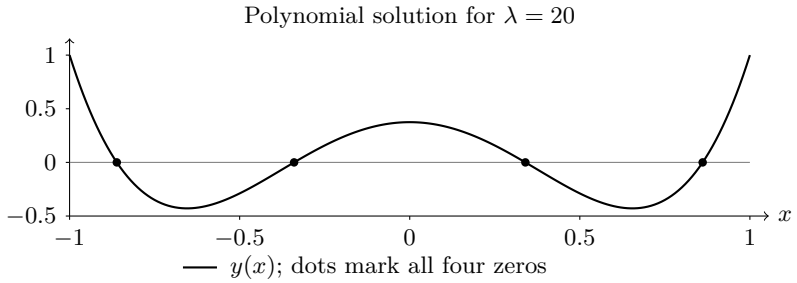
$$\boxed{y(x) = \frac{35x^4 - 30x^2 + 3}{8}.}$$

Here $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$. Substitution in $(1 - x^2)y'' - 2xy' + 20y$ cancels every coefficient. Initial data also hold, so uniqueness near 0 identifies the solution.

Step 4: Locate roots and extend the solution. Solving the quadratic in x^2 gives four distinct zeros:

$$\boxed{x = \pm\sqrt{\frac{15 - 2\sqrt{30}}{35}}, \quad x = \pm\sqrt{\frac{15 + 2\sqrt{30}}{35}}.}$$

They lie in $(-1, 1)$, and $y(-1) = y(1) = 1$. The polynomial satisfies the original equation identically for every real x, including ± 1. Singular normalized coefficients do not forbid this special solution from extending through them.



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Question 6

The leading coefficient vanishes at the proposed expansion center in

$$xy'' + 2y' + xy = 0.$$

A student therefore needs to check which initial data a Maclaurin solution can actually accommodate.

Tasks

1. Substitute $y = \sum_{n\geq 0} a_n x^n$. Find the exceptional constant-coefficient equation and the recurrence for all remaining coefficients.
2. Determine every analytic solution near 0 and identify its series in terms of a familiar function, including the value assigned at 0.
3. Decide whether $y(0) = 1$, $y'(0) = 1$ can be satisfied by an analytic solution or even by a twice continuously differentiable solution of the original equation near 0.
4. On $x > 0$, substitute $v = xy$ to find the full two-parameter solution. Determine which members extend boundedly to 0 and explain why only one free parameter survives in the analytic family.

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Question 6 – Solution

Strategy. Apply the original equation at its singular center before invoking any ordinary-point intuition.

Step 1: Retain the exceptional equation. The coefficient of x^0 is $2a_1 = 0$. For $n \geq 1$, the coefficient of x^n is $(n+1)(n+2)a_{n+1} + a_{n-1} = 0$. Equivalently,

$$\boxed{a_1 = 0, \qquad a_{m+2} = -\frac{a_m}{(m+2)(m+3)} \quad (m \geq 0).}$$

Thus a_0 is free but a_1 is not.

Step 2: Identify all analytic solutions. Every odd coefficient vanishes. The even chain gives

$$\boxed{y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 \frac{\sin x}{x} \quad (x \neq 0), \qquad y(0) = a_0.}$$

The ratio test gives infinite radius. Termwise substitution verifies the original equation, including $x = 0$. Conversely, every analytic solution must obey the recurrence, so the displayed family is exhaustive.

Step 3: Test the proposed initial values. The data $y(0) = 1$, $y'(0) = 1$ are incompatible with $a_1 = 0$. More generally, a C^2 solution satisfies the equation at 0, where it reduces to $2y'(0) = 0$. Even if the equation were initially imposed only at nearby nonzero points, continuity of y, y', y'' would give the same constraint by taking the limit. Such a C^2 solution cannot have the proposed data.

Step 4: Explain the lost second parameter. With $v = xy$, $v'' = xy'' + 2y'$, so $v'' + v = 0$ for $x > 0$. Consequently

$$\boxed{y(x) = A\frac{\sin x}{x} + B\frac{\cos x}{x} \quad (x > 0).}$$

The second term behaves like B/x as $x \to 0^+$; boundedness forces $B = 0$. The remaining term extends analytically with value A and slope 0. The ordinary-point theorem does not guarantee two arbitrary analytic initial coefficients at a center where the leading coefficient is zero.

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Question 7

Consider the nonlinear initial-value problem

$$y' = 1 + y^2, \qquad y(0) = 0.$$

Seek a Maclaurin solution $y = \sum_{n\geq 0} a_n x^n$. For the radius question, you may use these facts: sine and cosine are entire; the only complex zeros of cosine are $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$; and a Taylor series reaches its nearest nonremovable complex singularity.

Tasks

1. Derive the nonlinear coefficient recurrence using a Cauchy product. Compute all coefficients through degree seven.
2. Prove that the resulting solution is odd, and explain why the square y^2 cannot be handled by squaring individual coefficients separately.
3. Identify the exact solution by a method independent of the coefficient calculation. Determine its maximal real interval containing 0 and the exact Taylor radius.
4. For the degree-seven polynomial P_7, prove $0 < P_7(x) < y(x)$ when $0 < x < \pi/2$. Explain why a fixed polynomial cannot approximate this solution uniformly on that whole open interval.

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Question 7 – Solution

Strategy. Nonlinearity couples coefficients by convolution; an exact solution then validates convergence and reveals blow-up.

Step 1: Convolve rather than square termwise. The constant forcing contributes only at $n = 0$:

$$\boxed{(n+1)a_{n+1} = \delta_{n0} + \sum_{k=0}^{n} a_k a_{n-k}, \qquad a_0 = 0,}$$

where $\delta_{n0} = 1$ for $n = 0$ and 0 otherwise. This gives

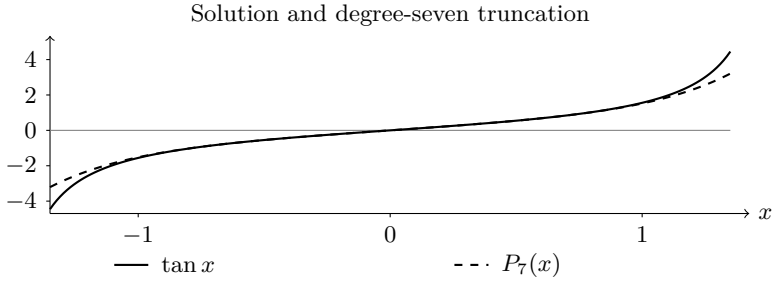
$$\boxed{P_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315},}$$

with $a_2 = a_4 = a_6 = 0$.

Step 2: Establish parity and coefficient signs. If y solves the IVP, $-y(-x)$ does too; local uniqueness gives oddness. The recurrence also proves by induction that all even coefficients vanish and all odd coefficients are positive: each required convolution contains positive odd-index products. For example, the coefficient of x^4 in y^2 is $2a_1a_3$, not a_2^2. Cross terms are essential.

Step 3: Identify the function and its radius. Separation yields $\arctan y = x$ near 0, so $\boxed{y = \tan x}$. It satisfies the equation and initial condition on $(-\pi/2, \pi/2)$ and blows up at both endpoints, making this the maximal real interval through 0. Since $\sin(\pm\pi/2) \neq 0$, the nearest zeros of cosine are nonremovable poles of the quotient. The supplied facts give $\boxed{R = \pi/2}$. Within this radius, the analytic solution's coefficients obey the recurrence, which uniquely determines them; hence it is the constructed series. At $x = \pi/2$ the positive-term series diverges: a finite endpoint sum would bound $\tan x$ below that endpoint, contradicting blow-up. Oddness gives divergence at $-\pi/2$ as well.

Step 4: Interpret truncation and blow-up. For $0 < x < \pi/2$, every omitted odd term is positive, so $0 < P_7(x) < \tan x$. But P_7 stays bounded as $x \to (\pi/2)^-$ while $\tan x \to +\infty$. Their difference is unbounded on that interval. Every fixed Taylor truncation has this defect; convergence on smaller compact intervals does not imply uniform approximation up to a blow-up boundary.



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Question 8

An unknown equation has the form

$$y'' + \alpha x y' + (\beta + \gamma x^2) y = 0,$$

where α, β, γ are real constants. Two analytic solutions have measured Taylor expansions

$$u(x) = 1 - x^2 + \tfrac{1}{2}x^4 - \tfrac{1}{6}x^6 + O(x^8), \qquad v(x) = x - x^3 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^7 + O(x^9).$$

The displayed coefficients are exact.

Tasks

1. Determine α, β, γ by substituting only the coefficients needed from these data. Show the recovery is unique.
2. Derive the full recurrence for a generic series solution of the recovered equation, including its conventions at the lowest indices.
3. Use the substitution $y = e^{-x^2} w$ to identify every solution exactly. Verify that the two measured series are a fundamental pair and determine their radii of convergence.
4. Predict the coefficients of x^8 in u and x^9 in v. Can an additional exact measurement assigning the value $1/20$ to the former coefficient be consistent with the equation and the original data?

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Question 8 – Solution

Strategy. Infer the differential operator from low-order data, then validate all orders by a transformation.

Step 1: Recover the three constants. The constant coefficient in the equation for u gives $-2+\beta=0$. The x coefficient for v gives $-6+\alpha+\beta=0$. The x^2 coefficient for u gives $6-2\alpha-\beta+\gamma=0$. These successive equations uniquely force

$$\boxed{\beta=2, \qquad \alpha=4, \qquad \gamma=4.}$$

Step 2: State a recurrence valid from the start. With $a_{-2}=a_{-1}=0$ for the shifted contribution,

$$\boxed{a_{n+2}=-\frac{(4n+2)a_n+4a_{n-2}}{(n+2)(n+1)} \quad (n\geq 0).}$$

The starting coefficients $a_0=y(0)$ and $a_1=y'(0)$ are free. In particular, $a_2=-a_0$ and $a_3=-a_1$; no negative-index coefficient is an additional parameter.

Step 3: Verify the recovered equation globally. Writing $E=e^{-x^2}$, differentiation gives

$$y'=E(w'-2xw), \qquad y''=E\big(w''-4xw'+(4x^2-2)w\big).$$

Thus $y''+4xy'+(2+4x^2)y=Ew''$. The equation becomes $w''=0$, yielding

$$\boxed{y=e^{-x^2}(C_1+C_2x), \quad u=e^{-x^2}, \quad v=xe^{-x^2}.}$$

Their initial data match the measured series, and their Wronskian is $E^2=e^{-2x^2}>0$. Both functions are entire, so both Taylor radii are infinite. The transformation also proves the entire measured series, not just the first displayed terms, are consistent.

Step 4: Test a new exact measurement. The exponential expansion gives $[x^8]u=1/4!=1/24$ and $[x^9]v=1/24$. The recurrence produces the same values. Since the recovered parameters and each solution's initial data are fixed uniquely, $1/20$ is incompatible; it cannot be accommodated by choosing later coefficients independently.

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Question 9

For

$$y'' + y = x, \qquad y(0) = y'(0) = 0,$$

a power-series approximation is to be certified on $[0, 1]$. For a trial function p, define its differential residual by $\mathcal{R}_p = p'' + p - x$.

Tasks

1. Derive the coefficient recurrence and construct the degree-seven Taylor polynomial P_7 of the exact solution.
2. Compute $\mathcal{R}_{P_7}$ exactly. Derive a definite-integral formula for the error $e = y - P_7$ from its differential equation and initial values.
3. Prove the signed error enclosure $-x^9/362880 \le e(x) \le 0$ on $[0, 1]$. Identify the exact solution and compare this certificate with the alternating Taylor remainder.
4. Show that the residual alone cannot control approximation error if initial data are not also checked. Construct trial functions with the same residual as P_7 but arbitrarily large error at $x = 1$.

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Question 9 – Solution

Strategy. Convert the residual into a forced error equation, keeping track of both initial errors.

Step 1: Construct the solution truncation. Matching powers gives $(n+2)(n+1)a_{n+2} + a_n = \delta_{n1}$, with $a_0 = a_1 = 0$. Thus all even coefficients vanish and

$$\boxed{P_7(x) = \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040}.}$$

Step 2: Derive the error equation. Direct differentiation yields $\mathcal{R}_{P_7} = x^7/5040$. Since P_7 satisfies both initial values, $e'' + e = -x^7/5040$ with $e(0) = e'(0) = 0$. The zero-data solution is

$$\boxed{e(x) = -\frac{1}{5040}\int_0^x \sin(x-t)t^7\,dt.}$$

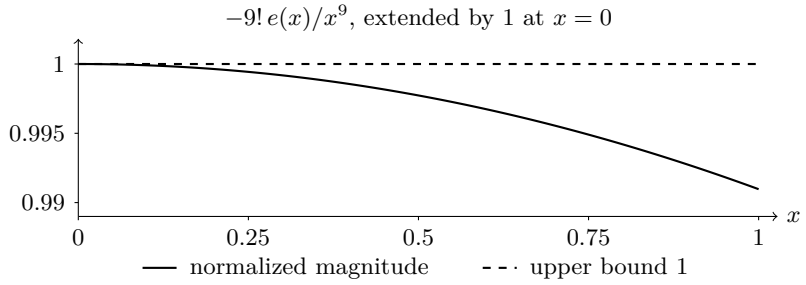
Differentiating this integral twice verifies the equation and the two initial conditions, so uniqueness proves the formula.

Step 3: Bound the signed integral. For $0 \le t \le x \le 1$, $0 \le \sin(x-t) \le x - t$. Therefore

$$-\frac{1}{5040}\int_0^x (x-t)t^7 dt = -\frac{x^9}{362880} \le e(x) \le 0.$$

The exact solution is $\boxed{y = x - \sin x}$, as substitution and initial values verify. Its convergent sine expansion has first omitted term $-x^9/9!$, so the alternating estimate agrees with the integral certificate. For $x > 0$ the integral is strictly positive before its minus sign, giving $e(x) < 0$.

Step 4: Expose the role of initial data. Let $p_M = P_7 + M\sin x$ for any real M. Since $(\sin x)'' + \sin x = 0$, every p_M has exactly the same residual. But $p_M'(0) = M$, and $y(1) - p_M(1) = e(1) - M\sin 1$ is unbounded as $|M| \to \infty$. Residual size certifies error only together with suitable initial or boundary errors and a stability estimate.



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Question 10

Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad y'' + y = f(x), \qquad y(0) = y'(0) = 0.$$

You may use the established facts that f is infinitely differentiable, even, positive away from 0, and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Tasks

1. Determine every derivative of the solution at 0. What Taylor series does formal coefficient matching produce?
2. Derive a definite-integral representation of the actual solution and verify the equation and both initial values. Explain its existence and uniqueness on the whole real line.
3. Prove that the solution is even and strictly positive for $0 < |x| < \pi$. What does this imply about whether its Taylor series represents it near 0?
4. A student argues that constant coefficients make 0 an ordinary point, so the Taylor method must work. Identify the missing hypothesis and distinguish a solution of the formally expanded equation from a solution of the original equation.

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Question 10 – Solution

Strategy. Test the forcing's analyticity before treating its Taylor coefficients as the forcing itself.

Step 1: Compute all center derivatives. The continuous forcing gives a C^2 solution; since f is smooth, repeated use of $y'' = f - y$ makes y smooth as well. Differentiation gives

$$y^{(n+2)}(0) = f^{(n)}(0) - y^{(n)}(0) = -y^{(n)}(0).$$

Starting with $y(0) = y'(0) = 0$, induction gives every derivative zero. Its Taylor series is therefore $\boxed{\sum_{n\geq 0} 0x^n = 0}$, with infinite radius as a series.

Step 2: Recover the actual forced solution. Variation of constants gives, for every real x,

$$\boxed{y(x) = \int_0^x \sin(x-t)f(t)\,dt.}$$

Here $y'(x) = \int_0^x \cos(x-t)f(t)\,dt$ and $y''(x) = f(x) - y(x)$. At 0 both integrals vanish. The formula is defined on every finite real segment; any difference between two solutions satisfies $z'' + z = 0$ with zero initial values, hence is zero. This proves global existence and uniqueness.

Step 3: Prove nonzero values despite zero Taylor data. Because f is even, $y(-x)$ solves the same IVP; uniqueness gives $y(-x) = y(x)$. For $0 < x < \pi$ and $0 < t < x$, both $\sin(x-t)$ and $f(t)$ are strictly positive, so the integral is positive. Evenness extends this conclusion to $0 < |x| < \pi$. The zero Taylor series cannot equal y on any neighborhood of 0, even though it converges everywhere.

Step 4: Locate the missing analyticity assumption. The leading coefficient is nonzero and the homogeneous coefficients are analytic, but the forcing is not analytic at 0: its Taylor sum is zero while it is positive nearby. The analytic ordinary-point theorem for a nonhomogeneous equation requires analytic forcing too. Replacing f by its Taylor series changes the equation to $y'' + y = 0$. The formal zero series solves that changed equation, not the original equation away from 0.

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Question 1

Consider the Euler initial-value problem

$$x^2y'' - 2xy' - 4y = 0, \qquad y(1) = 2, \quad y'(1) = 3.$$

A power x^m is a useful trial function, but the reason it works should also be understood.

Tasks

1. For $x > 0$, set $t = \ln x$ and $Y(t) = y(e^t)$. Derive the formulas for xy' and x^2y'' and transform the equation into one with constant coefficients.
2. Find a fundamental pair, compute its Wronskian, and solve the IVP exactly. Verify both initial values.
3. State the maximal real interval of this IVP and determine the behavior of its solution at each end of that interval.
4. Locate and classify every stationary point on that interval. Give the exact minimum value and explain how two monotone power modes can combine into a nonmonotone solution.

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Question 1 – Solution

Strategy. Logarithmic coordinates turn powers into exponentials and make the characteristic equation systematic.

Step 1: Transform derivatives correctly. The chain rule gives $Y' = xy'$ and $Y'' = xy' + x^2y''$, hence

$$\boxed{xy' = Y', \qquad x^2y'' = Y'' - Y'.}$$

The equation becomes $Y'' - 3Y' - 4Y = 0$, with characteristic polynomial $m^2 - 3m - 4 = (m-4)(m+1)$.

Step 2: Solve and verify the data. The powers x^4, x^{-1} form a pair with $W = x^4(-x^{-2}) - 4x^3x^{-1} = -5x^2 \neq 0$ for $x > 0$. Writing $y = C_1x^4 + C_2x^{-1}$ gives $C_1 + C_2 = 2$, $4C_1 - C_2 = 3$, so

$$\boxed{y = x^4 + x^{-1}.}$$

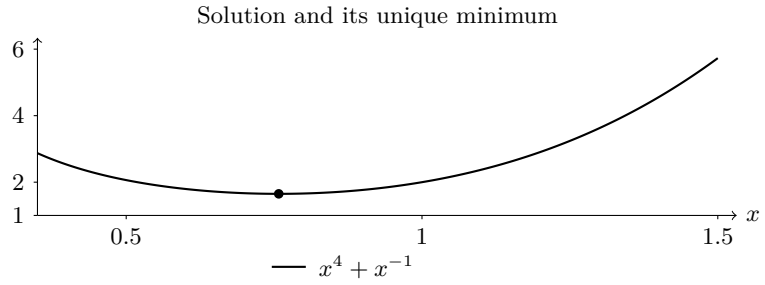
At 1, its value is 2 and derivative $4x^3 - x^{-2}$ is 3.

Step 3: Identify the maximal interval. The normalized coefficients are continuous on $(0, \infty)$, and the explicit solution exists throughout it. Since $y \to +\infty$ as $x \to 0^+$, no finite continuation through 0 is possible. Thus the maximal real interval through 1 is $\boxed{(0, \infty)}$. At infinity, $y \sim x^4 \to +\infty$; at zero, $y \sim x^{-1}$.

Step 4: Find the unique minimum. The equation $y' = 0$ is $4x^5 = 1$, giving $x_* = 4^{-1/5}$. Since $y'' = 12x^2 + 2x^{-3} > 0$, it is the unique global minimum. Using $x_*^{-1} = 4x_*^4$,

$$\boxed{y(x_*) = 5 \cdot 4^{-4/5}.}$$

One mode increases and the other decreases. Their derivatives cancel at exactly one point, although neither mode has a stationary point of its own.



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Question 2

For $x > 0$, consider

$$x^2y'' - 3xy' + 4y = 0, \qquad y(1) = 1, \quad y'(1) = 3.$$

A repeated characteristic exponent needs a second independent solution and careful treatment at the singular point.

Tasks

1. Derive the repeated exponent and obtain a second independent solution using logarithmic coordinates. Verify independence rather than merely listing two functions.
2. Solve the IVP and find every positive zero of its solution.
3. Determine whether this IVP solution admits a continuous, C^1, or C^2 extension to 0. Compute the relevant one-sided limits.
4. For the general positive-half-line solution, classify which choices permit a C^2 extension through 0 that solves the original equation. Explain whether the data $y(0) = y'(0) = 0$ determine that extension uniquely.

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Question 2 – Solution

Strategy. A repeated exponential in logarithmic time produces a logarithmic factor in the original variable.

Step 1: Construct a genuine fundamental pair. With $t = \ln x$, the equation becomes $Y'' - 4Y' + 4Y = 0$, whose characteristic polynomial is $(m-2)^2$. Thus

$$y = x^2(A + B\ln x).$$

For $u = x^2$, $v = x^2 \ln x$, the Wronskian is $W = x^2(2x\ln x + x) - 2x(x^2\ln x) = x^3 \neq 0$ on $x > 0$.

Step 2: Use both initial values. At 1, $y(1) = A$ and $y'(1) = 2A + B$. Hence $A = B = 1$, giving

$$\boxed{y = x^2(1 + \ln x).}$$

Because $x^2 > 0$, the only positive zero is $\boxed{x = e^{-1}}$.

Step 3: Test successive derivatives at zero. For this solution,

$$y' = x(3 + 2\ln x), \qquad y'' = 5 + 2\ln x.$$

As $x \to 0^+$, $y \to 0$ and $y' \to 0$, but $y'' \to -\infty$. Defining $y(0) = 0$ gives a right-hand C^1 extension with derivative 0: indeed $y(x)/x = x(1 + \ln x) \to 0$. It can be joined to zero on the negative side as a C^1 function, but no C^2 extension is possible.

Step 4: Classify classical extensions. In general, $y'' = 2A + B(2\ln x + 3)$, so a finite second-derivative limit requires $B = 0$. On $x < 0$ the general solution is $x^2(C + D\ln|x|)$. A C^2 join similarly forces $D = 0$, and matching second derivatives forces $C = A$. Thus precisely $\boxed{y = Ax^2}$ extend through 0 as C^2 solutions of the original equation. Each satisfies $y(0) = y'(0) = 0$, so these data do not give uniqueness at the singular point.

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Question 3

Study the logarithmically oscillating IVP

$$x^2y'' + xy' + 4y = 0, \qquad y(1) = 0, \quad y'(1) = 2, \qquad x > 0.$$

Oscillations in $\ln x$ behave differently from oscillations in x.

Tasks

1. Find the real general solution and the specified IVP solution.
2. Determine all its positive zeros and the ratio of consecutive zeros in increasing order. Explain where the zeros accumulate.
3. Decide whether the IVP solution is bounded near 0 and whether it has a continuous extension there. Extend the conclusion about limits to every nonzero real solution of this equation.
4. Set $z = xy'/2$. Determine the curve traced by (y, z) and its direction as x increases. Locate the initial point and distinguish this curve from the graph of y against x.

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Question 3 – Solution

Strategy. Use $t = \ln x$ to reveal a harmonic oscillator and then translate its time scale back to x.

Step 1: Solve the transformed oscillator. The transformed equation is $Y'' + 4Y = 0$, so $y = A\cos(2\ln x) + B\sin(2\ln x)$. At $x = 1$, $A = 0$ and $2B = y'(1) = 2$, hence

$$\boxed{y(x) = \sin(2\ln x), \qquad x > 0.}$$

The characteristic exponents are $m = \pm 2i$.

Step 2: Locate the geometric sequence of zeros. Zeros occur when $2\ln x = k\pi$:

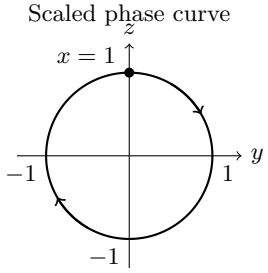
$$\boxed{x_k = e^{k\pi/2}, \quad k \in \mathbb{Z}, \qquad x_{k+1}/x_k = e^{\pi/2}.}$$

They accumulate at 0 as $k \to -\infty$ and grow without bound as $k \to \infty$. There is no smallest positive zero.

Step 3: Separate boundedness from a limit. The IVP solution has $|y| \le 1$, but sequences with $\ln x = \pi/4 - k\pi$ and $\ln x = 3\pi/4 - k\pi$ tend to 0 while giving values 1 and -1. Thus no limit at zero exists. Any nonzero real solution is a nonzero-amplitude sinusoid in $\ln x$ and likewise attains two distinct limiting values along sequences approaching zero. Only the zero solution extends continuously there.

Step 4: Read the scaled phase curve. For the IVP, $z = \cos(2\ln x)$, so $\boxed{y^2 + z^2 = 1}$. In logarithmic time, $\dot{y} = 2z$, $\dot{z} = -2y$; at $(y, z) = (0, 1)$ the motion is to the right. The circle is traversed clockwise as x increases, because $\ln x$ increases with x. The horizontal coordinate here is y, not x; equal scales preserve the unit circle.

Scaled phase curve



clockwise as x increases

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Question 4

Solve on the negative half-line:

$$x^2y'' + xy' - \tfrac{1}{4}y = 0, \qquad y(-4) = 3, \quad y'(-4) = -\tfrac{1}{8}.$$

Using fractional powers without specifying a real branch can invalidate a solution.

Tasks

1. Use $t = \ln(-x)$ to derive a real fundamental pair on $x < 0$. Explain why the positive-half-line expression $x^{1/2}$ is unsuitable as a real-valued basis there.
2. Translate both initial values into logarithmic coordinates and solve the IVP. Check the derivative sign directly in the original variable.
3. Give its maximal real interval and both endpoint behaviors.
4. Find its unique stationary point and minimum. Explain how increasing logarithmic time relates to increasing x on this negative interval.

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Question 4 – Solution

Strategy. Work with positive $-x$, retaining the sign in the chain rule.

Step 1: Choose a real branch. For $x = -e^t$, $dx/dt = x$, so $Y' = xy'$ and $Y'' = xy' + x^2y''$ still hold. The equation becomes $Y'' - Y/4 = 0$. A real pair is $\boxed{(-x)^{1/2},\ (-x)^{-1/2}}$ for $x < 0$. The expression $x^{1/2}$ has no real value on this interval; powers of $|x| = -x$ avoid that problem.

Step 2: Translate the slope correctly. At $t = \ln 4$, $Y = 3$ and $Y' = xy' = (-4)(-1/8) = 1/2$. Writing $y = A\sqrt{-x} + B/\sqrt{-x}$ yields $2A + B/2 = 3$ and $A - B/4 = 1/2$, hence $A = 1$, $B = 2$. Thus

$$\boxed{y = \sqrt{-x} + \frac{2}{\sqrt{-x}}, \qquad y' = -\frac{1}{2\sqrt{-x}} + \frac{1}{(-x)^{3/2}}.}$$

At -4 these give 3 and $-1/8$, as required.

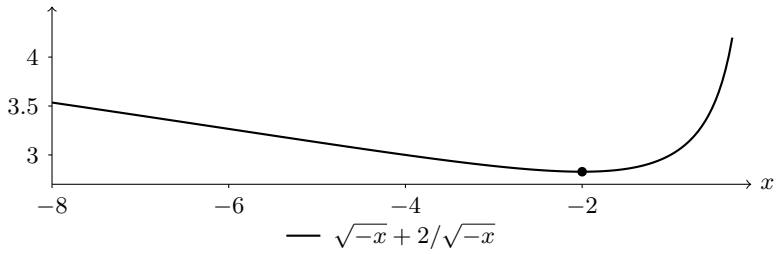
Step 3: State the maximal interval. The solution is smooth throughout $(-\infty, 0)$, and its divergence at 0 prevents extension through that point. This is the maximal real interval containing -4. As $x \to -\infty$, $y \sim \sqrt{-x} \to \infty$; as $x \to 0^-$, $y \sim 2/\sqrt{-x} \to \infty$.

Step 4: Locate the minimum and orient time. Put $r = -x > 0$. The derivative is $(2 - r)/(2r^{3/2})$, so it is negative for $x < -2$ and positive for $-2 < x < 0$. Therefore

$$\boxed{x_* = -2, \qquad y(x_*) = 2\sqrt{2}}$$

is the unique global minimum. Since $dt/dx = 1/x < 0$, increasing x corresponds to decreasing t on this half-line. This reverses the relation between the two horizontal directions, not the derivative formulas.

Real solution on the negative half-line



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Question 5

For $x > 0$, solve the resonantly forced Euler problem

$$x^2y'' - 3xy' + 4y = x^2 \ln x, \qquad y(1) = y'(1) = 0.$$

A trial expression $x^2(A + B \ln x)$ resembles the forcing but may not produce it.

Tasks

1. Transform the equation using $t = \ln x$ and then $Y = e^{2t}Z$. Derive the resulting equation for Z.
2. Find the general solution. Explain exactly why the proposed trial expression fails and why a cubic logarithmic term appears.
3. Apply the initial conditions and verify the final solution by computing the transformed residual.
4. Determine whether the IVP solution extends continuously, as a C^1 function, or as a C^2 function to 0. Give the relevant limiting expressions.

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Question 5 – Solution

Strategy. Remove the repeated characteristic exponential before integrating the forcing.

Step 1: Factor the logarithmic-time operator. The equation is $(D-2)^2Y = e^{2t}t$, where $D = d/dt$. Since $(D-2)(e^{2t}Z) = e^{2t}Z'$, a second application gives

$$\boxed{Z'' = t.}$$

Step 2: Integrate through the resonance. Two integrations yield $Z = t^3/6 + C_1 + C_2 t$, hence

$$\boxed{y = x^2\left(C_1 + C_2\ln x + \frac{(\ln x)^3}{6}\right).}$$

The proposed trial has $Z = A + Bt$ and therefore $Z'' = 0$: it belongs entirely to the homogeneous solution space. Integrating the degree-one forcing twice produces degree three, accounting for the cubic logarithm and its factor $1/6$.

Step 3: Impose and verify the initial data. At $x = 1$, $t = 0$, so $y(1) = C_1$ and $y'(1) = 2C_1 + C_2$. Thus both constants vanish and

$$\boxed{y = \frac{x^2(\ln x)^3}{6}.}$$

For this solution, $Z = t^3/6$ satisfies $Z'' = t$ exactly; therefore the original left-hand side is $e^{2t}t = x^2\ln x$. The two initial values are zero.

Step 4: Inspect regularity at the singular point. Writing $t = \ln x$,

$$y' = x\left(\frac{t^3}{3} + \frac{t^2}{2}\right), \qquad y'' = \frac{t^3}{3} + \frac{3t^2}{2} + t.$$

As $x \to 0^+$, the power of x dominates every logarithmic power, so $y \to 0$, $y' \to 0$, and $y(x)/x \to 0$. Assigning $y(0) = 0$ therefore gives a C^1 right extension, which may be joined to zero on the left as a C^1 function. But $y'' \to -\infty$, so no C^2 extension is possible.

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Question 6

For $x > 0$, consider

$$x^2 y'' + xy' = \frac{x^2}{1 + x^2}.$$

Instead of prescribing data at a positive point, seek the solution that is bounded near 0 and extends continuously with $y(0) = 0$.

Tasks

1. Integrate the equation to obtain its general solution in definite-integral form on $(0, \infty)$, displaying both arbitrary constants.
2. Impose boundedness and the specified value at zero. Prove that they select exactly one solution on the positive half-line.
3. Derive its power series about zero, give its first three nonzero terms, and determine the radius and both endpoint outcomes. Justify termwise integration.
4. Prove that the selected solution has a C^2 extension to zero, compute $y'(0)$ and $y''(0)$, and certify a two-term approximation on $[0, 1]$ using a rigorous remainder bound.

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Question 6 – Solution

Strategy. Integrate the weighted derivative first, then use regularity at zero to remove the logarithmic homogeneous mode.

Step 1: Integrate the Euler operator. Dividing by x gives $(xy')' = x/(1+x^2)$. Hence

$$xy' = \tfrac{1}{2}\ln(1+x^2) + C, \qquad y = \frac{1}{2}\int_1^x \frac{\ln(1+s^2)}{s}\,ds + C\ln x + D.$$

These formulas display both constants and solve the equation for every $x > 0$.

Step 2: Apply the singular-endpoint conditions. Since $\ln(1+s^2)/s \sim s$ at zero, its integral there converges. Boundedness therefore forces $C = 0$; the value $y(0) = 0$ fixes the remaining constant. The unique selected solution is

$$\boxed{y(x) = \frac{1}{2}\int_0^x \frac{\ln(1+s^2)}{s}\,ds.}$$

Any two bounded solutions differ by a constant, and their common value at zero makes that constant zero.

Step 3: Expand the integral with justification. The divided logarithmic series $\sum_{n\ge1}(-1)^{n-1}s^{2n-1}/n$ is uniformly convergent on $|s| \le \rho < 1$, by comparison with $\sum_{n\ge1}\rho^{2n-1}/n$. Termwise integration therefore yields, for $|x| < 1$,

$$\boxed{y(x) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n^2} = \frac{x^2}{4} - \frac{x^4}{16} + \frac{x^6}{36} - \cdots.}$$

The ratio of successive nonzero terms tends to x^2, so the radius is 1. At $x = \pm1$ the series converges absolutely by comparison with $\sum 1/n^2$. The series and the even integral extension agree there by continuity: the series is uniformly convergent on $[-1,1]$ by the same comparison.

Step 4: Recover derivatives and a certificate. The local power series supplies a smooth even extension with $y'(0) = 0$ and $y''(0) = 1/2$. With the forcing defined continuously as zero at 0, the original equation holds there as well. On $[0,1]$, the alternating term magnitudes decrease, so for $P_4 = x^2/4 - x^4/16$,

$$\boxed{0 \le y(x) - P_4(x) \le \frac{x^6}{36}.}$$

The bound includes the endpoint $x = 1$; it is a certificate, not a claim that the two-term approximation is highly accurate there.

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Question 7

Let $L > 0$ be fixed. Seek nonzero real solutions of the two-endpoint Euler problem

$$x^2y'' + xy' + \lambda y = 0, \qquad y(1) = y(e^L) = 0,$$

where λ is real. No general theory of boundary-value eigenproblems is required; solve the Euler equation directly.

Tasks

1. Transform the problem to $0 \le t \le L$ and prove that $\lambda \le 0$ permits only the zero solution.
2. Find every $\lambda > 0$ permitting a nonzero solution, and give the corresponding solution families.
3. For each admissible value, impose $y'(1) = 1$. Find the normalized solution and all its zeros, including the number strictly inside $(1, e^L)$.
4. Explain why uniqueness of ordinary initial-value problems does not imply uniqueness for these two endpoint conditions. For $L = \pi$, describe the first three normalized profiles in the coordinate $t = \ln x$.

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Question 7 – Solution

Strategy. In logarithmic coordinates, the two endpoint conditions reduce to elementary trigonometric or hyperbolic equations.

Step 1: Exclude nonpositive parameters. Set $Y(t) = y(e^t)$. Then $Y'' + \lambda Y = 0$, with $Y(0) = Y(L) = 0$. If $\lambda = 0$, $Y = A + Bt$ and both constants must vanish. If $\lambda = -\mu^2 < 0$, $Y = A\cosh(\mu t) + B\sinh(\mu t)$. The first condition gives $A = 0$; since $\sinh(\mu L) > 0$, the second gives $B = 0$.

Step 2: Find all nonzero families. For $\lambda = \omega^2 > 0$, the first condition leaves $Y = B\sin(\omega t)$. A nonzero B requires $\sin(\omega L) = 0$, so

$$\boxed{\lambda_n = (n\pi/L)^2, \qquad y = C\sin\left(\frac{n\pi \ln x}{L}\right), \quad n = 1, 2, \ldots.}$$

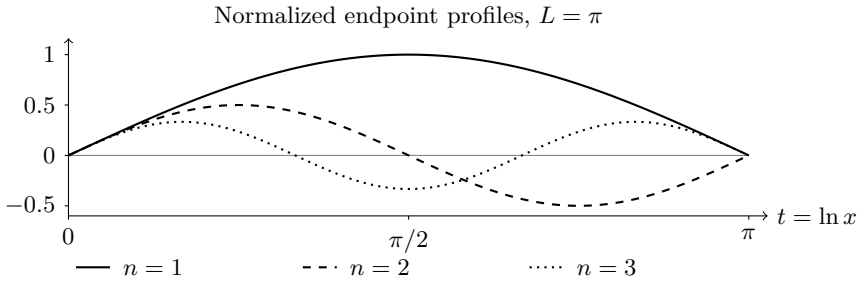
There are no other real parameter values admitting nonzero solutions.

Step 3: Normalize and locate the zeros. At $x = 1$, $y'(1) = Y'(0) = Cn\pi/L$. Thus

$$\boxed{y_n(x) = \frac{L}{n\pi}\sin\left(\frac{n\pi \ln x}{L}\right).}$$

Its zeros on the specified interval are $x_k = e^{kL/n}$ for $k = 0, \ldots, n$. Exactly $n - 1$ lie strictly inside; they are equally spaced in $\ln x$, not in x.

Step 4: Distinguish endpoint and initial data. The original endpoint conditions do not specify the initial slope. At an admissible parameter, every amplitude satisfies both endpoints, including zero; hence those data are not unique. Once $y'(1) = 1$ is imposed, the usual IVP uniqueness applies. For $L = \pi$, the first three normalized profiles are $Y_n(t) = \sin(nt)/n$, $n = 1, 2, 3$, with $0, 1, 2$ interior zeros, respectively. The figure uses logarithmic horizontal coordinates.



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Question 8

For fixed $a \in \mathbb{R}$ and parameter $\varepsilon \geq 0$, consider on $x > 0$

$$x^2y'' + (1 - 2a)xy' + (a^2 - \varepsilon^2)y = 0.$$

Two distinct power solutions become nearly identical when their characteristic exponents approach one another.

Tasks

1. Find the characteristic exponents for $\varepsilon > 0$ and explain why simply taking their two power solutions to the limit does not produce a fundamental pair at $\varepsilon = 0$.
2. Construct solutions $u_\varepsilon, v_\varepsilon$ satisfying $u_\varepsilon(1) = 1$, $u'_\varepsilon(1) = a$, $v_\varepsilon(1) = 0$, $v'_\varepsilon(1) = 1$.
3. Compute their Wronskian and prove independence for every $\varepsilon \geq 0$, using the limiting pair at zero.
4. Determine that limiting pair and justify uniform convergence of each basis function on every compact interval $[r, R] \subset (0, \infty)$ as $\varepsilon \to 0$. Explain how the logarithmic repeated-root solution emerges.

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Question 8 – Solution

Strategy. Normalize a symmetric sum and a divided difference before allowing the characteristic roots to merge.

Step 1: Identify the degenerating pair. The power equation is $(m-a)^2-\varepsilon^2=0$, giving $m=a\pm\varepsilon$. Both $x^{a+\varepsilon}$ and $x^{a-\varepsilon}$ tend to x^a as $\varepsilon\to 0$. Their separate limits are the same function, so they cannot remain a fundamental pair.

Step 2: Normalize before taking the limit. For $\varepsilon>0$, set $t=\ln x$ and define

$$\boxed{u_\varepsilon = x^a\cosh(\varepsilon t), \qquad v_\varepsilon = x^a\frac{\sinh(\varepsilon t)}{\varepsilon}.}$$

These are the half-sum and the difference divided by 2ε of the two powers. At $x=1$ they have the four required value/slope data by direct differentiation.

Step 3: Check the determinant exactly. Differentiation and $\cosh^2 s-\sinh^2 s=1$ give

$$\boxed{W(u_\varepsilon,v_\varepsilon)=x^{2a-1}>0.}$$

At $\varepsilon=0$, define $u_0=x^a$ and $v_0=x^a\ln x$. They solve the repeated-root equation, have the same normalized data, and their Wronskian is again x^{2a-1}. Independence holds for every parameter value.

Step 4: Justify the limit uniformly. On $[r,R]$, let $T=\max(|\ln r|,|\ln R|)$ and $M=\max_{[r,R]}x^a$. For $0<\varepsilon\le 1$, Taylor's theorem for cosh and sinh gives

$$|u_\varepsilon-u_0|\le\frac{M\varepsilon^2T^2\cosh T}{2}, \qquad |v_\varepsilon-v_0|\le\frac{M\varepsilon^2T^3\cosh T}{6}.$$

Both bounds tend to zero uniformly. The divided difference tends to $\partial_m x^m|_{m=a}=x^a\ln x$, recovering the independent logarithmic solution that the two unnormalized limits lose.

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Question 9

For real α and $\beta > 0$, consider a nonzero solution on $x > 0$ of

$$x^2y'' + (1 - 2\alpha)xy' + (\alpha^2 + \beta^2)y = 0.$$

Its power envelope may tend to zero even while its derivatives keep oscillating.

Tasks

1. Find the real general solution and show that its jth derivative has the form $x^{\alpha-j}$ times a sinusoid in $\ln x$ with nonzero amplitude.
2. For each integer $k \geq 0$, prove a necessary and sufficient condition on α for a nonzero solution to have a C^k extension to $x = 0$. Treat the equality case explicitly.
3. Apply your criterion to $y = x^{5/2}\sin(\ln x)$. Compute y'' and determine exactly which integer orders of smoothness are possible at zero.
4. Join this particular solution to zero for $x \leq 0$. Verify that the joined function is a classical C^2 solution of the corresponding Euler equation on the whole real line. Explain the implication for initial data at the singular point.

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Question 9 – Solution

Strategy. Track both the power envelope and the nonvanishing oscillation amplitude after every differentiation.

Step 1: Differentiate the complex powers. The exponents are $\alpha \pm i\beta$, so the real solutions are

$$y = x^{\alpha}\big(A\cos(\beta \ln x) + B\sin(\beta \ln x)\big).$$

The jth derivative of $x^{\alpha+i\beta}$ is $\prod_{\ell=0}^{j-1}(\alpha - \ell + i\beta)x^{\alpha-j+i\beta}$. Since $\beta > 0$, none of these factors vanishes. A nonzero real solution therefore retains a nonzero sinusoidal amplitude at every derivative order.

Step 2: Establish the exact regularity threshold. If $\alpha > k$, all derivatives through order k tend to zero. Their extensions are consistent derivatives: for $j < k$, $y^{(j)}(x)/x = O(x^{\alpha-j-1}) \to 0$. If $\alpha = k$, the kth derivative has constant nonzero amplitude and no limit. If $\alpha < k$, that derivative has an unbounded envelope and attains unbounded values along a sequence approaching zero. Hence

$$\boxed{\text{A nonzero solution extends as } C^k \text{ to } 0 \iff \alpha > k.}$$

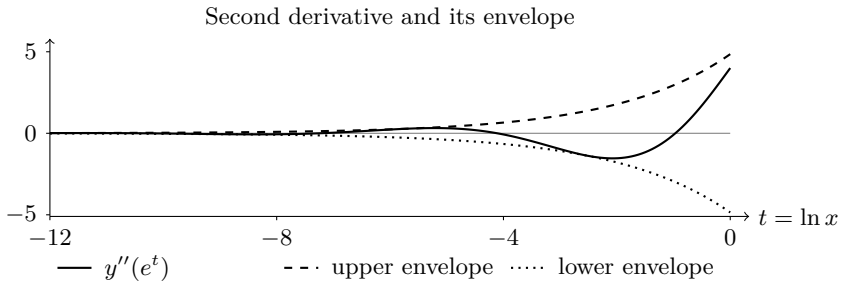
This includes $k = 0$; bounded oscillation at equality is not continuity.

Step 3: Inspect the concrete example. For $\alpha = 5/2$, $\beta = 1$,

$$\boxed{y'' = x^{1/2}\left(\frac{11}{4}\sin(\ln x) + 4\cos(\ln x)\right).}$$

The solution extends as C^0, C^1, C^2, with all three center values zero, but not as C^3 or any higher integer order. The plot shows $y''(e^t)$ and its decaying envelope as $t \to -\infty$.

Step 4: Verify the joined classical solution. The equation is $x^2y'' - 4xy' + (29/4)y = 0$. On $x > 0$ the formula solves it; on $x < 0$ the zero function does. The zero limits through the second derivative make the join C^2, and at 0 every term in the original equation is zero. Both this nonzero joined solution and the identically zero solution have $y(0) = y'(0) = 0$. Classical IVP uniqueness fails at this singular point.



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Question 10

For $x \geq 1$, consider a finite interval of forcing in an Euler equation:

$$x^2y'' - xy' + y = g(x), \qquad y(1) = y'(1) = 0, \qquad g(x) = \begin{cases} x, & 1 \leq x < e, \\ 0, & x > e. \end{cases}$$

Seek a solution that is C^1 across $x = e$ and solves the equation on each side. No classical second derivative at the switching point is assumed.

Tasks

1. Set $t = \ln x$ and $Y = e^t Z$. Derive a general definite-integral solution for zero initial data, valid for piecewise continuous forcing g.
2. Apply it to the specified forcing and give explicit formulas on $1 \leq x \leq e$ and $x \geq e$.
3. Verify continuity of y and y' at e. Compute the one-sided second derivatives and explain why a C^2 solution at that point is impossible.
4. Determine the large-x behavior after the forcing stops. Explain why the solution does not return to zero just because the right-hand side becomes zero.

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Question 10 – Solution

Strategy. Remove the repeated exponent, integrate in logarithmic time, and match the finite-forcing transition.

Step 1: Derive the integral response. With $D = d/dt$, the transformed equation is $(D-1)^2 Y = g(e^t)$, so $Z'' = e^{-t} g(e^t)$. The data at $t = 0$ give $Z(0) = Z'(0) = 0$. Thus

$$Z(t) = \int_0^t (t-s) e^{-s} g(e^s)\, ds, \qquad \boxed{y(x) = x \int_1^x \ln(x/r) \frac{g(r)}{r^2}\, dr.}$$

The second formula follows from $r = e^s$, including $ds = dr/r$.

Step 2: Evaluate the finite forcing interval. Here $Z'' = 1$ for $0 < t < 1$ and $Z'' = 0$ for $t > 1$. Integrating with the zero initial data gives $Z = t^2/2$ up to 1, then $Z = t - 1/2$. Therefore

$$\boxed{y(x) = \begin{cases} \frac{1}{2} x (\ln x)^2, & 1 \le x \le e, \\ x(\ln x - \frac{1}{2}), & x \ge e. \end{cases}}$$

Both formulas have the same value at e, and the first satisfies both initial conditions at 1.

Step 3: Check the join and the derivative jump. For $Y = e^t Z$, $y' = Z + Z'$ and $y'' = (Z' + Z'')/x$. At $t = 1$, both sides have $Z = 1/2$ and $Z' = 1$, so

$$y(e) = e/2, \qquad y'(e) = 3/2, \qquad y''(e^-) = 2/e, \quad y''(e^+) = 1/e.$$

The finite jump in forcing produces a jump $-1/e$ in y'', while y, y' remain continuous. The unequal one-sided limits rule out C^2 regularity, irrespective of any value assigned to $g(e)$.

Step 4: Interpret the surviving homogeneous motion. After e, the solution is the nonzero homogeneous combination $x \ln x - x/2$. Thus $\boxed{y(x)/(x \ln x) \to 1}$ as $x \to \infty$. The completed forcing interval leaves nonzero value and slope at the switch. Those become initial data for the homogeneous equation; zero subsequent forcing does not reset them to zero.

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Question 1

Consider the real differential equation

$$(x^2 - 1)y^{(4)} + 2xy''' - 3y' = \ln x.$$

A leading coefficient that vanishes at one point requires more care than simply counting derivatives.

Tasks

1. Classify the equation by order, linearity and homogeneity. Write its normalized form wherever that form is defined over the reals.
2. At $x_0 = 1/2$, prescribe $y = 1$, $y' = 0$, $y'' = -2$ and $y''' = 3$. State the largest interval containing x_0 on which the standard linear IVP theorem guarantees a unique solution, and explain why four data are needed.
3. A proposed C^4 solution near $x = 1$ has $y(1) = 0$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 1$. Test these data directly in the original equation.
4. Replace only the last datum by $y'''(1) = 3$. Explain what this compatibility check establishes and what it does not establish. Does a vanishing leading coefficient make the original equation third order on a neighborhood of 1?

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Question 1 – Solution

Strategy. Separate the equation's order from the intervals on which its highest derivative can be isolated.

Step 1: Classify and normalize. This is a fourth-order, linear, nonhomogeneous equation: coefficients depend only on x, and the forcing $\ln x$ is not identically zero. For $x > 0$, $x \neq 1$,

$$\boxed{y^{(4)} + \frac{2x}{x^2-1}y''' - \frac{3}{x^2-1}y' = \frac{\ln x}{x^2-1}.}$$

The coefficients of y'' and y are zero. The logarithm excludes $x \leq 0$; normalization additionally excludes 1.

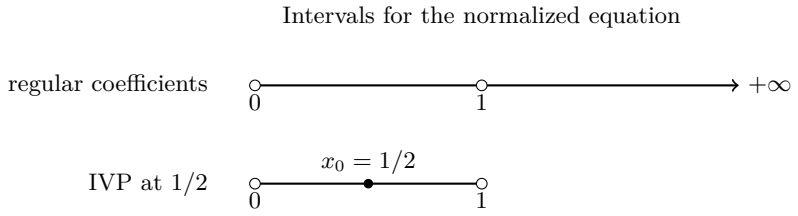
Step 2: Apply the theorem on its proper interval. All normalized coefficients and the forcing are continuous on $\boxed{(0,1)}$, the largest such interval through $1/2$. The linear IVP theorem gives one solution for arbitrary values of y, y', y'', y''' at that ordinary point. These four data specify the fourth-order initial state. The theorem does not assert that this particular solution cannot extend beyond the interval; it simply gives no guarantee across 1.

Step 3: Check the singular-point constraint. At $x = 1$, any C^4 solution must satisfy

$$\boxed{2y'''(1) - 3y'(1) = 0.}$$

The proposed data give $2 - 6 = -4$, not zero. They are impossible for a classical solution of the original equation near 1.

Step 4: Interpret compatibility without overclaiming. With $y'''(1) = 3$, the same test gives $6-6 = 0$. This removes that contradiction, but alone proves neither existence nor uniqueness; the normalized IVP theorem still does not apply at 1. The coefficient of $y^{(4)}$ is nonzero at nearby points other than 1, so the equation remains fourth order on a neighborhood. At the isolated point, its value imposes a lower-derivative constraint instead of determining $y^{(4)}(1)$.



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Question 2

Let $L[y] = y^{(4)}$ and consider $L[y] = 24x$ on $\mathbb{R}$. Define

$$p(x) = \frac{x^5}{5}, \qquad u(x) = p(x) + 1 + x^2, \qquad v(x) = p(x) + 2x - x^3.$$

For a nonhomogeneous equation, the phrase "superposition of solutions" needs a precise qualification.

Tasks

1. Verify that p, u, v solve the equation. Determine whether $u + v$, $u - v$ and $(u + v)/2$ solve the same equation or its homogeneous counterpart.
2. Find every triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ for which $\alpha u + \beta v + \gamma p$ solves $L[y] = 24x$.
3. Prove that the complete solution set is p plus the space of cubic polynomials. Explain why it is an affine set but not a vector space.
4. Find the unique solution with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 6$, $y'''(0) = 0$, and verify the forcing and all four data.

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Question 2 – Solution

Strategy. Apply the linear operator before deciding which combinations preserve the nonzero forcing.

Step 1: Track the right-hand side. Since $p^{(4)} = 24x$ and fourth derivatives of cubics vanish, $L[p] = L[u] = L[v] = 24x$. Linearity gives

$$L[u+v] = 48x, \qquad L[u-v] = 0, \qquad L[(u+v)/2] = 24x.$$

Thus the sum solves neither of the two stated equations, the difference solves the homogeneous equation, and the average solves the original one.

Step 2: Characterize every admissible combination. The proposed combination has image $24x(\alpha + \beta + \gamma)$. Because $24x$ is not identically zero on $\mathbb{R}$,

$$\boxed{\alpha u + \beta v + \gamma p \text{ solves the equation } \iff \alpha + \beta + \gamma = 1.}$$

The forcing's isolated zero at 0 does not remove this identity requirement.

Step 3: Describe the complete solution set. If y is any solution, $(y-p)^{(4)} = 0$. Four integrations show that $y - p$ is a polynomial of degree at most three. Conversely, adding any such polynomial to p preserves the equation. Hence

$$\boxed{y = p + c_0 + c_1 x + c_2 x^2 + c_3 x^3.}$$

This is a translate of a four-dimensional vector space, hence affine. It is not itself a vector space: the zero function does not satisfy the nonzero forcing, and sums generally double the forcing.

Step 4: Fit and check the initial state. The data give $c_0 = 1$, $c_1 = -2$, $2c_2 = 6$, $6c_3 = 0$, so

$$\boxed{y = \frac{x^5}{5} + 1 - 2x + 3x^2.}$$

The added cubic has fourth derivative zero. Its value and first three derivatives at 0 are $1, -2, 6, 0$; p contributes zero to all four. The coefficients are uniquely fixed.

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Question 3

For $y^{(4)} = 0$ on $\mathbb{R}$, consider the shifted functions

$$f_0 = 1, \qquad f_1 = x - 1, \qquad f_2 = \frac{(x-1)^2}{2}, \qquad f_3 = \frac{(x-1)^3}{6}.$$

Initial derivatives and polynomial coefficients coincide only for an appropriately normalized basis.

Tasks

1. Verify that the four functions solve the equation and compute their Wronskian for arbitrary x.
2. Construct the solution with $y(1) = 2$, $y'(1) = -1$, $y''(1) = 3$ and $y'''(1) = 4$. Explain why the coefficients in this basis can be read directly from the data.
3. Express the same solution in the unshifted basis $1, x, x^2, x^3$ and verify the four data in that form.
4. Prove that the map sending a solution to $(y(1), y'(1), y''(1), y'''(1))$ is a linear isomorphism onto $\mathbb{R}^4$. What does this say about the number of free constants and the effect of changing the basis?

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Question 3 – Solution

Strategy. Build the derivative matrix and use its normalization at the initial point.

Step 1: Verify solutions and a nonzero determinant. Every function is cubic or lower, so its fourth derivative is zero. With $h = x - 1$, their derivative matrix is

$$F(x) = \begin{pmatrix} 1 & h & h^2/2 & h^3/6 \\ 0 & 1 & h & h^2/2 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is triangular with unit diagonal, giving $\boxed{W = \det F = 1}$ at every x.

Step 2: Use the identity matrix at the center. Since $F(1) = I$, the coefficient vector is exactly the prescribed derivative vector. Therefore

$$\boxed{y = 2 - h + \frac{3}{2}h^2 + \frac{2}{3}h^3, \qquad h = x - 1.}$$

The factorials in the basis make its initial derivative matrix the identity; without them, derivative values would not equal coefficients.

Step 3: Change coordinates without changing the solution. Expansion gives

$$\boxed{y = \frac{23}{6} - 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3.}$$

Here $y' = -2 - x + 2x^2$, $y'' = -1 + 4x$, $y''' = 4$. At 1, these are $-1, 3, 4$, while $y(1) = 2$, verifying all data independently of the shifted expression.

Step 4: Establish the initial-data isomorphism. Differentiation and evaluation are linear. For any vector in $\mathbb{R}^4$, the corresponding combination of f_0, f_1, f_2, f_3 realizes it, proving surjectivity. If all four derivatives at 1 vanish, the coefficient vector in this basis is zero, proving injectivity. Thus the solution space has dimension four. A change of basis changes coordinates, not the number of free constants, the initial data themselves or the resulting function.

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Question 4

Four functions $y_1, \dots, y_4$ solve the same homogeneous equation

$$y^{(4)} - \frac{2x}{1+x^2} y''' + p_2(x) y'' + p_1(x) y' + p_0(x) y = 0$$

on $\mathbb{R}$, where p_0, p_1, p_2 are continuous. Their Wronskian at zero is $W(0) = -6$.

Tasks

1. Derive the Wronskian identity $W' = -p_3 W$ for a normalized fourth-order homogeneous equation, by differentiating the determinant row by row.
2. Determine $W(x)$ for the stated equation and decide whether the four functions form a fundamental set on the whole real line.
3. Explain why every four-component initial derivative vector at any finite x_0 is realized by exactly one linear combination of these functions.
4. A computation reports $\widetilde{W}(x) = -6(1+x^2)(1-x)$. It has the correct value at 0. Give two independent reasons why it cannot be the Wronskian of this common solution family.

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Question 4 – Solution

Strategy. The coefficient of the next-to-highest derivative controls the entire Wronskian, not just its value at one point.

Step 1: Differentiate the determinant. The Wronskian rows contain derivative orders $0, 1, 2, 3$. Differentiating any of the first three rows duplicates the following row, giving determinant zero. In the last-row term, substitute $y_j^{(4)} = -p_3 y_j''' - p_2 y_j'' - p_1 y_j' - p_0 y_j$ for every column. All terms except the multiple of the original last row again duplicate a row. Therefore

$$\boxed{W' = -p_3 W.}$$

Step 2: Determine the Wronskian globally. Here $p_3 = -2x/(1+x^2)$, so

$$W(x) = -6 \exp\left(\int_0^x \frac{2s}{1+s^2}\, ds \right) = \boxed{-6(1+x^2)}.$$

It never vanishes. The four solutions are linearly independent, and the regular fourth-order homogeneous solution space has dimension four; hence they form a fundamental set on $\mathbb{R}$.

Step 3: Interpret invertibility of the derivative matrix. At any finite x_0, the matrix with columns $(y_j, y_j', y_j'', y_j''')(x_0)$ has determinant $W(x_0) \neq 0$. It therefore maps one and only one coefficient vector to any prescribed initial derivative vector. The resulting combination solves the equation; the linear IVP theorem shows it is the unique solution for those data on $\mathbb{R}$.

Step 4: Reject a plausible-looking candidate. First, $\widetilde{W}(1) = 0$ contradicts the nonvanishing formula forced by $W(0) \neq 0$ on this regular connected interval. Second, it fails the differential identity even at the initial point: $\widetilde{W}'(0) = 6$, whereas $-p_3(0)\widetilde{W}(0) = 0$. Matching one determinant value is insufficient; a valid Wronskian must obey the identity throughout the interval.

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Question 5

You may use the fact that the function

$$\phi(s) = \begin{cases} \exp\left(-\dfrac{1}{1-4s^2}\right), & |s| < 1/2, \\ 0, & |s| \geq 1/2 \end{cases}$$

is C^∞ on $\mathbb{R}$, with every derivative zero at $s = \pm 1/2$. Define $f_1(x) = \phi(x+2)$, $f_2(x) = \phi(x)$ and $f_3(x) = \phi(x-2)$.

Tasks

1. Locate the supports of the three functions and prove that they are linearly independent on $\mathbb{R}$.
2. Prove that their three-by-three Wronskian is identically zero. Use their derivative supports, not a long symbolic expansion.
3. Could these functions all solve one equation $y''' + p_2 y'' + p_1 y' + p_0 y = 0$ with continuous coefficients on $\mathbb{R}$? Prove your answer using uniqueness of an initial-value problem.
4. State the extra hypothesis that makes a zero Wronskian imply dependence for solutions of a linear equation. Explain why the example does not contradict that theorem.

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Question 5 – Solution

Strategy. Compare independence of arbitrary smooth functions with independence inside one regular homogeneous solution space.

Step 1: Prove independence by separated values. The supports are $[-5/2, -3/2]$, $[-1/2, 1/2]$ and $[3/2, 5/2]$, respectively. They are disjoint, and each function has value e^{-1} at its center $-2, 0, 2$. If $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ identically, evaluation at those three centers gives $c_1 = c_2 = c_3 = 0$. The functions are therefore linearly independent.

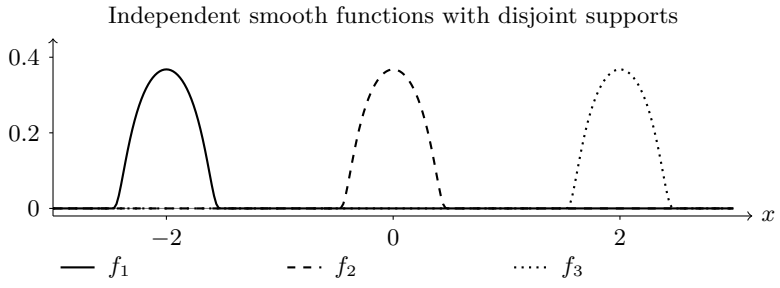
Step 2: Compute the determinant structurally. Outside a function's support, it and its derivatives vanish; at the support boundary, the supplied smoothness facts give the same conclusion. At every x, at most one column of

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}$$

can be nonzero. Thus $\boxed{W(f_1, f_2, f_3)(x) = 0 \text{ for all } x}$.

Step 3: Test the common-equation hypothesis. Suppose all three solved the stated regular homogeneous equation. The function f_1 has $f_1(0) = f_1'(0) = f_1''(0) = 0$. The zero solution has the same data, so uniqueness on $\mathbb{R}$ would force $f_1 \equiv 0$. But $f_1(-2) = e^{-1} > 0$, a contradiction. Such continuous coefficients cannot exist.

Step 4: State the theorem with its hypotheses. For three solutions of the same normalized third-order homogeneous linear equation with continuous coefficients on a connected interval, a zero Wronskian at one point gives a nonzero coefficient combination with zero initial derivative vector. Uniqueness then makes that combination identically zero, proving dependence. Our functions are arbitrary smooth functions, not solutions of one such equation. Their identically zero Wronskian alone therefore does not imply dependence.



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Question 6

For $y^{(4)} = 0$ on $[0, 1]$, prescribe four scalar endpoint conditions:

$$y(0) = 0, \qquad y(1) = 0, \qquad y'(0) + y'(1) = 0, \qquad y''(1) - y''(0) = \eta,$$

where η is a real parameter. Counting four conditions is not the same as verifying that they determine four constants independently.

Tasks

1. Write the general solution and translate all four conditions into a linear system for its coefficients. Determine the rank of the coefficient matrix.
2. Classify all η according to existence and uniqueness, and give every solution whenever solutions exist.
3. Replace the fourth condition by $y'(0) = 1$, keeping the first three. Find the resulting solution and prove it is unique.
4. Explain why none of these conclusions contradicts the standard fourth-order IVP theorem. Describe how the family in the compatible original problem varies while retaining all four endpoint conditions.

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Question 6 – Solution

Strategy. Test the endpoint functionals on the cubic solution space instead of relying on their number.

Step 1: Form the coefficient system. Write $y = a + bx + cx^2 + dx^3$. The conditions become

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}.$$

The fourth row is six times the third minus twelve times the second plus twelve times the first. The first three rows are independent, so the rank is $\boxed{3}$.

Step 2: Classify the parameter completely. The first three equations imply $a = 0$, $b + c + d = 0$ and $2b + 2c + 3d = 0$; subtracting twice the middle equation gives $d = 0$. Thus the last equation can hold only if $\eta = 0$. For that value, $c = -b$ and

$$\boxed{y = Kx(1 - x), \quad K \in \mathbb{R}.}$$

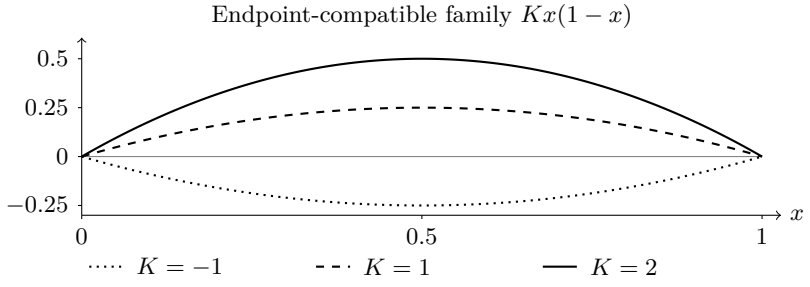
There are infinitely many solutions if $\eta = 0$ and none if $\eta \neq 0$. No value of η gives a unique solution under the original conditions.

Step 3: Replace the redundant information. The condition $y'(0) = 1$ gives $b = 1$. The first three conditions already imply $a = d = 0$ and $c = -b$, so

$$\boxed{y = x(1 - x)}$$

is the unique solution after replacement. The new condition fixes the one remaining free parameter.

Step 4: Distinguish endpoint data from a full initial state. The IVP theorem concerns y, y', y'', y''' specified at one ordinary point. These endpoint conditions mix values at two points and can be dependent or inconsistent on the solution space. In the compatible family, changing K changes height and curvature while preserving zero endpoint values, opposite endpoint slopes, and equal endpoint curvatures. The figure shows three such members; their different initial derivative vectors explain why IVP uniqueness is not violated.



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Question 7

Consider the singular third-order equation

$$xy''' - 2y'' = 0$$

on the whole real line. A classical solution here means a C^3 function satisfying the original equation at every point, including 0.

Tasks

1. Solve on each half-line by first setting $z = y''$. Display all constants before matching across zero.
2. Classify every global C^3 solution and determine the dimension of this solution space.
3. For prescribed $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$, determine exactly when solutions exist and how many free parameters remain.
4. Determine which global solutions are C^4. Does demanding this extra derivative restore uniqueness for compatible initial data at zero? Explain why the regular third-order dimension theorem cannot be used across this singular point.

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Question 7 – Solution

Strategy. Solve separately where the leading coefficient is nonzero, then impose the actual regularity required at the join.

Step 1: Solve on both regular components. For $x \neq 0$, $xz' - 2z = 0$ gives $z = Cx^2$. Two integrations, with the constant absorbed into the quartic coefficient, yield

$$y = \begin{cases} A_+x^4 + B_+x + C_+, & x > 0, \\ A_-x^4 + B_-x + C_-, & x < 0. \end{cases}$$

The constants on the two components are initially independent.

Step 2: Impose a classical join. Continuity of y and y' forces $C_+ = C_- = C$ and $B_+ = B_- = B$. The second and third derivatives are $12A_\pm x^2$ and $24A_\pm x$, both tending to zero, so C^3 regularity imposes no relation between A_+ and A_-. Every global solution is therefore

$$\boxed{y(x) = C + Bx + \begin{cases} A_+x^4, & x \geq 0, \\ A_-x^4, & x < 0, \end{cases}}$$

with four independent constants. At zero the equation reduces to $-2y''(0) = 0$, which these functions satisfy. This is a four-dimensional solution space.

Step 3: Classify the initial data. Every solution has $y(0) = C$, $y'(0) = B$ and $y''(0) = 0$. Thus $\boxed{\gamma = 0}$ is necessary and sufficient. When it holds, $C = \alpha$, $B = \beta$ and both A_+, A_- remain arbitrary. When $\gamma \neq 0$, no classical solution exists.

Step 4: Check the extra derivative. The one-sided fourth derivatives are $24A_+$ and $24A_-$. A C^4 join requires and is guaranteed by $A_+ = A_- = A$, giving the polynomial $C + Bx + Ax^4$. One free parameter A still survives for the compatible initial data, so uniqueness is not restored. The leading coefficient vanishes at 0, and normalization introduces $-2/x$. The hypotheses guaranteeing dimension three and arbitrary unique initial data hold only on the separate half-lines.

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Question 8

For the homogeneous equation $y''' + y' = 0$ on $\mathbb{R}$, consider the three functions $1, \sin x, \cos x$. A student believes that a nonzero solution cannot have both its value and its first derivative zero at the same point.

Tasks

1. Verify the three functions as solutions and compute their Wronskian. Explain why they form a fundamental set on $\mathbb{R}$.
2. Find the solution with $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$, and verify the full initial state.
3. Determine all zeros of this solution and their exact multiplicities. Decide whether finitely or infinitely many such zeros are possible here.
4. Correct the student's claim for a regular homogeneous equation of general order n. State which consecutive derivatives must vanish at a point to force the entire solution to be zero, and explain why the present example does not satisfy that condition.

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Question 8 – Solution

Strategy. Use the full initial derivative vector, rather than only the value and slope, to apply uniqueness.

Step 1: Verify a fundamental set. Each candidate satisfies $y''' + y' = 0$. Their Wronskian is

$$W = \det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} = \boxed{-1}.$$

Thus the three solutions are independent. The regular third-order equation has a three-dimensional homogeneous solution space, so they form a fundamental set.

Step 2: Determine the unique IVP solution. Write $y = A + B\sin x + C\cos x$. The data give $A + C = 0$, $B = 0$ and $-C = 1$. Hence

$$\boxed{y = 1 - \cos x.}$$

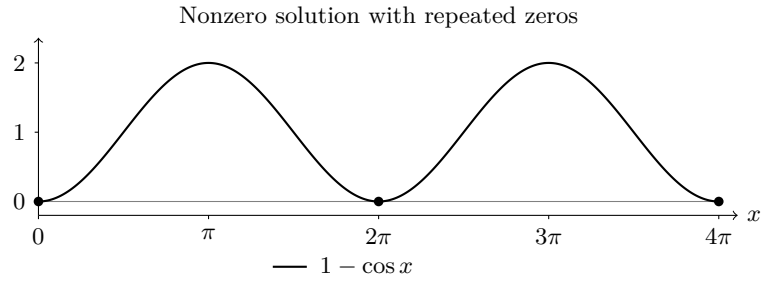
Its first two derivatives are $\sin x$ and $\cos x$, giving initial vector $(0, 0, 1)$; its third derivative $-\sin x$ verifies the equation.

Step 3: Classify all zero contacts. The equation $1 - \cos x = 0$ holds exactly at $x = 2k\pi$, $k \in \mathbb{Z}$. At each such point, $y' = 0$ but $y'' = 1 \neq 0$, so every zero has multiplicity exactly two. There are infinitely many, with no finite accumulation point. Their repeated tangency does not make the function identically zero.

Step 4: State the correct uniqueness consequence. For a normalized homogeneous linear equation of order n with continuous coefficients on a connected interval, if

$$\boxed{y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0,}$$

then uniqueness of the zero-data IVP forces $y \equiv 0$ throughout that interval. Here $n = 3$, and the second derivative at every zero is 1, so the required three-component vector is not zero. Value and slope alone are insufficient; an arbitrary number of separated double zeros is not excluded by the IVP theorem.



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Question 9

Consider the nonlinear third-order IVP

$$y''' = \sqrt{|y|}, \qquad y(0) = y'(0) = y''(0) = 0.$$

For each delay $\tau \geq 0$, investigate a candidate that is zero for $x \leq \tau$ and has the form $C(x-\tau)^p$ for $x > \tau$, with $C > 0$ and $p > 3$.

Tasks

1. Explain why the standard linear IVP theorem does not apply, despite a nonzero leading coefficient and a continuous right-hand side.
2. Find the exponent p and coefficient C that make the positive branch satisfy the equation.
3. Verify the joined function as a classical solution on $\mathbb{R}$ and check all three initial data for every $\tau \geq 0$. Determine its exact smoothness at the joining point.
4. Explain how this family proves nonuniqueness. Contrast it with the linear equation $y''' = y$ under the same initial data, and examine the ratio $\sqrt{h}/h$ as $h \to 0^+$ to identify a regularity issue in the nonlinear dependence on y.

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Question 9 – Solution

Strategy. Construct actual delayed solutions; continuity of a nonlinear right-hand side alone is not a uniqueness proof.

Step 1: Check linearity, not just the leading coefficient. The equation is third order but nonlinear because $\sqrt{|y|}$ is not a linear function of the unknown. A leading coefficient of 1 and continuity do not make the linear existence-and-uniqueness theorem applicable.

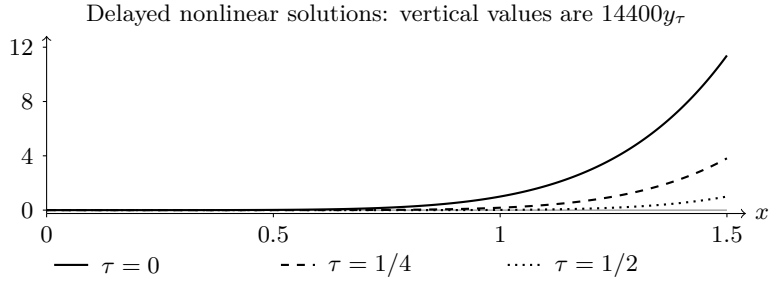
Step 2: Balance exponents and amplitudes. For $s = x - \tau > 0$, the proposed branch has $y''' = Cp(p-1)(p-2)s^{p-3}$ and $\sqrt{|y|} = \sqrt{C}\, s^{p/2}$. Since $p > 3$ and $C > 0$, equality requires $p - 3 = p/2$, giving $p = 6$. Then $120C = \sqrt{C}$, so $\boxed{C = 1/14400}$.

Step 3: Verify the join and the initial state. Define

$$\boxed{y_\tau(x) = \begin{cases} 0, & x \le \tau, \\ (x-\tau)^6/14400, & x > \tau. \end{cases}}$$

Derivatives through order five approach zero from the right and equal zero on the left; difference quotients at the join give the same derivatives. Thus $y_\tau \in C^5$, and the equation holds at the join too, since $y''' = \sqrt{|y|} = 0$ there. The sixth derivative jumps from 0 to $6!/14400 = 1/20$, so it is not C^6 at τ. Because $0 \le \tau$, all three initial data vanish, including when $\tau = 0$.

Step 4: Identify the precise contrast. Distinct delays give distinct solutions with identical initial data; the identically zero function is another solution. This proves nonuniqueness constructively. For the linear equation $y''' - y = 0$, continuous normalized coefficients and the same zero initial vector force the unique zero solution. Here $\sqrt{h}/h = 1/\sqrt{h} \to \infty$, so the nonlinear right-hand side is not locally Lipschitz in y at zero. Nonlinearity alone does not always cause nonuniqueness; the displayed family establishes the failure in this specific case.



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Question 10

Two solutions of $y^{(4)} = 0$ on $\mathbb{R}$ have initial derivative errors

$$\delta_j = \widetilde{y}^{(j)}(0) - y^{(j)}(0), \qquad |\delta_j| \le \varepsilon \quad (j = 0, 1, 2, 3),$$

where $\varepsilon \ge 0$. Let $e = \widetilde{y} - y$. We want a guaranteed error bound over a whole interval, not just at the initial point.

Tasks

1. Find $e(x)$ exactly in terms of the four initial errors.
2. For $T > 0$, prove the sharp uniform bound

   $$\max_{|x| \le T} |e(x)| \le \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6}\right).$$

   Show that the constant cannot be improved when only the stated error information is available.
3. At $T = 2$, find the largest ε guaranteeing solution error at most 10^{-2} for every admissible initial error vector. Evaluate the guarantee for $\varepsilon = 10^{-3}$.
4. If only $\delta_0, \delta_1, \delta_2$ are controlled, can any finite uniform guarantee depending on those three errors alone hold on $[-2, 2]$? Construct a decisive example and interpret what the result says about a complete initial state.

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Question 10 – Solution

Strategy. The linear initial-data map is explicit here, so its worst-case amplification can be determined exactly.

Step 1: Propagate all four initial errors. The difference satisfies $e^{(4)} = 0$ and $e^{(j)}(0) = \delta_j$. Therefore

$$\boxed{e(x) = \delta_0 + \delta_1 x + \frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}.}$$

The factorials are required because the data are derivatives, not monomial coefficients.

Step 2: Prove and attain the uniform bound. For $|x| \le T$, the triangle inequality gives

$$|e(x)| \le \varepsilon \left(1 + |x| + \frac{|x|^2}{2} + \frac{|x|^3}{6}\right) \le \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6}\right).$$

Choose all four $\delta_j = \varepsilon$. At $x = T$, every term has the same nonnegative sign, attaining equality. Thus the stated worst-case constant is exact, including the trivial case $\varepsilon = 0$.

Step 3: Translate the certificate into a tolerance. At $T = 2$, the amplification factor is $1+2+2+8/6 = 19/3$. Consequently the largest admissible tolerance is

$$\boxed{\varepsilon_{\max} = \frac{10^{-2}}{19/3} = \frac{3}{1900}.}$$

Any larger value fails for the equal-sign error vector from Step 2. For $\varepsilon = 10^{-3}$, the exact worst-case bound is $19/3000 \approx 0.0063333 < 0.01$.

Step 4: Show why one uncontrolled component matters. Take $y = 0$ and $\widetilde{y} = Mx^3/6$ for arbitrary real M. Both solve the equation and have the same value, first derivative and second derivative at zero, but their third-derivative difference is M. At $x = 2$, the solution error is $4|M|/3$, which is unbounded as $|M| \to \infty$. Thus controlling only three components cannot give the proposed finite guarantee. For the fourth-order IVP, the complete initial derivative vector is needed for this quantitative stability bound.

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Question 1

Write $D = d/dx$. Consider the constant-coefficient homogeneous equation

$$D^2(D-1)(D+2)\big((D+1)^2+4\big)y = 0$$

on $\mathbb{R}$. Long-term behavior depends on both the locations and the multiplicities of its characteristic roots.

Tasks

1. List every characteristic root with its multiplicity, state the order, and write the complete real general solution.
2. Characterize exactly which solutions are bounded on $[0,\infty)$. Give the dimension of that subspace.
3. Characterize exactly which solutions tend to zero as $x \to +\infty$. Give a bounded nonconstant solution that does not tend to zero.
4. Determine every solution bounded on the whole real line. Justify why terms cannot cancel to create additional bounded solutions.

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Question 1 – Solution

Strategy. Separate exponential growth, polynomial drift, decay and oscillation before imposing boundedness.

Step 1: Read all multiplicities. The roots are 0 (twice), 1, -2, and $-1 \pm 2i$. Their multiplicities total six, so a real fundamental family gives

$$\boxed{y = c_0 + c_1 x + Ae^{x} + Be^{-2x} + e^{-x}(C\cos 2x + S\sin 2x).}$$

The six constants are independent. In particular, the double zero root requires both 1 and x.

Step 2: Remove forward growth. If $A \neq 0$, the e^{x} term dominates as $x \to +\infty$. Once $A = 0$, the remaining exponential terms decay, so boundedness also requires $c_1 = 0$. Conversely those two conditions suffice:

$$\boxed{\text{bounded on } [0,\infty) \iff A = c_1 = 0.}$$

This is a four-dimensional subspace, with free constants c_0, B, C, S.

Step 3: Distinguish a limit from zero. Every forward-bounded solution tends to c_0. Thus the decaying subspace has $A = c_1 = c_0 = 0$ and dimension three. For example, $1 + e^{-2x}$ is bounded and nonconstant on $[0,\infty)$ but tends to 1.

Step 4: Impose the backward restriction. Start with a forward-bounded solution. Multiplication by e^{2x} and passage to $x \to -\infty$ shows $B = 0$, since bounded y would give a zero limit. If $(C, S) \neq (0, 0)$, write $C\cos 2x + S\sin 2x = R\cos(2x - \phi)$ with $R > 0$. Along a sequence tending to $-\infty$ at cosine maxima, Re^{-x} is unbounded. Hence $C = S = 0$ as well, leaving

$$\boxed{y(x) = c_0 \quad \text{as the entire-line bounded family.}}$$

Different growth rates and the explicit oscillatory subsequence rule out hidden cancellation.

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Question 2

Consider the fourth-order initial-value problem

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \qquad y(0) = y'(0) = 0, \quad y''(0) = 2, \quad y'''(0) = -6.$$

All characteristic roots can lie strictly in the left half-plane while a particular response first grows in magnitude.

Tasks

1. Factor the characteristic polynomial and write a complete real general solution.
2. Set $y = e^{-x}v$. Derive the equation and all four initial data for v, then solve the IVP without a four-by-four coefficient elimination.
3. Find every critical point of the IVP solution on $[0, \infty)$ and its exact global maximum there. Determine its limit at infinity.
4. Prove that every solution of the differential equation tends to zero as $x \to +\infty$. Explain why this does not require each solution's magnitude to decrease from its initial value.

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Question 2 – Solution

Strategy. Remove the repeated exponential factor, then analyze the resulting polynomial times exponential.

Step 1: Include the entire repeated-root chain. The polynomial is $(r+1)^4$, so

$$y = e^{-x}(c_0 + c_1x + c_2x^2 + c_3x^3).$$

Four distinct powers are needed for a root of multiplicity four.

Step 2: Transfer the initial derivatives. Since $(D+1)(e^{-x}v) = e^{-x}v'$, four applications give $v^{(4)} = 0$. Also $v = e^x y$, so the product rule gives

$$v(0) = 0, \quad v'(0) = y'(0) + y(0) = 0, \quad v''(0) = y''(0) + 2y'(0) + y(0) = 2,$$

$$v'''(0) = y'''(0) + 3y''(0) + 3y'(0) + y(0) = 0.$$

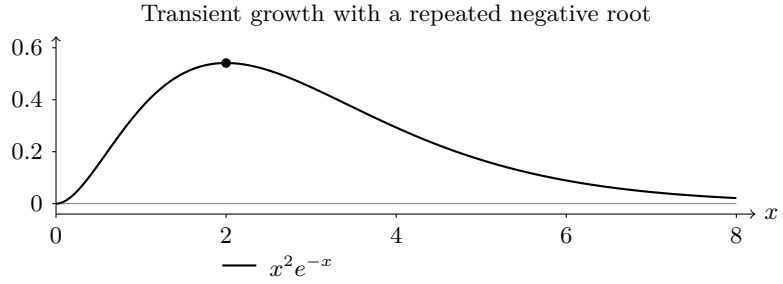
Therefore $v = x^2$ and $\boxed{y = x^2e^{-x}}$. Its initial value and first three derivatives are $0, 0, 2, -6$ as required; the shift identity verifies the equation.

Step 3: Locate the transient maximum. Here $y' = e^{-x}x(2-x)$. It vanishes at 0 and 2, is positive on $(0, 2)$ and negative on $(2, \infty)$. Thus

$$\boxed{\max_{x\geq 0} y(x) = 4e^{-2} \text{ at } x = 2, \qquad \lim_{x\to\infty} y(x) = 0.}$$

The endpoint 0 is a zero of the derivative and a minimum on the half-line.

Step 4: Separate eventual decay from monotonicity. Each x^ke^{-x}, $0 \leq k \leq 3$, tends to zero; hence every linear combination does too. Decay at infinity is an asymptotic statement. This IVP solution starts at zero and increases before decreasing, so neither its value nor its magnitude must decrease throughout the half-line.



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Question 3

Consider

$$y^{(4)} + 8y'' + 16y = 0, \qquad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 8.$$

A repeated imaginary root differs substantially from a simple imaginary root.

Tasks

1. Find the characteristic roots and write the full real general solution, taking account of multiplicities.
2. Determine the unique IVP solution and verify its four initial data.
3. Evaluate the solution and its first derivative at $x_n = n\pi/2$, $n = 1, 2, \ldots$. Use these values to prove unboundedness on $[0, \infty)$.
4. Characterize all solutions of the equation that are bounded on $[0, \infty)$. Prove that no nonzero solution tends to zero at infinity.

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Question 3 – Solution

Strategy. Retain the polynomial factors attached to repeated conjugate roots and test growth at suitable phases.

Step 1: Build the real basis. The polynomial is $(r^2+4)^2$, with roots $\pm 2i$, each twice. Thus

$$y = (A + Bx)\cos 2x + (C + Dx)\sin 2x.$$

Omitting the factors $x\cos 2x$ and $x\sin 2x$ would lose half the solution space.

Step 2: Fit the initial vector. At zero the four derivatives are

$$A, \qquad B + 2C, \qquad -4A + 4D, \qquad -12B - 8C.$$

The data give $A = D = 0$, $B = -1$ and $C = 1/2$, hence

$$\boxed{y = \tfrac{1}{2}\sin 2x - x\cos 2x.}$$

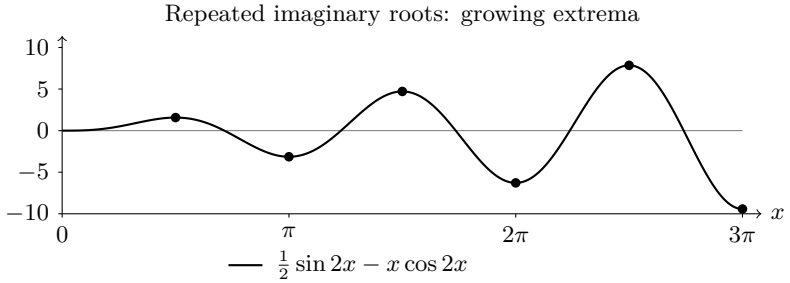
Indeed $y' = 2x\sin 2x$, $y'' = 2\sin 2x + 4x\cos 2x$, and $y''' = 8\cos 2x - 8x\sin 2x$, verifying all four data. Each basis term satisfies the factored equation.

Step 3: Exhibit growing extrema. For $x_n = n\pi/2$, we have

$$y'(x_n) = 0, \qquad y(x_n) = (-1)^{n+1}\frac{n\pi}{2}, \qquad y''(x_n) = 4x_n(-1)^n \neq 0.$$

These are alternating strict extrema with unbounded magnitude. In particular, purely imaginary roots alone do not imply boundedness when they are repeated.

Step 4: Classify the bounded family. If $(B, D) \neq (0, 0)$, choose a sequence $x \to \infty$ on which $B\cos 2x + D\sin 2x = \sqrt{B^2 + D^2} > 0$. The linear-in-$x$ part grows, whereas $A\cos 2x + C\sin 2x$ is bounded. Therefore boundedness is equivalent to $B = D = 0$, a two-dimensional subspace. A nonzero member of that subspace attains its positive amplitude infinitely often, so it cannot tend to zero. An unbounded member cannot tend to zero either.



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Question 4

An observed signal is known exactly on $\mathbb{R}$:

$$f(x) = e^{2x}(1 + x^2) + x \sin 3x.$$

Seek a real, monic, constant-coefficient homogeneous differential equation of the smallest possible order having this signal as a solution. Here monic means that the coefficient of the highest derivative is 1.

Tasks

1. Find a factored characteristic polynomial that annihilates f, carefully determining the multiplicity of every root.
2. Prove that no nonzero constant-coefficient differential operator of smaller order can annihilate f. State which independence fact prevents cancellation between its different exponential modes.
3. Write the complete real general solution of the resulting minimal equation and explain why its monic characteristic polynomial is unique.
4. Must this same equation also admit xe^{2x} and $\cos 3x$ as solutions? Explain why the two displayed summands of f do not mean that a second-order equation suffices. Restrict the conclusion to constant coefficients.

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Question 4 – Solution

Strategy. Read the highest polynomial degree at each distinct exponential frequency, including complex conjugates.

Step 1: Find sufficient root multiplicities. The shift identity $P(D)(e^{\lambda x}q) = e^{\lambda x}P(D+\lambda)q$ shows that a quadratic polynomial multiplying e^{2x} is killed by $(D-2)^3$. Also

$$x\sin 3x = \frac{x}{2i}(e^{3ix} - e^{-3ix}),$$

so each of $\pm 3i$ requires multiplicity two. A sufficient real monic polynomial is

$$\boxed{P(r) = (r-2)^3(r^2+9)^2, \qquad \deg P = 7.}$$

Step 2: Prove minimality, not just sufficiency. Suppose a polynomial Q has a root of multiplicity m at λ; write $Q(r) = (r-\lambda)^m R(r)$ with $R(\lambda) \neq 0$. Applied to $e^{\lambda x}q(x)$, the polynomial $R(D+\lambda)q$ retains the degree of q because its leading coefficient is multiplied by $R(\lambda)$. Thus D^m kills it exactly when $m > \deg q$.

Exponential-polynomial modes with distinct complex exponents are linearly independent on $\mathbb{R}$; this is the generalized characteristic-root independence theorem. Therefore the contributions at $2, 3i, -3i$ cannot cancel each other. Every annihilating Q must have multiplicities at least $3, 2, 2$, so must be divisible by P and have degree at least seven.

Step 3: Recover the entire solution space. The real general solution is

$$\boxed{y = e^{2x}(a_0 + a_1x + a_2x^2) + (b_0 + b_1x)\cos 3x + (c_0 + c_1x)\sin 3x.}$$

There are seven free real constants. A monic degree-seven annihilator divisible by the monic degree-seven P must equal P, proving uniqueness at minimal order.

Step 4: Interpret the forced companion modes. Both xe^{2x} and $\cos 3x$ belong to this solution space. The two written summands package several repeated-root modes: their number is not the differential order. The lower bound concerns nonzero constant-coefficient operators only; it does not assert a minimum order among variable-coefficient equations.

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Question 5

For the fourth-order equation

$$y^{(4)} - 5y'' + 4y = 0,$$

write the initial derivatives as $(y(0), y'(0), y''(0), y'''(0)) = (a, b, c, d)$. For each simple characteristic root λ, define the polynomial

$$\Pi_\lambda(r) = \prod_{\mu \neq \lambda} \frac{r - \mu}{\lambda - \mu},$$

where the product runs over the other characteristic roots.

Tasks

1. Find the roots and prove that $\Pi_\lambda(D)y$ isolates the single mode $A_\lambda e^{\lambda x}$ in the general solution.
2. Compute the coefficients of e^x and e^{2x} explicitly from a, b, c, d using these polynomials.
3. Give necessary and sufficient conditions on the initial derivatives for boundedness on $[0, \infty)$. Express them also as two relations obtained from $(D + 1)(D + 2)y = 0$.
4. Start from $(a, b, c, d) = (1, -1, 1, -1)$ and perturb only d by a nonzero amount ε. Find the limit

   $$\lim_{x \to \infty} e^{-2x} y_\varepsilon(x).$$

   Explain the long-term consequence of arbitrarily small such errors.

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Question 5 – Solution

Strategy. Use polynomial interpolation at the roots to extract growing modes directly from the initial data.

Step 1: Construct mode selectors. The polynomial factors as $(r-1)(r+1)(r-2)(r+2)$. Thus y is a linear combination of $e^{\lambda x}$ for $\lambda = -2, -1, 1, 2$. Since $Q(D)e^{\mu x} = Q(\mu)e^{\mu x}$ and $\Pi_\lambda(\mu)$ is 1 if $\mu = \lambda$ and 0 otherwise,

$$\Pi_\lambda(D)y = A_\lambda e^{\lambda x}.$$

Evaluating at zero reads off A_λ without solving all four coefficients simultaneously.

Step 2: Extract the unstable coefficients. The relevant polynomials are

$$\Pi_1(r) = -\frac{(r+2)(r+1)(r-2)}{6} = \frac{-r^3 - r^2 + 4r + 4}{6},$$

$$\Pi_2(r) = \frac{(r+2)(r+1)(r-1)}{12} = \frac{r^3 + 2r^2 - r - 2}{12}.$$

Hence

$$\boxed{A_1 = \frac{4a + 4b - c - d}{6}, \qquad A_2 = \frac{d + 2c - b - 2a}{12}.}$$

Step 3: Describe exactly the decaying initial states. Forward boundedness requires $A_2 = A_1 = 0$; distinct positive exponential rates cannot cancel. The remaining modes decay. Equivalently,

$$\boxed{c + 3b + 2a = 0, \qquad d + 3c + 2b = 0.}$$

These follow from $y'' + 3y' + 2y = 0$ and its derivative for the two negative-root modes. Conversely, substituting these relations into A_1, A_2 makes both zero, so the relations are sufficient as well.

Step 4: Quantify the loss of cancellation. The unperturbed solution is e^{-x}. Replacing $d = -1$ by $-1 + \varepsilon$ produces $A_1 = -\varepsilon/6$ and $A_2 = \varepsilon/12$. All modes other than e^{2x} disappear after multiplication by e^{-2x} in the limit, giving

$$\boxed{\lim_{x \to \infty} e^{-2x} y_\varepsilon(x) = \varepsilon/12.}$$

Every nonzero error of this form eventually yields exponential growth in magnitude. Exact decay requires precise cancellation of both positive-root modes.

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Question 6

Let $\alpha \in \mathbb{R}$ and consider the third-order homogeneous equation

$$(D+1)(D^2+2\alpha D+1)y=0$$

on $\mathbb{R}$. Call the equation *forward bounded* if every solution is bounded on $[0,\infty)$, and *forward decaying* if every solution tends to zero as $x \to +\infty$.

Tasks

1. Find the characteristic roots and identify all values of α where roots coincide, including coincidence with the fixed root -1.
2. Write a complete real general solution in every parameter regime. Treat $\alpha = 1$ and $\alpha = -1$ explicitly.
3. Classify exactly the parameters for forward boundedness and forward decay. Prove the boundary cases rather than deciding from root real parts alone.
4. At $\alpha = 0, 1, -1$, respectively, give a nondecaying bounded solution, a decaying solution with a nonconstant polynomial factor, and an unbounded solution. Explain the different effects of repeated negative and repeated positive roots.

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Question 6 – Solution

Strategy. Track the quadratic roots and separately check when either merges with the fixed root.

Step 1: Locate the collisions. Besides $r = -1$, the roots are $r_\pm = -\alpha \pm \sqrt{\alpha^2 - 1}$. They coincide when $\alpha = \pm 1$. Substituting $r = -1$ in the quadratic gives $2 - 2\alpha$, so coincidence with the fixed root occurs only at $\alpha = 1$, producing a triple root -1.

Step 2: State a complete basis in each case. For $|\alpha| > 1$, all three roots are real and distinct, and

$$y = Ce^{-x} + Ae^{r_+ x} + Be^{r_- x}.$$

For $|\alpha| < 1$, let $\beta = \sqrt{1 - \alpha^2} > 0$. Then

$$y = Ce^{-x} + e^{-\alpha x}(A\cos\beta x + B\sin\beta x).$$

The exceptional cases are

$$\begin{aligned} \alpha = 1: &\quad y = e^{-x}(C + Ax + Bx^2), \\ \alpha = -1: &\quad y = Ce^{-x} + (A + Bx)e^{x}. \end{aligned}$$

Each family has three free constants.

Step 3: Classify all forward behavior. If $\alpha > 1$, both quadratic roots are negative since $\sqrt{\alpha^2 - 1} < \alpha$. If $0 < \alpha < 1$, their real part is $-\alpha < 0$. At $\alpha = 1$, even the quadratic polynomial factor is dominated by e^{-x}. Thus every solution decays for every $\alpha > 0$.

At $\alpha = 0$, the modes are $e^{-x}, \cos x, \sin x$: all are bounded, but not all decay. If $-1 < \alpha < 0$, the oscillatory amplitude grows exponentially; choosing a cosine and its phase maxima proves unboundedness. At $\alpha = -1$ there are positive-root modes, and for $\alpha < -1$ both quadratic roots are positive. Consequently,

$$\boxed{\text{forward bounded} \iff \alpha \geq 0, \qquad \text{forward decaying} \iff \alpha > 0.}$$

Step 4: Test the special parameters concretely. The requested examples are $\cos x$ at $\alpha = 0$, $x^2 e^{-x}$ at $\alpha = 1$, and xe^{x} at $\alpha = -1$. A finite polynomial factor does not defeat exponential decay at a negative root. A positive root already permits growth, and repetition adds further polynomial amplification.

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Question 7

A real solution of

$$y^{(4)} + 5y'' + 4y = 0$$

is sampled at all times $x = n\pi$, $n = 0, 1, 2, \ldots$. The measured values are 3 for even n and 1 for odd n.

Tasks

1. Find the complete real general solution and determine exactly what the value samples reveal about its four coefficients.
2. Describe the full subspace of solutions invisible to these samples, meaning $y(n\pi) = 0$ for every non-negative integer n. Prove that infinitely many value samples still leave nonuniqueness.
3. Two additional measurements give $y'(0) = 1$ and $y'(\pi) = 3$. Recover the unique solution and verify every type of measurement.
4. Prove that, for any solution of this equation, the four measurements $y(0), y(\pi), y'(0), y'(\pi)$ uniquely determine it. Explain why the equally spaced value samples alone miss actual oscillations between sampling times.

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Question 7 – Solution

Strategy. Evaluate the real modes at the sampling grid before interpreting the amount of information in the data.

Step 1: Read the visible coefficients. The polynomial is $(r^2+1)(r^2+4)$, so

$$y = A\cos x + B\sin x + C\cos 2x + D\sin 2x.$$

At the sample times, $y(n\pi) = A(-1)^n + C$. Thus $A + C = 3$ and $-A + C = 1$, giving $A = 1$ and $C = 2$. These samples impose no restriction on B, D.

Step 2: Find the invisible subspace. Zero values at both even and odd sample indices force $A = C = 0$, while the sine modes vanish at every $n\pi$. Therefore

$$\boxed{\{y : y(n\pi) = 0 \ \forall n \geq 0\} = \operatorname{span}\{\sin x, \sin 2x\}.}$$

It has dimension two. Adding any member to one fitting signal leaves every recorded value unchanged, even though the functions differ between samples.

Step 3: Recover the missing oscillations. The derivative samples give $B + 2D = 1$ and $-B + 2D = 3$, hence $B = -1, D = 1$. The unique signal is

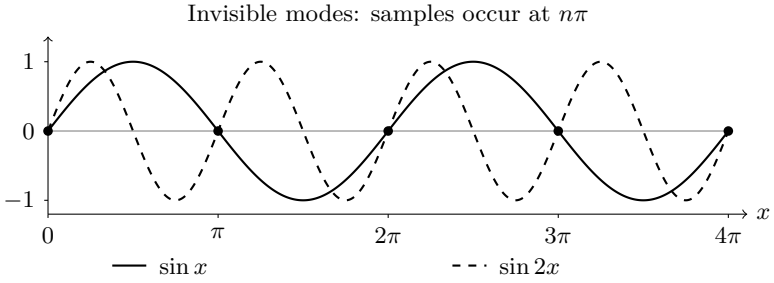
$$\boxed{y = \cos x - \sin x + 2\cos 2x + \sin 2x.}$$

Its value samples are $(-1)^n + 2$ and its derivative samples at $0, \pi$ are $1, 3$, respectively. Every term solves the equation.

Step 4: Prove general identifiability. For arbitrary measurements $u = y(0)$, $v = y(\pi)$, $p = y'(0)$ and $q = y'(\pi)$,

$$A = \frac{u - v}{2}, \quad C = \frac{u + v}{2}, \quad B = \frac{p - q}{2}, \quad D = \frac{p + q}{4}.$$

These formulas exist and are unique for every real measurement vector. Value samples alone repeat only two linear constraints; their number does not restore the sine information. The figure displays the two independent invisible modes and their common sample zeros.



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Question 8

Consider two homogeneous constant-coefficient operators

$$P(D) = (D-1)^2(D+2)(D^2+1), \qquad Q(D) = (D-1)(D+2)^2(D^2+4).$$

A *common solution* must satisfy both $P(D)y = 0$ and $Q(D)y = 0$ on $\mathbb{R}$.

Tasks

1. Write the complete real general solution of each equation separately.
2. Determine the full common solution space and its dimension. Justify why the smaller multiplicity at each shared root is the relevant one.
3. Find the common solution with $y(0) = 3$ and $y'(0) = 0$. Compute the higher initial derivatives $y''(0)$ and $y'''(0)$ that are then forced.
4. Find the monic real constant-coefficient operator of smallest order that annihilates every solution of either equation. Explain why this last requirement uses larger multiplicities and differs from finding common solutions.

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Question 8 – Solution

Strategy. Compare each exponent and its polynomial degree, distinguishing an intersection of solution spaces from a space containing both.

Step 1: List the two five-dimensional families. For P, the roots are 1 twice, -2, and $\pm i$, giving

$$y_P = (a + bx)e^x + ce^{-2x} + d\cos x + e\sin x.$$

For Q, the roots are 1, -2 twice, and $\pm 2i$, giving

$$y_Q = Ae^x + (B + Cx)e^{-2x} + E\cos 2x + F\sin 2x.$$

All constants in each display are independent.

Step 2: Retain only common modes. Exponential-polynomial modes at different exponents are independent. Equality of the two displays therefore removes both oscillatory pairs, xe^x, and xe^{-2x}. Precisely the simple shared modes survive:

$$\boxed{y = Ae^x + Be^{-2x}.}$$

They are independent, so the common space has dimension two. In polynomial language, the common characteristic factor is the greatest common divisor $(r-1)(r+2)$: a mode must satisfy both multiplicity restrictions and hence the smaller one.

Step 3: Fit the common initial data. We need $A + B = 3$ and $A - 2B = 0$, so $A = 2, B = 1$. Thus

$$\boxed{y = 2e^x + e^{-2x}, \qquad y''(0) = 6, \qquad y'''(0) = -6.}$$

Each exponential is a shared mode, proving both equations directly. The first two data fix the solution within the common space; higher data cannot be chosen independently there.

Step 4: Build the smallest operator containing both kernels. An operator annihilating every solution of either equation must include both repeated real-root chains and both oscillatory pairs. Its characteristic polynomial must be divisible by both P and Q. Their monic least common multiple is

$$\boxed{R(r) = (r-1)^2(r+2)^2(r^2+1)(r^2+4), \qquad \deg R = 8.}$$

The multiplicity of each root is now the larger of its two multiplicities. This polynomial is sufficient by the characteristic-root rule and necessary because each listed mode must be killed. Thus its order is minimal and its monic choice unique. Its kernel contains both five-dimensional spaces, whereas the common space is their two-dimensional intersection.

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Question 9

Multiplication by x is an operator: $(D - x)u = u' - xu$. On $\mathbb{R}$, consider

$$(D - x)D^2y = 0.$$

Unlike constant-coefficient factors, these factors need not commute.

Tasks

1. Expand the equation. Set $z = y''$ and find its complete real solution space, allowing definite integrals rather than elementary antiderivatives.
2. Define $F(x) = \int_0^x (x - t)e^{t^2/2}\,dt$. Prove that $1, x, F$ form a fundamental set by computing their Wronskian.
3. Solve the IVP $y(0) = y'(0) = 0$, $y''(0) = 1$. Prove that its solution is even and satisfies $y(x) \geq x^2/2$ for every real x, with strict inequality when $x \neq 0$.
4. Expand the reversed product $D^2(D - x)y$. Determine all functions that solve both the original and reversed equations, and explain why a characteristic polynomial cannot be used here as if x were constant.

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Question 9 – Solution

Strategy. Solve the factors in their given order and use the product rule when reversing them.

Step 1: Reduce the original equation. Expansion gives $y''' - xy'' = 0$. Thus $z' - xz = 0$ and $z = Ce^{x^2/2}$. Integrating twice from zero yields

$$\boxed{y = A + Bx + CF(x), \qquad F(x) = \int_0^x (x-t)e^{t^2/2}\,dt.}$$

This formula holds for negative x as well, with the usual oriented integral.

Step 2: Verify the fundamental set. Differentiating the integral gives $F' = \int_0^x e^{t^2/2}\,dt$ and $F'' = e^{x^2/2}$, so $F''' = xF''$. The derivative matrix is triangular:

$$W(1, x, F) = \det \begin{pmatrix} 1 & x & F \\ 0 & 1 & F' \\ 0 & 0 & e^{x^2/2} \end{pmatrix} = \boxed{e^{x^2/2} > 0}.$$

These are three independent solutions of a regular third-order linear equation, hence a fundamental set on $\mathbb{R}$.

Step 3: Fit and analyze the initial data. Since $F(0) = F'(0) = 0$ and $F''(0) = 1$, the IVP gives $\boxed{y = F}$. Its second derivative is even, and its value and slope at zero vanish; integrating shows that F is even. For $x > 0$,

$$F(x) - \frac{x^2}{2} = \int_0^x (x-t)(e^{t^2/2} - 1)\,dt > 0.$$

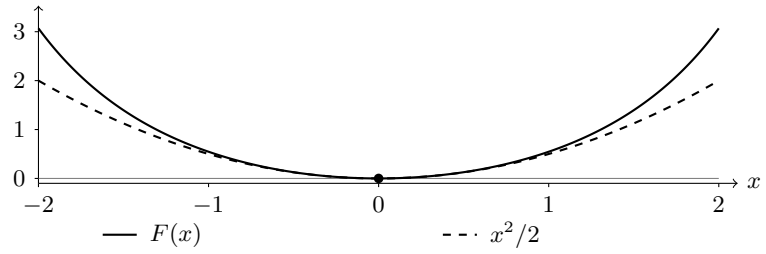
The inequality follows because the integrand is positive for $0 < t < x$. Evenness proves the same strict inequality for $x < 0$; equality holds at zero.

Step 4: Respect the order of the factors. The product rule gives

$$D^2(y' - xy) = y''' - xy'' - 2y'.$$

Subtracting the two equations forces $y' = 0$ for any common solution. Conversely, every constant solves both, so their common space consists exactly of constants. The variable coefficient creates the extra $-2y'$ term; treating x as a fixed characteristic root would discard it and give an invalid method.

Integral solution and its quadratic lower bound



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Question 10

For the fourth-order equation

$$y^{(4)} + 4y = 0,$$

define along a real solution

$$E(x) = y'''(x)y'(x) - \frac{1}{2}\big(y''(x)\big)^2 + 2y(x)^2.$$

A quantity that is constant along solutions need not control their size.

Tasks

1. Find the characteristic roots and complete real general solution.
2. Differentiate E and prove that it is constant along every solution.
3. Verify that $y = e^x \cos x$ is an unbounded solution with $E = 0$. Give a specific sequence of times proving its unboundedness.
4. Characterize all solutions bounded on $[0, \infty)$ and all solutions bounded on $\mathbb{R}$. Explain why conservation of E does not contradict these classifications and cannot yield a bound of the form $|y(x)|^2 \le CE(0)$ with a fixed positive C for all solutions.

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Question 10 – Solution

Strategy. Compare the root-based behavior with what the conserved expression actually measures.

Step 1: Find growing and decaying oscillatory modes. Since $r^4 + 4 = (r^2 - 2r + 2)(r^2 + 2r + 2)$, the roots are $1 \pm i$ and $-1 \pm i$. Thus

$$\boxed{y = e^x(A\cos x + B\sin x) + e^{-x}(C\cos x + D\sin x).}$$

All roots are simple and the four real modes are independent.

Step 2: Verify the conservation law. The product rule gives exact cancellation:

$$E' = y^{(4)}y' + y'''y'' - y''y''' + 4yy' = y'(y^{(4)} + 4y) = 0.$$

Hence $E(x) = E(0)$ for every x.

Step 3: Test a zero-level growing solution. For $y = e^x \cos x$, the successive derivatives are

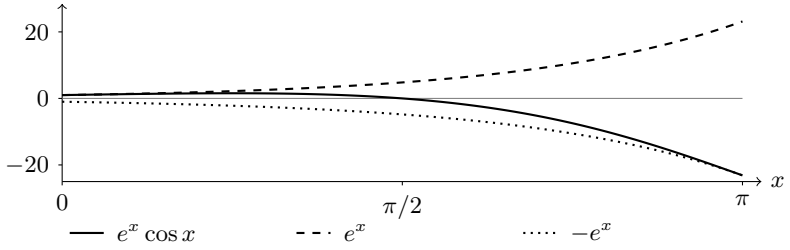
$$y' = e^x(\cos x - \sin x), \quad y'' = -2e^x \sin x, \quad y''' = -2e^x(\sin x + \cos x), \quad y^{(4)} = -4y.$$

At zero its initial vector is $(1, 1, 0, -2)$, so $E(0) = -2 + 2 = 0$. Conservation yields $E \equiv 0$. Yet $y(2\pi n) = e^{2\pi n} \to \infty$, proving unboundedness.

Step 4: Identify the limits of the invariant. On $[0, \infty)$ boundedness requires $A = B = 0$: any nonzero growing sinusoid attains its amplitude along an unbounded sequence, while the other term decays. These conditions also suffice, giving a two-dimensional decaying family. Requiring boundedness backward as well forces $C = D = 0$ by the same phase argument, so the only entire-line bounded solution is zero.

The expression E is not positive definite in the four initial derivatives: it has a mixed product and a negative square. In particular a nonzero state can have $E = 0$. The displayed solution has $|y(0)|^2 = 1$ and $E(0) = 0$, immediately refuting the proposed bound for every positive C. Constancy of an indefinite expression does not guarantee bounded motion.

A growing solution on the conserved level $E = 0$



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Question 1

Write $D = d/dx$ and let

$$L = D^2(D-1)^3(D^2+4)^2.$$

Consider on $\mathbb{R}$ the equation

$$Ly = 3x^2 + (x+1)e^x + x\cos 2x + e^{-x}\sin x.$$

This question concerns designing and justifying the coefficient trial, not carrying out a large elimination.

Tasks

1. State the order and complete real homogeneous solution, with all root multiplicities.
2. Write the standard undetermined-coefficient trial for a particular solution, including all polynomial degrees, both trigonometric partners, and every resonance factor. Count its unknown coefficients.
3. Explain why a trial containing only cosine terms for $x\cos 2x$, or only sine terms for $e^{-x}\sin x$, is not the appropriate full trial. Explain why one common resonance factor for all four forcing terms is inappropriate.
4. Prove that substitution determines exactly one member of your trial space. Use the effect of L on each polynomial-exponential family and its intersection with the homogeneous solution space. You need not compute the eleven coefficients.

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Question 1 – Solution

Strategy. Treat each distinct exponential frequency separately and remove only its own homogeneous overlap.

Step 1: List the nine homogeneous modes. The roots are 0 twice, 1 three times, and $\pm 2i$ twice each. Thus L has order nine and

$$y_h = c_0 + c_1 x + e^x(c_2 + c_3 x + c_4 x^2) + (c_5 + c_6 x)\cos 2x + (c_7 + c_8 x)\sin 2x.$$

Step 2: Build the trial block by block. A standard trial is

$$\begin{aligned} y_p = {} & x^2(a_2 x^2 + a_1 x + a_0) + x^3 e^x(b_1 x + b_0) \\ & + x^2\big((d_1 x + d_0)\cos 2x + (e_1 x + e_0)\sin 2x\big) \\ & + e^{-x}(f\cos x + g\sin x). \end{aligned}$$

There are $3 + 2 + 4 + 2 = \boxed{11}$ unknowns. The resonance powers are respectively $2, 3, 2, 0$, because $-1 \pm i$ are not roots of L.

Step 3: Keep the derivative families complete. Differentiation mixes sine and cosine, and differentiating polynomial factors also generates lower powers of x. Therefore the full paired families must be allowed; coefficients that ultimately vanish should emerge from the equations. Resonance is tied to the particular exponent, not to the equation's total order. A common factor would either retain homogeneous overlap in some blocks or unnecessarily alter others.

Step 4: Establish solvability and uniqueness in the trial. For a root λ of multiplicity m, write the characteristic polynomial as $(r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. The shift identity gives

$$L(e^{\lambda x} x^m q) = e^{\lambda x} D^m R(D + \lambda)(x^m q).$$

For $\deg q \leq d$, the resulting polynomial has degree at most d; its top coefficient is the top coefficient of q times $R(\lambda)(m + d)!/d!$, which is nonzero when that coefficient is nonzero. Successive coefficient matching is therefore invertible. Real and imaginary parts give the same conclusion for each paired block.

Equivalently, the eleven-dimensional trial maps into the eleven-dimensional forcing family and contains no nonzero homogeneous member: each block starts beyond its root's allowed homogeneous degree. Its kernel is zero, so this map is bijective. Hence the stated forcing determines exactly one trial member, although other particular solutions can differ by y_h.

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Question 2

Consider the resonant third-order equation

$$(D-1)^3 y = e^x(x^2+1)$$

on $\mathbb{R}$.

Tasks

1. Explain why $e^x(Ax^2+Bx+C)$ cannot be a particular-solution trial. Give a complete resonance-corrected trial.
2. Use $y = e^x v$ to determine a particular solution by coefficient matching and write the full real general solution.
3. Solve the IVP $y(0) = y'(0) = y''(0) = 0$. Verify all three initial data and the original forcing.
4. For an arbitrary solution, calculate $\lim_{x\to\infty} e^{-x}y(x)/x^5$. Prove that no choice of homogeneous constants makes a solution bounded on $[0,\infty)$.

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Question 2 – Solution

Strategy. Shift the triple root to zero, where the coefficient calculation becomes polynomial differentiation.

Step 1: Remove the homogeneous overlap. The characteristic root 1 has multiplicity three. Every proposed term $e^x(Ax^2+Bx+C)$ is already homogeneous and gives zero under $(D-1)^3$. The corrected trial is

$$y_p = x^3e^x(Ax^2+Bx+C).$$

Step 2: Solve the shifted coefficient equations. Since $(D-1)(e^xv) = e^xv'$, the equation becomes $v''' = x^2+1$. For $v_p = Ax^5+Bx^4+Cx^3$, matching gives

$$60Ax^2+24Bx+6C = x^2+1, \qquad A=\frac{1}{60}, \quad B=0, \quad C=\frac{1}{6}.$$

The complete solution is consequently

$$\boxed{y = e^x\left(c_0+c_1x+c_2x^2+\frac{x^3}{6}+\frac{x^5}{60}\right).}$$

Step 3: Impose and verify the initial data. With $v = e^{-x}y$, the initial values are

$$v(0)=y(0)=0, \quad v'(0)=y'(0)-y(0)=0, \quad v''(0)=y''(0)-2y'(0)+y(0)=0.$$

Thus $c_0=c_1=c_2=0$. The polynomial $v = x^3/6+x^5/60$ and its first two derivatives vanish at zero, so the corresponding initial data for $y=e^xv$ vanish as well. Its third derivative is $v''' = 1+x^2$, and the shift identity verifies the forcing exactly.

Step 4: Identify unavoidable resonant growth. For every choice of c_0, c_1, c_2,

$$\boxed{\lim_{x\to\infty}\frac{e^{-x}y(x)}{x^5} = \frac{1}{60}.}$$

The lower-degree terms disappear after division by x^5. Hence $y(x)$ is eventually positive and asymptotic to $e^xx^5/60$, so it is unbounded. Homogeneous terms have polynomial degree at most two after removing e^x and cannot cancel the forced degree-five term.

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Question 3

Consider the initially resting response of a fourth-order equation:

$$(D^2+1)^2 y = \cos x, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Tasks

1. Identify the resonance multiplicity and choose a full real particular-solution trial. Explain why multiplying the ordinary sine-cosine trial by only x still fails.
2. Determine a particular solution by direct substitution, then add the complete homogeneous family.
3. Impose the four initial data and verify the resulting solution explicitly.
4. Evaluate the IVP solution at $x = 2\pi n$ and prove quadratic growth along this sequence. Explain why no homogeneous correction can make any solution bounded on $[0, \infty)$.

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Question 3 – Solution

Strategy. Account for the double conjugate roots before imposing the initial state.

Step 1: Use the full resonance multiplicity. The roots $\pm i$ both have multiplicity two. The functions $\cos x, \sin x, x\cos x, x\sin x$ are all homogeneous, so an x-weighted trial still gives zero. Use $y_p = x^2(A\cos x + B\sin x)$.

Step 2: Match the forcing directly. A first application gives $(D^2+1)(x^2\cos x) = 2\cos x - 4x\sin x$, and a second gives $-8\cos x$. Likewise, $(D^2+1)^2(x^2\sin x) = -8\sin x$. Thus $A = -1/8$, $B = 0$, and

$$y = -\frac{x^2\cos x}{8} + (a+bx)\cos x + (c+dx)\sin x.$$

Step 3: Recover the initially resting solution. The particular term has initial vector $(0, 0, -1/4, 0)$. The homogeneous term has vector $(a, b+c, -a+2d, -3b-c)$. Setting their sum to zero gives $a = b = c = 0, d = 1/8$, hence

$$\boxed{y = \frac{x\sin x - x^2\cos x}{8}.}$$

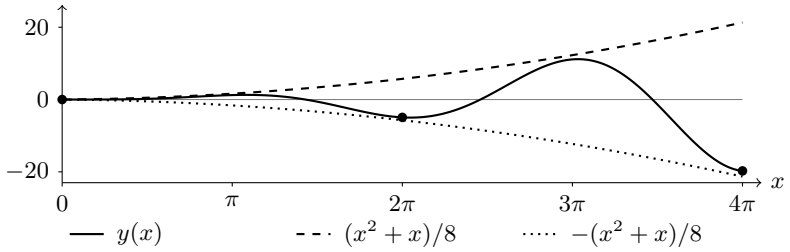
The two numerator terms cancel to order two: $x\sin x - x^2\cos x = x^4/3 + O(x^6)$ near zero. Thus y and its first three derivatives vanish there. The term $x\sin x/8$ is homogeneous, while the already checked particular term produces $\cos x$.

Step 4: Separate quadratic forcing from linear freedom. At $x = 2\pi n$,

$$\boxed{y(2\pi n) = -\frac{(2\pi n)^2}{8}.}$$

This proves unbounded quadratic growth in magnitude along that sequence. Every homogeneous term is $O(x)$ on the positive half-line, so no correction can cancel these quadratic values. The figure shows the response and the valid envelopes $\pm(x^2+x)/8$.

Double resonance: response and magnitude bounds



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Question 4

On $\mathbb{R}$, consider

$$y^{(4)} - y = 6x^2 + 8e^{-x} + 10\cos x.$$

Its three forcing terms require different coefficient calculations.

Tasks

1. Factor the characteristic polynomial, write the full real homogeneous solution, and select a separate particular-solution trial for each forcing term.
2. Determine all coefficients and assemble one particular solution.
3. Verify each forcing contribution directly using $L = D^4 - 1$, rather than relying only on the trial-selection rule. Write the complete general solution.
4. Decide whether any solution is bounded on $[0, \infty)$. Prove your answer even when homogeneous constants are chosen to cancel all removable growing terms.

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Question 4 – Solution

Strategy. Use linearity to separate polynomial forcing from two distinct resonant frequencies.

Step 1: Identify roots and trials. The characteristic roots are $1, -1, i, -i$, all simple, so

$$y_h = Ae^x + Be^{-x} + C\cos x + D\sin x.$$

Use $ax^2 + bx + c$ for $6x^2$, kxe^{-x} for $8e^{-x}$, and $x(m\cos x + n\sin x)$ for $10\cos x$. Only the last two blocks are resonant.

Step 2: Compute the coefficients. The fourth derivative of a quadratic is zero, giving $a = -6, b = c = 0$. For a simple root λ of $P(r) = r^4 - 1$, the shift identity gives $P(D)(xe^{\lambda x}) = P'(\lambda)e^{\lambda x}$. Since $P'(-1) = -4$, we obtain $k = -2$.

For the trigonometric terms, $L(x\cos x) = 4\sin x$ and $L(x\sin x) = -4\cos x$. Therefore $m = 0, n = -5/2$, and

$$\boxed{y_p = -6x^2 - 2xe^{-x} - \frac{5}{2}x\sin x.}$$

Step 3: Verify all three residuals. Direct differentiation gives

$$L(-6x^2) = 6x^2, \qquad D^4(xe^{-x}) = e^{-x}(x-4),$$

$$D^4(x\sin x) = x\sin x - 4\cos x.$$

Thus the second and third terms produce $8e^{-x}$ and $10\cos x$ exactly. The full solution is $\boxed{y = y_h + y_p}$; no homogeneous constants are determined without additional data.

Step 4: Rule out bounded solutions. If $A \neq 0$, the Ae^x term dominates every other term, so boundedness is impossible. If $A = 0$, division by x^2 gives

$$\lim_{x\to\infty} \frac{y(x)}{x^2} = -6,$$

because the remaining oscillatory growth is only $O(x)$ and the other exponential terms decay. Hence even after removing the positive exponential, quadratic forcing remains. No solution is bounded on the positive half-line.

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Question 5

Consider

$$(D+1)^2(D-2)y = e^{-x}.$$

An annihilator can simplify the forcing while introducing extra solutions that must later be rejected.

Tasks

1. Apply a minimal constant-coefficient annihilator of the forcing and write the full solution of the resulting fourth-order homogeneous equation.
2. Substitute that enlarged family into the original third-order equation. Find the constraint that removes the extra freedom and write the actual general solution.
3. Find the solution with $y(0) = y'(0) = y''(0) = 0$. Determine the coefficient of its growing exponential and its limit after multiplication by e^{-2x}.
4. Instead require $y(0) = y'(0) = 0$ and decay as $x \to +\infty$. Find this solution and the initial value $y''(0)$ it requires. Explain why decaying forcing does not guarantee that the zero-data response decays.

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Question 5 – Solution

Strategy. Retain the original equation as a constraint after annihilating its right-hand side.

Step 1: Solve the enlarged equation. The forcing is annihilated by $D+1$. Applying it gives $(D+1)^3(D-2)y=0$, whose general solution is

$$y = e^{-x}(A + Bx + Cx^2) + Ee^{2x}.$$

This family has four constants, although the original equation has order three.

Step 2: Restore the original forcing. Under $y = e^{-x}v$, the original operator becomes $e^{-x}D^2(D-3)v$. For $v = A + Bx + Cx^2$ its value is $-6Ce^{-x}$; the Ee^{2x} term is homogeneous. Thus $C = -1/6$, leaving

$$\boxed{y = e^{-x}(A + Bx - x^2/6) + Ee^{2x}.}$$

Annihilation alone only ensures that the original residual is a multiple of e^{-x}, not that its coefficient equals one.

Step 3: Fit the zero initial vector. The three equations are $A + E = 0$, $B - A + 2E = 0$, and $A - 2B - 1/3 + 4E = 0$. They yield $E = 1/27$, $A = -1/27$, $B = -1/9$, so

$$\boxed{y_0 = \frac{e^{2x}}{27} - e^{-x}\left(\frac{1}{27} + \frac{x}{9} + \frac{x^2}{6}\right), \qquad \lim_{x\to\infty} e^{-2x}y_0 = \frac{1}{27}.}$$

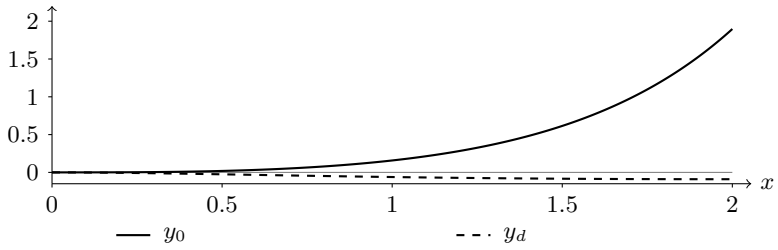
The displayed coefficient equations verify the initial data, and Step 2 verifies the forcing.

Step 4: Replace curvature data by a future condition. Decay requires $E = 0$; with $y(0) = y'(0) = 0$, this gives $A = B = 0$. Hence

$$\boxed{y_d = -x^2e^{-x}/6, \qquad y_d''(0) = -1/3.}$$

This is the unique decaying response with those first two data. The zero-data response must add a growing homogeneous mode to cancel the particular solution's initial curvature. Decay of the input alone does not suppress that mode.

Decaying input: two choices of initial curvature



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Question 6

Let $\omega \geq 0$ and consider the zero-data response

$$(D^2+1)(D^2+4)y_\omega = \cos(\omega x), \qquad y_\omega(0) = y'_\omega(0) = y''_\omega(0) = y'''_\omega(0) = 0.$$

Tasks

1. For $\omega \neq 1, 2$, find a particular solution and then the full zero-data response by adding homogeneous modes.
2. Solve the IVP separately at each resonant frequency $\omega = 1$ and $\omega = 2$, using resonance-corrected trials.
3. Rewrite the nonresonant answer using differences of cosines and prove that it tends to the resonant answer as $\omega \to 1$ for every fixed x.
4. Classify which zero-data responses are bounded on $[0, \infty)$. Explain why the fixed-x limit in the previous task does not provide a bound uniform over all $x \geq 0$ as $\omega \to 1$.

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Question 6 – Solution

Strategy. Enforce the initial data before taking a resonant limit; separate divergent terms can cancel.

Step 1: Solve away from resonance. Set $K = [(1-\omega^2)(4-\omega^2)]^{-1}$. The trial $K\cos\omega x$ has the required residual. Odd initial derivatives vanish; adding $A\cos x + C\cos 2x$ and matching the value and second derivative gives

$$\boxed{y_\omega = K\cos\omega x - \frac{\cos x}{3(1-\omega^2)} + \frac{\cos 2x}{3(4-\omega^2)}, \quad \omega \neq 1, 2.}$$

Indeed $A + C = -K$ and $A + 4C = -K\omega^2$, which verify the remaining two data.

Step 2: Solve at the two simple resonances. For $L = (D^2+1)(D^2+4)$, direct differentiation gives $L(x\sin x) = 6\cos x$ and $L(x\sin 2x) = -12\cos 2x$. Adding homogeneous cosine terms to enforce zero value and curvature yields

$$\boxed{y_1 = \frac{x\sin x}{6} + \frac{\cos 2x - \cos x}{9}, \qquad y_2 = -\frac{x\sin 2x}{12} + \frac{\cos x - \cos 2x}{9}.}$$

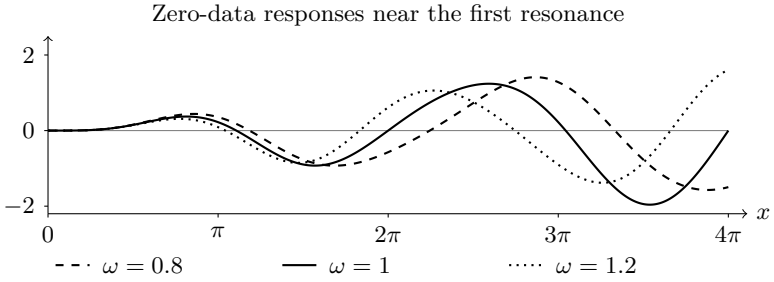
Both are even and vanish at zero. Their second derivatives at zero vanish by cancellation of $1/3$ with $-1/3$ in each case, so all four initial data hold.

Step 3: Take the limit of the combined response. Partial fractions give

$$y_\omega = \frac{1}{3}\left(\frac{\cos\omega x - \cos x}{1-\omega^2} - \frac{\cos\omega x - \cos 2x}{4-\omega^2}\right).$$

For fixed x, the first ratio tends to $x\sin x/2$ by differentiating numerator and denominator with respect to ω. The second tends to $(\cos x - \cos 2x)/3$. The resulting limit is exactly y_1.

Step 4: Distinguish the two limit questions. Every fixed nonresonant response is a bounded sum of cosines. At resonance, the linear sine term has unbounded values along its phase maxima, and the added cosines remain bounded. Thus boundedness holds exactly when $\omega \notin \{1, 2\}$. A common bound for all $x \geq 0$ and all nonresonant ω sufficiently near 1 would pass to the fixed-x limit and bound y_1 everywhere, a contradiction.



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Question 7

For real parameters a, b, c, consider

$$y^{(4)} + y'' = a + b\cos x + c\cos 2x.$$

A forcing with zero average over a period need not admit a periodic response.

Tasks

1. Use undetermined coefficients to find a particular solution for all a, b, c, and write the full real general solution.
2. Give necessary and sufficient conditions on the parameters and homogeneous constants for boundedness on $[0, \infty)$.
3. Determine exactly when the equation admits a 2π-periodic solution. Verify the necessary restrictions independently by integrating the equation against 1 and $\cos x$ over one period.
4. Give a forcing in this family with zero average but no periodic solution, and another nonzero forcing that does admit periodic solutions. Explain the different obstructions caused by a nonzero mean and by a resonant harmonic.

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Question 7 – Solution

Strategy. Track the zero-frequency resonance and the oscillatory resonance separately.

Step 1: Calculate the resonant and nonresonant pieces. The characteristic polynomial is $r^2(r^2+1)$. Direct substitution gives

$$L(x^2/2) = 1, \qquad L(x\sin x) = -2\cos x, \qquad L(\cos 2x) = 12\cos 2x.$$

Consequently,

$$\boxed{y = \frac{ax^2}{2} - \frac{b}{2}x\sin x + \frac{c}{12}\cos 2x + k_0 + k_1 x + A\cos x + B\sin x.}$$

The factors x^2 and x reflect the double root at zero and simple roots at $\pm i$, respectively.

Step 2: Classify boundedness exactly. If $a \neq 0$, division by x^2 gives a nonzero limit, so boundedness fails. Once $a = 0$, a bounded solution would satisfy

$$0 = \lim_{x\to\infty} \frac{y(x)}{x}, \qquad \frac{y(x)}{x} = k_1 - \frac{b}{2}\sin x + o(1).$$

Along sequences with sine equal to 1 and -1, this requires both $k_1 - b/2 = 0$ and $k_1 + b/2 = 0$. Hence

$$\boxed{\text{bounded on } [0,\infty) \iff a = b = k_1 = 0.}$$

These conditions suffice because all remaining terms are bounded.

Step 3: Check periodic compatibility independently. With $a = b = 0$, choosing $k_1 = 0$ gives a 2π-periodic solution for every c. Conversely, for a 2π-periodic C^4 solution, integration gives

$$\int_0^{2\pi} (y^{(4)} + y'')\,dx = 0 = 2\pi a.$$

Integration by parts transfers the even derivatives onto $\cos x$ without boundary terms. Since $(D^4+D^2)\cos x = 0$, orthogonality then gives $0 = b\pi$. Thus $a = b = 0$ is necessary and sufficient for existence of a 2π-periodic response.

Step 4: Explain why zero mean is insufficient. The forcing $\cos x$ has zero average but produces the unavoidable term $-x\sin x/2$, so no solution is periodic. In contrast, $\cos 2x$ admits $y = \cos 2x/12$. A nonzero mean excites quadratic drift through the double zero root; a first harmonic excites linear oscillatory growth. Removing only the mean addresses just one of these two resonances.

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Question 8

Consider the initially resting response

$$(D^2+1)(D^2+9)y = \cos^3 x, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

A nonlinear expression in the independent variable can conceal several resonant harmonics while the differential equation remains linear in y.

Tasks

1. Expand the forcing into elementary harmonics. Identify all resonances and select a complete real particular-solution trial.
2. Determine a particular solution by coefficient matching, and write the full homogeneous correction.
3. Impose the initial data and verify the resulting response, including the forcing and the initial curvature.
4. Evaluate the IVP solution at $x_n = \pi/2 + 2\pi n$, $n = 0, 1, \ldots$, to prove unboundedness. Derive an explicit envelope for its magnitude valid for every $x \geq 0$.

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Question 8 – Solution

Strategy. Expand the trigonometric product before choosing resonance factors.

Step 1: Expose both forcing frequencies. The identity $\cos^3 x = (3\cos x + \cos 3x)/4$ reveals simple resonance at both $\pm i$ and $\pm 3i$. A full trial is

$$x(A\cos x + B\sin x + C\cos 3x + D\sin 3x).$$

There is no nonlinear dependence on the unknown y; the right-hand side is a known function of x.

Step 2: Match each harmonic. For $L = (D^2+1)(D^2+9)$, differentiation gives $L(x\sin x) = 16\cos x$ and $L(x\sin 3x) = -48\cos 3x$. The cosine-weighted partners generate sine forcing and therefore have zero coefficients. Thus

$$y_p = \frac{3x\sin x}{64} - \frac{x\sin 3x}{192}, \qquad y_h = a\cos x + b\sin x + c\cos 3x + d\sin 3x.$$

The residual is $3\cos x/4 + \cos 3x/4 = \cos^3 x$.

Step 3: Correct the initial curvature. The particular term is even, has value zero, and has $y_p''(0) = 3/32 - 1/32 = 1/16$. The zero data require $a + c = 0$, $-a - 9c = -1/16$, and $b + 3d = -b - 27d = 0$. Hence

$$\boxed{y = \frac{x(9\sin x - \sin 3x)}{192} + \frac{\cos 3x - \cos x}{128}.}$$

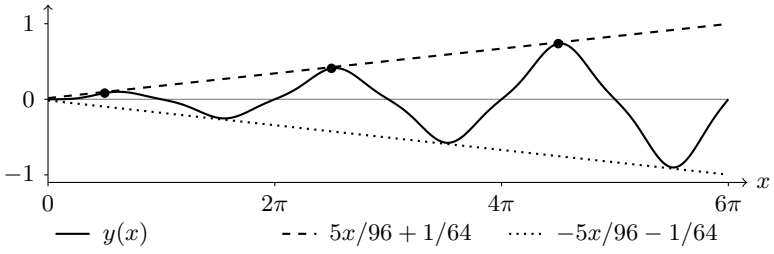
Here $a = -1/128, c = 1/128, b = d = 0$. The cosine correction has second derivative $-1/16$ at zero and cancels the particular curvature; evenness verifies the odd initial derivatives.

Step 4: Prove growth and a global envelope. At $x_n = \pi/2 + 2\pi n$, the sine values are $1, -1$ and both cosines vanish, so $\boxed{y(x_n) = 5x_n/96 \to \infty}$. Also $9\sin x - \sin 3x = 6\sin x + 4\sin^3 x$, whose absolute value is at most 10. Therefore

$$\boxed{|y(x)| \le \frac{5x}{96} + \frac{1}{64} \quad (x \ge 0).}$$

The displayed envelopes are bounds, not assertions that equality holds at every point.

Hidden resonant harmonics: response and bounds



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Question 9

For the operator $L = (D+1)^3$, consider three separate forcing functions

$$f_1(x) = x^2 e^{-x}, \qquad f_2(x) = \sin^2 x, \qquad f_3(x) = e^{x^2}.$$

The standard finite undetermined-coefficient method uses polynomial-exponential-trigonometric families closed under differentiation, with resonance corrections when required.

Tasks

1. Decide which forcings admit this standard finite method. Give complete real trials for each admissible forcing, first simplifying trigonometric products.
2. Compute and verify one particular solution for each admissible forcing. State the common homogeneous family to be added in both cases.
3. Prove that the derivatives of e^{x^2} span an infinite-dimensional space: write $D^k e^{x^2} = P_k(x)e^{x^2}$ and determine the degree and leading coefficient of P_k.
4. Explain why this rules out the standard finite trial method for f_3, while not ruling out existence or uniqueness of the IVP with any prescribed $y(0), y'(0), y''(0)$. Do not use variation of parameters to solve it here.

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Question 9 – Solution

Strategy. Test closure under differentiation, then distinguish a method's scope from the equation's solvability.

Step 1: Select the admissible trial families. The root -1 has multiplicity three. For f_1, use $y_{p,1} = x^3e^{-x}(Ax^2 + Bx + C)$. For $f_2 = (1 - \cos 2x)/2$, none of the frequencies $0, \pm 2i$ is a characteristic root, so use $y_{p,2} = a + b\cos 2x + c\sin 2x$. Both have finite derivative families. The third forcing does not, as shown below.

Step 2: Match and check the admissible cases. For $y = e^{-x}v$, the equation becomes $v''' = x^2$ in the first case. Thus

$$\boxed{y_{p,1} = x^5e^{-x}/60.}$$

Its shifted third derivative is exactly x^2. In the second case,

$$L(1) = 1, \quad L(\cos 2x) = -11\cos 2x + 2\sin 2x, \quad L(\sin 2x) = -2\cos 2x - 11\sin 2x.$$

The coefficient equations are $a = 1/2$, $-11b - 2c = -1/2$, $2b - 11c = 0$. They give

$$\boxed{y_{p,2} = \frac{1}{2} + \frac{11}{250}\cos 2x + \frac{1}{125}\sin 2x.}$$

Each complete solution adds $e^{-x}(c_0 + c_1x + c_2x^2)$.

Step 3: Prove infinite derivative dimension. Start with $P_0 = 1$. Differentiation gives $P_{k+1} = P_k' + 2xP_k$. If P_k has degree k and leading coefficient 2^k, the second term has degree $k+1$ and leading coefficient 2^{k+1}, whereas the derivative has lower degree. Induction proves these claims for every k. Polynomials of distinct degrees are linearly independent; multiplying by the never-zero e^{x^2} preserves independence. Hence no nonzero constant-coefficient operator annihilates f_3.

Step 4: Interpret the obstruction correctly. Every standard finite trial, even after multiplication by a resonance power, lies in a finite-dimensional differentiation-invariant family. Applying L keeps it in that family. It therefore cannot produce e^{x^2}, whose derivative family is infinite-dimensional. This is a limitation of the specified method. The normalized equation has constant coefficients and smooth forcing on $\mathbb{R}$, so the linear IVP theorem still gives a unique global solution for any three initial data.

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Question 10

A forcing pulse acts on a third-order equation for $x \geq 0$:

$$(D+1)^3 y = \begin{cases} e^{-x}p(x), & 0 < x < 1, \\ 0, & x > 1, \end{cases} \qquad p(x) = 120x^3 + ax^2 + bx + c.$$

Initially $y(0) = y'(0) = y''(0) = 0$. Require y, y', y'' to be continuous at $x = 1$; the equation is imposed on the two open intervals, not at the switch itself.

Tasks

1. Set $y = e^{-x}v$. Write the shifted equations and identify the conditions at $x = 1$ equivalent to having $y(x) = 0$ for every $x \geq 1$.
2. Use a polynomial undetermined-coefficient solution on $[0, 1]$ to find the unique a, b, c producing this complete cancellation. Give y on both intervals.
3. Prove that no nonzero polynomial pulse of degree at most two can achieve the same cancellation under the same zero initial data.
4. Verify all initial and joining conditions and the forcing. Determine whether the resulting solution is C^3 across $x = 1$, and locate the minimum of the scaled response $e^x y$ on $[0, 1]$.

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Question 10 – Solution

Strategy. Convert the pulse into a polynomial third derivative and enforce triple zeros at both ends.

Step 1: Translate the initial and terminal states. The shift gives $v''' = p$ on $(0,1)$ and $v''' = 0$ on $(1,\infty)$. Multiplication by the smooth, nonzero e^{-x} preserves matching of the first three state components. Thus $v(0) = v'(0) = v''(0) = 0$, and complete cancellation requires $v(1) = v'(1) = v''(1) = 0$. These terminal conditions also suffice: the post-pulse quadratic solution is then zero.

Step 2: Determine the polynomial exactly. The zero-data polynomial solution of $v''' = p$ has degree six and leading coefficient 1, since $D^3x^6 = 120x^3$. The three zero conditions at each endpoint require divisibility by $x^3(x-1)^3$. Equal degrees and leading coefficients therefore force

$$v = x^3(x-1)^3 = x^6 - 3x^5 + 3x^4 - x^3.$$

Differentiating three times gives $p = 120x^3 - 180x^2 + 72x - 6$. Hence

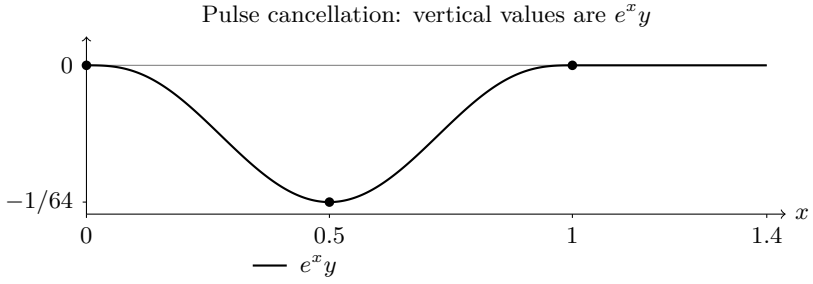
$$\boxed{(a,b,c) = (-180, 72, -6), \qquad y(x) = \begin{cases} e^{-x}x^3(x-1)^3, & 0 \le x \le 1, \\ 0, & x \ge 1. \end{cases}}$$

The formulas agree at the shared endpoint and uniquely determine the pulse.

Step 3: Prove the degree is necessary. For a polynomial p of degree at most two, the zero-data solution v has degree at most five. The terminal requirements would still make it divisible by the degree-six polynomial $x^3(x-1)^3$. It must therefore be zero, and so must $p = v'''$. No nonzero lower-degree polynomial pulse works.

Step 4: Check the join and the scaled shape. The triple endpoint zeros verify all six state conditions. The computed third derivative verifies the forcing before the switch; the zero branch verifies it afterward. At 1, the left third derivative of y is $e^{-1}v'''(1) = 6/e$, while the right third derivative is zero. Thus y is C^2 but not C^3 across the switch, exactly as allowed.

On $[0,1]$, $e^x y = -[x(1-x)]^3$. Since $x(1-x)$ has maximum $1/4$ at $x = 1/2$, the scaled response has its unique minimum $\boxed{-1/64}$ there.



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Question 1

Consider the fourth-order IVP

$$(1+x^2)y^{(4)} = 1, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the normalized homogeneous basis $1, x, x^2/2, x^3/6$.

Tasks

1. Normalize the equation and verify that the stated basis is fundamental. Write the four equations for the parameter derivatives in variation of parameters.
2. Solve those equations. Integrate each parameter from zero and combine the result into one definite integral for $y(x)$.
3. Verify the equation and all initial data by differentiating the combined integral. Explain exactly what goes wrong if the leading coefficient is not divided out.
4. Prove upper and lower bounds for $y(x)$ when $x \geq 0$ using the largest and smallest values of $1/(1+t^2)$ on $[0, x]$. State when equality holds.

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Question 1 – Solution

Strategy. Normalize first, then combine the four parameter integrals into one positive kernel.

Step 1: Form the normalized parameter system. Put $g(x) = 1/(1+x^2)$. The derivative matrix of the given basis is triangular with unit diagonal, so its Wronskian is 1. The parameter derivatives satisfy

$$\begin{pmatrix} 1 & x & x^2/2 & x^3/6 \\ 0 & 1 & x & x^2/2 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0' \\ u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g \end{pmatrix}.$$

The first three zero rows remove derivative terms introduced by varying the constants; the last row supplies the normalized forcing.

Step 2: Solve and combine. Back substitution gives $(u_0', u_1', u_2', u_3') = g(-x^3/6, x^2/2, -x, 1)$. Choose each $u_j(0) = 0$. In $y = \sum u_j f_j$, the combined numerator is $(x-t)^3$, yielding

$$\boxed{y(x) = \frac{1}{6}\int_0^x \frac{(x-t)^3}{1+t^2}\,dt.}$$

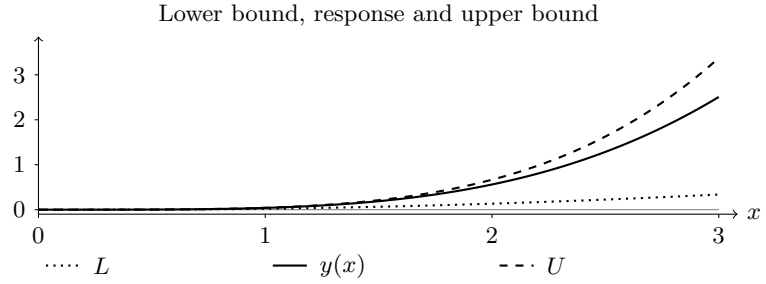
Oriented integration also defines the solution for negative x.

Step 3: Verify the initial state and residual. Successive derivatives have kernels $(x-t)^2/2$, $x-t$, and 1, all divided by $1+t^2$. The fourth derivative is $1/(1+x^2)$. The first four state components vanish at zero, and multiplication by $1+x^2$ gives the original forcing. If one incorrectly uses $g = 1$, the same construction gives $x^4/24$, whose original residual is $1+x^2$, not 1.

Step 4: Bound the positive integral. For $0 \le t \le x$, $1/(1+x^2) \le 1/(1+t^2) \le 1$. Hence

$$\boxed{\frac{x^4}{24(1+x^2)} \le y(x) \le \frac{x^4}{24} \quad (x \ge 0).}$$

Both inequalities are strict for $x > 0$, since their integrands differ on an interval of positive length where $(x-t)^3 > 0$. Both become equalities at $x = 0$. The figure labels the lower and upper bounds L and U, respectively.



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Question 2

For $x \geq 0$, consider

$$y''' - y' = e^{-x^2}.$$

Use the homogeneous basis $1, e^x, e^{-x}$; definite integrals may be left unevaluated.

Tasks

1. Verify the basis and derive the three parameter derivatives by variation of parameters.
2. Find the solution y_0 with $y_0(0) = y_0'(0) = y_0''(0) = 0$, expressed as one integral. Verify its kernel and initial data.
3. Determine $\lim_{x\to\infty} e^{-x} y_0(x)$ and prove that it is positive. Justify convergence of every improper integral used.
4. Keep $y(0) = y'(0) = 0$ but choose $y''(0)$ so that the response is bounded on $[0, \infty)$. Prove uniqueness of this choice and determine the finite limit of the resulting solution.

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Question 2 – Solution

Strategy. Use parameter integrals to isolate the growing homogeneous mode and the initial curvature needed to cancel it.

Step 1: Solve the derivative system. The functions solve $y''' - y' = 0$ and have Wronskian 2. With $g = e^{-x^2}$, the equations are

$$u_0' + e^x u_+' + e^{-x} u_-' = 0, \quad e^x u_+' - e^{-x} u_-' = 0, \quad e^x u_+' + e^{-x} u_-' = g.$$

Thus $u_0' = -g$, $u_+' = e^{-x} g/2$, and $u_-' = e^x g/2$.

Step 2: Construct the zero-data response. Integrating from zero and combining gives

$$\boxed{y_0(x) = \int_0^x [\cosh(x-t) - 1] e^{-t^2}\, dt.}$$

For $G(s) = \cosh s - 1$, we have $G(0) = G'(0) = 0$, $G''(0) = 1$, and $G''' - G' = 0$. Leibniz differentiation therefore gives $y_0''' - y_0' = g$ and the three zero initial data.

Step 3: Identify the growing coefficient. Define

$$A(x) = \int_0^x e^{-t^2-t}\, dt, \quad B(x) = \int_0^x e^{-t^2+t}\, dt, \quad C(x) = \int_0^x e^{-t^2}\, dt.$$

Their limits $A_\infty, B_\infty, C_\infty$ are finite: for $t \geq 2$, each integrand is at most $e^{-t^2/2} \leq e^{-t}$. All are positive. Since $y_0 = e^x A(x)/2 + e^{-x} B(x)/2 - C(x)$,

$$\boxed{\lim_{x\to\infty} e^{-x} y_0(x) = A_\infty/2 > 0.}$$

Step 4: Cancel growth without losing the first two data. Any other solution with the same value and slope is $y = y_0 + k(\cosh x - 1)$, where $k = y''(0)$. Boundedness requires $k = -A_\infty$, and no other choice cancels the growing coefficient. For this choice,

$$y = -\tfrac{1}{2} e^x (A_\infty - A(x)) + \tfrac{1}{2} e^{-x} (B(x) - A_\infty) + A_\infty - C(x).$$

The first term tends to zero: for $x > 0$, $e^x \int_x^\infty e^{-t^2-t}\, dt \leq e^{-x^2}$. The second also tends to zero. Therefore the choice is sufficient, and

$$\boxed{y''(0) = -A_\infty, \qquad \lim_{x\to\infty} y = A_\infty - C_\infty.}$$

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Question 3

On $x > 0$, consider the Euler-type equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x.$$

Use the proposed homogeneous basis x, x^2, x^3 and variation of parameters, rather than a guessed logarithmic trial.

Tasks

1. Verify the three homogeneous solutions and their Wronskian. Normalize the equation and write the parameter system with the correct right-hand side.
2. Solve for the parameter derivatives and integrate to obtain one particular solution. Write the complete general solution.
3. Impose $y(1) = y'(1) = y''(1) = 0$. Verify all three initial data and the forcing directly.
4. State the interval containing 1 on which this construction and the standard linear IVP theorem apply. Explain why evaluating the same parameter system at $x = 0$ is invalid, even if some expressions have finite limits there.

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Question 3 – Solution

Strategy. Normalize the Euler operator before solving the derivative equations, and retain its positive-domain restriction.

Step 1: Verify the basis and normalization. For $y = x^m$, the left side is $(m-1)(m-2)(m-3)x^m$. Thus the proposed functions are homogeneous. Their Wronskian is $2x^3 \neq 0$ for $x > 0$. Division by x^3 gives normalized forcing $x \ln x$, so

$$\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \ln x \end{pmatrix}.$$

Step 2: Integrate the solved parameter derivatives. Elimination gives $u_1' = x^2 \ln x/2$, $u_2' = -x \ln x$, $u_3' = \ln x/2$. Convenient antiderivatives are

$$u_1 = \frac{x^3 \ln x}{6} - \frac{x^3}{18}, \quad u_2 = -\frac{x^2 \ln x}{2} + \frac{x^2}{4}, \quad u_3 = \frac{x \ln x}{2} - \frac{x}{2}.$$

Combining $xu_1 + x^2u_2 + x^3u_3$ gives

$$\boxed{y_p = x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right), \qquad y = y_p + c_1 x + c_2 x^2 + c_3 x^3.}$$

Integration constants are already represented by the homogeneous terms.

Step 3: Fit and check the initial vector. The particular solution has vector $(-11/36, -19/18, -5/2)$ at 1. The three linear equations for its cancellation give

$$\boxed{y = \frac{x}{18} - \frac{x^2}{4} + \frac{x^3}{2} + x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right).}$$

The added terms have vector $(11/36, 19/18, 5/2)$, verifying the data. For a direct residual check, let $\theta = xD$. The operator is $(\theta-1)(\theta-2)(\theta-3)$; on $x^4(a \ln x + b)$ it gives $x^4(6a \ln x + 11a + 6b)$. With $a = 1/6, b = -11/36$, this is exactly $x^4 \ln x$.

Step 4: Respect the regular interval. The normalized coefficients, forcing and nonsingular derivative matrix are valid throughout $(0, \infty)$, the largest such interval containing 1. At zero the leading coefficient and Wronskian vanish and $\ln x$ is undefined. Finite limits of selected terms do not make that parameter system invertible or extend the theorem across zero.

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Question 4

Let g be continuous on $\mathbb{R}$ and consider $y''' = g(x)$. Two homogeneous bases are

$$f = (1, x, x^2/2), \qquad \widetilde{f} = (1 + x,\ x + x^2/2,\ 1 + x^2/2).$$

Regard these as row vectors of functions and write $y_p = fu = \widetilde{f}\, v$, with column vectors of parameters.

Tasks

1. Find the constant matrix M such that $\widetilde{f} = fM$. Verify that both bases are fundamental.
2. Derive u' by variation of parameters, then obtain v' from the change of basis. Explain why the lower-limit-zero constructions produce exactly the same particular solution.
3. Replace the lower integration limit 0 by an arbitrary real b. Find the difference between the two resulting particular solutions and prove that it is homogeneous.
4. Explain why changing basis or choosing separate integration constants does not create more than three free constants in the general solution. Distinguish unique parameter coordinates in a chosen basis from the nonuniqueness of a particular solution.

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Question 4 – Solution

Strategy. Track the constant coordinate transformation and isolate the effect of the integration base point.

Step 1: Identify the basis transformation. The coefficient columns give

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \det M = 2.$$

All six displayed functions are quadratic or lower and solve $y''' = 0$. The Wronskian of f is 1, and that of $\widetilde{f}$ is $\det M = 2$, so both triples are fundamental.

Step 2: Transform the coordinates. The standard derivative system gives $u' = g(x)(x^2/2, -x, 1)^T$. Since the derivative matrices satisfy $\widetilde{F} = FM$, their parameter equations imply $Mv' = u'$, hence

$$\boxed{v' = g(x) \begin{pmatrix} x^2/4 - x/2 - 1/2 \\ -x^2/4 - x/2 + 1/2 \\ x^2/4 + x/2 + 1/2 \end{pmatrix}.}$$

With $u(0) = v(0) = 0$, integration gives $u = Mv$ at every x. Thus $fu = fMv = \widetilde{f}\, v$ exactly, not merely up to a homogeneous term. Both equal

$$Y_0(x) = \frac{1}{2} \int_0^x (x - t)^2 g(t)\, dt.$$

Step 3: Change the integration base point. The construction based at b gives $Y_b(x) = \frac{1}{2} \int_b^x (x - t)^2 g(t)\, dt$. Therefore

$$\boxed{Y_b - Y_0 = -\frac{x^2}{2} \int_0^b g(t)\, dt + x \int_0^b t g(t)\, dt - \frac{1}{2} \int_0^b t^2 g(t)\, dt.}$$

The three integrals are constants with respect to x, even when $b < 0$. The difference is a quadratic polynomial and hence homogeneous.

Step 4: Count genuine solution freedom. Once a basis and lower limit are fixed, the invertible derivative matrix uniquely fixes the parameter derivatives, and the specified parameter values fix their integrals. Allowing arbitrary integration constants adds a constant linear combination of the three homogeneous basis functions. Adding a separate homogeneous triple afterward repeats that same freedom rather than creating six independent constants. Particular solutions need not coincide, but their differences are homogeneous; the full solution space of $y''' = g$ is one particular solution plus the three-dimensional homogeneous space.

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Question 5

On the positive half-line, consider

$$y''' = \frac{1}{x}, \qquad y(1) = y'(1) = y''(1) = 0.$$

Use the homogeneous basis $1, x, x^2/2$ and integrate the variation parameters from 1.

Tasks

1. Derive the parameter derivatives and compute the complete IVP solution.
2. Verify the original equation and all initial data. Find the one-sided limits of y, y', y'' as $x \to 0^+$.
3. Determine whether the solution admits a C^1 or C^2 extension to the endpoint 0. Explain why a divergent parameter does not necessarily force y itself to diverge.
4. Test the attempted lower-limit-zero formula $\frac{1}{2}\int_0^x (x-t)^2/t\,dt$. Decide whether it converges for fixed $x > 0$, and explain why finite endpoint values of y, y' do not justify using zero as a regular IVP base point.

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Question 5 – Solution

Strategy. Separate parameter divergence, solution limits and the regularity needed for initial data.

Step 1: Integrate from the regular point. The triangular parameter system for $y''' = g$ gives $(u_0', u_1', u_2') = (x^2 g/2, -xg, g)$. Here $g = 1/x$, so

$$u_0 = (x^2 - 1)/4, \qquad u_1 = 1 - x, \qquad u_2 = \ln x.$$

Combining $u_0 + xu_1 + x^2 u_2/2$ yields

$$\boxed{y = \frac{x^2 \ln x}{2} - \frac{3x^2}{4} + x - \frac{1}{4} \quad (x > 0).}$$

Step 2: Check derivatives and endpoint behavior. Direct differentiation gives

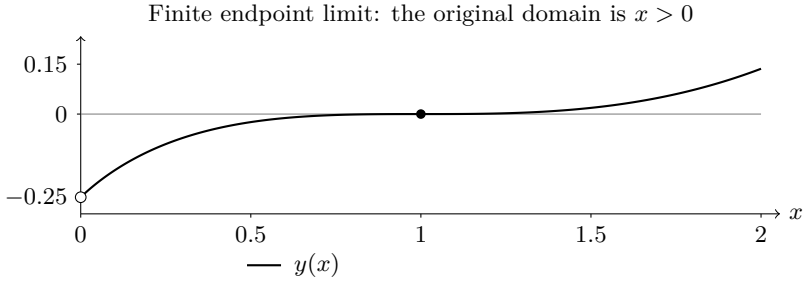
$$y' = x \ln x - x + 1, \qquad y'' = \ln x, \qquad y''' = 1/x.$$

These formulas and $\ln 1 = 0$ verify all three data at 1. Since $x \ln x$ and $x^2 \ln x$ tend to zero at the positive endpoint,

$$\boxed{\lim_{x \to 0^+} y = -1/4, \qquad \lim_{x \to 0^+} y' = 1, \qquad \lim_{x \to 0^+} y'' = -\infty.}$$

Step 3: Classify the extension. Defining $y(0) = -1/4$ gives right derivative 1, because $(y(x)+1/4)/x \to 1$. This derivative is continuous from the right, so the solution extends as a C^1 function to $[0, \infty)$; it may also be continued to the left with matching value and slope. No C^2 extension is possible because y'' diverges. Although $u_2 = \ln x$ diverges, it is multiplied by $x^2/2$ in y, whose contribution tends to zero.

Step 4: Reject the singular base point. For fixed $x > 0$ and $0 < t < x/2$, $(x - t)^2/(2t) \geq x^2/(8t)$. The proposed improper integral therefore diverges to $+\infty$. Its lower limit cannot simply be moved to zero. The equation is singular there, and even the solution's second derivative lacks finite initial data. Two finite endpoint limits do not supply a regular third-order initial state.



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Question 6

For continuous f on $[0, \infty)$, consider

$$(D+1)^4 y = f(x), \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the factorial-normalized homogeneous basis $e^{-x}, xe^{-x}, x^2e^{-x}/2, x^3e^{-x}/6$.

Tasks

1. Derive the variation-parameter equations or an equivalent triangular system, and combine their integrals into one response kernel.
2. Verify that the kernel gives the original equation and all four zero initial data for every continuous forcing.
3. If $|f(x)| \le M$ for $M \ge 0$, derive an explicit bound on $|y(x)|$ for each $x \ge 0$. Prove its sharpness and deduce an all-time bound.
4. Apply the same reasoning to the difference of two responses with identical initial data and forcing discrepancy at most ε. Explain why bounded forcing need not produce a response tending to zero.

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Question 6 – Solution

Strategy. Use the repeated-root basis to produce a positive kernel whose integral is explicit.

Step 1: Solve the shifted parameter system. Write $y = e^{-x}v$. The equation becomes $v^{(4)} = e^x f(x)$. The original basis becomes $1, x, x^2/2, x^3/6$ for v, with parameter derivatives

$$(u_0', u_1', u_2', u_3') = e^x f(x)(-x^3/6, x^2/2, -x, 1).$$

The homogeneous basis is fundamental: its Wronskian is $e^{-4x} \neq 0$. Integrating from zero gives

$$\boxed{y(x) = \int_0^x G(x-t)f(t)\,dt, \qquad G(s) = \frac{s^3 e^{-s}}{6}.}$$

Step 2: Verify the response kernel. The function G satisfies $(D+1)^4 G = 0$ for $s > 0$, and $G(0) = G'(0) = G''(0) = 0$, $G'''(0) = 1$. Repeated Leibniz differentiation therefore adds no endpoint terms until the fourth derivative, where it adds exactly $f(x)$. Combining derivatives in $(D+1)^4 y$ cancels the integral terms. All four initial data vanish because the relevant integrals have zero length.

Step 3: Integrate the positive kernel sharply. For $s \geq 0$, $G(s) \geq 0$. Repeated integration by parts yields

$$\int_0^x G(s)\,ds = 1 - e^{-x}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) = H(x).$$

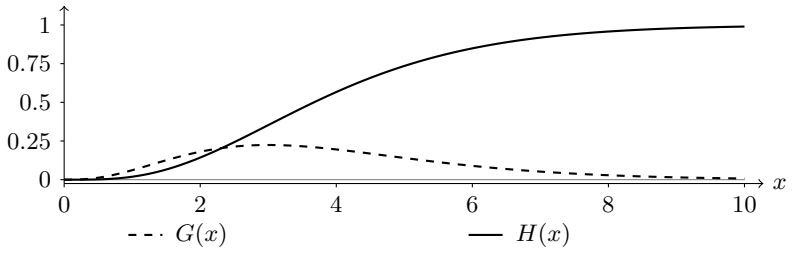
Consequently,

$$\boxed{|y(x)| \leq MH(x) \leq M \quad (x \geq 0).}$$

The forcing $f \equiv M$ gives $y = MH$ and attains the pointwise bound. Also $H(x) \to 1$, so no smaller all-time constant than M works for all admissible forcings when $M > 0$. The case $M = 0$ gives the zero solution.

Step 4: Bound input error without claiming decay. For identical initial data, the response difference has zero initial vector and forcing difference bounded by ε. Thus $|y_1 - y_2| \leq \varepsilon H(x) \leq \varepsilon$. But $f \equiv 1$ gives $y = H(x) \to 1$, not zero. A positive integrable kernel controls size; it does not force the output to vanish for a persistent input.

Positive kernel and its accumulated response



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Question 7

For the variable-coefficient IVP

$$y''' - xy'' = 1, \qquad y(0) = y'(0) = y''(0) = 0,$$

define $F(x) = \int_0^x (x-t)e^{t^2/2}\,dt$ and use the homogeneous basis $1, x, F(x)$.

Tasks

1. Verify the basis and its Wronskian, then derive the three variation-parameter derivatives.
2. Combine the lower-limit-zero parameter integrals into $y(x) = \int_0^x K(x,t)\,dt$. Write K both using F, F' and as a definite integral for $x \geq t$.
3. Differentiate to verify the IVP. Prove that y is strictly convex and $y(x) > x^3/6$ for every $x > 0$.
4. Determine whether $K(x,t)$ can depend only on $x - t$. Use a derivative of the kernel to give a precise obstruction to treating this variable-coefficient response as a translation-invariant convolution.

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Question 7 – Solution

Strategy. Exploit the triangular derivative matrix while retaining both kernel variables.

Step 1: Solve the parameter equations. We have $F(0) = F'(0) = 0$, $F'' = e^{x^2/2}$ and $F''' = xF''$. Thus $1, x, F$ solve the homogeneous equation and have Wronskian $e^{x^2/2} > 0$. The triangular system gives

$$u_2' = e^{-x^2/2}, \quad u_1' = -F'e^{-x^2/2}, \quad u_0' = (xF' - F)e^{-x^2/2}.$$

Step 2: Combine the parameter integrals. Multiplying by $1, x, F(x)$ and integrating from zero yields

$$\boxed{K(x,t) = e^{-t^2/2}\big[F(x) - F(t) - (x-t)F'(t)\big].}$$

Taylor's integral identity, obtained by integrating F'' twice, gives for $x \geq t$

$$K(x,t) = e^{-t^2/2}\int_t^x (x-s)e^{s^2/2}\,ds, \qquad y(x) = \int_0^x K(x,t)\,dt.$$

The factor involving t is essential; it comes from the inverse derivative matrix evaluated at the integration variable.

Step 3: Verify the equation and inequalities. At the diagonal, $K(t,t) = K_x(t,t) = 0$, while

$$K_{xx}(x,t) = e^{(x^2-t^2)/2}, \qquad y''(x) = e^{x^2/2}\int_0^x e^{-t^2/2}\,dt.$$

Hence $y''' = xy'' + 1$ and all three initial data vanish. For $x > 0$, $y'' > 0$, proving strict convexity. Also $y''' = 1 + xy'' > 1$ there. Integrating this strict inequality three times from the zero initial vector gives $\boxed{y(x) > x^3/6}$ for every $x > 0$.

Step 4: Test translation invariance. If $K(x,t) = H(x-t)$, then $K_{xx} = H''(x-t)$ would have the same value at all pairs with the same difference. But for any fixed $h > 0$,

$$K_{xx}(t+h,t) = e^{th+h^2/2},$$

which varies with t. Therefore this kernel is not a function of $x-t$ alone. The variable coefficient distinguishes the absolute source time from the elapsed time.

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Question 8

Let g be continuous and nonnegative on $[0, \infty)$, and consider

$$y''' + y' = g(x), \qquad y(0) = y'(0) = y''(0) = 0.$$

Use the homogeneous basis $1, \cos x, \sin x$.

Tasks

1. Derive the variation parameters and combine them into a single response integral.
2. Prove that $y(x) \geq 0$ for every $x \geq 0$. Give a sufficient condition for strict positivity at a specified $x > 0$ and justify it analytically.
3. For the continuous pulse $g(x) = \sin^2 x$ on $[0, \pi]$ and $g(x) = 0$ for $x \geq \pi$, compute $y(x)$ for all $x \geq \pi$ using the integrals of g, $g \cos x$ and $g \sin x$.
4. Determine the exact minimum and maximum of this post-pulse response. Prove that a nonnegative forcing need not make y monotone, even though the solution stays nonnegative.

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Question 8 – Solution

Strategy. Distinguish a nonnegative response kernel from the sign of its derivative.

Step 1: Compute the response kernel. The derivative matrix of $1, \cos x, \sin x$ has Wronskian 1. Its parameter equations give

$$u_0' = g, \qquad u_c' = -g\cos x, \qquad u_s' = -g\sin x.$$

Integrating from zero and combining by the cosine difference identity yields

$$\boxed{y(x) = \int_0^x [1 - \cos(x - t)]g(t)\,dt.}$$

The kernel and its first derivative vanish at zero, its second derivative is 1 there, and its third plus first derivative vanishes. These facts verify the equation and zero data.

Step 2: Prove positivity without a monotonicity assumption. Both factors in the integrand are nonnegative. If $g(t_0) > 0$ for some $0 \le t_0 < x$, continuity gives an interval of positive forcing inside $[0, x)$. The zeros of $1 - \cos(x - t)$ are isolated, so the integral is strictly positive. In particular, a continuous nonnegative forcing that is not identically zero on $[0, x)$ suffices.

Step 3: Evaluate the post-pulse moments. For the stated pulse,

$$\int_0^\pi \sin^2 t\,dt = \frac{\pi}{2}, \qquad \int_0^\pi \sin^2 t\cos t\,dt = 0, \qquad \int_0^\pi \sin^3 t\,dt = \frac{4}{3}.$$

The first follows from the double-angle identity, the second from the derivative of $\sin^3 t/3$, and the third by substituting $u = \cos t$. Thus

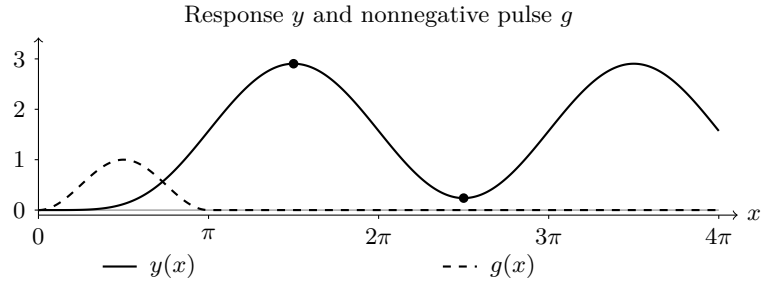
$$\boxed{y(x) = \frac{\pi}{2} - \frac{4}{3}\sin x \quad (x \ge \pi).}$$

For plotting the initial interval, evaluating the same kernel gives $y = x/2 - 2\sin x/3 + \sin 2x/12$ on $[0, \pi]$; its state matches the displayed branch at π.

Step 4: Separate positivity from monotonicity. The post-pulse range is

$$\boxed{\frac{\pi}{2} - \frac{4}{3} \le y(x) \le \frac{\pi}{2} + \frac{4}{3}.}$$

Both bounds are attained infinitely often; the lower bound is positive. Yet $y' = -4\cos x/3$ changes sign, and $y'(2\pi) = -4/3 < 0$. Thus the nonnegative input gives a nonnegative response that continues oscillating after the pulse. Positivity of $1 - \cos s$ does not imply positivity of its derivative $\sin s$.



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Question 9

On $x > 0$, consider

$$x^4 y^{(4)} = 1, \qquad y(1) = y'(1) = y''(1) = y'''(1) = 0.$$

A proposed response kernel for $x \geq t > 0$ is

$$\widehat{K}(x,t) = \frac{(x-t)^3}{6x^4}.$$

The placement of the leading-coefficient factor must be checked, not guessed.

Tasks

1. Normalize the equation and derive the correct kernel from variation of parameters using $1, x, x^2/2, x^3/6$.
2. For fixed t, verify the correct kernel's homogeneous equation when $x > t$ and its first four diagonal data, through the third x-derivative.
3. Integrate the correct kernel from 1 to x to find the IVP solution in elementary form. Verify its initial data and forcing.
4. Show that $\widehat{K}$ has the same diagonal data but is still wrong. Compute the response it would predict and evaluate $x^4 \widehat{y}^{(4)}$ at $x = 2$. Explain which additional kernel requirement fails.

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Question 9 – Solution

Strategy. Evaluate the normalized forcing at the integration variable and check the kernel away from its diagonal.

Step 1: Derive the correctly normalized kernel. The equation is $y^{(4)} = x^{-4}$. For the given basis, variation of parameters gives $u' = x^{-4}(-x^3/6, x^2/2, -x, 1)^T$. Integrating from 1 and combining yields

$$\boxed{K(x,t) = \frac{(x-t)^3}{6t^4}, \qquad y(x) = \int_1^x K(x,t)\,dt.}$$

The denominator is evaluated at the source variable t. The same polynomial expression and oriented integral apply for $0 < x < 1$.

Step 2: Verify both kernel requirements. For fixed $t > 0$, K is a cubic polynomial in x, so $x^4 K_{xxxx} = 0$ for $x > t$. Its diagonal data are

$$K(t,t) = K_x(t,t) = K_{xx}(t,t) = 0, \qquad K_{xxx}(t,t) = 1/t^4.$$

Leibniz differentiation gives $y^{(4)}(x) = 1/x^4$, because the fourth kernel derivative is zero and the third diagonal derivative supplies exactly that term.

Step 3: Evaluate and check the elementary response. Expanding $(x-t)^3/t^4$ and integrating its four powers of t gives

$$\boxed{y = \frac{x^3}{18} - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln x}{6} - \frac{11}{36}.}$$

Its derivatives are $y' = x^2/6 - x/2 + 1/2 - 1/(6x)$, $y'' = x/3 - 1/2 + 1/(6x^2)$, and $y''' = 1/3 - 1/(3x^3)$. These and y vanish at 1; one further derivative gives $y^{(4)} = x^{-4}$.

Step 4: Expose the misleading diagonal agreement. The factor $(x-t)^3$ ensures that $\widehat{K}$ has the same three zero diagonal values and $\widehat{K}_{xxx}(t,t) = 1/t^4$. But its proposed response is

$$\widehat{y} = \frac{1}{6x^4}\int_1^x (x-t)^3\,dt = \frac{(x-1)^4}{24x^4}.$$

It satisfies the zero initial data, yet

$$x^4\widehat{y}^{(4)} = -\frac{4}{x} + \frac{30}{x^2} - \frac{60}{x^3} + \frac{35}{x^4}, \qquad x^4\widehat{y}^{(4)}\Big|_{x=2} = \boxed{3/16 \neq 1}.$$

The missing requirement is the homogeneous equation in x away from $x = t$. Correct diagonal data alone do not make a valid response kernel.

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Question 10

For the zero-data IVP

$$y''' + y' = e^x, \qquad y(0) = y'(0) = y''(0) = 0,$$

a proposed approximation on $[0, 1]$ is $q(x) = x^3/6 + x^4/24$.

Tasks

1. Compute the residual $r = q''' + q' - e^x$ and check the approximation's initial data. Express the residual using the remainder after the cubic Taylor polynomial of e^x.
2. Use variation of parameters to represent the error $E = y - q$ by a definite integral with zero initial data. Prove whether the approximation lies above or below the true solution on $(0, 1]$.
3. Use Taylor's remainder theorem and $1 - \cos s \le s^2/2$ to prove $0 \le E(x) \le \mathrm{e}x^7/5040$ for $0 \le x \le 1$. Decide whether an absolute-error tolerance of 10^{-3} is certified throughout this interval.
4. Independently verify $Y(x) = (e^x + \cos x - \sin x)/2 - 1$ as the exact IVP solution. Compare its error at $x = 1$ with the certified upper bound, and explain why direct agreement at finitely many sample points is weaker than the integral certificate.

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Question 10 – Solution

Strategy. Treat the residual as an error forcing and bound its full response, not just sampled values.

Step 1: Compute the forcing error. We have $q''' = 1 + x$ and $q' = x^2/2 + x^3/6$, so

$$r = 1 + x + x^2/2 + x^3/6 - e^x = -R_4(x),$$

where $R_4 = e^x - (1 + x + x^2/2 + x^3/6)$. The values $q(0), q'(0), q''(0)$ are zero. Thus $E = y - q$ has a zero initial vector and satisfies $E''' + E' = R_4$.

Step 2: Derive the positive error integral. For the basis $1, \cos x, \sin x$, the variation derivatives for this error equation are $(R_4, -R_4 \cos x, -R_4 \sin x)$. Integrating from zero gives

$$\boxed{E(x) = \int_0^x [1 - \cos(x - t)] R_4(t)\, dt.}$$

For $0 < t < x \leq 1$, both factors are strictly positive. Hence $E(x) > 0$ on $(0, 1]$: the approximation underestimates the true solution.

Step 3: Certify a uniform error bound. Taylor's theorem gives $0 \leq R_4(t) \leq \mathrm{e}t^4/24$ on $[0, 1]$. Therefore

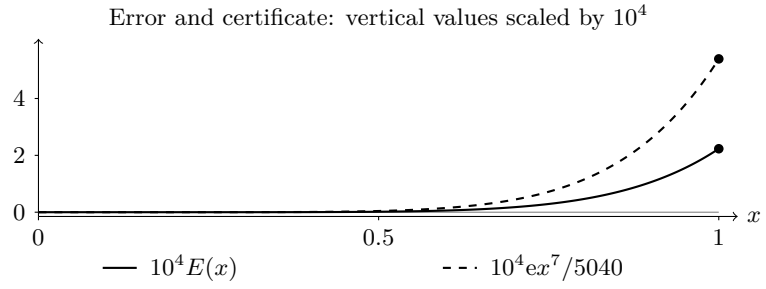
$$0 \leq E(x) \leq \frac{\mathrm{e}}{48} \int_0^x (x - t)^2 t^4\, dt = \boxed{\frac{\mathrm{e}x^7}{5040}},$$

where the polynomial integral equals $x^7/105$. The maximum certified error is $\mathrm{e}/5040 \approx 0.00053934 < 10^{-3}$, so the tolerance holds throughout $[0, 1]$.

Step 4: Compare with an exact independent check. For $Y = (e^x + \cos x - \sin x)/2 - 1$, direct differentiation gives $Y''' + Y' = e^x$ and its initial vector is zero. Uniqueness therefore implies $Y = y$. At 1, $q(1) = 5/24$, so the actual error is

$$\boxed{E(1) = \frac{\mathrm{e} + \cos 1 - \sin 1}{2} - \frac{29}{24}.}$$

Numerically, $E(1) \approx 0.00022324$, positive and smaller than the certified bound. Exact evaluation or sample agreement is useful corroboration, but finitely many values cannot exclude larger errors between them. The integral inequality proves the bound for every point of the interval.



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Question 1

Consider the initial-value problem, for $t \geq 0$,

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \qquad (y, y', y'', y''')(0) = (1, 0, 0, 0).$$

A student writes $(s+1)^4 Y(s) = s^3$ and concludes that the initial displacement can be represented by a single term from the fourth derivative.

Tasks

1. Derive the correct transformed equation, displaying the initial terms contributed by every derivative. Identify the student's error.
2. Find $Y(s)$ and invert it by organizing the numerator in powers of $s + 1$.
3. Verify the differential equation and all four initial values directly. Determine whether the response ever increases or changes sign.
4. Find a half-plane of convergence and the initial and final limits. Explain why a quadruple pole does not prevent decay, and sketch the response.

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Question 1 – Solution

Strategy. Collect every initial contribution before exploiting the repeated shifted pole.

Step 1: Transform all derivatives. The derivative transforms are $s^4Y - s^3$, $s^3Y - s^2$, $s^2Y - s$, and $sY - 1$. Thus

$$(s+1)^4Y = s^3 + 4s^2 + 6s + 4.$$

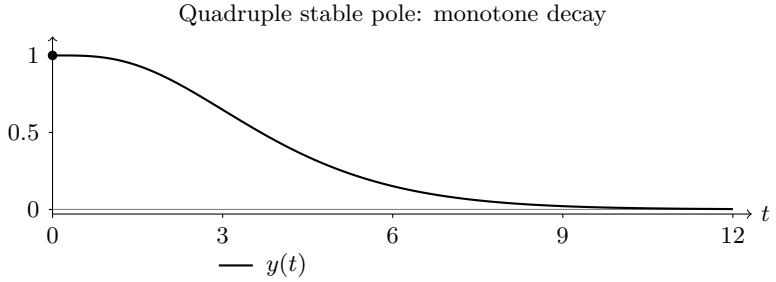
The student omitted the initial terms from $4y''' + 6y'' + 4y'$.

Step 2: Use shifted powers. Writing $z = s + 1$ makes the numerator $z^3 + z^2 + z + 1$. Therefore

$$Y = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}, \qquad \boxed{y(t) = e^{-t}\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right).}$$

Step 3: Check the state and shape. For $y = e^{-t}p(t)$, $(D+1)^4 y = e^{-t}p^{(4)} = 0$. Moreover $y' = -t^3e^{-t}/6$, so $y'(0) = y''(0) = y'''(0) = 0$ and $y(0) = 1$. Every term in p is positive on $t \geq 0$; hence $y > 0$ and $y' < 0$ for $t > 0$.

Step 4: Check convergence and limits. The transform converges absolutely for $\operatorname{Re} s > -1$. Also $\lim_{s\to\infty} sY = 1$ and $\lim_{s\to 0} sY = 0$. The final-value theorem applies because all poles of sY lie at -1. Polynomial growth from multiplicity is dominated by e^{-t}, so $y(t) \to 0$.



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Question 2

A sixth-order initial-value problem is

$$(D^2+1)^3 y = 0, \qquad y^{(j)}(0) = 0 \ (0 \le j \le 4), \qquad y^{(5)}(0) = 1,$$

where $D = d/dt$ and $t \ge 0$. To invert a repeated quadratic pole, introduce $H_a(s) = 1/(s^2+a^2)$ for $a > 0$.

Tasks

1. Expand the differential operator and transform the problem. Explain why only one initial term survives.
2. Starting with $\mathcal{L}^{-1}H_a = \sin(at)/a$, use differentiation with respect to a twice to invert $(s^2+a^2)^{-3}$. Then set $a = 1$.
3. Verify the equation and initial state without relying on the transform calculation. Identify the first nonzero Taylor term.
4. Describe the pole multiplicities and a convergence half-plane. Decide whether the solution is bounded and whether the final-value theorem can be used, giving a decisive argument for each claim.

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Question 2 – Solution

Strategy. Parameter differentiation resolves repeated quadratic poles while a Taylor check tests the high-order initial data.

Step 1: Transform the sixth-order problem. The equation is $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$. Only $\mathcal{L}(y^{(6)}) = s^6Y - 1$ has a nonzero initial subtraction, so

$$\boxed{Y(s) = \frac{1}{(s^2+1)^3}.}$$

Step 2: Differentiate a transform family. Since $\partial_a(s^2+a^2)^{-m} = -2ma(s^2+a^2)^{-m-1}$,

$$h_2(t) = \frac{\sin(at) - at\cos(at)}{2a^3}, \qquad h_3(t) = -\frac{1}{4a}\partial_a h_2(t) = \frac{3\sin(at) - 3at\cos(at) - a^2t^2\sin(at)}{8a^5}.$$

Differentiation under the integral is valid for a in a compact positive interval and $\operatorname{Re} s > 0$, since the derivatives have polynomial growth. Hence

$$\boxed{y(t) = \frac{3\sin t - 3t\cos t - t^2\sin t}{8}.}$$

Step 3: Verify independently. Each term is the real or imaginary part of e^{it} times a polynomial of degree at most two; $(D-i)^3$ annihilates these complex modes, and the conjugate factor annihilates their conjugates. Thus $(D^2+1)^3 y = 0$. Expanding the displayed trigonometric expression gives

$$y(t) = \frac{t^5}{120} - \frac{t^7}{1680} + O(t^9).$$

This proves the five zero initial derivatives and $y^{(5)}(0) = 1$.

Step 4: Distinguish oscillation from boundedness. There are triple poles at $s = \pm i$; $\operatorname{Re} s > 0$ gives absolute convergence. Along $t_n = \pi/2 + 2\pi n$, $y(t_n) = (3 - t_n^2)/8 \to -\infty$, so the response is unbounded. The poles of sY on the imaginary axis invalidate the final-value theorem. The formal value $\lim_{s\to 0} sY = 0$ is not a time limit.

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Question 3

Let $u(t-a)$ denote the unit step at a. Solve the delayed-ramp problem

$$(D+2)^3 y = u(t-1)(t-1), \qquad (y, y', y'')(0) = (0,0,0), \qquad t \geq 0.$$

The value assigned to $u(0)$ does not affect an ordinary Laplace transform.

Tasks

1. Transform the initial-value problem, taking care to shift the entire ramp.
2. Obtain a closed form for the response on each side of $t = 1$. Display a partial-fraction decomposition for the unshifted transform.
3. Check the equation on the two open time intervals and determine exactly which derivative first jumps at $t = 1$. State the global smoothness of y.
4. Prove the response is positive after switching, find its large-time linear asymptote, and sketch it together with that asymptote on $t \geq 1$.

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Question 3 – Solution

Strategy. Solve a zero-state ramp response, then translate time and check the switching jet.

Step 1: Transform the shifted input. All initial terms vanish. The second shifting theorem gives

$$Y(s) = \frac{e^{-s}}{s^2(s+2)^3}, \qquad \operatorname{Re} s > 0.$$

Step 2: Invert before translating. The unshifted fraction decomposes as

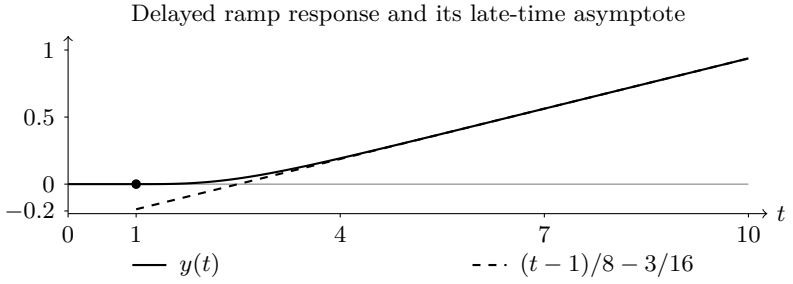
$$\frac{1}{s^2(s+2)^3} = -\frac{3}{16s} + \frac{1}{8s^2} + \frac{3}{16(s+2)} + \frac{1}{4(s+2)^2} + \frac{1}{4(s+2)^3}.$$

Thus, with $r = t - 1$,

$$g(r) = \frac{r}{8} - \frac{3}{16} + e^{-2r}\left(\frac{3}{16} + \frac{r}{4} + \frac{r^2}{8}\right), \qquad \boxed{y(t) = \begin{cases} 0, & 0 \le t \le 1, \\ g(t-1), & t > 1. \end{cases}}$$

Step 3: Verify forcing and regularity. The exponential term is annihilated by $(D+2)^3$; applying it to $r/8 - 3/16$ gives r. Also $g(r) = r^4/24 + O(r^5)$ as $r \downarrow 0$. Consequently y, y', y'', y''' match zero at switching, while $y^{(4)}(1+) - y^{(4)}(1-) = 1$. The solution is C^3 but not C^4; the original equation also holds at $t = 1$.

Step 4: Establish shape and asymptote. Convolution gives $g(r) = \int_0^r (r-v)v^2 e^{-2v}/2\, dv > 0$ for $r > 0$. The exponential correction tends to zero, so the asymptote is $y = (t-1)/8 - 3/16$, approached from above. It is drawn only for $t \ge 1$.



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Question 4

Consider the causal, distributional equation

$$y^{(4)} + 2y''' + y'' = 3\delta(t-1) + \delta'(t-1).$$

The response is zero for $t < 0$, with no impulse at the origin. Here δ' is the distributional derivative of δ, and $[v]_1 = v(1+) - v(1-)$ denotes a jump.

Tasks

1. Find the transformed response, using the zero prehistory and both delayed distribution transforms.
2. First invert $H(s) = 1/[s^2(s+1)^2]$, then use differentiation to obtain a real-time formula for y.
3. Derive all four jumps $[y]_1, [y']_1, [y'']_1, [y''']_1$ by matching the coefficients of distributional derivatives. Verify them from your formula.
4. Determine the motion before the impulse, the eventual linear asymptote and the smoothness at switching. Explain why assigning an isolated value at $t = 1$ cannot remove the derivative jump. Sketch the response and its asymptote.

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Question 4 – Solution

Strategy. Separate the impulse kernel from the input derivative, then match distributions rather than imposing false continuity conditions.

Step 1: Transform the distributions. For positive delay, $\mathcal{L}\delta(t-1) = e^{-s}$ and $\mathcal{L}\delta'(t-1) = se^{-s}$. Zero prehistory therefore gives

$$Y = e^{-s}(s+3)H(s), \qquad H(s) = \frac{1}{s^2(s+1)^2}, \qquad \operatorname{Re} s > 0.$$

Step 2: Invert the kernel. Since $H = -2/s + 1/s^2 + 2/(s+1) + 1/(s+1)^2$, $h(r) = r - 2 + (r+2)e^{-r}$ and $h(0) = 0$. Thus sH transforms h', without an additional origin impulse. For $r = t-1$,

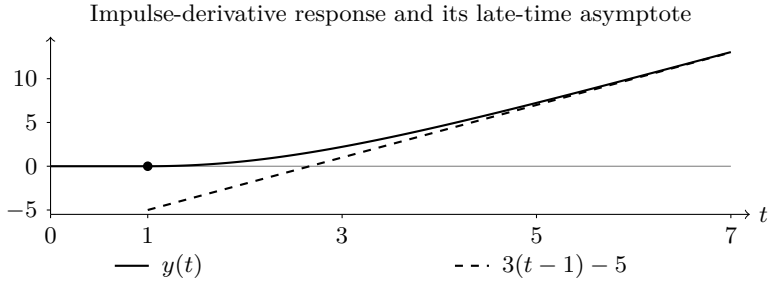
$$\boxed{y(t) = u(t-1)k(r), \qquad k(r) = 3r - 5 + (2r+5)e^{-r}.}$$

Step 3: Match every singular coefficient. Put $J_j = [y^{(j)}]_1$. The coefficients of $\delta''', \delta'', \delta', \delta$ in the left side are, respectively,

$$J_0, \quad J_1 + 2J_0, \quad J_2 + 2J_1 + J_0, \quad J_3 + 2J_2 + J_1.$$

Matching $0, 0, 1, 3$ gives $\boxed{(J_0, J_1, J_2, J_3) = (0, 0, 1, 1)}$. Directly, $(k, k', k'', k''')(0) = (0, 0, 1, 1)$; away from $r = 0$, $D^2(D+1)^2 k = 0$, completing the distributional verification.

Step 4: Interpret the motion. The response vanishes before $t = 1$ and approaches $3(t-1)-5$ afterward. It is C^1, but its one-sided second derivatives differ. Changing a function at one point cannot change those one-sided limits or its distribution.



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Question 5

An unstable homogeneous mode may or may not appear in a forced response. For real parameters a, b, consider

$$y''' + y'' - 2y' = e^{-t}, \qquad y(0) = 0, \quad y'(0) = a, \quad y''(0) = b, \qquad t \geq 0.$$

Tasks

1. Derive $Y(s)$ and identify all candidate poles before cancellation.
2. Find the necessary and sufficient relation between a, b for a bounded response. Give a complete inverse transform that proves both necessity and sufficiency.
3. Among the bounded responses, find the unique initial pair for which $y(t) \to 0$. Compute that solution and verify its equation and initial data.
4. An experiment perturbs only b from this special pair to $b + \varepsilon$. Find the exact response error. Explain how an arbitrarily small nonzero error can invalidate boundedness, and state precisely when a final-value calculation is justified.

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Question 5 – Solution

Strategy. Inspect the residue of the positive pole before using a final-value theorem.

Step 1: Keep the initial-data numerator. Transformation yields

$$P(s)Y = a(s+1) + b + \frac{1}{s+1}, \qquad P(s) = s(s-1)(s+2).$$

The candidate poles are $1, 0, -1, -2$; $\operatorname{Re} s > 1$ is a common convergence half-plane before any cancellation.

Step 2: Classify bounded responses. Residues give

$$y(t) = -\frac{a+b+1}{2} + \frac{2a+b+1/2}{3}e^{t} + \frac{1}{2}e^{-t} + \frac{-a+b-1}{6}e^{-2t}.$$

Only e^t is unbounded, and no other mode can cancel it asymptotically. Therefore

$$\boxed{y \text{ is bounded } \iff b = -2a - \tfrac{1}{2}.}$$

For this line of data, $\lim y = (a - 1/2)/2$.

Step 3: Remove the constant mode as well. Zero limit requires $a = 1/2$, $b = -3/2$, giving

$$\boxed{y_*(t) = \tfrac{1}{2}(e^{-t} - e^{-2t}).}$$

Its first three initial values are $(0, 1/2, -3/2)$. Since $P(-1) = 2$ and $P(-2) = 0$, $P(D)y_* = e^{-t}$, directly checking the equation. Its transform converges for $\operatorname{Re} s > -1$.

Step 4: Quantify sensitivity and theorem use. A change $b \mapsto b + \varepsilon$ gives

$$\boxed{y - y_* = \varepsilon\left(-\frac{1}{2} + \frac{e^{t}}{3} + \frac{e^{-2t}}{6}\right).}$$

This error has initial state $(0, 0, \varepsilon)$ and is asymptotic to $\varepsilon e^t/3$. Every nonzero perturbation is eventually unbounded. Only after the positive pole cancels do all poles of sY lie strictly left of the imaginary axis; then $\lim y = \lim_{s\to 0} sY$ is valid. For unstable data, the finite algebraic value at $s = 0$ does not represent a time limit.

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Question 6

A variable coefficient makes the transform satisfy a differential equation. Seek a real-analytic solution at $t = 0$ of

$$ty''' + 3y'' + ty' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0,$$

for $t \geq 0$, with a Laplace transform. A third initial value is not specified.

Tasks

1. Determine the only compatible value of $y''(0)$. Explain why the usual three-free-data theorem for a regular third-order equation does not apply at the origin.
2. Use $\mathcal{L}\{tf(t)\} = -F'(s)$ to derive and solve an equation for $Y(s)$ on real $s > 0$. Fix the integration constant from the behavior as $s \to \infty$.
3. Invert the result by writing it as an integral of elementary cosine transforms. Extend the answer to $t = 0$ and verify the original differential equation.
4. Derive the power-series coefficient recurrence and prove uniqueness among analytic solutions with the prescribed data. Check the first three terms of the large-s expansion of Y against the initial derivatives.

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Question 6 – Solution

Strategy. Use the singular-endpoint constraint before transforming; the transform's boundary condition replaces its integration constant.

Step 1: Enforce compatibility. At $t = 0$, continuity of the equation gives $3y''(0) + 1 = 0$, so $\boxed{y''(0) = -1/3}$. The leading coefficient t vanishes at the origin; normalizing by it would produce singular coefficients there.

Step 2: Transform carefully. Writing $c = y''(0)$, the four transforms are

$$-\frac{d}{ds}(s^3Y - s^2 - c), \quad 3(s^2Y - s), \quad -\frac{d}{ds}(sY - 1), \quad Y.$$

Their sum is $-(s^3 + s)Y' - s = 0$. Thus $Y' = -1/(1 + s^2)$ for $s > 0$. Because the transform of a continuous exponential-order function tends to zero as $s \to \infty$, the constant is fixed:

$$\boxed{Y(s) = \int_s^\infty \frac{dv}{1 + v^2} = \arctan(1/s), \qquad s > 0.}$$

Step 3: Invert an integral representation. Since $Y(s) = \int_0^1 s/(s^2 + u^2)\, du$, absolute convergence for $\operatorname{Re} s > 0$ permits interchange of integrals. Therefore

$$\boxed{y(t) = \int_0^1 \cos(ut)\, du = \frac{\sin t}{t} \ (t > 0), \qquad y(0) = 1.}$$

For $t > 0$, this function satisfies $ty'' + 2y' + ty = 0$. Differentiating gives the required third-order equation; its analytic extension satisfies the equation at zero too.

Step 4: Prove analytic uniqueness. For $y = \sum_{n \geq 0} c_n t^n$, coefficient matching gives

$$(n + 1)\big((n + 2)(n + 3)c_{n+2} + c_n\big) = 0 \quad (n \geq 0).$$

The seeds $c_0 = 1, c_1 = 0$ uniquely give $c_{2k} = (-1)^k/(2k + 1)!$ and $c_{2k+1} = 0$, an everywhere convergent series. Ordinary uniqueness on $t > 0$ extends local uniqueness to the whole positive axis. Finally $Y = s^{-1} - (3s^3)^{-1} + (5s^5)^{-1} + O(s^{-7})$, agreeing with $y(0) = 1$, $y''(0) = -1/3$, and $y^{(4)}(0) = 1/5$ in $Y \sim \sum y^{(j)}(0)/s^{j+1}$.

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Question 7

A third-order equation has an exponentially fading memory:

$$(D+1)^3 y(t) = \int_0^t e^{-(t-v)} y(v)\, dv + e^{-t}, \qquad (y, y', y'')(0) = (0, 0, 0).$$

All functions are defined for $t \geq 0$.

Tasks

1. Transform the convolution equation and solve algebraically for $Y(s)$. Factor the resulting fourth-degree denominator.
2. Invert the transform in a real form, displaying a useful decomposition in $z = s + 1$.
3. Derive an equivalent ordinary fourth-order initial-value problem. Explain why differentiation requires an additional initial condition, and prove that your fourth-order solution really satisfies the original memory equation.
4. Determine the long-time limit and justify any final-value theorem used. Explain how the feedback changes the zero-state response from what it would be if the memory integral were omitted.

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Question 7 – Solution

Strategy. Convert memory into a rational factor, then guard against the extra solutions that differentiation can introduce.

Step 1: Transform and factor. The convolution theorem gives

$$(s+1)^3 Y = \frac{Y}{s+1} + \frac{1}{s+1}, \qquad \boxed{Y = \frac{1}{(s+1)^4 - 1}.}$$

The denominator is $s(s+2)((s+1)^2+1)$, with roots $0, -2, -1 \pm i$. A sufficient convergence half-plane is $\operatorname{Re} s > 0$.

Step 2: Invert real quadratic factors. For $z = s+1$,

$$\frac{1}{z^4-1} = \frac{1}{2}\left(\frac{1}{z^2-1} - \frac{1}{z^2+1}\right), \qquad \boxed{y(t) = \tfrac{1}{2}e^{-t}(\sinh t - \sin t).}$$

Equivalently, $y = (1 - e^{-2t})/4 - e^{-t}\sin t/2$.

Step 3: Prove equivalence, including the lost condition. If $m(t) = \int_0^t e^{-(t-v)}y(v)\,dv$, then $(D+1)m = y$ and $m(0) = 0$. Applying $D+1$ to the original equation gives

$$((D+1)^4 - 1)y = 0, \qquad (y, y', y'', y''')(0) = (0, 0, 0, 1).$$

The fourth value follows from the original equation at zero. The displayed solution has $y = t^3/6 + O(t^4)$ and lies in the four characteristic modes, so it satisfies this fourth-order IVP. Conversely its residual $R = (D+1)^3 y - m - e^{-t}$ obeys $R' + R = 0$ and $R(0) = 0$. Hence $R \equiv 0$, proving equivalence rather than just necessity.

Step 4: Interpret the surviving mode. The explicit formula gives $\boxed{\lim_{t\to\infty} y(t) = 1/4}$. This also equals $\lim_{s\to 0} sY$; after cancellation, the poles of sY are $-2, -1 \pm i$, all strictly stable. Without memory, the transform is $1/(s+1)^4$ and the response is $t^3 e^{-t}/6 \to 0$. Feedback introduces a zero characteristic root and thus a nonzero constant component, although the external forcing itself decays.

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Question 8

A stable third-order filter receives identical unit impulses at times $T, 2T, 3T, \ldots$, where $T > 0$:

$$(D+1)^3 y = \sum_{n=1}^{\infty} \delta(t - nT).$$

Assume zero prehistory and zero initial state. Interpret the equation in the sense of distributions; there is no impulse at $t = 0$.

Tasks

1. Find $Y(s)$, stating a convergence half-plane. Give the exact finite-sum response at any finite time.
2. For $t = NT + x$ with $0 \le x < T$, find the limiting periodic profile as $N \to \infty$. Sum it explicitly using $q = e^{-T}$.
3. Determine the continuity and derivative jumps across each impulse, including the join in the limiting profile. Compute the mean of the limiting profile over a period.
4. Decide whether $y(t)$ has a pointwise limit as $t \to \infty$. Explain the failure or validity of a final-value argument, and sketch the transient and limiting profile for $T = 2$.

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Question 8 – Solution

Strategy. Sum delayed impulse kernels and distinguish a periodic limiting profile from a constant final value.

Step 1: Sum the transformed impulses. For $\operatorname{Re} s > 0$,

$$Y = \frac{e^{-sT}}{(1-e^{-sT})(s+1)^3}, \qquad y(t) = \sum_{1 \le n \le \lfloor t/T \rfloor} h(t-nT), \quad h(v) = \tfrac{1}{2} v^2 e^{-v}.$$

The sum is empty before T; including the current impulse adds $h(0) = 0$.

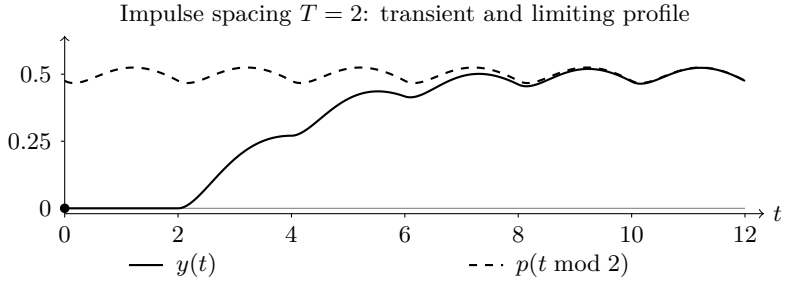
Step 2: Sum the limiting profile. At $t = NT + x$ the sum is $\sum_{k=0}^{N-1} h(x + kT)$. Geometric-series identities give

$$\boxed{p(x) = \frac{e^{-x}}{2}\left[\frac{x^2}{1-q} + \frac{2xTq}{(1-q)^2} + \frac{T^2 q(1+q)}{(1-q)^3}\right], \quad q = e^{-T}.}$$

The omitted tail is uniformly $O(N^2 q^N)$ for $0 \le x \le T$ and fixed T, so $y(NT + x) \to p(x)$ uniformly on a period.

Step 3: Check joins and mean. Since $(h, h', h'')(0) = (0, 0, 1)$, y, y' are continuous and y'' jumps upward by one. Reindexing the convergent profile series gives the same periodic joins: $p(0) = p(T-)$, $p'(0) = p'(T-)$, $p''(0) - p''(T-) = 1$. Also $\int_0^T p(x)\,dx = \int_0^\infty h(v)\,dv = 1$, so its mean is $1/T$.

Step 4: Reject a constant final value. The profile is nonconstant: on each open gap $(D+1)^3 p = 0$, so a constant profile would be zero, contradicting its integral. Distinct phases therefore have distinct subsequential limits. Poles at $s = 2\pi i k/T$, $k \neq 0$, remain in sY; the formal limit $\lim_{s \to 0} sY = 1/T$ is a mean, not a final value.



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Question 9

An experiment supplies the exact transform

$$Y(s) = \frac{s^4 + 3s^3 + 7s^2 + 11s + 18}{(s+3)(s^2+1)(s^2+4)}$$

of a solution to

$$y^{(4)} + 5y'' + 4y = Ae^{-3t}, \qquad t \geq 0.$$

The forcing amplitude A and all four initial values are unknown real numbers.

Tasks

1. Recover A and $(y, y', y'', y''')(0)$ by polynomial division of $(s^2+1)(s^2+4)Y$. Prove that this recovery is unique.
2. Invert Y completely using real sine, cosine and exponential terms.
3. Verify the recovered forcing and all four initial values directly from your answer, and state an absolute-convergence half-plane.
4. Find the monic constant-coefficient homogeneous differential operator of least order that annihilates this particular response. Explain why its order differs from that of the forced equation, and prove minimality.

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Question 9 – Solution

Strategy. Separate the polynomial encoding of initial data from the proper rational forcing term.

Step 1: Recover data uniquely. Write $P(s) = s^4 + 5s^2 + 4$. Polynomial division gives

$$P(s)Y = s^3 + 7s - 10 + \frac{48}{s+3}.$$

For initial values $c_j = y^{(j)}(0)$, transformation of the equation gives

$$PY = c_0 s^3 + c_1 s^2 + (c_2 + 5c_0)s + (c_3 + 5c_1) + \frac{A}{s+3}.$$

Thus $\boxed{A = 48, \quad (c_0, c_1, c_2, c_3) = (1, 0, 2, -10)}$. Polynomial division has a unique polynomial part and remainder; the displayed triangular relation then determines every c_j uniquely.

Step 2: Resolve the real modes. Real partial fractions yield

$$Y = \frac{(2/5)s + 22/15}{s^2+1} + \frac{(3/13)s - 14/39}{s^2+4} + \frac{24/65}{s+3},$$

so

$$\boxed{y = \frac{2}{5}\cos t + \frac{22}{15}\sin t + \frac{3}{13}\cos(2t) - \frac{7}{39}\sin(2t) + \frac{24}{65}e^{-3t}.}$$

Step 3: Check the recovered experiment. The oscillatory terms are annihilated by $P(D)$ and $P(-3) = 130$, so the exponential contributes $130(24/65)e^{-3t} = 48e^{-3t}$. At zero the four derivatives are

$$\begin{aligned} y(0) &= 2/5 + 3/13 + 24/65 = 1, \\ y'(0) &= 22/15 - 14/39 - 72/65 = 0, \\ y''(0) &= -2/5 - 12/13 + 216/65 = 2, \\ y'''(0) &= -22/15 + 56/39 - 648/65 = -10. \end{aligned}$$

Absolute convergence holds for $\operatorname{Re} s > 0$.

Step 4: Prove the minimum homogeneous order. All five complex modes at $i, -i, 2i, -2i, -3$ have nonzero coefficients. Their linear independence forces every annihilating polynomial to vanish at all five distinct roots. Hence the least monic operator is $\boxed{(D+3)(D^2+1)(D^2+4)}$, of order five. The fourth-order equation leaves the exponential as a nonzero forcing; annihilating that forcing adds the fifth factor.

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Question 10

For $\varepsilon \geq 0$, let h_ε be the causal impulse response of

$$(D+1)^2(D+1+\varepsilon)^2 h_\varepsilon = \delta(t),$$

with zero prehistory. The impulse is at the origin; use the causal transform convention $\mathcal{L}\delta(t) = 1$. Ordinary formulas below describe $t \geq 0$.

Tasks

1. Find the transform and invert it for $\varepsilon > 0$ by partial fractions in $z = s + 1$.
2. Derive a convolution formula with a nonnegative integrand. Use it to obtain h_0 and prove $h_\varepsilon \to h_0$ without subtracting divergent coefficients.
3. Prove two-sided bounds for $h_\varepsilon(t)$ and an explicit bound of the form $\sup_{t\geq 0} |h_\varepsilon(t) - h_0(t)| \leq C\varepsilon$ with a numerical constant C.
4. Explain the change in pole multiplicities and verify the four right-hand initial derivatives required by a unit impulse. Discuss reliable evaluation for small εt, and sketch the kernels for $\varepsilon = 0, 1/2, 2$.

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Question 10 – Solution

Strategy. Use partial fractions for a closed form and convolution for a stable coalescing-pole limit.

Step 1: Invert distinct repeated poles. For $\varepsilon > 0$, $H_\varepsilon = 1/[z^2(z+\varepsilon)^2]$ and

$$H_\varepsilon = \frac{1}{\varepsilon^2}\left(\frac{1}{z^2} + \frac{1}{(z+\varepsilon)^2}\right) + \frac{2}{\varepsilon^3}\left(\frac{1}{z+\varepsilon} - \frac{1}{z}\right).$$

Hence $h_\varepsilon = e^{-t}\{t(1+e^{-\varepsilon t})/\varepsilon^2 - 2(1-e^{-\varepsilon t})/\varepsilon^3\}$.

Step 2: Take the limit through convolution. Convolving te^{-t} with $te^{-(1+\varepsilon)t}$ gives

$$\boxed{h_\varepsilon(t) = e^{-t}\int_0^t v(t-v)e^{-\varepsilon v}\,dv, \qquad h_0(t) = \frac{t^3e^{-t}}{6}.}$$

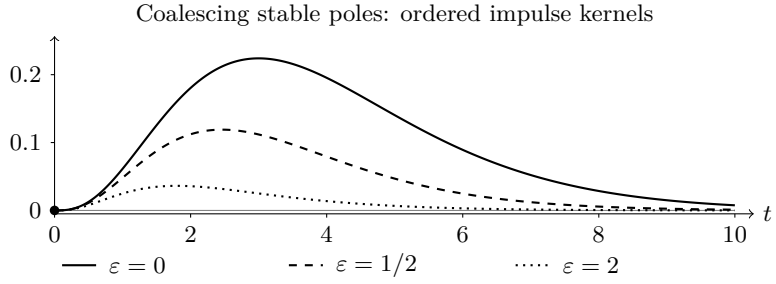
For fixed t, dominated convergence justifies the limit, including $t = 0$.

Step 3: Prove global error bounds. Bounding the integrand and using $1 - e^{-\varepsilon v} \le \varepsilon v$ yields

$$e^{-(1+\varepsilon)t}\frac{t^3}{6} \le h_\varepsilon(t) \le e^{-t}\frac{t^3}{6}, \qquad 0 \le h_0 - h_\varepsilon \le \frac{\varepsilon t^4 e^{-t}}{12}.$$

The maximum of t^4e^{-t} is $256e^{-4}$ at $t = 4$. Thus $\boxed{\sup_{t\ge 0}|h_\varepsilon - h_0| \le 64\varepsilon/(3e^4)}$.

Step 4: Check poles, impulses and evaluation. Two double poles merge into a quadruple pole at -1; absolute convergence holds for $\operatorname{Re} s > -1$. Convolution gives $h_\varepsilon = t^3/6 + O(t^4)$, so its right-hand jet is $(0,0,0,1)$, producing exactly a unit δ. For small εt, use the integral or its exponential series; the closed form subtracts large nearly equal terms and can lose significant digits.



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Question 1

Consider the variable-coefficient initial-value problem on the real line,

$$(1+t^2)y''' - 2ty'' + 2y' = 0, \qquad (y, y', y'')(0) = (1, 0, 2).$$

A state representation must preserve the scalar equation's leading coefficient and all three initial data.

Tasks

1. Set $X = (y, y', y'')^T$ and derive $X' = A(t)X$. State the initial vector and the largest real interval on which the conversion is regular.
2. Find the solution by testing a polynomial of degree at most two. Verify every component of the system and explain why this trajectory is unique.
3. Let $\Phi' = A\Phi$, $\Phi(0) = I$. Determine $\det \Phi(t)$ without finding the full matrix. Interpret the result as a change in three-dimensional state volume.
4. Decide whether the increasing volume factor forces every individual trajectory to grow. Give an explicit counterexample or proof, and sketch all three components of the prescribed trajectory on $[-2, 2]$.

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Question 1 – Solution

Strategy. Normalize first, and distinguish the evolution of a family of states from that of one trajectory.

Step 1: Preserve the derivative chain. Since $1 + t^2 > 0$ for every real t,

$$\boxed{A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2/(1+t^2) & 2t/(1+t^2) \end{pmatrix}, \qquad X(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.}$$

The conversion is regular on $\mathbb{R}$, and its first two rows enforce $X_2 = X_1'$, $X_3 = X_1''$, so it is reversible.

Step 2: Verify the complete state. The initial data select $y = 1 + t^2$ among quadratic polynomials. Its derivatives are $2t, 2, 0$, and the scalar residual is $-4t + 4t = 0$. Thus

$$\boxed{X(t) = (1 + t^2,\, 2t,\, 2)^T.}$$

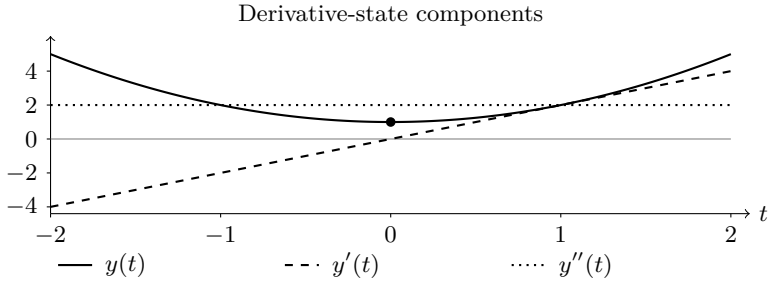
The system's last row gives $(-4t + 4t)/(1 + t^2) = 0 = X_3'$; the other rows and initial vector also agree. Continuity of $A(t)$ gives uniqueness on $\mathbb{R}$.

Step 3: Compute the state-volume factor. Liouville's formula gives

$$\det \Phi(t) = \exp\left(\int_0^t \frac{2v}{1+v^2}\, dv \right) = \boxed{1 + t^2}.$$

An initial parallelepiped of states has its oriented volume multiplied by this positive factor. The factor increases with $|t|$, not with t on the entire line.

Step 4: Test the individual-growth claim. For any constant c, $X = (c, 0, 0)^T$ is a constant solution. Hence expansion of state volume does not force every trajectory to grow. The prescribed components are shown below; their different behaviors also caution against that inference.



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Question 2

A scalar measurement of a three-state system is

$$X' = AX, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \qquad y = X_1 + 2X_2.$$

The measured initial derivatives are $y(0) = 1$, $y'(0) = 0$, $y''(0) = -12$. All coefficients are constant and $t \in \mathbb{R}$.

Tasks

1. Form the matrix O such that $(y, y', y'')^T = OX$. Determine whether the three measured derivatives uniquely determine the state.
2. Derive a scalar third-order equation for y and explicit reconstruction formulas for all three state components in terms of y, y', y''.
3. Recover $X(0)$ and find the measured response $y(t)$ explicitly.
4. Prove that every solution of your scalar equation reconstructs a solution of the original system. Verify the recovered trajectory and explain why differentiating the measurement has lost no state information.

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Question 2 – Solution

Strategy. Invert the map from state coordinates to the measurement's derivative coordinates.

Step 1: Test the measurement map. With $c = (1, 2, 0)$, the rows of O are c, cA, cA^2. Hence

$$O = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -12 & -22 & -11 \end{pmatrix}, \qquad \det O = -15 \neq 0.$$

The derivative triple determines the state uniquely.

Step 2: Convert and reconstruct. The identity $A^3 + 6A^2 + 11A + 6I = 0$ gives

$$\boxed{y''' + 6y'' + 11y' + 6y = 0.}$$

Solving $OX = (y, y', y'')^T$ gives

$$\boxed{\begin{aligned} X_1 &= (-33y - 22y' - 4y'')/15, \\ X_2 &= (24y + 11y' + 2y'')/15, \\ X_3 &= (-12y + 2y' - y'')/15. \end{aligned}}$$

Step 3: Solve the recovered IVP. The measured data give $X(0) = (1, 0, 0)^T$. The scalar roots are $-1, -2, -3$; imposing the measured derivative triple yields

$$\boxed{y = -3e^{-t} + 9e^{-2t} - 5e^{-3t}.}$$

The state is $X = (f, f', f'')^T$, where $f = 3e^{-t} - 3e^{-2t} + e^{-3t}$. In particular, $f + 2f' = y$ and $(f, f', f'')(0) = (1, 0, 0)$.

Step 4: Prove full equivalence. If $Z = (y, y', y'')^T$, then $Z' = BZ$ with last row of B equal to $(-6, -11, -6)$. Direct multiplication gives $OA = BO$, so for any scalar solution, $X = O^{-1}Z$ satisfies $X' = O^{-1}BZ = AX$. Its measurement is y by the first row of O. Conversely every state solution produces this scalar equation. The recovered f satisfies $f''' + 6f'' + 11f' + 6f = 0$, checking the original last row directly. Invertibility of O proves that no hidden state was discarded.

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Question 3

Only one coordinate of the following system is observed:

$$X' = AX, \qquad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad y = X_1, \qquad t \geq 0.$$

A proposed scalar model is obtained by replacing the characteristic polynomial of A with a differential operator.

Tasks

1. Derive the lowest-order monic constant-coefficient homogeneous scalar equation satisfied by every possible measurement y. Prove that this order is minimal.
2. For a prescribed measurement, reconstruct all compatible states. Determine exactly which initial states are bounded on $[0, \infty)$.
3. Write the third-order scalar equation suggested by the characteristic polynomial. Identify its extra scalar mode, and find the initial constraint that excludes that mode.
4. Decide whether an arbitrarily long, exact record of y can certify boundedness of the full state. Exhibit two states with identical measurements and different boundedness behavior.

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Question 3 – Solution

Strategy. Separate the measured Jordan block from the invisible unstable coordinate.

Step 1: Find the true measurement equation. From $y' = -y + X_2$ and $X_2' = -X_2$,

$$\boxed{y'' + 2y' + y = 0, \qquad y = (a + bt)e^{-t}.}$$

Both e^{-t} and te^{-t} occur as measurements and are independent. Any constant-coefficient annihilator must have a double root at -1, so order two is minimal.

Step 2: Recover the visible and hidden states. Every compatible state is

$$\boxed{X(t) = ((a + bt)e^{-t},\, be^{-t},\, ce^{2t})^T, \qquad c \in \mathbb{R}.}$$

The first two coordinates always decay. Therefore the full state is bounded if and only if $c = X_3(0) = 0$. The measurement does not determine c.

Step 3: Remove the spurious scalar mode. The characteristic polynomial gives

$$(D - 2)(D + 1)^2 y = 0, \quad \text{equivalently} \quad y''' - 3y' - 2y = 0.$$

Its general solution adds de^{2t} to the actual measurement family. Put $r = (D + 1)^2 y$. Then $r' = 2r$ and $r = 9de^{2t}$, so the precise exclusion condition is

$$\boxed{y''(0) + 2y'(0) + y(0) = 0.}$$

The characteristic-polynomial equation is necessary, but without this constraint it is not sufficient for a scalar function to be a measurement.

Step 4: Test what observation can certify. The initial states $(1, 0, 0)^T$ and $(1, 0, 1)^T$ both produce $y = e^{-t}$ forever. Their full trajectories are $(e^{-t}, 0, 0)^T$ and $(e^{-t}, 0, e^{2t})^T$; one is bounded and the other is not. Thus even an exact infinite measurement record cannot certify boundedness without additional information about $X_3(0)$.

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Question 4

A smooth system on the entire real line is

$$x' = z, \qquad z' = tw, \qquad w' = 0.$$

Eliminating z, w appears to produce a third-order scalar equation for x. The elimination must be examined at $t = 0$.

Tasks

1. Eliminate z, w for $t \neq 0$. Explain why the resulting scalar equation is singular at the origin although the original system is regular there.
2. Solve the original system for arbitrary data $(x, z, w)(0) = (a, b, c)$. Determine which derivatives of x at zero encode these three constants.
3. Classify all $C^3(\mathbb{R})$ solutions of the scalar equation. Prove that each reconstructs a unique smooth system solution, explicitly treating the origin.
4. Decide whether $x(0), x'(0), x''(0)$ alone determine a unique scalar solution. Give a one-parameter counterexample if needed, and sketch three members sharing the zero derivative triple on $[-1, 1]$.

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Question 4 – Solution

Strategy. Retain the state coordinate that division by t conceals at the singular point.

Step 1: Eliminate on the regular subintervals. Since $x'' = tw$, for $t \neq 0$ we have $w = x''/t$. Differentiating gives

$$\boxed{tx''' - x'' = 0.}$$

The leading coefficient vanishes at zero. This singularity came from the chosen scalar coordinate; the original system matrix is continuous everywhere.

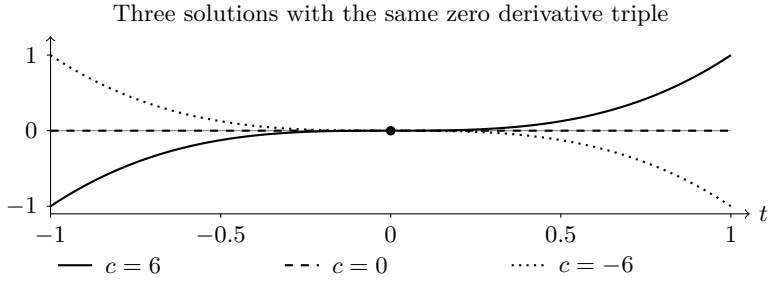
Step 2: Solve before dividing. Integrating successively gives

$$\boxed{w = c, \qquad z = b + ct^2/2, \qquad x = a + bt + ct^3/6.}$$

Thus $(x, x', x'')(0) = (a, b, 0)$, while $x'''(0) = c$. The missing state coordinate is encoded in the third derivative, not in the usual second derivative.

Step 3: Check global equivalence. On either side of zero, $(x''/t)' = 0$, so $x = a_\pm + b_\pm t + c_\pm t^3/6$. Continuity of x, x' matches $a_\pm, b_\pm$, and continuity of x''' matches $c_\pm$. All C^3 solutions therefore have the single global polynomial form above. Reconstruct $z = x'$ and $w = x''/t$ off zero, setting $w(0) = x'''(0)$. These are smooth and satisfy all three system equations. At zero, the scalar equation itself requires $x''(0) = 0$.

Step 4: Explain the apparent nonuniqueness. The family $x = ct^3/6$ has $(x, x', x'')(0) = (0, 0, 0)$ for every c. It corresponds to distinct system data $(0, 0, c)$, so it does not contradict uniqueness for the regular system. The plotted members use $c = -6, 0, 6$.



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Question 5

Two coupled displacements satisfy

$$x'' + 2x - z = 0, \qquad z'' + 2z - x = 0,$$

with $(x, x', z, z')(0) = (1, 0, 0, 0)$. Time is measured in units that make the coefficients dimensionless.

Tasks

1. Write a four-state first-order system using the order (x, x', z, z'). Eliminate z to obtain a fourth-order scalar equation for x and translate all four initial values.
2. Introduce $u = x + z$, $v = x - z$. Solve the resulting independent equations and recover x, z.
3. Prove that every solution of the scalar fourth-order equation reconstructs a unique solution of the coupled equations. Verify the prescribed solution and its initial data.
4. Derive a positive conserved energy, express it in u, v, and use it to prove boundedness. Sketch both displacements for $0 \le t \le 12$ and explain how two different frequencies appear.

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Question 5 – Solution

Strategy. Compare the derivative-state, scalar-elimination and normal-mode descriptions of the same four-dimensional motion.

Step 1: Convert and transfer the initial state. For $X = (x, x', z, z')^T$,

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} X.$$

Since $z = x'' + 2x$, substitution into the second equation gives

$$\boxed{x^{(4)} + 4x'' + 3x = 0, \qquad (x, x', x'', x''')(0) = (1, 0, -2, 0).}$$

Step 2: Solve the normal modes. Adding and subtracting give $u'' + u = 0$, $v'' + 3v = 0$, with initial displacements one and velocities zero. Thus

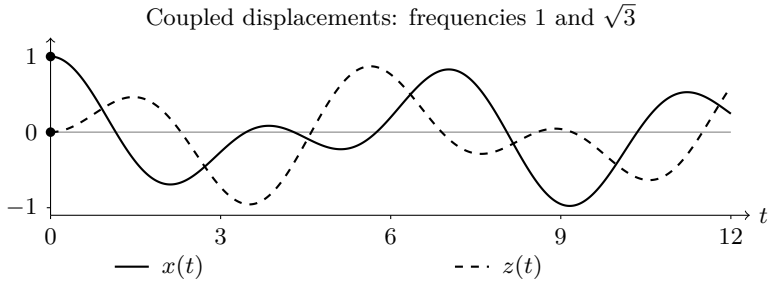
$$\boxed{x = \tfrac{1}{2}(\cos t + \cos(\sqrt{3}t)), \qquad z = \tfrac{1}{2}(\cos t - \cos(\sqrt{3}t)).}$$

Step 3: Prove reversibility. For any scalar solution, define $z = x'' + 2x$. The first coupled equation is then an identity, and the second has residual $x^{(4)} + 4x'' + 3x = 0$. This formula also proves uniqueness of z. The displayed cosines satisfy the scalar factors $(D^2 + 1)(D^2 + 3)$ and give $z(0) = z'(0) = 0$ as required.

Step 4: Check energy and boundedness. Multiplying the equations by x', z' and adding proves conservation of

$$E = \tfrac{1}{2}(x'^2 + z'^2) + x^2 + z^2 - xz = \tfrac{1}{4}(u'^2 + u^2 + v'^2 + 3v^2) = 1.$$

Since $x^2 + z^2 - xz \geq (x^2 + z^2)/2$, both displacements are bounded; the explicit formulas sharpen this to $|x|, |z| \leq 1$. Each displacement mixes the frequencies 1 and $\sqrt{3}$.



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Question 6

Let $D = d/dt$. Ordered differential factors act from right to left in

$$(D+1)(D-t)(D-1)y = 0, \qquad (y, y', y'')(0) = (0, 0, 1).$$

Here $(D-t)f = f' - tf$. Multiplication by t does not commute with differentiation.

Tasks

1. Define $q_1 = y$, $q_2 = (D-1)y$, $q_3 = (D-t)q_2$. Derive the first-order system for q and its initial vector.
2. Expand the scalar third-order equation and show that the change between (y, y', y'') and q is invertible for every real t.
3. Solve the triangular system by successive integrating factors. Give an explicit nested-integral formula for y and verify the initial data and equation through the state relations.
4. Compare the given operator with $(D-t)(D+1)(D-1)$. Compute their difference, and exhibit a function that solves the reordered equation but not the original one.

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Question 6 – Solution

Strategy. Keep the factors in their stated order and regard intermediate derivatives as state coordinates.

Step 1: Form the ordered cascade. The definitions give

$$\boxed{q_1' = q_1 + q_2, \qquad q_2' = tq_2 + q_3, \qquad q_3' = -q_3, \qquad q(0) = (0,0,1)^T.}$$

In particular, $q_3 = y'' - (t+1)y' + ty$; using only y'' as this coordinate would lose the variable-coefficient terms.

Step 2: Expand and invert the coordinate map. Differentiating the expression for q_3 and adding q_3 yields

$$\boxed{y''' - ty'' - 2y' + (t+1)y = 0.}$$

The coordinate matrix is

$$q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ t & -(t+1) & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \qquad \det = 1.$$

Its inverse relations are $y = q_1$, $y' = q_1 + q_2$ and $y'' = q_1 + (t+1)q_2 + q_3$, so the conversion is globally reversible.

Step 3: Integrate in triangular order. Successive integrating factors give

$$q_3 = e^{-t}, \qquad q_2 = e^{t^2/2} \int_0^t e^{-v - v^2/2}\, dv,$$

$$\boxed{y(t) = e^t \int_0^t e^{-u + u^2/2} \left(\int_0^u e^{-v - v^2/2}\, dv \right) du.}$$

These oriented integrals apply to every real t. Their derivatives give the three cascade equations. At zero the state is $(0,0,1)$, and the inverse relations give $(y, y', y'') = (0,0,1)$. Finally $q_3' + q_3 = 0$ verifies the original ordered product directly.

Step 4: Quantify the order error. The commutator $(D+1)(D-t) - (D-t)(D+1)$ is multiplication by -1. Thus the original operator minus the reordered one equals $-(D-1)$. The reordered operator is $D^3 - tD^2 - D + t$, whereas the original has the extra $-D + 1$. For $y = e^{-t}$ the reordered product vanishes, but the original gives $2e^{-t} \neq 0$. Reordering these factors changes the solution space.

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Question 7

A controlled third derivative moves a state from rest to rest in unit time:

$$y''' = u(t), \qquad (y, y', y'')(0) = (0, 0, 0), \qquad (y, y', y'')(1) = (1, 0, 0).$$

For square-integrable controls u on $[0, 1]$, interpret the derivative chain through its integral solution. The effort is $J(u) = \int_0^1 u(t)^2 \, dt$.

Tasks

1. Write the first-order state system and its transition matrix. Derive the three terminal constraints as integrals of u.
2. Find the unique quadratic control satisfying the terminal conditions. Integrate it to obtain the displacement.
3. Prove this quadratic is the unique minimum-effort control among all square-integrable admissible controls, up to equality almost everywhere. Compute the minimum effort.
4. Verify both endpoint states and explain why the minimization proof covers controls that are not polynomials. Sketch the displacement and $u/60$, explicitly labeling the scaling.

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Question 7 – Solution

Strategy. Express the terminal state as three moments, then use orthogonality to prove optimality over the full control class.

Step 1: Derive the terminal moments. For $X = (y, y', y'')^T$, $X' = NX + e_3 u$, where N has ones just above the diagonal and zeros elsewhere. Since $N^3 = 0$,

$$e^{Nr} = \begin{pmatrix} 1 & r & r^2/2 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \qquad X(1) = \int_0^1 \begin{pmatrix} (1-t)^2/2 \\ 1-t \\ 1 \end{pmatrix} u(t)\, dt.$$

The required three moments are $(1, 0, 0)^T$.

Step 2: Solve within the quadratic space. For $u = a + bt + ct^2$, the constraints are

$$a/6 + b/24 + c/60 = 1, \quad a/2 + b/6 + c/12 = 0, \quad a + b/2 + c/3 = 0.$$

Their solution is

$$\boxed{u_* = 60 - 360t + 360t^2, \qquad y_* = 10t^3 - 15t^4 + 6t^5.}$$

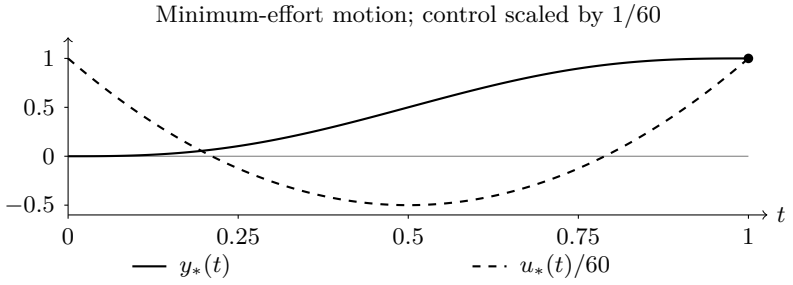
The moment kernels span all quadratics. A quadratic in the homogeneous null space is orthogonal to itself and hence zero, proving uniqueness here.

Step 3: Prove global minimum effort. For any admissible $u = u_* + v$, all three terminal moments of v vanish. Because u_* is in their span, $\int_0^1 u_* v = 0$. Therefore

$$\boxed{J(u) = J(u_*) + \int_0^1 v^2\, dt \geq J(u_*) = 720.}$$

Equality holds exactly when $v = 0$ almost everywhere. This argument only uses square integrability and moment constraints, not polynomial form for v.

Step 4: Verify the endpoint motion. We have $y_*' = 30t^2(1-t)^2$ and $y_*'' = 60t(1-t)(1-2t)$, so the two endpoint states are exactly $(0, 0, 0)$ and $(1, 0, 0)$; differentiating again gives u_*. The graph uses $u_*/60$ so the control and displacement are both readable.



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Question 8

The functions $1, t, e^t$ form a scalar basis for $y''' - y'' = 0$. For the derivative state $X = (y, y', y'')^T$, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A fundamental matrix need not equal the identity at the chosen base time.

Tasks

1. Form the fundamental matrix $F(t)$ from the given scalar basis and its derivatives. Check $F' = AF$ and compute its determinant.
2. Derive the transition matrix $\Phi(t, s) = F(t)F(s)^{-1}$ explicitly. Verify its normalization, composition law and determinant.
3. Use this transition matrix to solve $y''' - y'' = 1$ with $(y, y', y'')(1) = (0, 0, 0)$ for every real t. Treat the integral orientation when $t < 1$.
4. Explain why using $F(t)$ directly as the transition from time zero gives wrong initial data in general. Prove that replacing F by FC, for any constant invertible C, leaves the true transition matrix unchanged.

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Question 8 – Solution

Strategy. Normalize a scalar-derived fundamental matrix at the actual base time before transporting states or forcing.

Step 1: Lift the scalar basis. The derivative columns give

$$F(t) = \begin{pmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{pmatrix}, \qquad \det F = e^t \neq 0.$$

Differentiating the rows gives $F' = AF$, including the last row because $(e^t)' = e^t$. Thus F is fundamental on the entire real line.

Step 2: Normalize and compose. With $r = t - s$, direct multiplication gives

$$\boxed{\Phi(t,s) = \begin{pmatrix} 1 & r & e^r - 1 - r \\ 0 & 1 & e^r - 1 \\ 0 & 0 & e^r \end{pmatrix}.}$$

It equals I at $t = s$, and $\det \Phi = e^{t-s}$. The identity $F(t)F(s)^{-1}F(s)F(a)^{-1} = F(t)F(a)^{-1}$ proves $\Phi(t,s)\Phi(s,a) = \Phi(t,a)$ for all real base times.

Step 3: Transport the forcing in both directions. The zero initial vector at time one gives

$$X(t) = \int_1^t \Phi(t,v)e_3\,dv, \qquad \boxed{y(t) = e^{t-1} - 1 - (t-1) - \tfrac{1}{2}(t-1)^2.}$$

Indeed its first component integrates $e^{t-v} - 1 - (t-v)$. For $t < 1$ the integral is oriented: it is minus the integral from t to 1. Writing $r = t - 1$, we obtain $y' = e^r - 1 - r$, $y'' = e^r - 1$, $y''' = e^r$. These give the zero state at $t = 1$ and residual $y''' - y'' = 1$ on all of $\mathbb{R}$.

Step 4: Separate basis from transition. Here $F(0) \neq I$. For example, $F(t)e_3 = (e^t, e^t, e^t)^T$ starts at $(1,1,1)^T$, not at e_3. The correct expression is $F(t)F(0)^{-1}e_3$. For any constant invertible C,

$$(F(t)C)(F(s)C)^{-1} = F(t)CC^{-1}F(s)^{-1} = \Phi(t,s).$$

A basis change alters coordinates of solution columns, but not the transport of a specified physical state.

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Question 9

A fourth-order oscillator has a repeated quadratic factor:

$$y^{(4)} + 2y'' + y = 0, \qquad t \geq 0.$$

Introduce the state

$$W = (y,\ y',\ y'' + y,\ y''' + y')^T.$$

All four scalar initial derivatives are arbitrary real numbers.

Tasks

1. Derive the system for W and show that this coordinate change is invertible. Identify the autonomous oscillator inside the system.
2. Write the full solution in terms of $a = y(0)$, $b = y'(0)$, $A = y''(0) + y(0)$ and $B = y'''(0) + y'(0)$.
3. Give necessary and sufficient conditions on the four initial derivatives for boundedness. Prove necessity even when the secular terms oscillate and have infinitely many zeros.
4. Find a conserved quadratic expression involving only the last two state coordinates. Decide whether the full four-state system can have a conserved positive-definite quadratic form, and justify your answer.

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Question 9 – Solution

Strategy. Expose an internal oscillator that resonantly forces the displacement oscillator.

Step 1: Reveal the oscillator cascade. The original equation gives

$$\boxed{W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} W.}$$

The coordinate map is triangular with determinant one; its inverse is $(y, y', y'', y''') = (W_1, W_2, W_3 - W_1, W_4 - W_2)$. The last pair satisfies $W_3' = W_4$, $W_4' = -W_3$ independently.

Step 2: Solve the resonant forcing. We have $W_3 = A\cos t + B\sin t$ and $y'' + y = W_3$. A resonant particular solution is $(A/2)t\sin t - (B/2)t\cos t$. Enforcing the first two data gives

$$\boxed{y = a\cos t + (b + B/2)\sin t + (A/2)t\sin t - (B/2)t\cos t.}$$

It has $y''(0) = A - a$ and $y'''(0) = B - b$, checking all four data. Applying $D^2 + 1$ twice verifies the fourth-order equation.

Step 3: Classify bounded data. If $A = B = 0$, only bounded sine and cosine remain. If $(A, B) \neq (0, 0)$, choose a phase θ with $A\sin\theta - B\cos\theta = \sqrt{A^2 + B^2}$. Along $t_n = \theta + 2\pi n \geq 0$, the secular part grows as $t_n\sqrt{A^2 + B^2}/2$, while the other terms remain bounded. Consequently

$$\boxed{y \text{ bounded} \iff y''(0) = -y(0), \quad y'''(0) = -y'(0).}$$

The whole state is bounded exactly on this same two-dimensional subspace.

Step 4: Distinguish two kinds of conserved energy. The internal oscillator has

$$E_{\text{int}} = \tfrac{1}{2}(W_3^2 + W_4^2) = \tfrac{1}{2}(A^2 + B^2), \qquad E_{\text{int}}' = W_3W_4 - W_4W_3 = 0.$$

This is not positive definite on the full state space. If a conserved positive- definite form W^TPW existed, its smallest eigenvalue would bound $\|W(t)\|^2$ by a constant for every initial state. The unbounded solutions just constructed contradict that conclusion. Purely imaginary characteristic roots with repeated factors therefore do not imply boundedness of all trajectories.

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Question 10

For the stable scalar equation $(D+1)^3 y = 0$, introduce scaled state coordinates

$$q_1 = y, \qquad q_2 = (y' + y)/4, \qquad q_3 = (y'' + 2y' + y)/16.$$

Let $\|\cdot\|_2$ be the Euclidean norm and consider $t \geq 0$.

Tasks

1. Derive the constant system for q and the inverse map to (y, y', y''). Explain why all characteristic roots are still -1.
2. Solve the system for $q(0) = (0, 0, 1)^T$. Show explicitly that its Euclidean norm can exceed its initial value even though every solution tends to zero.
3. Set $w = (q_1, 4q_2, 16q_3)^T$. Prove a quadratic norm estimate that is strictly decreasing for nonzero states, using
$$2w_2(w_1 + w_3) \leq \sqrt{2}\|w\|_2^2.$$
Obtain an exponential bound for $\|q(t)\|_2$ valid for every initial state.
4. Explain why transient growth is consistent with asymptotic stability. For the prescribed initial state, sketch $\|q(t)\|_2$ and $\|w(t)\|_2/16$, labeling the normalization, and distinguish the two quantities.

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Question 10 – Solution

Strategy. Compare Euclidean growth in scaled coordinates with decay in a suitable weighted norm.

Step 1: Derive the scaled cascade. With N having ones just above the diagonal,

$$\boxed{q' = (-I + 4N)q, \qquad y = q_1, \quad y' = 4q_2 - q_1, \quad y'' = 16q_3 - 8q_2 + q_1.}$$

The coordinate map is invertible, so this is a similarity transformation of the scalar companion system. Its triangular matrix has three eigenvalues -1.

Step 2: Compute an actual transient. Since $N^3 = 0$, $e^{(-I+4N)t} = e^{-t}(I + 4tN + 8t^2N^2)$. Therefore

$$\boxed{q = e^{-t}(8t^2, 4t, 1)^T, \qquad \|q(t)\|_2 = e^{-t}\sqrt{64t^4 + 16t^2 + 1}.}$$

The initial norm is one, but $\|q(1)\|_2 = 9/e > 1$. Every state still tends to zero because its components are polynomials times e^{-t}.

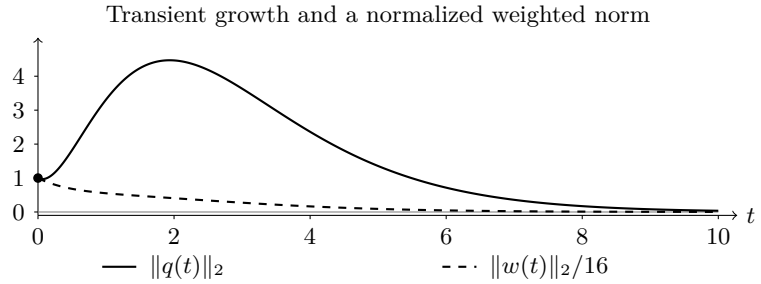
Step 3: Prove a uniform weighted estimate. The new state satisfies $w' = (-I + N)w$, so

$$\frac{d}{dt}\|w\|_2^2 = -2\|w\|_2^2 + 2w_2(w_1 + w_3) \le -(2 - \sqrt{2})\|w\|_2^2.$$

The stated inequality follows from Cauchy–Schwarz and $2\sqrt{2}|w_2|\sqrt{w_1^2 + w_3^2} \le \sqrt{2}\|w\|_2^2$. With $\alpha = 1 - 1/\sqrt{2} > 0$, integration and $\|q\|_2 \le \|w\|_2 \le 16\|q\|_2$ give

$$\boxed{\|q(t)\|_2 \le 16e^{-\alpha t}\|q(0)\|_2.}$$

Step 4: Interpret coordinate-dependent growth. Asymptotic stability allows a finite transient; it does not require every norm to decrease at every instant. For this trajectory, $\|w\|_2/16 = e^{-t}\sqrt{t^4/4 + t^2 + 1}$ decreases strictly, whereas $\|q\|_2$ has a pronounced transient. They measure the same state with different weights.



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Question 1

Consider the entire-coefficient problem

$$y^{(4)} = x^2 y'' + 2xy', \qquad (y, y', y'', y''')(0) = (0, 1, 0, 0).$$

Write $y = \sum_{n \geq 0} a_n x^n$ using ordinary power-series coefficients.

Tasks

1. Derive the recurrence, including its exceptional first index. Identify all potentially nonzero coefficient classes.
2. Find the terms through degree thirteen and verify the initial data and the recurrence relations determining those terms.
3. Prove that the solution series is entire. For the degree-nine truncation p_9, give a rigorous uniform error bound below 10^{-6} on $[-1, 1]$.
4. Determine the sign of $y - p_9$ on either side of zero. Plot the error on $[0, 1]$ together with a pointwise certificate, explicitly scaling the vertical values by 10^6.

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Question 1 – Solution

Strategy. Follow the surviving coefficient chain and bound its tail by a geometric majorant.

Step 1: Match every coefficient. The coefficient of x^n on the right is $n(n+1)a_n$, so

$$\boxed{a_{n+4} = \frac{n\,a_n}{(n+4)(n+3)(n+2)} \quad (n \geq 0).}$$

In particular $a_4 = 0$, not a freely chosen coefficient. The initial coefficients are $(a_0, a_1, a_2, a_3) = (0, 1, 0, 0)$; only indices $1 \pmod 4$ survive.

Step 2: Compute the first terms. Successive use of $n = 1, 5, 9$ gives

$$\boxed{y = x + \frac{x^5}{60} + \frac{x^9}{6048} + \frac{x^{13}}{1153152} + \cdots.}$$

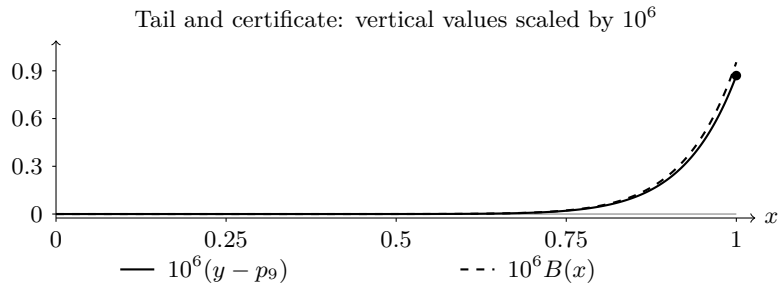
The missing coefficients are zero. The first four derivatives at zero match the data, and substitution into the recurrence verifies the displayed chain.

Step 3: Prove convergence and certify the tail. For fixed x, successive nonzero terms have ratio in magnitude $|x|^4 n/[(n+4)(n+3)(n+2)] \to 0$, so the series is entire and may be differentiated termwise. For $n \geq 9$, the coefficient ratio is at most $1/(n+2)^2 < 1/100$. Consequently, for $|x| \leq 1$,

$$|y - p_9| \leq \frac{100|x|^{13}}{99 \cdot 1153152} \leq B(x) := \frac{11|x|^{13}}{10 \cdot 1153152} \leq \frac{11}{10 \cdot 1153152} < 10^{-6}.$$

This bound concerns the full infinite tail, not just its first omitted term.

Step 4: Check the error's sign and graph. Every surviving coefficient is positive and every surviving degree is odd. Thus $y - p_9$ is positive for $x > 0$, negative for $x < 0$, and zero at zero. The figure compares $10^6(y - p_9)$ and $10^6 B$ on $[0, 1]$.



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Question 2

For the fourth-order problem

$$y^{(4)} + y'' + y = e^x, \qquad y^{(j)}(0) = 0 \quad (0 \le j \le 3),$$

let $b_n = y^{(n)}(0)$ and $a_n = b_n/n!$. The ordinary generating function $B(z) = \sum_{n\ge 0} b_n z^n$ is not the solution's Taylor series.

Tasks

1. Derive a recurrence for b_n and prove that the derivative sequence is periodic. Give one full period.
2. Find $B(z)$ and its radius of convergence. Write the actual solution series using a_n and determine its radius separately.
3. Verify the differential equation directly from the solution series and explain why treating b_n as ordinary Taylor coefficients would give a wrong answer.
4. Determine $\lim_{x\to+\infty} e^{-x}y(x)$ without explicitly solving for all homogeneous constants. Justify why the homogeneous contribution disappears in this limit.

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Question 2 – Solution

Strategy. Keep derivative values separate from coefficients divided by factorials.

Step 1: Derive periodic derivative data. Differentiating the equation n times at zero gives

$$b_{n+4} + b_{n+2} + b_n = 1 \quad (n \geq 0).$$

The first six values are $(0, 0, 0, 0, 1, 1)$. Subtracting this equation from its version at $n + 2$ gives $b_{n+6} = b_n$. Thus $b_n = 1$ precisely when $n \equiv 4, 5 \pmod 6$, and otherwise $b_n = 0$.

Step 2: Compare the two generating functions. The ordinary generating function of the derivatives is

$$\boxed{B(z) = \frac{z^4 + z^5}{1 - z^6}, \qquad R_B = 1.}$$

There is a genuine pole at $z = 1$. In contrast, the solution is

$$\boxed{y(x) = \sum_{k=0}^{\infty} \left(\frac{x^{6k+4}}{(6k+4)!} + \frac{x^{6k+5}}{(6k+5)!} \right), \qquad R_y = \infty.}$$

Absolute convergence follows by comparison with $e^{|x|}$.

Step 3: Verify the equation and normalization. The coefficient of $x^n/n!$ in $y^{(4)} + y'' + y$ is $b_{n+4} + b_{n+2} + b_n = 1$, giving e^x. The first four derivative values vanish. Using b_n as ordinary coefficients would make the fourth derivative at zero 24, whereas the equation and initial values require $y^{(4)}(0) = 1$.

Step 4: Find the dominant exponential. A particular solution is $e^x/3$, since $1^4 + 1^2 + 1 = 3$. The characteristic polynomial factors as

$$r^4 + r^2 + 1 = (r^2 + r + 1)(r^2 - r + 1).$$

Its distinct roots have real parts $-1/2$ or $1/2$. Every homogeneous term is therefore $O(e^{x/2})$ as $x \to +\infty$, regardless of its constants. It follows that $\boxed{\lim_{x \to +\infty} e^{-x} y(x) = 1/3}$.

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Question 3

For a real parameter λ, investigate polynomial solutions of

$$y^{(4)} - xy' + \lambda y = 0.$$

A polynomial solution is required to be nonzero; its degree is not prescribed.

Tasks

1. Derive the ordinary coefficient recurrence and use the highest-degree term to restrict λ.
2. Prove that your restriction is sufficient. For each allowed parameter, determine the dimension of the polynomial solution space and give a monic formula.
3. Find the monic solution when $\lambda = 8$ and verify it directly in the differential equation.
4. Explain why the other three coefficient chains cannot supply additional independent polynomials for the same parameter. Decide what remains of the polynomial classification when λ is negative or nonintegral.

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Question 3 – Solution

Strategy. Combine a highest-degree argument with the exact termination mechanism in the recurrence.

Step 1: Identify the only possible parameters. Coefficient matching gives

$$\boxed{a_{n+4} = \frac{(n-\lambda)a_n}{(n+4)(n+3)(n+2)(n+1)}.}$$

If a nonzero polynomial has degree m, its leading term in $-xy' + \lambda y$ is $(\lambda - m)a_m x^m$; $y^{(4)}$ has smaller degree. Hence $\lambda = m$ must be a nonnegative integer.

Step 2: Construct and count the terminating chain. For $\lambda = m$, normalize $a_m = 1$. Descending the recurrence gives

$$\boxed{p_m(x) = \sum_{j=0}^{\lfloor m/4 \rfloor} \frac{(-1)^j m!}{4^j j!(m-4j)!}\, x^{m-4j}.}$$

At $n = m$ the next coefficient is zero and the chain terminates. The displayed finite sum satisfies every coefficient equation, so it is a solution. All its coefficients are fixed by the leading coefficient; thus the polynomial solution space, including zero, is one-dimensional.

Step 3: Check a concrete eighth-degree solution. For $m = 8$,

$$\boxed{p_8 = x^8 - 420x^4 + 1260.}$$

We have $p_8^{(4)} = 1680x^4 - 10080$ and $-xp_8' + 8p_8 = -1680x^4 + 10080$, so the residual is identically zero.

Step 4: Rule out hidden terminating chains. The recurrence preserves the index modulo four. A nonzero chain can terminate only when its numerator $n - m$ vanishes, which occurs solely in the class containing m. Any nonzero seed in another class continues forever and cannot be part of a polynomial. Equivalently, a second polynomial of lower degree would require that degree to equal m, a contradiction. For negative or nonintegral λ, only the zero polynomial solves the equation.

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Question 4

Near $x = 0$, consider

$$(1 - x^2)y''' - 6xy'' - 6y' = 0, \qquad y(0) = a_0, \quad y'(0) = a_1, \quad y''(0) = 2a_2.$$

The leading coefficient vanishes at $x = \pm 1$, but this does not by itself prove that every solution is singular there.

Tasks

1. Derive the recurrence for the Taylor coefficients and sum the resulting series on $|x| < 1$.
2. Classify exactly which initial triples give analytic continuation through $x = 1$, through $x = -1$, and through both points.
3. Determine the Taylor radius at zero for every initial triple. Explain how an entire solution can coexist with singular coefficients in the normalized equation.
4. For the data $(a_0, a_1, a_2) = (0, 1, -1)$, find the continued value and first two derivatives at $x = 1$. Verify the original equation at that point, and give the maximal real interval on which this solution is analytic.

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Question 4 – Solution

Strategy. Sum the two coefficient tails, then check cancellation of each possible pole separately.

Step 1: Derive and sum the recurrence. The coefficient equation simplifies to

$$(n+3)(n+2)(n+1)(a_{n+3}-a_{n+1})=0 \quad (n\geq 0).$$

Thus the odd tail repeats a_1 and the positive even tail repeats a_2:

$$\boxed{y=a_0+\frac{a_1x+a_2x^2}{1-x^2}, \qquad |x|<1.}$$

As an independent check, the differential equation is exactly $\big((1-x^2)y\big)'''=0$.

Step 2: Classify pole cancellation. The numerator of the fractional term vanishes at 1 exactly when $a_1+a_2=0$, and at -1 exactly when $-a_1+a_2=0$. Because the denominator's zeros are simple, these are also sufficient for analytic continuation through the respective point. Continuation through both requires $a_1=a_2=0$, yielding the constant solution.

Step 3: Determine every convergence radius. If $a_1=a_2=0$, the Taylor series is constant and its radius is infinite. Otherwise at least one genuine pole remains at distance one, so the radius is exactly one, even if the other pole cancels. Singularities of the normalized coefficients constrain the general theory; they need not be singularities of a particular solution after cancellation.

Step 4: Verify passage through a singular point. For the stated data, $y=x/(1+x)$ after cancellation. Therefore

$$\boxed{y(1)=\tfrac{1}{2}, \qquad y'(1)=\tfrac{1}{4}, \qquad y''(1)=-\tfrac{1}{4}.}$$

At $x=1$, the original equation reduces to $-6y''-6y'=0$, which these values satisfy. The continued solution is analytic on $(-1,\infty)$, its maximal real analytic interval containing zero. Its Taylor series about zero still has radius one because the pole at -1 remains.

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Question 5

A formal power series is an algebraic coefficient expansion without an assumed positive radius of convergence. Investigate

$$x^2y''' - y'' = -x, \qquad y(0) = y'(0) = y''(0) = 0.$$

For a possible one-sided smooth solution, you may use

$$z(x) = \int_0^\infty e^{-t}\frac{x}{1-xt}\,dt, \qquad x \le 0.$$

Tasks

1. Put $z = y''$ and derive its formal coefficient recurrence. Find the first four nonzero terms of the formal series for y.
2. Prove that the formal series for y has radius zero. Deduce whether an analytic solution at zero can satisfy the equation and data.
3. Verify that the given integral defines a smooth function from the left at zero and satisfies $x^2z' - z = -x$. Recover a one-sided smooth solution y with the required initial data.
4. Explain how this smooth solution can have the divergent formal series as its Taylor series. Identify the hypothesis that prevents application of the ordinary-point analytic existence theorem at zero.

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Question 5 – Solution

Strategy. Separate formal coefficient solvability from convergence and from one-sided smooth existence.

Step 1: Compute the formal coefficients. For $z = \sum b_n x^n$, coefficient matching gives $b_0 = 0$, $b_1 = 1$, and $b_n = (n-1)b_{n-1}$ for $n \geq 2$. Thus $b_n = (n-1)!$ for $n \geq 1$. Two integrations with zero constants give

$$\boxed{y \sim \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)(n+2)} x^{n+2} = \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{10} + \frac{x^6}{5} + \cdots .}$$

Step 2: Prove divergence. The ratio of consecutive displayed coefficients is $n(n+1)/(n+3) \to \infty$. For every fixed nonzero x, the terms eventually fail to tend to zero. The radius is zero. Any analytic solution would have exactly these forced coefficients, so no analytic solution at zero exists.

Step 3: Construct and verify a smooth solution. For $x \leq 0$, $1 - xt \geq 1$. For $n \geq 1$,

$$\partial_x^n \frac{x}{1-xt} = \frac{n! t^{n-1}}{(1-xt)^{n+1}}.$$

An integrable majorant permits differentiation of every order up to the left endpoint, giving $z(0) = 0$ and $z^{(n)}(0-) = n!(n-1)!$. Also

$$x^2 z' - z = x \int_0^{\infty} \frac{d}{dt}\left(\frac{e^{-t}}{1-xt}\right) dt = -x.$$

Hence $\boxed{y(x) = \int_0^x (x-v) z(v)\, dv}$, with oriented integration, is smooth on $(-\infty, 0]$, satisfies the equation there and has the zero initial triple. The equality at zero follows by continuity.

Step 4: Resolve the apparent contradiction. Its derivatives give exactly the formal coefficients above, but smoothness does not require the Taylor series to converge or represent the function. The leading coefficient x^2 vanishes at zero, so this is not an ordinary point. The analytic existence theorem for a normalized regular equation does not apply.

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Question 6

The series solution of $y''' + y' = 0$ with $(y, y', y'')(0) = (0, 1, 0)$ describes an oscillator derivative pair (y', y''). A proposed rational approximation is

$$U(x) = \frac{1 - x^2/4}{1 + x^2/4}, \qquad V(x) = -\frac{x}{1 + x^2/4}.$$

Its designer claims that correct initial values and exact conservation of $U^2 + V^2$ are enough to make it the true derivative pair.

Tasks

1. Derive the series recurrence and identify the exact solution. Give its terms through degree seven.
2. Expand U, V through the first coefficient at which each differs from the true pair. Check their initial values and quadratic invariant.
3. Test the necessary derivative relation $U' = V$. Compute its defect exactly and decide whether any function can have both $y' = U$ and $y'' = V$ on an interval containing zero.
4. Find the phase $\tau(x)$ for which $(U, V) = (\cos\tau, -\sin\tau)$. Explain the actual evolution law and plot the phase lag $x - \tau(x)$ beside its leading cubic approximation on $[0, 2]$.

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Question 6 – Solution

Strategy. Test derivative compatibility in addition to an invariant and a few Taylor coefficients.

Step 1: Solve the series IVP. Coefficient matching gives $a_{n+3} = -a_{n+1}/[(n+3)(n+2)]$. The data $(a_0, a_1, a_2) = (0, 1, 0)$ give the entire series

$$\boxed{y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots.}$$

The true pair is $(\cos x, -\sin x)$.

Step 2: Check local accuracy and the invariant. Near zero,

$$U = 1 - x^2/2 + x^4/8 + O(x^6), \qquad V = -x + x^3/4 + O(x^5).$$

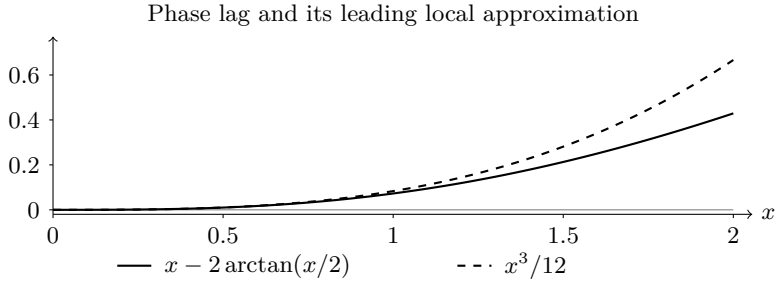
The correct coefficients are $x^4/24$ and $x^3/6$, respectively. Nevertheless $(U, V)(0) = (1, 0)$, and direct squaring gives $U^2 + V^2 = 1$ exactly.

Step 3: Expose the derivative defect. Writing $d = 1 + x^2/4$, we obtain

$$\boxed{U' - V = \frac{x^3}{4d^2}.}$$

This is nonzero for $x \neq 0$. If $y' = U$, differentiation forces $y'' = U'$, so no such function has both proposed derivatives on a nontrivial interval containing zero. Conservation alone does not enforce the differential equation.

Step 4: Identify the altered clock. The half-angle identities give $\tau = 2\arctan(x/2)$ and $\tau' = 1/d$. Thus $U' = V/d$, $V' = -U/d$: the pair traverses the correct circle with a different speed. The phase lag is $x - \tau = x^3/12 + O(x^5)$ and is positive for $x > 0$ because its derivative is $x^2/(4 + x^2) > 0$.



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Question 7

Use series to investigate the directional boundedness of solutions to

$$y''' - 2xy'' - 4y' = 0, \qquad y(0) = a, \quad y'(0) = c, \quad y''(0) = 2b.$$

You may use $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$.

Tasks

1. Derive the coefficient recurrence and construct three entire basis functions normalized by the three initial coefficients.
2. Sum the even chain and express the odd chain as an integral involving e^{-t^2}. Verify the resulting general solution and its initial values.
3. Give necessary and sufficient conditions for boundedness as $x \to +\infty$, as $x \to -\infty$, and in both directions. Justify cancellation limits with an explicit Gaussian-tail bound.
4. For $a = b = -\sqrt{\pi}/2$, $c = 1$, determine the limit at $+\infty$ and the sign of the response for $x \geq 0$. Sketch that one-sided bounded response on $[0, 4]$.

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Question 7 – Solution

Strategy. Build an entire basis from the recurrence, then examine its exponentially amplified coefficients at each end of the real line.

Step 1: Follow the even and odd chains. Matching x^n gives $a_{n+3} = 2a_{n+1}/(n+3)$ for $n \geq 0$. The independent seeds are $a_0 = a$, $a_1 = c$, $a_2 = b$. Ratio tests along each chain give infinite radius. A normalized basis is

$$1, \quad e^{x^2} - 1, \quad H(x) = x + \frac{2x^3}{3} + \frac{4x^5}{15} + \cdots .$$

Step 2: Identify and verify the odd basis. The function $H = e^{x^2} \int_0^x e^{-t^2}\, dt$ satisfies $H' = 2xH + 1$; differentiating twice proves the third-order equation and its initial triple $(0, 1, 0)$. The other two basis functions also satisfy the equation. Thus

$$\boxed{y = a + b(e^{x^2} - 1) + ce^{x^2} \int_0^x e^{-t^2}\, dt.}$$

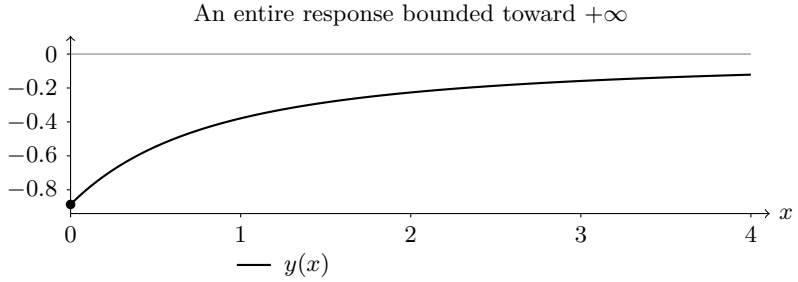
Its initial coefficients are exactly (a, c, b) in degrees zero, one and two.

Step 3: Classify both ends. Put $K = \sqrt{\pi}/2$. At $+\infty$, a nonzero $b + cK$ forces exponential growth. If $b + cK = 0$, the remaining tail tends to zero because, for $x > 0$,

$$0 \leq e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{2x}.$$

The inequality follows by replacing 1 with t/x in the integral. Therefore boundedness at $+\infty$ is equivalent to $b + cK = 0$, with limit $a - b$. Oddness of the integral gives the condition $b - cK = 0$ at $-\infty$, also with limit $a - b$. Both conditions hold exactly when $b = c = 0$, leaving constants.

Step 4: Check the selected response. Here $\boxed{y = -e^{x^2} \int_x^\infty e^{-t^2} dt < 0}$ for $x \geq 0$, and the tail estimate gives $y \to 0$ as $x \to +\infty$.



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Question 8

Investigate the forced singular equation

$$xy''' - y'' = x, \qquad x > 0.$$

One wishes to continue a solution to $x = 0$. Analytic continuation, twice continuous differentiability and three times continuous differentiability are different requirements here.

Tasks

1. Attempt an ordinary power series at zero. Identify the precise coefficient equation that obstructs a formal Taylor solution.
2. Solve the equation on $x > 0$ by first setting $z = y''$. Give the complete three-constant family.
3. Determine the highest integer k for which the solutions extend as C^k functions to $[0, \infty)$. Explain why no choice of the homogeneous constants removes the obstruction.
4. Select the constants giving $y = x^3 \log x/6$. Find its endpoint limit, its minimum on $(0, 1]$ and its value at one. Sketch it, showing zero as an excluded endpoint of the original domain.

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Question 8 – Solution

Strategy. Let the failed coefficient equation identify a logarithmic term forced by resonance.

Step 1: Locate the formal obstruction. For $y = \sum a_n x^n$, matching x^n gives

$$(n+2)(n+1)(n-1)a_{n+2} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

At $n = 0$ this requires $a_2 = 0$, but at $n = 1$ it requires $0 = 1$. No formal ordinary power series, and hence no analytic solution, exists at zero.

Step 2: Solve the reduced equation. For $z = y''$, $(z/x)' = 1/x$, giving $z = x \log x + Cx$. Integrating twice yields

$$\boxed{y = A + Bx + \frac{x^3 \log x}{6} + \left(\frac{C}{6} - \frac{5}{36}\right) x^3.}$$

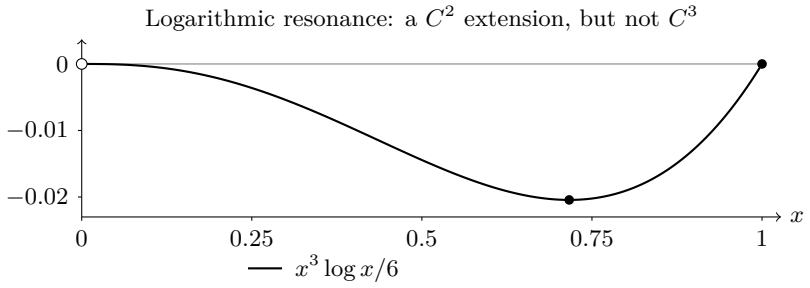
The three homogeneous modes are $1, x, x^3$; the forced logarithm is additional.

Step 3: Classify endpoint smoothness. As $x \downarrow 0$, $y \to A$, $y' \to B$ and $y'' = x \log x + Cx \to 0$. These limits give a C^2 extension. But $y''' = \log x + 1 + C \to -\infty$, so no C^3 extension exists. Constants can change finite terms, not cancel the coefficient of $\log x$. This is a formal inconsistency, unlike a solvable but divergent coefficient recurrence.

Step 4: Verify the illustrated member. Set $A = B = 0$, $C = 5/6$. Then $y' = x^2(3 \log x + 1)/6$, so

$$\boxed{x_{\min} = e^{-1/3}, \qquad y_{\min} = -\frac{1}{18e}, \qquad y(1) = 0, \quad \lim_{x \downarrow 0} y(x) = 0.}$$

The derivative changes from negative to positive at this unique minimum. The open marker records the original domain $x > 0$, even though a C^2 extension to zero is possible.



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Question 9

A solution is known to satisfy a constant-coefficient homogeneous equation

$$y''' = Ay'' + By' + Cy, \qquad (y, y', y'')(0) = (0, 0, 1),$$

where A, B, C are unknown real constants. Its measured Taylor expansion is

$$y(x) = \frac{x^2}{2} + \frac{x^3}{3} + \frac{7x^4}{24} + \frac{x^5}{5} + O(x^6).$$

Tasks

1. Convert the measured coefficients into derivatives and recover A, B, C uniquely. Show the triangular recovery equations.
2. Explain why degree five is the lowest Taylor degree that suffices in general. Predict the coefficient of x^6 from the recovered equation.
3. Derive the full coefficient recurrence and justify that it determines an entire solution with the given data. Rule out any lower-order monic constant-coefficient homogeneous equation for this same solution.
4. Explain why the stated model assumption is essential. Construct analytic functions with the same measured expansion through degree five that do not solve the recovered equation.

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Question 9 – Solution

Strategy. Recover constant coefficients from the earliest derivatives at which they enter, while keeping the scope of identification explicit.

Step 1: Recover the constants. The measured derivatives are $b_2 = 1$, $b_3 = 2$, $b_4 = 7$, $b_5 = 24$, where $b_n = y^{(n)}(0)$. The equation and its first two derivatives at zero give

$$b_3 = A, \qquad b_4 = A^2 + B, \qquad b_5 = A^3 + 2AB + C.$$

Thus $\boxed{A = 2,\ B = 3,\ C = 4}$. The triangular equations prove uniqueness.

Step 2: Check the information threshold. Through degree four, only A, B are fixed; arbitrary C gives the same earlier derivatives. Degree five first reveals C, so it is necessary and sufficient in general. The next derivative is

$$b_6 = 2b_5 + 3b_4 + 4b_3 = 77, \qquad \boxed{a_6 = 77/720}.$$

Step 3: Extend the series and check minimal order. For ordinary coefficients,

$$a_{n+3} = \frac{2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + 4a_n}{(n+3)(n+2)(n+1)} \quad (n \geq 0).$$

The three seeds determine every coefficient. Equivalently, its constant companion system has the everywhere-convergent matrix exponential, so the series is entire. A homogeneous regular equation of order one or two with $y(0) = y'(0) = 0$ would have only the zero solution by uniqueness, contradicting $y''(0) = 1$. Thus the least possible monic homogeneous order is three.

Step 4: Show the model assumption matters. Let y be the recovered solution and $y_\varepsilon = y + \varepsilon x^6$. For any nonzero ε, this is entire and has the same measured coefficients through degree five. Yet for $L = D^3 - 2D^2 - 3D - 4$,

$$Ly_\varepsilon = \varepsilon(120x^3 - 60x^4 - 18x^5 - 4x^6) \not\equiv 0.$$

A finite Taylor record identifies the equation within the stipulated model class; it does not establish that an arbitrary analytic function belongs to that class.

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Question 10

For $\varepsilon > 0$, consider the fourth-order initial-value problem

$$\varepsilon y^{(4)} + y'' = 0, \qquad (y, y', y'', y''')(0) = (0, 0, 1, 0).$$

A Taylor approximation accurate for each fixed parameter need not be accurate uniformly as the parameter tends to zero.

Tasks

1. Derive the Taylor recurrence, sum the solution series, and determine its radius for each fixed $\varepsilon > 0$.
2. Prove uniform convergence of y_ε and y'_ε on the whole real line as $\varepsilon \downarrow 0$. Show that y''_ε has no limit at any fixed nonzero x.
3. At fixed $x \neq 0$, investigate the limit of every fixed even-degree Taylor truncation of degree at least two. Explain why taking a parameter limit and truncating the series give incompatible conclusions.
4. Use $\xi = x/\sqrt{\varepsilon}$ to obtain a parameter-independent profile. Give a uniform Taylor error bound for bounded ξ, and plot the profile and its degree-six polynomial on $0 \leq \xi \leq \pi$.

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Question 10 – Solution

Strategy. Resolve the shrinking oscillation scale before judging a series truncation.

Step 1: Derive and sum the series. The recurrence is $a_{n+4} = -a_{n+2}/[\varepsilon(n+4)(n+3)]$, with $a_0 = a_1 = a_3 = 0$, $a_2 = 1/2$. Therefore

$$\boxed{y_\varepsilon = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j}}{(2j)!\varepsilon^{j-1}} = \varepsilon\big(1 - \cos(x/\sqrt{\varepsilon})\big).}$$

The radius is infinite for each fixed positive parameter. The formula verifies all four initial derivatives and the equation directly.

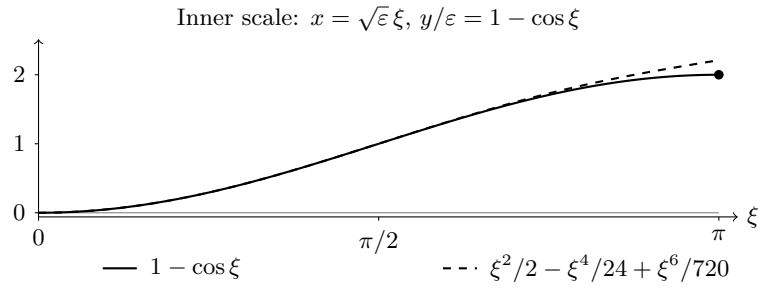
Step 2: Distinguish levels of derivative convergence. The bounds $|y_\varepsilon| \le 2\varepsilon$ and $|y'_\varepsilon| \le \sqrt{\varepsilon}$ prove uniform convergence to zero on $\mathbb{R}$. But $y''_\varepsilon = \cos(x/\sqrt{\varepsilon})$. For fixed $x \neq 0$, the sequences $\varepsilon_n = x^2/(2\pi n)^2$ and $\widetilde{\varepsilon}_n = x^2/((2n+1)\pi)^2$ give values 1 and -1. No pointwise limit exists there. The reduced equation $y'' = 0$ cannot retain the original second-derivative datum.

Step 3: Test fixed truncations. The degree-two polynomial is $x^2/2$, which fails to approach zero at fixed nonzero x. For any fixed degree $2m \ge 4$, its highest term is a nonzero constant times ε^{1-m} and dominates the lower powers as $\varepsilon \downarrow 0$. Thus the truncation diverges while the full sum tends to zero. The Taylor remainder is not uniform in this parameter at fixed x.

Step 4: Rescale and certify. With $y_\varepsilon/\varepsilon = 1 - \cos\xi$, Taylor's theorem gives, for $|\xi| \le R$,

$$\left| 1 - \cos\xi - \sum_{j=1}^{m} \frac{(-1)^{j+1}\xi^{2j}}{(2j)!} \right| \le \frac{R^{2m+2}}{(2m+2)!}.$$

Multiplying by ε bounds the original displacement error. For the graph, $m = 3$, $R = \pi$, so the profile error is at most $\pi^8/40320$.



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Question 1

A boundary-value problem prescribes values at two different points:

$$y'' = -6x, \qquad y(0) = 1, \qquad y(1) = 2.$$

Compare it with the family of initial-value problems having $y(0) = 1$ and $y'(0) = s$, where s is a freely adjustable initial slope.

Tasks

1. Solve the initial-value family and determine the unique slope that meets the right boundary condition. Explain the distinction between shooting for an endpoint and prescribing an initial slope.
2. Verify the resulting differential equation and both endpoint values. Find the maximum and minimum on $[0, 1]$ and determine whether endpoint values bound the entire response.
3. Replace the boundary values by arbitrary α, β. Prove existence and uniqueness for every pair and give the solution explicitly.
4. If the endpoint values change by δ_0, δ_1 while the forcing stays fixed, find the exact change in the solution and a sharp uniform bound. Sketch the original solution and the line joining its endpoints.

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Question 1 – Solution

Strategy. Solve the initial-value family first, then use the endpoint equation to select its slope.

Step 1: Select the shooting slope. Integrating twice gives $y_s = 1 + sx - x^3$. The right endpoint is $y_s(1) = s$, so $s = 2$ is the unique successful slope. An initial-value problem fixes s directly; the boundary-value problem determines it from a condition at another point.

Step 2: Verify the solution and its extrema. The solution $\boxed{y = 1 + 2x - x^3}$ has $y'' = -6x$, $y(0) = 1$, $y(1) = 2$. Since $y' = 2 - 3x^2$, its maximum occurs at $r = \sqrt{2/3}$:

$$\boxed{y_{\max} = 1 + \frac{4}{3}\sqrt{\frac{2}{3}} > 2, \qquad y_{\min} = y(0) = 1.}$$

Thus the endpoint values do not bound this forced response from above.

Step 3: Allow arbitrary endpoint values. Direct integration gives

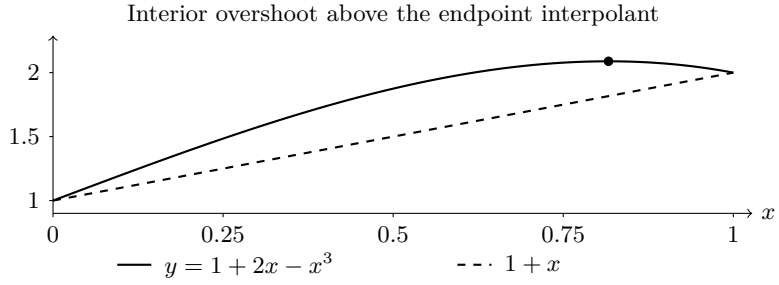
$$\boxed{y_{\alpha,\beta}(x) = \alpha + (\beta - \alpha + 1)x - x^3.}$$

This works for every α, β. The difference of two solutions with the same data satisfies $w'' = 0$, $w(0) = w(1) = 0$, so it is zero, proving uniqueness.

Step 4: Quantify boundary-data sensitivity. The exact response change is $\Delta y = (1 - x)\delta_0 + x\delta_1$. Therefore

$$\boxed{\max_{0 \le x \le 1} |\Delta y| = \max(|\delta_0|, |\delta_1|).}$$

The inequality follows from convex weights; equality holds at an endpoint. The original curve exceeds its endpoint line $1 + x$ by $x - x^3 > 0$ inside $(0, 1)$.



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Question 2

Consider the fixed-interval problem

$$y'' + y = 1, \qquad 0 \le x \le \pi, \qquad y(0) = \alpha, \quad y(\pi) = \beta.$$

All functions are real. Counting two endpoint conditions does not settle whether a second-order boundary-value problem has exactly one solution.

Tasks

1. Find the complete solution family before applying the boundary conditions. Derive the necessary and sufficient compatibility condition on α, β.
2. Classify the number of solutions for every pair of boundary values. Explain the role of the nonzero homogeneous solution that vanishes at both endpoints.
3. When the problem is compatible, impose $\int_0^\pi y(x)\,dx = m$. Determine whether this extra condition selects a unique solution for every real m.
4. Replace the right endpoint by $\pi/2$, retaining prescribed values at both ends. Explain why uniqueness returns. For the original interval and $\alpha = \beta = 1$, sketch three distinct solutions with identical endpoint values.

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Question 2 – Solution

Strategy. Test whether the endpoint equations independently determine the homogeneous constants.

Step 1: Derive compatibility. The general solution is $y = 1 + A\cos x + B\sin x$. The left endpoint gives $A = \alpha - 1$, while the right endpoint gives $\beta = 1 - A = 2 - \alpha$. Thus

$$\boxed{\text{A solution exists exactly when } \alpha + \beta = 2.}$$

Step 2: Count the compatible solutions. When compatible, the entire family is

$$\boxed{y = 1 + (\alpha - 1)\cos x + B\sin x, \qquad B \in \mathbb{R}.}$$

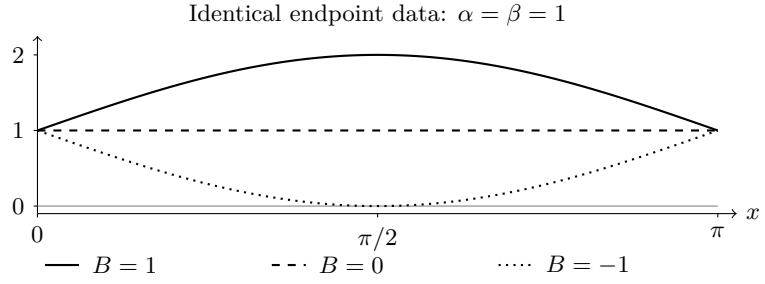
There are infinitely many solutions; otherwise there are none. There is never exactly one with just these two conditions. The nonzero homogeneous solution $\sin x$ vanishes at both endpoints and can be added without changing the data.

Step 3: Use an integral normalization. Since the cosine integrates to zero and the sine integrates to two,

$$\int_0^{\pi} y\,dx = \pi + 2B = m, \qquad \boxed{B = (m - \pi)/2.}$$

For compatible endpoints, every real m selects exactly one solution. If the endpoints are incompatible, an extra integral condition cannot fix that.

Step 4: Change the interval and compare. On $[0, \pi/2]$, the right endpoint gives $B = \beta - 1$ independently of $A = \alpha - 1$, so every pair gives one solution. Independence of the endpoint equations, not their number alone, makes the difference. The graph on the original interval shows $y = 1 + B\sin x$ for $B = -1, 0, 1$.



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Question 3

Let the signed flux be $q(x) = -(1+x)y'(x)$ on $[0,1]$. A variable-conductivity boundary-value problem is

$$-\big((1+x)y'\big)' = 2x - 1, \qquad q(0) = q_0, \qquad q(1) = q_1.$$

The sign convention uses the positive x direction at both endpoints.

Tasks

1. Derive the necessary and sufficient compatibility condition on the two prescribed fluxes. Interpret it as a balance over the interval.
2. For $q_0 = q_1 = 0$, find every solution and verify the equation and both flux conditions directly.
3. Add the normalization $\int_0^1 y(x)\,dx = 0$. Find the resulting constant exactly and prove uniqueness of the normalized solution.
4. Determine whether this normalized solution is increasing, and compute its endpoint values. Explain why flux data alone cannot fix its absolute level and why incompatible fluxes cannot be repaired by adding a constant.

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Question 3 – Solution

Strategy. Integrate the flux balance before trying to determine the additive level of the response.

Step 1: Check the interval balance. The equation is $q' = 2x - 1$, so

$$q(x) = q_0 + x^2 - x, \qquad q_1 - q_0 = \int_0^1 (2x - 1)\, dx = 0.$$

Thus $\boxed{q_1 = q_0}$ is necessary. It is sufficient because division by $1 + x > 0$ and one integration produce a solution for every such pair.

Step 2: Solve the zero-flux case. Here $y' = (x - x^2)/(1 + x) = -x + 2 - 2/(1 + x)$, hence

$$\boxed{y = C - x^2/2 + 2x - 2\log(1 + x).}$$

Multiplication by $-(1 + x)$ gives $q = x^2 - x$, whose derivative is $2x - 1$ and whose values at zero and one are zero. This checks the equation and flux data.

Step 3: Fix the additive constant. Using $\int_0^1 \log(1 + x)\, dx = 2\log 2 - 1$, we obtain

$$\int_0^1 y\, dx = C + \frac{17}{6} - 4\log 2.$$

Thus $\boxed{C = 4\log 2 - 17/6}$. The difference of two solutions with the same fluxes has $((1 + x)w')' = 0$ and zero endpoint flux, so $w' = 0$. It is constant; the zero-mean condition forces that constant to vanish.

Step 4: Interpret the resulting profile. Since $y' = x(1 - x)/(1 + x) > 0$ for $0 < x < 1$, the response is strictly increasing. Its endpoint values are

$$\boxed{y(0) = 4\log 2 - 17/6, \qquad y(1) = 2\log 2 - 4/3.}$$

Fluxes involve derivatives and therefore leave a constant level undetermined. Adding a constant changes neither flux nor the integrated balance, so it cannot repair a violation of $q_1 = q_0$.

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Question 4

A mixed boundary condition couples endpoint value and slope:

$$y'' = 2, \qquad y(0) = 0, \qquad y'(1) - h\,y(1) = b,$$

where h, b are real parameters.

Tasks

1. Solve the problem for $h \neq 1$ and verify all data. Classify the number of solutions at $h = 1$ for every b.
2. For fixed $h \neq 1$, determine the exact response to a perturbation $b \mapsto b + \delta$. Find its uniform size on $[0, 1]$.
3. Follow the path $b = 1$ as $h \to 1$. Explain how a well-defined limit along this path coexists with nonuniqueness at the limiting parameters.
4. For arbitrary real k, use $h_n = 1 - 1/n$ and $b_n = 1 + k/n$ to find the limiting solution. Decide whether the boundary-value solution extends continuously as a single-valued function of (h, b) at $(1, 1)$.

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Question 4 – Solution

Strategy. Inspect the denominator in the endpoint equation and distinguish a chosen parameter path from a unique limiting response.

Step 1: Solve and classify the singular case. The left boundary gives $y = x^2 + cx$. The mixed condition becomes $(1-h)c = b - 2 + h$. For $h \neq 1$,

$$\boxed{y = x^2 + \left(\frac{b-1}{1-h} - 1\right) x.}$$

Its second derivative is two, its left value is zero, and substitution verifies the mixed condition. At $h = 1$, the condition reduces to $b = 1$ independently of c: infinitely many solutions if $b = 1$, and none otherwise.

Step 2: Quantify sensitivity. At fixed $h \neq 1$, the change in the response is

$$\boxed{\Delta y(x) = \frac{\delta x}{1-h}, \qquad \max_{[0,1]} |\Delta y| = \frac{|\delta|}{|1-h|}.}$$

Thus arbitrarily small boundary errors can be strongly amplified near $h = 1$.

Step 3: Examine one compatible path. If $b = 1$ and $h \neq 1$, then $c = -1$, so every solution on this path is $y = x^2 - x$. Its limit is one member of the infinite family at $(h, b) = (1, 1)$. The path selects this member; the limiting boundary conditions alone do not.

Step 4: Compare different approaches. For the proposed sequences, $(b_n - 1)/(1 - h_n) = k$, so

$$\boxed{y_n = x^2 + (k-1)x \quad \text{for every } n.}$$

All parameter pairs approach $(1, 1)$, but their solution limits depend on k. They range over the entire limiting family. Consequently no choice of a single solution at $(1, 1)$ can make the solution map continuous there along all paths.

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Question 5

A dimensionless fourth-order beam model has constant load $y^{(4)} = 24$ on $[0, 1]$. Compare two endpoint designs:

$$\begin{aligned} &\text{Clamped:} && y(0) = y'(0) = y(1) = y'(1) = 0, \\ &\text{Simply supported:} && y(0) = y''(0) = y(1) = y''(1) = 0. \end{aligned}$$

Tasks

1. Find the solution for each design by integrating the fourth-order equation. Verify all four conditions separately for each solution.
2. Prove uniqueness for both designs by applying integration by parts to the difference of two solutions. Track the boundary terms explicitly.
3. Find each maximum displacement and its location. Explain why symmetry alone would not be a complete proof that the midpoint is a maximum.
4. Compare the two profiles pointwise and compute the ratio of their maximum displacements. Sketch both on the same axes and explain why changing derivative conditions changes the response under the same load.

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Question 5 – Solution

Strategy. Determine the integration constants from the correct derivative data, then use an energy identity to check uniqueness.

Step 1: Solve both sets of endpoint equations. Starting from $y = x^4 + ax^3 + bx^2 + cx + d$, the two solutions are

$$\boxed{y_C = x^2(1-x)^2, \qquad y_S = x^4 - 2x^3 + x.}$$

Both fourth derivatives equal 24 and both endpoint values are zero. Also $y_C' = 2x(1-x)(1-2x)$ vanishes at both ends, while $y_S'' = 12x(x-1)$ vanishes at both ends. These verify the distinct boundary data.

Step 2: Prove uniqueness in each case. A difference w has $w^{(4)} = 0$ and homogeneous boundary conditions. Twice integrating by parts gives

$$0 = \int_0^1 w w^{(4)} dx = [w w''' - w' w'']_0^1 + \int_0^1 (w'')^2 dx.$$

The boundary term is zero for clamped data because $w = w' = 0$, and for simply supported data because $w = w'' = 0$. Thus $w'' = 0$; its zero endpoint values imply $w = 0$. Each design has exactly one solution.

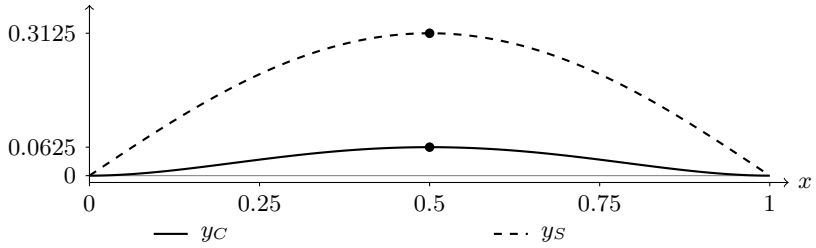
Step 3: Locate and compare the maxima. The derivative of y_C changes from positive to negative at $1/2$. For the other design, $y_S' = (2x-1)(2x^2 - 2x - 1)$, whose second factor is negative on $[0, 1]$. Thus both have their unique maximum at $1/2$, with

$$\boxed{y_C(1/2) = 1/16, \qquad y_S(1/2) = 5/16.}$$

The derivative signs establish maxima; symmetry by itself only suggests a stationary midpoint.

Step 4: Compare the complete profiles. The difference is $y_S - y_C = x(1-x) > 0$ inside the interval, and the maximum ratio is five. The same differential load is compatible with different responses because fixing slopes differs from fixing second derivatives at the boundaries.

Changing the supports changes the deflection



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Question 6

A third-order model is to satisfy four endpoint measurements:

$$y''' = q, \qquad y(0) = y'(0) = 0, \qquad y(1) = a, \quad y'(1) = b.$$

The constant load q may either be prescribed or be an unknown to infer.

Tasks

1. For a fixed prescribed q, derive the necessary and sufficient relation between a, b for a solution to exist. Determine the number of compatible solutions.
2. Instead regard q as unknown. Recover both the load and the complete profile uniquely for arbitrary real a, b.
3. Apply the inverse problem to $a = 1, b = 0$. Verify the load and all four endpoint conditions, and prove that the resulting displacement is increasing on $(0, 1)$.
4. For a nominal load $q = 6$ and right displacement $a = 1$, determine the required right slope. Explain why the fourth boundary condition can be inconsistent when q is fixed but informative when q is unknown.

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Question 6 – Solution

Strategy. Count the integration constants together with any unknown model parameter, and then solve the actual endpoint equations.

Step 1: Find compatibility for a fixed load. The left conditions give $y = qx^3/6 + cx^2/2$. The right conditions require $a = q/6 + c/2$ and $b = q/2 + c$. Subtraction yields

$$\boxed{b - 2a = q/6.}$$

This condition is necessary and sufficient: if it holds, $c = 2a - q/3$ gives the unique solution; otherwise no solution satisfies all the prescribed data.

Step 2: Recover an unknown load. When q is free, the same two equations determine both q and c:

$$\boxed{q = 6(b - 2a), \qquad y = (b - 2a)x^3 + (3a - b)x^2.}$$

There is one solution for every pair (a, b). Direct differentiation verifies the constant load and the two endpoint slopes and values.

Step 3: Check the displacement from rest to rest. For $a = 1, b = 0$,

$$\boxed{q = -12, \qquad y = 3x^2 - 2x^3.}$$

Its third derivative is -12, its endpoint values are zero and one, and $y' = 6x(1 - x)$ vanishes at both ends. Since this derivative is positive inside, the displacement is strictly increasing on $(0, 1)$.

Step 4: Interpret the extra measurement. For $q = 6, a = 1$, compatibility requires $b-2 = 1$, so $\boxed{b = 3}$ and $y = x^3$. Any other prescribed right slope is inconsistent with those fixed data. The left conditions already remove two of the three integration constants; with fixed q, the remaining constant cannot fit arbitrary two right data. When q is unknown, it supplies one more adjustable quantity, which the fourth measurement identifies. Counting conditions is useful only with this distinction.

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Question 7

Consider a singular-endpoint problem on $0 < x \leq 1$:

$$(x^2 y')' = 0, \qquad y(1) = 1.$$

An additional condition at zero is to be imposed through a limit or regularity requirement. The differential equation is initially required only for $x > 0$.

Tasks

1. Find every solution on $(0, 1]$ satisfying the right boundary condition. Identify why zero is not a regular endpoint of the normalized equation.
2. Select the solutions that remain bounded as $x \downarrow 0$, and separately those with finite weighted energy $\int_0^1 x^2 (y')^2 dx$. Determine whether these two selections agree.
3. Replace boundedness by $\lim_{x \downarrow 0} xy(x) = k$, with k prescribed. Find the unique solution for every real k and its limiting weighted derivative $\lim_{x \downarrow 0} x^2 y'(x)$.
4. Decide whether a continuous solution can satisfy $y(0) = 0$ and $y(1) = 1$. Explain why two stated endpoint values do not justify invoking the usual regular boundary-value theory at this singular endpoint.

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Question 7 – Solution

Strategy. Solve on the open interval first, then apply each proposed endpoint condition to the possible singular term.

Step 1: Find the family on the actual domain. Integration gives $x^2y' = C$ and $y = A - C/x$. Incorporating $y(1) = 1$, write

$$\boxed{y = 1 - k + k/x, \qquad k \in \mathbb{R}.}$$

For $x > 0$ the normalized equation is $y'' + (2/x)y' = 0$. Its coefficient $2/x$ is singular at zero; the original leading coefficient also vanishes there.

Step 2: Test boundedness and energy. Boundedness at zero forces $k = 0$, giving $y \equiv 1$. Since $y' = -k/x^2$,

$$\int_0^1 x^2(y')^2 dx = \lim_{t\downarrow 0} k^2 \left(\frac{1}{t} - 1\right).$$

This is infinite for $k \neq 0$ and zero for $k = 0$. Thus boundedness and finite weighted energy select the same single solution in this family.

Step 3: Prescribe a weighted endpoint value. For the displayed family,

$$\lim_{x\downarrow 0} xy(x) = k, \qquad \boxed{\lim_{x\downarrow 0} x^2y'(x) = -k.}$$

Therefore each real prescribed weighted limit selects exactly one solution, including unbounded solutions when $k \neq 0$. A weighted boundary condition is not the same as an ordinary finite value $y(0)$.

Step 4: Test the proposed continuous boundary data. A continuous extension is bounded, so it must be $y \equiv 1$ and must have $y(0) = 1$. Consequently no continuous solution satisfies $y(0) = 0, y(1) = 1$. One cannot assign an unrelated value at zero and retain continuity. Regular endpoint theory requires hypotheses on the normalized coefficients that fail here; the explicit family and its limits decide this problem instead.

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Question 8

A linear differential equation is paired with a nonlinear boundary condition:

$$y'' = 0, \qquad y(0) = 0, \qquad y'(1) = y(1)^2 + \lambda,$$

where $\lambda \in \mathbb{R}$.

Tasks

1. Reduce the boundary-value problem to an algebraic equation and classify the number of real solutions for every λ.
2. At $\lambda = 0$, test whether the average of the two solutions is also a solution. Explain which part of the full problem prevents superposition.
3. Set $\lambda = 1/4 - \varepsilon$ with $\varepsilon > 0$. Measure the uniform distances of the two solutions from the unique critical solution at $\lambda = 1/4$. Decide whether a local Lipschitz estimate in the parameter can hold there.
4. Plot the admissible slopes $c = y'(0)$ against λ, clearly indicating the critical point and where no real solution exists. Explain why uniqueness at a single parameter need not imply stable dependence on nearby parameters.

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Question 8 – Solution

Strategy. Solve the differential equation completely, then retain the nonlinear structure of its endpoint constraint.

Step 1: Classify the slope equation. Every solution of $y'' = 0, y(0) = 0$ is $y = cx$. Its remaining boundary condition is $c = c^2 + \lambda$, so

$$\boxed{c_\pm = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}.}$$

There are two real solutions for $\lambda < 1/4$, one for $\lambda = 1/4$, and none for $\lambda > 1/4$. The critical solution is $y_* = x/2$.

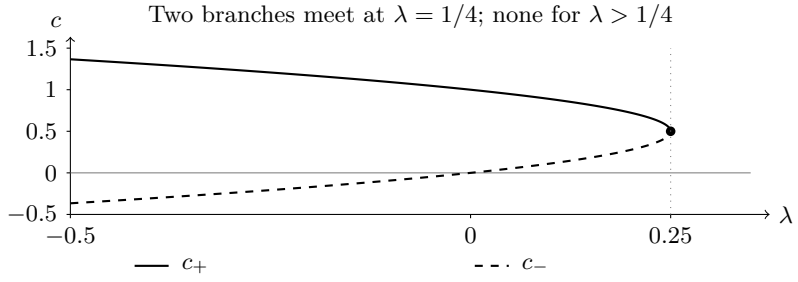
Step 2: Check the superposition claim. At $\lambda = 0$, the solutions are 0 and x. Their average $x/2$ has $y'(1) = 1/2$ but $y(1)^2 = 1/4$, so it fails the boundary condition. The differential equation is linear, but the square in the boundary operator makes the full boundary-value problem nonlinear.

Step 3: Quantify critical sensitivity. At $\lambda = 1/4 - \varepsilon$, the slopes are $1/2 \pm \sqrt{\varepsilon}$. Hence

$$\boxed{\max_{[0,1]} |y_\pm - y_*| = \sqrt{\varepsilon}.}$$

A Lipschitz bound with fixed constant K would require $\sqrt{\varepsilon} \le K\varepsilon$ for all sufficiently small positive ε, which is impossible. Both branches approach y_* continuously from this side, but not at a rate bounded linearly by the parameter error.

Step 4: Read the complete branch picture. The branches meet at $(\lambda, c) = (1/4, 1/2)$ and have no real continuation for larger λ. For $\lambda < 1/4$, $dc_\pm/d\lambda = \mp 1/\sqrt{1 - 4\lambda}$, whose magnitude diverges at the join. Critical uniqueness therefore does not guarantee a locally unique family or a Lipschitz-stable response.



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Question 9

Let $L[y] = -y'' + q(x)y$ on $[0, 1]$, where $q \geq 0$ is continuous. All solutions below satisfy $y(0) = y(1) = 0$. A useful comparison statement is: if $L[w] \geq 0$ and $w(0), w(1) \geq 0$, then $w \geq 0$ on the interval.

Tasks

1. Prove that $L[y] = f$ has exactly one solution for every continuous f. Use an energy identity for uniqueness and a family of initial-value solutions for existence.
2. Prove the stated comparison result. To handle the case where q can vanish, apply the interior-minimum test first to $w + \eta[1 + x(1 - x)]$ with $\eta > 0$, then let $\eta \downarrow 0$.
3. If an approximation v has zero endpoint values and $|L[v] - f| \leq \epsilon$, prove a pointwise error bound using $\phi = x(1 - x)/2$. Give a sharp constant for the uniform error estimate over all such problems.
4. Apply the result to $q = x^2$, $f = 1$, $v = \phi$. Compute the maximum residual exactly and give rigorous upper and lower bounds for the unknown exact solution, without claiming the approximation solves the equation.

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Question 9 – Solution

Strategy. Combine an energy identity, a strict comparison argument and a simple barrier to certify the boundary-value error.

Step 1: Establish existence and uniqueness. A homogeneous difference w with zero endpoint values satisfies $0 = \int_0^1 wL[w]dx = \int_0^1 ((w')^2 + qw^2)dx$, hence $w' = 0$ and $w = 0$. For existence, let p solve $L[p] = f$ with $p(0) = p'(0) = 0$, and let h solve $L[h] = 0$ with $h(0) = 0, h'(0) = 1$. Regular IVP theory supplies both. If $h(1) = 0$, uniqueness just proved would force $h = 0$, a contradiction. Thus $y = p - p(1)h/h(1)$ is the required solution.

Step 2: Make comparison strict before taking a limit. Put $\psi = 1 + x(1 - x) > 0$. Then $L[\psi] = 2 + q\psi > 0$, so $L[w + \eta\psi] > 0$ and its endpoint values are positive. At a nonpositive interior minimum, its second derivative is nonnegative and $q \geq 0$ would give $L[w + \eta\psi] \leq 0$, a contradiction. Thus $w + \eta\psi > 0$. Letting $\eta \downarrow 0$ proves $w \geq 0$.

Step 3: Bound the error with a barrier. Since $L[\phi] = 1 + q\phi \geq 1$, for $e = y - v$ we have $L[\epsilon\phi \pm e] \geq 0$, with zero endpoints. Comparison yields

$$\boxed{|y(x) - v(x)| \leq \epsilon\phi(x), \qquad \max_{[0,1]} |y - v| \leq \epsilon/8.}$$

The constant $1/8$ is sharp: take $q = 0$, $v = 0$, $f = \epsilon$, for which $y = \epsilon\phi$ attains equality at $x = 1/2$.

Step 4: Certify the proposed approximation. Here $L[\phi] - 1 = x^2\phi = x^3(1 - x)/2$. Its derivative is $x^2(3 - 4x)/2$, so its maximum is $\epsilon = 27/512$ at $x = 3/4$. Also $L[v - y] \geq 0$, hence $y \leq v$. Combining this sign with the error bound gives

$$\boxed{\left(1 - \frac{27}{512}\right)\phi \leq y \leq \phi, \qquad \max_{[0,1]} |y - \phi| \leq \frac{27}{4096}.}$$

The residual is not zero; these are certified bounds, not an exact formula for y.

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Question 10

A positive point load acts at an unknown location $a \in (0,1)$:

$$-y'' = F\delta(x-a), \qquad y(0) = y(1) = 0, \qquad F > 0.$$

Seek a continuous, piecewise linear displacement. The distributional equation requires $y'(a+) - y'(a-) = -F$; no jump in y is allowed.

Tasks

1. Derive the solution on each side of a using the endpoint values, continuity and derivative jump. Verify the distributional equation.
2. Find the peak height and the total area $\int_0^1 y(x)\,dx$. For fixed F, determine which load location maximizes the height.
3. Suppose only the peak height and total area are measured, while both F and a are unknown. Decide whether these measurements determine the load uniquely, and characterize all compatible loads for a positive measured height H.
4. Instead measure the endpoint slopes $m_L = y'(0)$ and $m_R = y'(1)$. Recover F, a and state their admissibility conditions. Solve the case $m_L = 2, m_R = -1$, and sketch it alongside the reflected load at $1-a$ with the same F.

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Question 10 – Solution

Strategy. Use the boundary conditions to determine two lines, then distinguish independent measurements from redundant ones.

Step 1: Match the two linear pieces. Write $y = mx$ to the left and $y = n(1-x)$ to the right. Continuity requires $ma = n(1-a)$, and the slope jump gives $-n - m = -F$. Thus

$$\boxed{y(x) = \begin{cases} F(1-a)x, & 0 \le x \le a, \\ Fa(1-x), & a \le x \le 1. \end{cases}}$$

Both endpoint values vanish. The continuous join produces no δ' term; the derivative jump $-F$ gives exactly $-y'' = F\delta(x-a)$, with zero ordinary second derivative elsewhere.

Step 2: Compute the peak and area. The left slope is positive and the right slope negative, so the peak is at a:

$$\boxed{H = Fa(1-a), \qquad \int_0^1 y\,dx = H/2.}$$

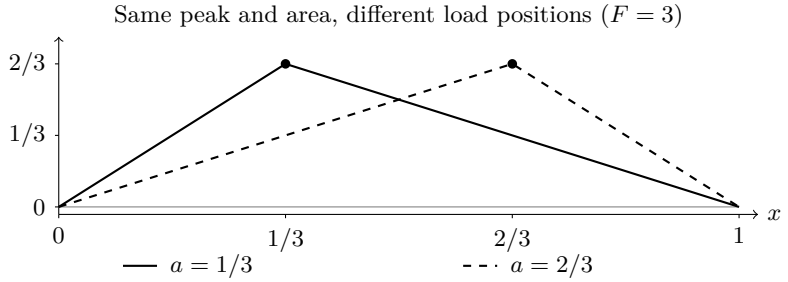
The area is that of a triangle with base one and height H. For fixed F, $a(1-a) = 1/4 - (a-1/2)^2$ is maximized at $a = 1/2$, giving height $F/4$.

Step 3: Test whether the measurements are independent. If the measured area differs from $H/2$, no load fits the model. If it equals $H/2$, every $a \in (0,1)$ is compatible with $\boxed{F = H/[a(1-a)]}$. The two measurements supply only one independent number, so there are infinitely many compatible location-strength pairs.

Step 4: Recover the load from endpoint slopes. Since $m_L = F(1-a)$ and $m_R = -Fa$,

$$\boxed{F = m_L - m_R, \qquad a = \frac{-m_R}{m_L - m_R}.}$$

The precise admissibility conditions are $m_L > 0$ and $m_R < 0$. For $(m_L, m_R) = (2, -1)$, $F = 3, a = 1/3$, with height $2/3$ and area $1/3$. The reflected load has the same height and area but different endpoint slopes.



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Question 1

For a fixed $L > 0$, consider

$$-y'' = \lambda y, \qquad 0 \le x \le L, \qquad y(0) = y(L) = 0.$$

An eigenvalue is a real λ for which a nonzero solution exists; any such solution is an eigenfunction. The identically zero solution does not qualify.

Tasks

1. Treat $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, and find every eigenvalue and its full eigenspace.
2. Choose each eigenfunction to have $\int_0^L y^2 dx = 1$ and $y'(0) > 0$. Explain why these two conditions fix its amplitude and sign uniquely.
3. Locate all interior zeros of the nth eigenfunction. For $L = 1$, sketch the first three normalized modes and identify their interior zero counts.
4. If the interval length is doubled, determine the change in each eigenvalue and in the normalization factor. Explain why a function solving the differential equation alone need not be an eigenfunction.

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Question 1 – Solution

Strategy. Enforce both homogeneous endpoint conditions in each sign case before normalizing a nonzero solution.

Step 1: Classify all real parameters. For $\lambda = -\mu^2 < 0$, $y(0) = 0$ gives $y = A\sinh(\mu x)$; since $\sinh(\mu L) > 0$, the right condition forces $A = 0$. For $\lambda = 0$, the left condition gives $y = Ax$ and the right gives $A = 0$. For $\lambda = k^2 > 0$, $y = A\sin(kx)$, so a nonzero A requires $kL = n\pi$. Thus

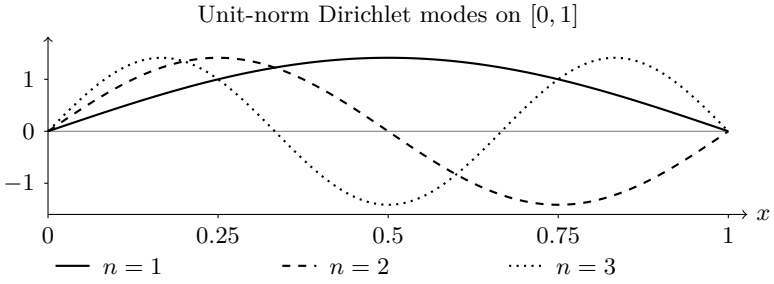
$$\boxed{\lambda_n = (n\pi/L)^2, \quad E_n = \operatorname{span}\{\sin(n\pi x/L)\}, \quad n = 1, 2, \ldots.}$$

Step 2: Fix amplitude and sign. The identity $\sin^2 t = (1 - \cos 2t)/2$ gives $\int_0^L \sin^2(n\pi x/L)dx = L/2$. The unit norm requires $|A| = \sqrt{2/L}$, and $y'(0) = An\pi/L > 0$ selects the positive sign:

$$\boxed{\phi_n(x) = \sqrt{2/L}\sin(n\pi x/L).}$$

Step 3: Count the interior zeros. The zeros are $x = jL/n$, $j = 0, \ldots, n$, of which exactly $n - 1$ lie inside. They are simple because the derivative there is nonzero. The first three modes on the unit interval therefore have zero, one and two interior zeros.

Step 4: Rescale the interval. Replacing L by $2L$ divides each eigenvalue by four and the normalization factor by $\sqrt{2}$. For example, $\cos(kx)$ solves the differential equation with $\lambda = k^2$ but fails the left endpoint condition. The equation, both boundary conditions and nontriviality are all part of the eigenproblem.



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Question 2

Study the derivative boundary conditions

$$-y'' = \lambda y, \qquad 0 \le x \le L, \qquad y'(0) = y'(L) = 0, \quad L > 0.$$

A calculation that divides by $\sqrt{\lambda}$ can lose an important mode.

Tasks

1. Find every real eigenvalue and eigenspace, including a separate check at zero and an exclusion of negative eigenvalues.
2. Normalize each mode by $\int_0^L y^2 dx = 1$ and $y(0) > 0$. Explain why the zero mode has a different normalization factor.
3. Integrate the differential equation to prove that every nonzero-eigenvalue mode has zero mean. Decide what happens to the zero eigenspace if zero mean is imposed as an extra constraint.
4. For $L = 2$, test the claims that x is a zero-mode eigenfunction and that $1+\cos(\pi x/2)$ is an eigenfunction. Distinguish satisfying both endpoint conditions from belonging to a single eigenspace.

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Question 2 – Solution

Strategy. Preserve the constant solution by checking the zero parameter before dividing by a frequency.

Step 1: Find the complete spectrum. Integration by parts for any eigenfunction gives $\lambda \int_0^L y^2 dx = \int_0^L (y')^2 dx \geq 0$, so negative eigenvalues are impossible. At zero, $y = A + Bx$ and the derivative conditions force $B = 0$; nonzero constants are eigenfunctions. For $\lambda = k^2 > 0$, $y = A\cos(kx) + B\sin(kx)$; the left derivative condition forces $B = 0$ and the right requires $\sin(kL) = 0$. Thus

$$\boxed{\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\cos(n\pi x/L)\}, \quad n = 0, 1, 2, \ldots.}$$

Step 2: Normalize the modes. The constant has squared integral L, while every positive-frequency cosine has squared integral $L/2$. The positive value at zero fixes the signs:

$$\boxed{\phi_0 = 1/\sqrt{L}, \qquad \phi_n = \sqrt{2/L}\cos(n\pi x/L) \quad (n \geq 1).}$$

Step 3: Interpret the mean constraint. Integrating gives $\lambda \int_0^L y\, dx = -y'(L) + y'(0) = 0$. Thus all modes with $\lambda \neq 0$ have zero mean. The only constant with zero mean is the zero function, so the additional constraint removes the zero eigenspace from the admissible nonzero functions; it preserves every higher mode.

Step 4: Test the two proposed functions. The function x has derivative one at both ends, so it fails the boundary conditions. The function $g = 1 + \cos(\pi x/2)$ satisfies both derivative conditions, but $-g'' = (\pi^2/4)\cos(\pi x/2)$ is not any constant multiple of g. Indeed, matching the constant term would require $\lambda = 0$, which fails for the cosine term. A sum from different eigenspaces need not be an eigenfunction.

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Question 3

An unknown interval length $L > 0$ has mixed boundary conditions:

$$-y'' = \lambda y, \qquad y(0) = 0, \qquad y'(L) = 0.$$

Two measurements are reported as consecutive eigenvalues, in increasing order: $9\pi^2/16$ and $25\pi^2/16$. The report does not say they are the first two.

Tasks

1. Derive the complete real spectrum and rule out zero and negative values. Index the positive eigenvalues starting at $n = 0$.
2. Recover L and the indices of the two measured modes. Explain why taking the difference of the square roots is useful.
3. Find the true lowest eigenvalue and normalize its eigenfunction by $\int_0^L y^2 dx = 1$ and $y'(0) > 0$.
4. Drop the claim that the measurements are consecutive. Characterize every possible length and pair of indices consistent with the same two numbers. Explain exactly what information the consecutiveness assumption supplied.

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Question 3 – Solution

Strategy. Use equally spaced frequencies, not equally spaced eigenvalues, to interpret the measurements.

Step 1: Classify the mixed spectrum. The endpoint term vanishes in integration by parts, so $\lambda \int y^2 = \int (y')^2 > 0$ for a nonzero admissible function; equality would force a constant and then $y = 0$. With $\lambda = k^2 > 0$, the left condition gives $y = A\sin(kx)$ and the right requires $\cos(kL) = 0$. Hence

$$\boxed{\lambda_n = ((n + 1/2)\pi/L)^2, \qquad n = 0, 1, \ldots.}$$

Each eigenspace consists of multiples of the corresponding sine.

Step 2: Recover the consecutive indices. Consecutive positive frequencies differ by π/L. The measured difference is $5\pi/4 - 3\pi/4 = \pi/2$, giving $\boxed{L = 2}$. The first measured frequency satisfies $(n + 1/2)\pi/2 = 3\pi/4$, so its index is $n = 1$ and the next is $n = 2$. They were not the first two eigenvalues.

Step 3: Recover the missing lowest mode. The index-zero eigenvalue is $\boxed{\lambda_0 = \pi^2/16}$. Since $\int_0^2 \sin^2(\pi x/4)dx = 1$ and its initial slope is positive, the requested eigenfunction is $\boxed{\phi_0 = \sin(\pi x/4)}$.

Step 4: Identify the remaining inverse ambiguity. Let the measured indices be $r < s$. Their frequency ratio implies $5(2r + 1) = 3(2s + 1)$. Since three and five are coprime, all solutions are

$$2r + 1 = 3d, \quad 2s + 1 = 5d, \quad \boxed{L = 2d, \quad r = (3d - 1)/2, \quad s = (5d - 1)/2,}$$

where d is any positive odd integer. Conversely, substitution verifies every such choice. The index gap is $s - r = d$; consecutiveness forces $d = 1$ and removes this infinite ambiguity.

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Question 4

Let $h \in \mathbb{R}$ control the endpoint condition

$$-y'' = \lambda y, \qquad 0 \le x \le 1, \qquad y(0) = 0, \quad y'(1) = h\, y(1).$$

Unlike fixed zero endpoint values, this boundary condition can permit a negative eigenvalue.

Tasks

1. Determine precisely when zero is an eigenvalue and identify its eigenspace.
2. For $\lambda = -\mu^2 < 0$, derive the equation for μ. Prove that there is exactly one negative eigenvalue when $h > 1$ and none when $h \le 1$.
3. For $\lambda = k^2 > 0$, derive the frequency equation without losing or adding modes by division. Locate all its roots by intervals between multiples of π, including the special first interval.
4. Derive the energy identity and explain why it does not always imply nonnegative eigenvalues. For $h = 2$, bound the negative eigenvalue using $1 < \mu < 2$, and sketch the equation locating μ.

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Question 4 – Solution

Strategy. Separate all three signs and prove root counts by monotonicity on the correct domains.

Step 1: Check the zero parameter directly. At $\lambda = 0$, $y = Ax$ and the right condition is $A = hA$. Thus zero is an eigenvalue exactly when $h = 1$, with eigenspace $\operatorname{span}\{x\}$.

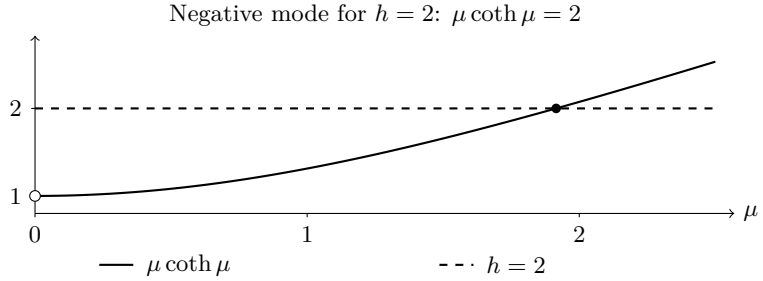
Step 2: Count the negative modes. For $\lambda = -\mu^2$, the nonzero solution is a multiple of $\sinh(\mu x)$, and its endpoint condition is $\boxed{\mu \coth \mu = h}$. The function $g(\mu) = \mu \coth \mu$ tends to one as $\mu \downarrow 0$ and to infinity as $\mu \to \infty$. Its derivative has numerator $\sinh \mu \cosh \mu - \mu > 0$, since this numerator starts at zero and has derivative $2 \sinh^2 \mu > 0$. Thus precisely $h > 1$ gives one negative mode.

Step 3: Locate every positive mode. Here $y = A \sin(kx)$ and $k \cos k = h \sin k$. A positive multiple of π cannot solve this equation, so division by $\sin k$ is safe at a root:

$$\boxed{k \cot k = h.}$$

Its derivative is $(\sin k \cos k - k)/\sin^2 k < 0$ for $k > 0$ away from poles. On each $(n\pi, (n+1)\pi)$, $n \geq 1$, it decreases from $+\infty$ to $-\infty$, giving exactly one root for every h. On $(0, \pi)$ it decreases from the unattained limit one to $-\infty$: one root if $h < 1$, none if $h \geq 1$.

Step 4: Retain the endpoint energy term. Integration by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 (y')^2 dx - h y(1)^2$, which can be negative. For $h = 2$, $g(1) < 2 < g(2)$: the first inequality follows from $e^2 > 3$ and $g(2) > 2$ follows from $\coth 2 > 1$. Thus $1 < \mu < 2$ and $\boxed{-4 < \lambda_- < -1}$. The graph locates the unique crossing.



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Question 5

Consider a ring represented by the interval $[0, 2\pi]$ with matching values and slopes at its ends:

$$-y'' = \lambda y, \qquad y(0) = y(2\pi), \qquad y'(0) = y'(2\pi).$$

Here the eigenspace dimension need not be one. No Fourier expansion is required.

Tasks

1. Rule out negative eigenvalues using the boundary terms in an energy identity, and determine the zero eigenspace directly.
2. Find all positive eigenvalues and the dimension of each eigenspace. Explain why imposing one endpoint equation alone would be insufficient.
3. At a positive eigenvalue n^2, impose $y(0) = 0$ and $y'(0) = 1$. Find the resulting eigenfunction and decide whether these normalization data are possible at zero.
4. For the normalized positive modes, test the additional measurements $y(\pi) = 0$ and $y'(\pi) = y'(0)$. Which measurement adds no information, and which selects only part of the spectrum?

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Question 5 – Solution

Strategy. Treat value and derivative matching together as a two-dimensional endpoint map.

Step 1: Handle nonpositive parameters. The boundary term $[yy']_0^{2\pi}$ is zero because both values and slopes match. Thus $\lambda \int y^2 = \int (y')^2 \geq 0$. At zero, $y = A + Bx$; value matching forces $B = 0$ and slope matching is then automatic. Hence $\boxed{E_0 = \text{span}\{1\}}$, of dimension one.

Step 2: Find the positive eigenspaces. For $\lambda = k^2 > 0$, write $y = A\cos(kx) + B\sin(kx)$. With $\theta = 2\pi k$, the two conditions are

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is $2(1 - \cos\theta)$, so nonzero solutions require $k = n$, a positive integer. Then the entire matrix is zero and both constants are free:

$$\boxed{\lambda_n = n^2, \quad E_n = \text{span}\{\cos nx, \sin nx\}, \quad n \geq 1.}$$

Each positive eigenspace has dimension two. One equation alone would generally leave a nonzero vector even when the other endpoint equation fails.

Step 3: Apply the initial normalization. The data $y(0) = 0, y'(0) = 1$ require $A = 0, nB = 1$, giving $\boxed{y_n = \sin(nx)/n}$. At zero every eigenfunction is constant, so its initial derivative cannot be one. These extra data select a representative from each positive eigenspace, not from the zero eigenspace.

Step 4: Interpret the midpoint measurements. For every positive integer n, $y_n(\pi) = \sin(n\pi)/n = 0$, so that measurement adds no restriction. But $y_n'(\pi) = \cos(n\pi) = (-1)^n$ equals $y_n'(0) = 1$ exactly for even n. The second measurement selects eigenvalues $\boxed{4, 16, 36, \ldots}$ from the original positive spectrum.

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Question 6

A variable-coefficient eigenproblem uses the positive interval $[1, e^2]$:

$$-(xy')' = \lambda \frac{y}{x}, \qquad y(1) = y(e^2) = 0.$$

The natural squared norm for this problem is $\int_1^{e^2} y(x)^2\, dx/x$.

Tasks

1. Use $t = \log x$ and $y(x) = u(t)$ to transform the equation and both boundary conditions. Show every derivative factor.
2. Find all real eigenvalues and eigenspaces, checking all parameter signs. Normalize each mode to have weighted norm one and $y'(1) > 0$.
3. Locate the first mode's maximum and all interior zeros of the nth mode. Explain why equal spacing in the transformed coordinate does not imply equal spacing in x.
4. Replace e^2 by a general right endpoint $b > 1$. Give the eigenvalues and weighted normalization factor as functions of b. Sketch the first two normalized modes for the original interval.

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Question 6 – Solution

Strategy. Transform the differential equation and its weight together, not just its apparent frequency.

Step 1: Carry out the logarithmic substitution. Since $dt/dx = 1/x$, $y' = u'/x$, $xy' = u'$ and $(xy')' = u''/x$. Multiplying the equation by x gives

$$\boxed{-u'' = \lambda u, \qquad u(0) = u(2) = 0.}$$

Also $dx/x = dt$, so the weighted squared norm is exactly $\int_0^2 u^2 dt$.

Step 2: Classify and normalize. At zero the affine function with two zero endpoints is zero. At a negative parameter, the left-zero hyperbolic sine cannot vanish at the right endpoint. For $\lambda = k^2 > 0$, $u = A\sin(kt)$ and $\sin(2k) = 0$. Thus

$$\boxed{\lambda_n = (n\pi/2)^2, \qquad \phi_n(x) = \sin\left(\frac{n\pi \log x}{2}\right), \quad n \geq 1.}$$

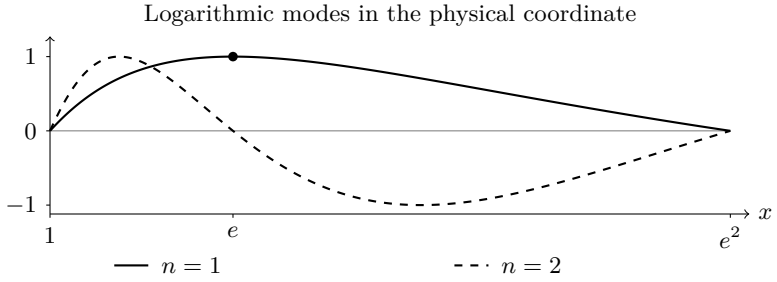
The eigenspace is the span of ϕ_n. Its weighted squared norm is one, and $\phi_n'(1) = n\pi/2 > 0$, fixing both amplitude and sign.

Step 3: Read locations in the original coordinate. For $n = 1$, the sine attains its unique maximum one when $t = 1$, hence $\boxed{x = e}$, the geometric rather than arithmetic midpoint of the endpoints. For mode n, the interior zeros are $\boxed{x_j = e^{2j/n},\ j = 1, \ldots, n-1}$. They form a geometric progression: logarithmic spacing becomes multiplicative spacing in x.

Step 4: Change the endpoint. For $[1, b]$ the transformed length is $\ell = \log b > 0$. Consequently

$$\boxed{\lambda_n = (n\pi/\log b)^2, \qquad \phi_n = \sqrt{\frac{2}{\log b}}\sin\left(\frac{n\pi \log x}{\log b}\right).}$$

The graph for $b = e^2$ uses the physical coordinate x, with the landmark $x = e$.



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Question 7

For $a \in \mathbb{R}$ and $L > 0$, study

$$-y'' + 2ay' = \lambda y, \qquad y(0) = y(L) = 0.$$

The first derivative makes this look different from a sine eigenproblem. Use the weighted norm $\int_0^L e^{-2ax} y^2 dx$ when normalizing.

Tasks

1. Substitute $y = e^{ax}u$ and derive the transformed equation and endpoint conditions. Find every real eigenvalue and eigenspace.
2. Give the weighted unit-norm eigenfunction with positive initial derivative. Explain how the exponential factor affects shape without changing the zeros.
3. For $a = 1, L = \pi$, locate the maximum of the first eigenfunction with amplitude chosen as $y = e^x \sin x$. Sketch it and the underlying sine on the same axes.
4. Suppose the first two eigenvalues are known to be 5 and 8 but both a and L are unknown. Determine everything these data identify, and exhibit the remaining ambiguity.

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Question 7 – Solution

Strategy. Remove the first derivative with an invertible exponential change and keep track of the shifted spectral parameter.

Step 1: Transform the operator completely. The derivatives are $y' = e^{ax}(u' + au)$ and $y'' = e^{ax}(u'' + 2au' + a^2u)$. Therefore

$$-y'' + 2ay' = e^{ax}(-u'' + a^2u), \qquad \boxed{-u'' = (\lambda - a^2)u, \quad u(0) = u(L) = 0.}$$

The exponential is never zero, so this is a bijection of solution spaces. The shifted parameter must be positive: zero or negative gives only the zero Dirichlet solution. All eigenvalues and eigenfunctions are thus

$$\boxed{\lambda_n = a^2 + (n\pi/L)^2, \quad E_n = \operatorname{span}\{e^{ax}\sin(n\pi x/L)\}, \quad n \geq 1.}$$

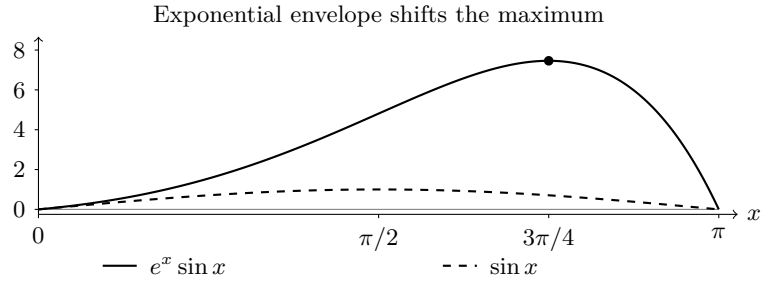
Step 2: Normalize with the correct weight. The factors e^{-2ax} and e^{2ax} cancel in the squared norm. Hence $\boxed{\phi_n = \sqrt{2/L}\, e^{ax}\sin(n\pi x/L)}$ has unit weighted norm and positive initial derivative. The envelope changes heights and the locations of extrema, but its positivity leaves the zeros jL/n unchanged.

Step 3: Locate the displaced maximum. For $y = e^x \sin x$, $y' = e^x(\sin x + \cos x)$ is positive up to $3\pi/4$ and negative afterwards on $(0, \pi)$. Thus

$$\boxed{x_{\max} = 3\pi/4, \qquad y_{\max} = e^{3\pi/4}/\sqrt{2}.}$$

The sine itself peaks at $\pi/2$; multiplying by the increasing envelope shifts the peak without moving either zero.

Step 4: Solve the inverse spectral data. The gap $8 - 5 = 3$ equals $3\pi^2/L^2$, so $L = \pi$. Then $a^2 + 1 = 5$, giving $\boxed{a = 2 \text{ or } a = -2}$. Both choices give the entire same spectrum $4 + n^2$, yet their exponential envelopes point in opposite directions. The eigenvalues alone cannot recover the sign of the drift parameter.



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Question 8

Compare an eigenproblem with its forced counterpart on $[0, 1]$:

$$-y'' - \lambda y = \sin(\pi x), \qquad y(0) = y(1) = 0.$$

The homogeneous Dirichlet eigenvalues of $-d^2/dx^2$ are $n^2\pi^2$, $n \geq 1$. The identity

$$\int_0^1 \sin(n\pi x)\sin(\pi x)dx = 0 \qquad (n \geq 2)$$

may be verified by $2\sin A \sin B = \cos(A - B) - \cos(A + B)$.

Tasks

1. Solve the forced problem for every real λ outside the homogeneous spectrum and prove uniqueness.
2. At $\lambda = \pi^2$, prove nonexistence using integration by parts against $\sin(\pi x)$, rather than merely observing a vanishing denominator.
3. At $\lambda = n^2\pi^2$ with $n \geq 2$, find all solutions. Determine which is selected by $\int_0^1 y\sin(n\pi x)dx = 0$.
4. Compare the uniform size of the solution as $\lambda \to \pi^2$ with its limit as $\lambda \to 4\pi^2$ through nonspectral values. Explain why a finite limiting particular solution does not imply uniqueness at the limiting parameter.

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Question 8 – Solution

Strategy. Separate the existence of a particular solution from the nontrivial homogeneous space at a spectral parameter.

Step 1: Solve away from the spectrum. Since $-\big(\sin \pi x\big)'' = \pi^2 \sin \pi x$,

$$\boxed{y = \frac{\sin \pi x}{\pi^2 - \lambda}}$$

solves the equation and both endpoints whenever $\lambda \neq \pi^2$. If λ is outside the entire spectrum, any difference of two solutions is a zero-endpoint homogeneous solution and is therefore zero. This proves uniqueness, including all zero and negative parameter values.

Step 2: Prove failure of compatibility at the first eigenvalue. Let $s = \sin \pi x$. For a proposed solution at $\lambda = \pi^2$, twice integrating by parts gives

$$\int_0^1 s(-y'' - \pi^2 y)dx = [-sy' + s'y]_0^1 + \int_0^1 (-s'' - \pi^2 s)y \, dx = 0.$$

Both s and y vanish at the endpoints. The forcing instead requires this integral to equal $\int_0^1 s^2 dx = 1/2$, a contradiction. No solution exists.

Step 3: Find every compatible resonant solution. For $n \geq 2$, the particular solution above is valid and the homogeneous space is spanned by $\sin(n\pi x)$. Therefore all solutions are

$$\boxed{y = \frac{\sin \pi x}{(1 - n^2)\pi^2} + C \sin(n\pi x), \quad C \in \mathbb{R}.}$$

The supplied product identity and $\int_0^1 \sin^2(n\pi x)dx = 1/2$ show that the extra integral condition is $C/2 = 0$, selecting exactly $C = 0$.

Step 4: Distinguish two limiting behaviors. Outside the spectrum, $\max |y| = 1/|\pi^2 - \lambda|$, which diverges at the first eigenvalue. As $\lambda \to 4\pi^2$, the same solution converges uniformly to $-\sin(\pi x)/(3\pi^2)$. This is only the $C = 0$ member of the infinite family at $4\pi^2$. A finite chosen limit does not remove the homogeneous freedom.

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Question 9

An inverse design seeks a continuous potential q on $[0,1]$ for

$$-y'' + q(x)y = \lambda y, \qquad y(0) = y(1) = 0.$$

For the constructive part, prescribe $\lambda = \pi^2$ and $y = \sin \pi x + \frac{1}{4} \sin 2\pi x$. Eigenfunctions are required to be C^2.

Tasks

1. Show that the alternative target $y = x(1-x)$ cannot be an eigenfunction for any continuous q and any finite real λ. Explain the endpoint obstruction.
2. For the trigonometric target, recover q explicitly and extend it continuously to both endpoints. Compute $q(0)$ and $q(1)$ and verify the differential equation.
3. Prove that the target is positive on $(0,1)$ and has simple endpoint zeros. For any C^2 test function v with zero endpoints, put $z = v/y$ inside the interval and derive $\int_0^1 [v'^2 + (q-\lambda)v^2]dx = \int_0^1 y^2 z'^2 dx$.
4. Use this identity to prove that no real eigenvalue can lie below π^2, and that every eigenfunction with eigenvalue π^2 is a multiple of the designed target. Address the boundary term in the identity.

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Question 9 – Solution

Strategy. Recover the potential where the target is nonzero, then use endpoint regularity and a factored energy identity to certify the design.

Step 1: Reject an incompatible target. For $y = x(1-x)$, division inside the interval would require $q = \lambda + y''/y = \lambda - 2/[x(1-x)]$, which is unbounded at both ends. Equivalently, continuity of the equation at zero would require $-y''(0) + q(0)y(0) = 2 = \lambda y(0) = 0$. No continuous potential can work.

Step 2: Construct a regular potential. Write $c = \cos \pi x$. The target is $y = \sin \pi x(1 + c/2)$ and $y'' = -\pi^2 \sin \pi x(1 + 2c)$. Consequently

$$\boxed{q(x) = -\frac{3\pi^2 c}{2+c}, \qquad q(0) = -\pi^2, \quad q(1) = 3\pi^2.}$$

Since $2 + c \geq 1$, the formula is continuous on the closed interval. Substitution gives $qy = (\lambda y + y'')$, verifying the equation inside; continuity verifies it at the endpoints as well.

Step 3: Factor the energy. The factor $1 + c/2$ lies between $1/2$ and $3/2$, so $y > 0$ inside. The endpoint derivatives are $y'(0) = 3\pi/2$ and $y'(1) = -\pi/2$, both nonzero. With $v = yz$ and $y'' = (q - \lambda)y$, expansion gives

$$v'^2 + (q - \lambda)v^2 = y^2 z'^2 + (yy'z^2)'.$$

For zero-endpoint C^2 functions v, the simple zeros of y imply that $z = v/y$ and z' stay bounded near either endpoint, by Taylor expansion. Thus $[yy'z^2]_0^1 = 0$ and integration proves the requested identity.

Step 4: Certify the lowest eigenvalue and its eigenspace. If v is a nonzero eigenfunction with eigenvalue ν, integration by parts and the identity yield

$$(\nu - \pi^2) \int_0^1 v^2 dx = \int_0^1 y^2 z'^2 dx \geq 0.$$

Thus $\nu \geq \pi^2$, and the constructed nonzero y attains this bound. Equality forces $z' = 0$ inside, hence $v = Cy$. Therefore π^2 is the lowest real eigenvalue and its eigenspace is precisely $\operatorname{span}\{y\}$.

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Question 10

A two-material interval has stiffness

$$p(x) = \begin{cases} 1, & 0 < x < 1/2, \\ 4, & 1/2 < x < 1, \end{cases} \qquad -(py')' = \lambda y, \qquad y(0) = y(1) = 0.$$

Require y and the flux py' to be continuous at $x = 1/2$. Solutions are C^2 on each side. Ordinary slopes need not be continuous across the interface.

Tasks

1. Use integration by parts on the two subintervals to show that every real eigenvalue is positive. Explain cancellation of the interface terms.
2. Set $k = \sqrt{\lambda}$ and write the endpoint-compatible sine on each side. Derive the two interface equations for their amplitudes and a determinant condition.
3. With $t = k/4$, factor the determinant and characterize all positive eigenvalues. Treat $\sin t = 0$ separately to avoid losing modes when solving for the amplitude ratio.
4. Identify the smallest eigenvalue and give its eigenfunction with left amplitude one. Verify continuity and the slope ratio at the interface, and sketch the mode with its physical corner.

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Question 10 – Solution

Strategy. Match displacement and stiffness-weighted slope, then preserve every branch of the determinant equation.

Step 1: Exclude nonpositive eigenvalues. Adding the two integrations by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 p(y')^2 dx$: endpoint values vanish and the interface terms cancel because both y and py' match. The right side is positive for a nonzero admissible function; equality would make it constant on both sides, hence identically zero. Therefore $\lambda > 0$.

Step 2: Form the interface system. The endpoint-compatible pieces are $A\sin(kx)$ on the left and $B\sin(k(1-x)/2)$ on the right. With $t = k/4$, matching gives

$$A\sin 2t = B\sin t, \qquad A\cos 2t = -2B\cos t.$$

The second equation uses flux, not ordinary derivative. The determinant vanishes exactly when $2\sin 2t\cos t + \cos 2t\sin t = 0$.

Step 3: Preserve both spectral branches. Factoring yields $\sin t(6\cos^2 t - 1) = 0$. Thus the complete spectrum is

$$\boxed{\lambda = 16t^2, \qquad t > 0, \quad t = m\pi \ (m \geq 1) \text{ or } \cos^2 t = 1/6.}$$

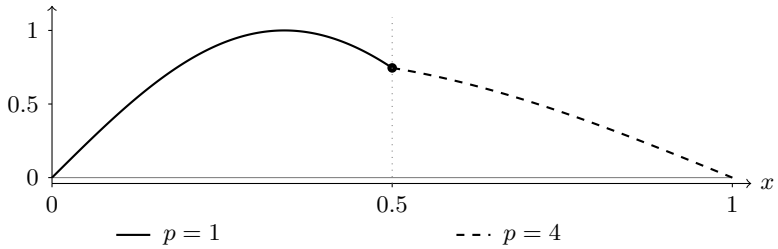
At $t = m\pi$, the first matching equation is zero and the second gives $B = (-1)^{m+1}A/2$. On the other branch, $\sin t \neq 0$ and $B = 2A\cos t$. Each branch gives a nonzero solution and a one-dimensional eigenspace. The apparent root $t = 0$ is excluded by Step 1.

Step 4: Construct the first mode and its corner. The smallest positive root is $t_0 = \arccos(1/\sqrt{6}) \in (0, \pi/2)$. Hence $\boxed{\lambda_1 = 16t_0^2}$ and, taking $A = 1$,

$$\boxed{y = \begin{cases} \sin(4t_0 x), & 0 \leq x \leq 1/2, \\ \sqrt{2/3}\sin(2t_0(1-x)), & 1/2 \leq x \leq 1. \end{cases}}$$

At the join both values equal $\sqrt{5}/3$. The one-sided slopes are $-8t_0/3$ and $-2t_0/3$, so $y'(1/2-) = 4y'(1/2+)$ as required by flux continuity. The slope change is physical; smoothing it would violate the model.

First interface mode: continuous flux, unequal slopes



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Question 1

A positive number T is a period of f if $f(x+T) = f(x)$ for every real x. A fundamental period is the smallest positive period, when one exists. Consider

$$f(x) = \sin 6x + \cos 10x.$$

You may use linear independence of sines and cosines at distinct positive integer frequencies; justify that fact here by integrating products on $[0, 2\pi]$.

Tasks

1. Find a common period of the two summands. Explain why a common period alone does not prove that the sum has that fundamental period.
2. Expand $f(x+T) - f(x)$ and use independence to characterize every positive period of f. Determine the fundamental period exactly.
3. Find the fundamental period of $g(x) = 3f(2x-1) + 7$. Explain the effects of translation, nonzero amplitude scaling and a constant offset.
4. Decide whether a nonzero constant function and the zero function have fundamental periods. Explain why the definition cannot simply assign either function period zero as a fundamental period.

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Question 1 – Solution

Strategy. Prove that a period of the sum must preserve each of its distinct frequency components.

Step 1: Find a candidate period. The summands have periods $\pi/3$ and $\pi/5$, and π is a common period. Thus π is a period of f. In general, cancellation can give a sum a shorter period or even make it constant, so minimality still needs proof.

Step 2: Prove necessity and minimality. Product-to-sum identities show that distinct positive-integer sine/cosine frequencies have zero product integral on $[0, 2\pi]$, while each squared integral equals π. Multiplying a zero linear combination by each component and integrating therefore forces every coefficient to vanish. Expanding the shift of f now gives

$$\cos 6T = 1, \quad \sin 6T = 0, \quad \cos 10T = 1, \quad \sin 10T = 0.$$

Equivalently, $6T = 2\pi m$ and $10T = 2\pi n$ for integers m, n. Then $5m = 3n$, so $m = 3j, n = 5j$ and $T = j\pi$. Conversely every such positive shift works. Hence

$$\boxed{\text{Positive periods: } j\pi \ (j = 1, 2, \ldots), \qquad T_0 = \pi.}$$

Step 3: Transform the independent variable. Since $2x - 1$ ranges over all real numbers, a shift $T > 0$ preserves g exactly when $2T$ preserves f. Thus $\boxed{T_g = \pi/2}$. Translation, multiplication by a nonzero constant and addition of a constant do not change the set of periods; the argument dilation rescales it.

Step 4: Treat constant functions separately. Every positive number is a period of a constant function, including the zero function. There is no smallest positive number, so neither has a fundamental period. Zero is excluded by the definition and cannot be used to repair this lack of a least positive period.

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Question 2

Consider a signal with incommensurate frequencies:

$$f(x) = \cos x + \cos(\sqrt{2}\, x).$$

A graph over a long interval may suggest repeated patterns without establishing an exact period. Recall that $\sqrt{2}$ is irrational.

Tasks

1. Show that $f(x) \leq 2$ and determine every point where equality holds. Use this to prove that f has no positive period.
2. For a second proof, apply $D^2 + 2$ to f, where $D = d/dx$. Explain why a period of a smooth function is also a period of this derivative combination, and derive a contradiction.
3. Set $T = 140\pi$. Prove a uniform bound on $|f(x+T) - f(x)|$ using $99^2 - 2 \cdot 70^2 = 1$ and $|\cos(u+v) - \cos u| \leq |v|$.
4. Decide whether this small uniform discrepancy makes T an exact period. Contrast the result with $h(x) = \cos x + \cos(3x/2)$, and determine the fundamental period of h.

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Question 2 – Solution

Strategy. Distinguish exact simultaneous phase return from a uniformly small phase mismatch.

Step 1: Use the exact maximum. Each cosine is at most one. Equality $f(x) = 2$ requires $x = 2\pi m$ and $\sqrt{2}\,x = 2\pi n$ for integers m, n. Irrationality forces $m = n = 0$, so the maximum is attained only at $x = 0$. If $T > 0$ were a period, then $f(T) = f(0) = 2$, contradicting this uniqueness.

Step 2: Isolate the frequencies by differentiation. Direct differentiation gives $(D^2 + 2)f = \cos x$. Differentiating an identity $f(x+T) = f(x)$ twice preserves it, so T would also be a period of $\cos x$. Subtracting this cosine from f shows that T preserves $\cos(\sqrt{2}x)$ too. Evaluating both shifted cosines at zero requires $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$, impossible for positive T.

Step 3: Bound an approximate return. At $T = 140\pi$, the first cosine returns exactly. The second phase shift is $2\pi(70\sqrt{2}) = 2\pi \cdot 99 - \delta$, where

$$\delta = 2\pi(99 - 70\sqrt{2}) = \frac{2\pi}{99 + 70\sqrt{2}}.$$

The stated cosine inequality therefore yields

$$\boxed{\sup_{x\in\mathbb{R}} |f(x + 140\pi) - f(x)| \le \frac{2\pi}{99 + 70\sqrt{2}} < 0.032.}$$

Step 4: Contrast approximate and exact recurrence. The discrepancy bound is positive and does not make T a period; Step 1 rules out every positive exact period. For h, any period must send its value two at zero to another value two. Thus $T = 2\pi m$ and $3T/2 = 2\pi n$, forcing m even. Conversely 4π preserves both summands. Hence $\boxed{T_h = 4\pi}$, an exact fundamental period.

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Question 3

Define the two-periodic extension of a ramp by the unique representation

$$x = 2m + r, \qquad m \in \mathbb{Z}, \quad -1 \le r < 1, \qquad s(x) = r.$$

The half-open convention specifies the actual function values at the joins. Also let p be the one-periodic extension of $x(1 - x)$ from $[0, 1)$.

Tasks

1. Prove that s has fundamental period two. Find its value and one-sided limits at every odd integer, and sketch it on $[-3, 3]$ with correct open and closed points.
2. Determine whether $s(-x) = -s(x)$ holds everywhere. Change only its join values to produce an odd, two-periodic function, and decide whether that change can make it continuous.
3. Prove that $\int_a^{a+2} s(x)dx = 0$ for every real a. Explain why changing the join values does not affect this statement.
4. Determine the continuity and differentiability of p at integers. Explain why matching function values at the ends of a defining interval is weaker than matching first derivatives.

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Question 3 – Solution

Strategy. Separate periodicity, assigned endpoint values, one-sided limits and integral properties.

Step 1: Locate the joins and establish minimality. The representation gives $s(x+2) = s(x)$. Its discontinuities occur precisely at odd integers. Any period must preserve this set, so a positive period is a positive even integer; two is therefore fundamental. At $c = 2m+1$,

$$\boxed{s(c) = -1, \qquad s(c-) = 1, \qquad s(c+) = -1.}$$

Each increasing segment has its left endpoint included and its right endpoint excluded. The graph also includes the assigned value $s(3) = -1$.

Step 2: Repair oddness, not continuity. Away from odd integers, $s(-x) = -s(x)$. At an odd integer both $s(x)$ and $s(-x)$ equal -1, so pointwise oddness fails. Assigning zero at every odd integer produces an odd function and preserves two-periodicity. No assigned join value can reconcile the unequal one-sided limits, so the jumps remain.

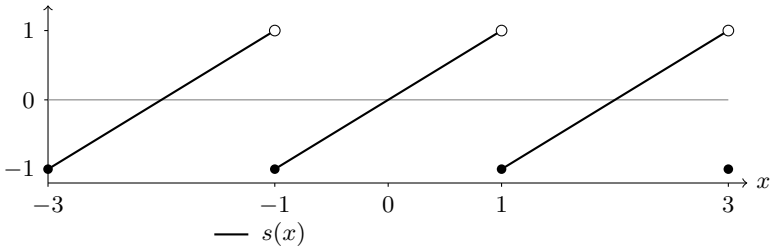
Step 3: Integrate over any full period. The integral on $[-1,1]$ is $\int_{-1}^{1} x\,dx = 0$. Reduce a modulo two to $b \in [-1,1)$. Splitting at one and shifting the remaining piece by two gives

$$\int_a^{a+2} s = \int_b^1 s + \int_1^{b+2} s = \int_b^1 s + \int_{-1}^{b} s = 0.$$

Changing finitely many join values in any bounded interval leaves these Riemann integrals unchanged. Integral cancellation does not require pointwise oddness.

Step 4: Check a continuous periodic join. For p, both endpoint limits of $x(1-x)$ are zero, matching the assigned value at each integer. Thus p is continuous everywhere. The left derivative at an integer is -1, while the right derivative is 1, so it is not differentiable there. Value matching alone does not give derivative matching.

Half-open ramp: left endpoints included, right limits excluded



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Question 4

Orthogonality depends on the interval used in the inner product. For $L > 0$, let

$$\langle u, v \rangle_{a,L} = \int_a^{a+L} u(x)v(x)dx.$$

Examine $u = \sin x$ and $v = \cos x$.

Tasks

1. Compute $\langle u, v \rangle_{a,L}$ exactly. Characterize all lengths L for which u and v are orthogonal for every starting point a.
2. When $L = \pi/2$, find all starting points that give orthogonality. Explain why the same length can work at one location and fail at another.
3. On $[0, \pi/2]$, compute both squared norms and the inner product of the individually normalized functions. Decide whether normalization creates orthogonality.
4. On a window of length $m\pi$, m a positive integer, give an orthonormal pair obtained by rescaling u, v. Sketch their product on $[0, \pi]$ to explain cancellation without treating the graph as proof.

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Question 4 – Solution

Strategy. Compute the product integral before deciding whether a pair is orthogonal or merely normalized.

Step 1: Determine the window dependence. Using $\sin x \cos x = \frac{1}{2}\sin 2x$ gives

$$\langle u, v\rangle_{a,L} = \frac{\cos 2a - \cos(2a+2L)}{4} = \boxed{\frac{1}{2}\sin L \sin(2a+L).}$$

It vanishes for all a exactly when $\sin L = 0$. The positive lengths are $\boxed{L = m\pi,\ m = 1, 2, \dots}$. If $\sin L \neq 0$, one can choose a with $\sin(2a+L) = 1$, proving necessity.

Step 2: Locate the shorter orthogonal windows. For $L = \pi/2$, orthogonality requires $\sin(2a+\pi/2) = 0$. Hence

$$\boxed{a = -\pi/4 + j\pi/2, \qquad j \in \mathbb{Z}.}$$

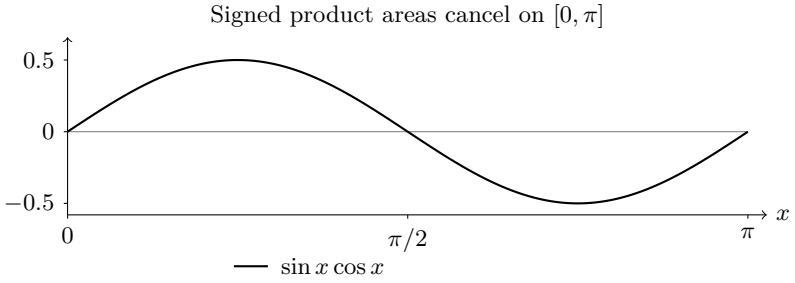
The product changes sign; shifting the window can balance its signed areas. At $a = 0$, however, the product is positive inside the interval.

Step 3: Test individual normalization. On $[0, \pi/2]$, $\|u\|^2 = \|v\|^2 = \pi/4$ and $\langle u, v\rangle = 1/2$. The unit-norm functions are $\widehat{u} = 2\sin x/\sqrt{\pi}$ and $\widehat{v} = 2\cos x/\sqrt{\pi}$, with

$$\boxed{\langle \widehat{u}, \widehat{v}\rangle = 2/\pi \neq 0.}$$

Nonzero scaling cannot turn a nonzero inner product into zero.

Step 4: Normalize on the correct windows. Over any length $m\pi$, the oscillatory term in either squared integral cancels, so both squared norms equal $m\pi/2$. Thus $\boxed{\sqrt{2/(m\pi)}\sin x,\ \sqrt{2/(m\pi)}\cos x}$ are orthonormal on every such window. The graph shows equal positive and negative product areas on $[0, \pi]$; the integral calculation establishes their exact cancellation.



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Question 5

Use $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Construct monic orthogonal polynomials from $1, x, x^2, x^3$; monic means the leading coefficient is one.

Tasks

1. Explain which inner products vanish by parity. Starting with $p_0 = 1$ and $p_1 = x$, find monic p_2, p_3 so that all four polynomials are mutually orthogonal.
2. Compute their squared norms exactly and turn them into an orthonormal set. Distinguish monic normalization from unit-norm normalization.
3. Find the unique quadratic-or-lower polynomial minimizing $\int_{-1}^{1}[x^3 - q(x)]^2 dx$. Give the exact minimum error and justify uniqueness.
4. A proposed shortcut merely divides each monomial by its norm. Give an explicit inner product showing why this does not produce an orthogonal family.

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Question 5 – Solution

Strategy. Use parity to eliminate unnecessary projections, then compute the remaining moments exactly.

Step 1: Construct the monic family. An odd integrand integrates to zero on $[-1, 1]$, so every even polynomial is orthogonal to every odd polynomial. For $p_2 = x^2 - c$, the equation $\langle p_2, 1\rangle = 2/3 - 2c = 0$ gives $c = 1/3$. For $p_3 = x^3 - dx$, $\langle p_3, x\rangle = 2/5 - 2d/3 = 0$ gives $d = 3/5$. Thus

$$\boxed{p_0 = 1, \quad p_1 = x, \quad p_2 = x^2 - 1/3, \quad p_3 = x^3 - 3x/5.}$$

Parity and these two moment equations verify every distinct pair.

Step 2: Compute and apply the norms. Integrating the squares term by term yields

$$\boxed{\|p_0\|^2 = 2, \quad \|p_1\|^2 = 2/3, \quad \|p_2\|^2 = 8/45, \quad \|p_3\|^2 = 8/175.}$$

An orthonormal set is therefore $\boxed{p_0/\sqrt{2},\ \sqrt{3/2}\,p_1,\ \sqrt{45/8}\,p_2,\ \sqrt{175/8}\,p_3}$. Monic normalization fixes the leading coefficient; unit normalization fixes the squared integral and generally changes that coefficient.

Step 3: Find the best quadratic approximation. The decomposition $x^3 = 3x/5 + p_3$ has its residual orthogonal to every quadratic. For any other quadratic q, the cross term vanishes, giving

$$\|x^3 - q\|^2 = \|p_3\|^2 + \|3x/5 - q\|^2.$$

Hence $\boxed{q_* = 3x/5, \quad \min \|x^3 - q\|^2 = 8/175}$. Equality forces the continuous polynomial $q - 3x/5$ to have zero norm and therefore vanish identically, proving uniqueness.

Step 4: Reject normalization without orthogonalization. The normalized versions of 1 and x^2 are $1/\sqrt{2}$ and $\sqrt{5/2}\,x^2$. Their inner product is $\boxed{\sqrt{5}/3 \neq 0}$. Normalizing lengths alone does not remove projections onto the preceding functions.

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Question 6

On continuous real functions on $[0, 1]$, define

$$\langle f, g\rangle_w = \int_0^1 xf(x)g(x)dx.$$

The weight vanishes at one endpoint. Seek a monic quadratic $q(x) = x^2 + ax + b$ orthogonal to both 1 and x in this inner product.

Tasks

1. Prove positive definiteness despite the zero weight at $x = 0$. Explain the role of continuity of the functions.
2. Determine a, b from the two weighted moment equations. Verify both equations and prove uniqueness of the monic quadratic.
3. Compute $\|q\|_w^2$, give its unit-norm version with positive leading coefficient, and locate its roots. Sketch q on $[0, 1]$.
4. Test whether the same q is orthogonal to 1 for unweighted integration. Also explain why replacing the weight by $x - 1/2$ would fail to define an inner product on this space.

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Question 6 – Solution

Strategy. Include the weight in every moment, and distinguish an isolated zero of a nonnegative weight from a sign-changing weight.

Step 1: Establish positive definiteness. Symmetry and bilinearity follow from integration, and $\int_0^1 xf^2dx \geq 0$. If a continuous f is not identically zero, continuity supplies an interval of positive length inside $(0, 1]$ on which $f^2 > 0$, even if the first known nonzero value is at zero. The weighted integral is then positive. Thus zero weighted norm forces $f = 0$ everywhere.

Step 2: Solve the weighted moment equations. The two conditions are

$$\frac{1}{4} + \frac{a}{3} + \frac{b}{2} = 0, \qquad \frac{1}{5} + \frac{a}{4} + \frac{b}{3} = 0.$$

Their coefficient determinant is $1/9 - 1/8 = -1/72 \neq 0$, so the solution is unique:

$$\boxed{a = -6/5, \qquad b = 3/10, \qquad q = x^2 - 6x/5 + 3/10.}$$

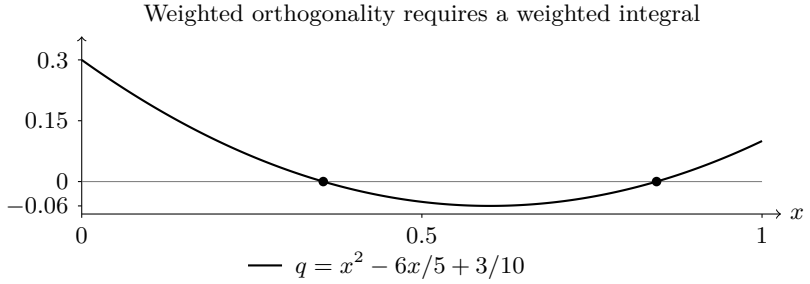
Substitution gives $1/4 - 2/5 + 3/20 = 0$ and $1/5 - 3/10 + 1/10 = 0$.

Step 3: Normalize and locate the sign changes. Expansion and integration give

$$\int_0^1 xq^2dx = \frac{1}{6} - \frac{12}{25} + \frac{51}{100} - \frac{6}{25} + \frac{9}{200} = \boxed{\frac{1}{600}}.$$

Thus the unit-norm version is $\boxed{\sqrt{600}\, q}$. The roots are $\boxed{(6 - \sqrt{6})/10,\ (6 + \sqrt{6})/10}$, both strictly inside $(0, 1)$. The quadratic is positive outside these roots and negative between them.

Step 4: Check two changes of weight. Without the weight, $\int_0^1 q\, dx = 1/3 - 3/5 + 3/10 = \boxed{1/30 \neq 0}$. Orthogonality therefore changes with the weight. For the proposed signed weight, the nonzero constant function one would have squared "norm" $\int_0^1 (x - 1/2)dx = 0$. This violates positive definiteness, so that bilinear form is not an inner product on the stated space.



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Question 7

Let $p \in C^1[a,b]$ be positive, let q, w be continuous with $w > 0$, and let u, v be nonzero real C^2 eigenfunctions:

$$-(pu')' + qu = \lambda wu, \qquad -(pv')' + qv = \mu wv.$$

At each endpoint $c = a, b$, both satisfy the same condition $A_c y(c) + B_c p(c) y'(c) = 0$, with real $(A_c, B_c) \neq (0,0)$.

Tasks

1. Derive an identity relating $(\lambda - \mu) \int_a^b wuv\,dx$ to endpoint terms by multiplying and subtracting the differential equations.
2. Prove the endpoint terms vanish for all the stated boundary conditions, including $B_c = 0$. Deduce weighted orthogonality when $\lambda \neq \mu$.
3. Apply the result to $u = \sin \pi x$ and $v = \sin 2\pi x$ on $[0,1]$ with zero endpoint values. Give an orthonormal pair.
4. Explain why equal eigenvalues do not force orthogonality. On $[0, 2\pi]$ with periodic value-and-slope matching, use $u = \sin x$ and $v = \sin x + \cos x$ as a counterexample, then orthogonalize and normalize this pair.

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Question 7 – Solution

Strategy. Track the boundary expression before dividing by an eigenvalue difference.

Step 1: Derive the boundary identity. Multiply the equation for u by v and that for v by u, then subtract. The q terms cancel, and integration gives

$$\boxed{(\lambda - \mu)\int_a^b wuv\,dx = [p(uv' - u'v)]_a^b.}$$

This identity is valid before any orthogonality conclusion is drawn.

Step 2: Use the actual endpoint conditions. At an endpoint with $B_c = 0$, $A_c \neq 0$ and both function values are zero, so the boundary expression is zero. If $B_c \neq 0$, then $pu' = -A_c u/B_c$ and $pv' = -A_c v/B_c$; substitution again gives zero. Therefore $\lambda \neq \mu$ implies $\boxed{\int_a^b wuv\,dx = 0}$. Equal eigenvalues give only the identity $0 = 0$.

Step 3: Normalize two distinct Dirichlet modes. Here $p = w = 1, q = 0$, with eigenvalues π^2 and $4\pi^2$. The theorem gives orthogonality, and direct integration of each square gives $1/2$. Thus $\boxed{\sqrt{2}\sin \pi x,\ \sqrt{2}\sin 2\pi x}$ are orthonormal. The normalization is separate from the boundary proof of orthogonality.

Step 4: Handle a repeated eigenvalue. Both $u = \sin x$ and $v = \sin x + \cos x$ have eigenvalue one and satisfy periodic matching on $[0, 2\pi]$. Nevertheless, $\langle u, v\rangle = \pi \neq 0$. Since $\|u\|^2 = \pi$, subtracting the projection gives $v - \langle v, u\rangle u/\|u\|^2 = \cos x$. The resulting orthonormal pair is

$$\boxed{\sin x/\sqrt{\pi}, \qquad \cos x/\sqrt{\pi}.}$$

Distinct eigenvalues ensure orthogonality under the hypotheses; a repeated eigenspace may instead require an orthogonal choice of basis.

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Question 8

Approximate $f(x) = |x|$ on $[-1, 1]$ by a polynomial $g(x) = a + bx^2$. Use the squared error $\|f - g\|_2^2 = \int_{-1}^{1}(f - g)^2 dx$. For this inner product, 1 and $p_2 = x^2 - 1/3$ are orthogonal, with squared norms 2 and 8/45 respectively.

Tasks

1. Compute the projection coefficients onto $1, p_2$ and find the unique polynomial g_* minimizing the squared integral error.
2. Verify directly that the residual is orthogonal to both 1 and x^2. Prove the error decomposition that establishes the minimum and uniqueness.
3. Compute the exact minimum squared error. Sketch f and g_*, and identify where the largest absolute residual occurs.
4. Does the least-squares optimum also minimize the maximum absolute error? Compare it with $h(x) = 1/8 + x^2$ and justify your conclusion by exact uniform error bounds.

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Question 8 – Solution

Strategy. Use an orthogonal basis for the integral objective, then test the distinct uniform-error objective separately.

Step 1: Compute the projection. The needed moments are $\langle f, 1\rangle = 1$ and $\langle f, p_2\rangle = 1/2 - 1/3 = 1/6$. Thus

$$g_* = \frac{1}{2} + \frac{1/6}{8/45}(x^2 - 1/3) = \boxed{\frac{3}{16} + \frac{15}{16}x^2.}$$

Step 2: Verify residual orthogonality and uniqueness. For $r = f - g_*$,

$$\int_{-1}^{1} r\,dx = 1 - \frac{3}{8} - \frac{5}{8} = 0, \qquad \int_{-1}^{1} rx^2 dx = \frac{1}{2} - \frac{1}{8} - \frac{3}{8} = 0.$$

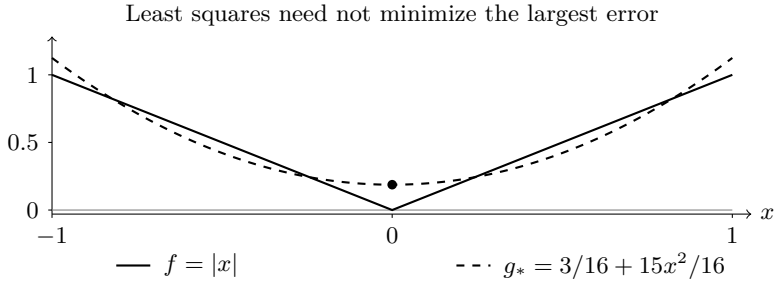
For any other $g = a + bx^2$, the cross term vanishes, so $\|f - g\|_2^2 = \|r\|_2^2 + \|g_* - g\|_2^2$. The second term is zero only when the two continuous polynomials agree, proving both optimality and uniqueness.

Step 3: Evaluate the two errors for the optimum. Subtracting the squared projection norm from $\|f\|_2^2 = 2/3$ gives

$$\boxed{\|r\|_2^2 = \frac{2}{3} - \frac{1}{2} - \frac{(1/6)^2}{8/45} = \frac{1}{96}.}$$

On $[0, 1]$, $r = x - 3/16 - 15x^2/16$ has maximum $19/240$ at $x = 8/15$, and endpoint values $-3/16, -1/8$. By evenness, the largest absolute error is $\boxed{\|r\|_\infty = 3/16}$, attained at zero.

Step 4: Disprove uniform optimality. For $h = 1/8 + x^2$, put $t = |x| \in [0, 1]$. Then $f - h = t(1-t) - 1/8 \in [-1/8, 1/8]$, with both extremes attained. Hence $\boxed{\|f - h\|_\infty = 1/8 < 3/16}$. This competitor proves that the unique least-squares solution is not a uniform best approximation. Different error objectives need not select the same function.



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Question 9

A continuous signal f on $[-\pi, \pi]$ is known only through three moments:

$$\int_{-\pi}^{\pi} f\,dx = 2\pi, \qquad \int_{-\pi}^{\pi} f\cos x\,dx = \pi, \qquad \int_{-\pi}^{\pi} f\sin x\,dx = -2\pi.$$

Use the ordinary integral inner product. The three measurement functions $1, \cos x, \sin x$ are mutually orthogonal.

Tasks

1. Find the unique function in their span that matches all three measurements. Compute its squared norm.
2. Characterize every continuous signal with those measurements as a determined part plus an undetermined residual. Prove a sharp lower bound for $\int_{-\pi}^{\pi} f^2 dx$ and characterize equality.
3. Exhibit infinitely many distinct compatible signals explicitly, and compute their squared norms. Explain why linear independence of the measurement functions does not imply recovery of an arbitrary signal.
4. Suppose a fourth measurement gives $\int_{-\pi}^{\pi} f\cos 2x\,dx = 0$. Does that make the signal unique? Give an explicit counterexample and distinguish a finite orthogonal family from a complete representation.

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Question 9 – Solution

Strategy. Orthogonal measurements determine a finite projection, while all perpendicular components remain unseen.

Step 1: Recover the measured component. The squared norms of $1, \cos x, \sin x$ are $2\pi, \pi, \pi$. Dividing each moment by its corresponding squared norm gives

$$\boxed{p(x) = 1 + \cos x - 2\sin x, \qquad \|p\|^2 = 7\pi.}$$

Orthogonality makes these coefficients unique within the three-dimensional span.

Step 2: Derive the sharp energy bound. Every compatible f has the form $\boxed{f = p + r}$, where r is continuous and orthogonal to all three measurement functions. Conversely every such residual preserves the measurements. Since $\langle p, r\rangle = 0$,

$$\boxed{\|f\|^2 = 7\pi + \|r\|^2 \geq 7\pi.}$$

Equality holds exactly when $r = 0$ everywhere: a continuous function with zero squared integral is identically zero. Thus p is the unique minimum-energy signal.

Step 3: Exhibit the invisible freedom. Product-to-sum identities give zero inner products between $\cos 2x$ and each of $1, \cos x, \sin x$. Therefore

$$\boxed{f_C = p + C\cos 2x, \quad C \in \mathbb{R}, \qquad \|f_C\|^2 = 7\pi + C^2\pi.}$$

The measurement functions are independent, but their span does not contain all continuous functions. The data determine the projection, not its orthogonal complement.

Step 4: Test the fourth measurement. The added measurement removes the particular $\cos 2x$ freedom above, but $\boxed{p + D\sin 2x}$ satisfies all four conditions for every real D. The required product integrals again vanish by parity or product-to-sum. A finite orthogonal family supplies independent coordinates for its own span; it is not a complete representation of all continuous signals on the interval.

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Question 10

Let $u(x) = \sin x$ and $v(x) = \sin 5x$ on $[0, 2\pi]$. Compare the continuous average inner product with its equally spaced sampling version:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f g\, dx, \qquad \langle f, g\rangle_N = \frac{1}{N}\sum_{j=0}^{N-1} f(2\pi j/N) g(2\pi j/N).$$

The sampled form may fail to be positive definite on functions even though it is positive definite on the vectors of sampled values.

Tasks

1. Compute the continuous Gram matrix of u, v; a Gram matrix has entries consisting of all pairwise inner products.
2. For $N = 4$, list both sample vectors and compute the sampled Gram matrix. Exhibit a nonzero continuous function in their span with zero sampled norm, and sketch the curves with the common sampled points.
3. For integer m, derive $S_m = N^{-1}\sum_{j=0}^{N-1}\cos(2\pi mj/N)$ using a finite geometric sum. Express every entry of the sampled Gram matrix in terms of S_2, S_4, S_6, S_{10}.
4. Find the smallest integer $N \geq 3$ for which the sampled Gram matrix equals the continuous one. Check every smaller candidate and explain why exactness for this pair does not imply exactness for all continuous functions.

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Question 10 – Solution

Strategy. Distinguish continuous orthogonality from orthogonality of a finite set of sampled values.

Step 1: Compute the continuous matrix. Product-to-sum gives $\langle u, v\rangle = 0$, while each sine has average square $1/2$. Hence $\boxed{G = \frac{1}{2} I_2}$.

Step 2: Identify the four-point alias. At $0, \pi/2, \pi, 3\pi/2$, both functions give $(0, 1, 0, -1)$. Thus

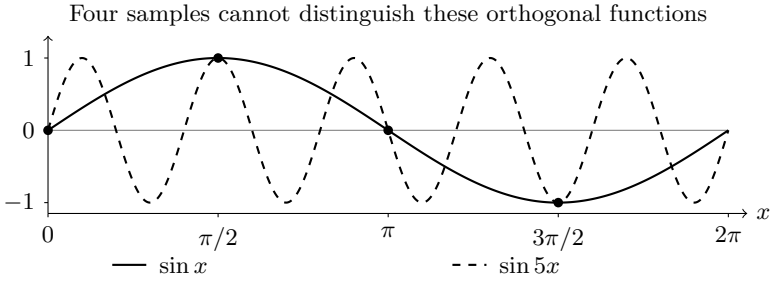
$$\boxed{G_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.}$$

The nonzero function $u - v$ vanishes at every sample, so its sampled norm is zero, although its continuous squared norm is one. The samples have lost the distinction.

Step 3: Derive the exact sampling formulas. Put $z = e^{2\pi i m/N}$. If N divides m, every summand equals one. Otherwise $z \neq 1$ and $\sum_{j=0}^{N-1} z^j = (1 - z^N)/(1 - z) = 0$. Taking real parts yields $S_m = 1$ when $N \mid m$, and $S_m = 0$ otherwise. Product-to-sum now gives

$$\boxed{G_N = \frac{1}{2}\begin{pmatrix} 1 - S_2 & S_4 - S_6 \\ S_4 - S_6 & 1 - S_{10} \end{pmatrix}.}$$

Step 4: Find the smallest exact sample count. For $N = 3$ the off-diagonal entry is $-1/2$; for $N = 4$ it is $1/2$. For $N = 5$, the second diagonal entry is zero. For $N = 6$ the off-diagonal entry is $-1/2$. None matches G. For $N = 7$, none of $2, 4, 6, 10$ is divisible by seven, so $\boxed{G_7 = G}$ and seven is the smallest allowed count. This is exact only for the tested products: for example, $\sin 7x$ vanishes at all seven sample points but has continuous average square $1/2$.



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Question 1

Let $f(x) = x$ on $[0, \pi]$. Define its sine coefficients and partial sums by

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx, \qquad S_N = \sum_{n=1}^{N} b_n \sin(nx).$$

For a piecewise smooth odd periodic extension, the Fourier series converges to the average of its one-sided limits, including at periodic joins.

Tasks

1. Derive b_n by integration by parts. Describe the odd 2π-periodic extension, including its jump at odd multiples of π.
2. Determine the series sum at every $x \in [0, \pi]$. Distinguish the assigned endpoint value $f(\pi)$ from the series limit.
3. Decide whether S_N converges uniformly to f on $[0, \pi]$, or to x on the open interval $(0, \pi)$. Give a quantitative obstruction for both claims.
4. Use the value at $x = \pi/2$ to evaluate $1 - 1/3 + 1/5 - \cdots$. Sketch S_{12} with f and explain why the endpoint mismatch does not invalidate the interior representation.

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Question 1 – Solution

Strategy. Determine the odd extension before applying the convergence theorem at its endpoints.

Step 1: Compute the coefficients and extension. Integration by parts yields

$$\int_0^\pi x\sin(nx)dx = [-x\cos(nx)/n]_0^\pi + [\sin(nx)/n^2]_0^\pi = -\pi(-1)^n/n.$$

Hence $\boxed{b_n = 2(-1)^{n+1}/n}$. The odd extension equals x on $(-\pi,\pi)$ and repeats every 2π. At an odd multiple of π its left and right limits are π and $-\pi$. The join value may be chosen zero for an odd periodic representative.

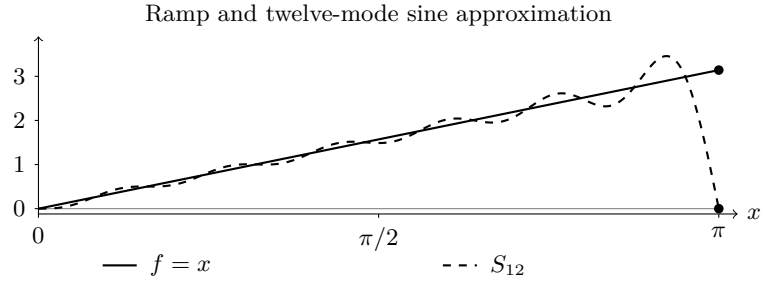
Step 2: Apply the convergence theorem. The series equals x for $0 < x < \pi$. At zero it equals zero by continuity of the extension, and at π it equals the jump average zero:

$$\boxed{\sum_{n\geq 1} \frac{2(-1)^{n+1}}{n}\sin(nx) = x \ (0 \leq x < \pi), \qquad \text{sum at } \pi = 0.}$$

This does not reproduce the separately assigned value $f(\pi) = \pi$.

Step 3: Rule out uniform convergence near the join. On the closed interval, $|f(\pi) - S_N(\pi)| = \pi$ for every N. On the open interval, continuity of each finite sum gives $\lim_{x\uparrow\pi} |x - S_N(x)| = \pi$, so the supremum error is at least π. Neither asserted uniform convergence holds, despite pointwise convergence inside.

Step 4: Evaluate an interior numerical series. At $x = \pi/2$, only odd modes remain and their sine values alternate: $\pi/2 = 2(1 - 1/3 + 1/5 - \cdots)$. Therefore the convergent series equals $\boxed{\pi/4}$. The graph displays a finite approximation; the convergence theorem, not endpoint agreement of the original data, justifies the interior sum.



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Question 2

Let $f(x) = 1$ for $0 < x < \pi$, with assigned endpoint values zero. The odd 2π-periodic extension is a square wave. Define its first M nonzero modes by

$$P_M(x) = \frac{4}{\pi} \sum_{j=0}^{M-1} \frac{\sin((2j+1)x)}{2j+1}.$$

You may use convergence to the average of the one-sided limits for piecewise smooth periodic functions.

Tasks

1. Derive the sine coefficients and verify the definition of P_M. State the limiting values inside the interval and at its endpoints.
2. Sum the finite cosine progression to prove $P'_M(x) = (2/\pi)\sin(2Mx)/\sin x$ for $0 < x < \pi$. Locate the first local maximum to the right of zero.
3. Express the height of that maximum as an integral. Rescale the integral and prove that the heights tend to $(2/\pi)\int_0^{\pi} (\sin t)/t \, dt$.
4. Prove that this limiting height is greater than one, using $\sin t/t \geq 1 - t^2/6$ for $0 \leq t \leq \pi/2$ and $1/t \geq 1/\pi$ for $\pi/2 \leq t \leq \pi$. Sketch P_{10} and explain why pointwise convergence does not remove the near-jump overshoot.

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Question 2 – Solution

Strategy. Analyze a moving maximum of a finite sum rather than substitute a moving point into a pointwise convergence claim.

Step 1: Find the nonzero modes. Direct integration gives $b_n = 2(1-(-1)^n)/(\pi n)$: zero for even n and $4/(\pi n)$ for odd n. Thus P_M is the stated truncation. The odd extension has jump averages zero at both endpoints; its sum is one on $(0,\pi)$.

Step 2: Locate the first maximum. A finite geometric sum, or telescoping $2\sin x\cos((2j+1)x)$, gives $\sum_{j=0}^{M-1}\cos((2j+1)x) = \sin(2Mx)/(2\sin x)$. Differentiating the finite sum is legitimate, so

$$\boxed{P_M'(x) = \frac{2}{\pi}\frac{\sin(2Mx)}{\sin x}, \qquad x_M = \frac{\pi}{2M}.}$$

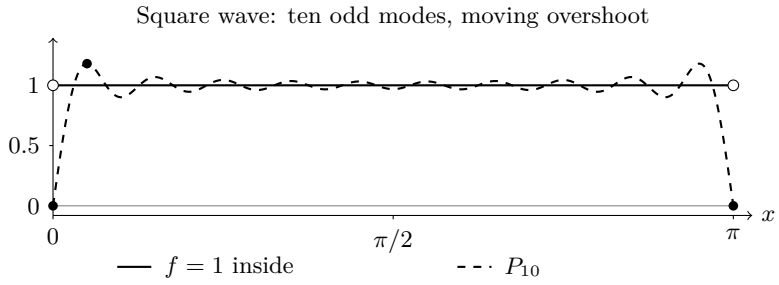
The derivative is positive before x_M and negative just after it, for $M \geq 1$.

Step 3: Pass to the moving-peak limit correctly. Since $P_M(0) = 0$, substitution $t = 2Mx$ gives

$$P_M(x_M) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{2M\sin(t/(2M))}\,dt \longrightarrow \boxed{\frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt}.$$

Write the integrand as $(\sin t/t)(z/\sin z)$, where $z = t/(2M)$. Since $z/\sin z$ extends continuously with value one at zero, this factor tends uniformly to one for $0 \leq t \leq \pi$. Also $|\sin t/t| \leq 1$. These facts justify the integral limit; its value is approximately 1.179.

Step 4: Prove a persistent excess. The supplied inequalities imply $\int_0^{\pi}\sin t/t\,dt \geq \pi/2 - \pi^3/144 + 1/\pi > \pi/2$, since $\pi^4 < 144$. The limiting peak is strictly above one. The peak location moves toward the jump, so convergence at each fixed interior point is consistent with this persistent overshoot. The square wave jumps by two across zero.



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Question 3

A tent on $[0,1]$ has height $H>0$ and peak at $a\in(0,1)$:

$$f(x)=\begin{cases} Hx/a, & 0\le x\le a,\\ H(1-x)/(1-a), & a\le x\le 1.\end{cases}$$

Its sine coefficients are $b_n=2\int_0^1 f(x)\sin(n\pi x)dx$.

Tasks

1. Integrate separately on the two sides of the corner to derive b_n. Explain why the corner contributes even though both endpoint values vanish.
2. For $a=1/3, H=1$, identify all missing modes and prove that the sine series converges uniformly to the tent on the entire closed interval.
3. Suppose the only known coefficient data are $b_1>0$ and $b_2/b_1=1/4$, and the signal is known to be a tent of this form. Recover a, H uniquely in terms of b_1.
4. Compare a tent with its reflection $f(1-x)$. State how each coefficient changes and explain why the absolute values of all coefficients cannot distinguish the two peak locations. Sketch the $a=1/3, H=1$ tent and its first twelve-mode sum.

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Question 3 – Solution

Strategy. Retain the change in slope at the interior corner when integrating by parts.

Step 1: Compute the corner contribution. Put $k = n\pi$. One integration by parts removes the zero endpoint terms; integrating each constant slope then gives

$$\int_0^1 f \sin(kx)dx = \frac{1}{k^2}\left(\frac{H}{a} + \frac{H}{1-a}\right)\sin(ka).$$

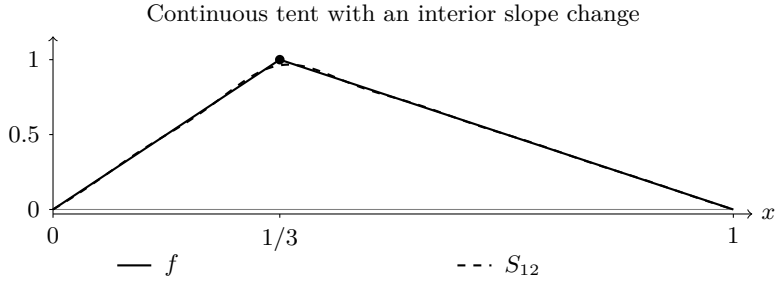
Thus $\boxed{b_n = 2H\sin(n\pi a)/[\pi^2 a(1-a)n^2]}$. The nonzero slope change, not an endpoint displacement, supplies these terms.

Step 2: Identify missing modes and convergence. For $a = 1/3, H = 1$, $\boxed{b_n = 9\sin(n\pi/3)/(\pi^2 n^2)}$. Exactly the multiples of three vanish. The bound $|b_n| \le 9/(\pi^2 n^2)$ gives uniform absolute convergence by the Weierstrass test. The continuous, piecewise linear odd extension satisfies the Fourier convergence theorem, so the sum is the tent at every point, including both endpoints and the corner.

Step 3: Recover the unknown tent. Because $\sin(\pi a) > 0$, $b_2/b_1 = \sin(2\pi a)/(4\sin\pi a) = \frac{1}{2}\cos\pi a$. The measured ratio gives $\cos\pi a = 1/2$, hence $\boxed{a = 1/3}$ uniquely on $(0,1)$. Substituting into b_1 gives

$$\boxed{H = \frac{2\pi^2 b_1}{9\sqrt{3}}.}$$

Step 4: Interpret reflection and lost sign information. Under $x \mapsto 1-x$, $\sin(n\pi(1-x)) = (-1)^{n+1}\sin(n\pi x)$. A change of variable gives $\boxed{b_n^{\text{reflected}} = (-1)^{n+1} b_n}$. Odd modes are unchanged and even modes reverse sign. All magnitudes therefore coincide, so they cannot distinguish a from $1-a$ when those locations differ.



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Question 4

On $[0, \pi]$, let f equal one on $(\pi/3, 2\pi/3)$ and zero elsewhere, including the two jump points. Let S_N denote its first N sine modes.

Tasks

1. Derive every sine coefficient from the defining integral. Use symmetry about $\pi/2$ to explain why all even modes vanish.
2. Determine whether any odd modes vanish. State the sine-series limit at every point of $[0, \pi]$, paying attention to the assigned values at the jumps.
3. Change the assigned values at the two jumps to arbitrary real numbers. Explain what happens to the coefficients, the series and its pointwise limits.
4. Prove that no continuous function can approximate this assigned target with uniform error less than $1/2$. Explain the implication for sine partial sums and sketch S_{15} alongside the pulse.

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Question 4 – Solution

Strategy. Use symmetry for coefficient selection, but use one-sided limits to determine convergence at jumps.

Step 1: Compute and simplify the coefficients. Integration over the nonzero portion gives

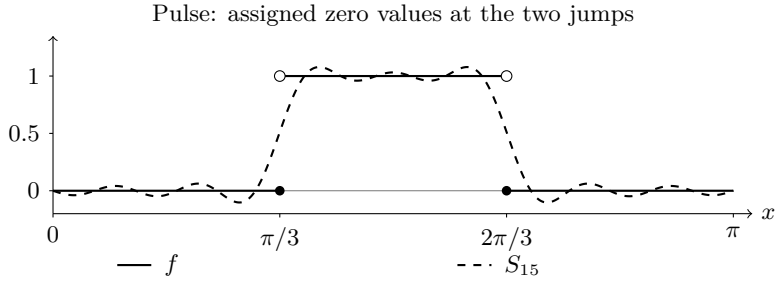
$$\boxed{b_n = \frac{2}{\pi n}\left[\cos(n\pi/3) - \cos(2n\pi/3)\right].}$$

The reflection $f(\pi - x) = f(x)$ implies $b_n = (-1)^{n+1} b_n$, so even modes are zero. For odd n, $\cos(2n\pi/3) = -\cos(n\pi/3)$, giving $b_n = 4\cos(n\pi/3)/(\pi n)$.

Step 2: Account for every mode and every limit. For odd integers n, $\cos(n\pi/3)$ is $1/2$ or -1, never zero. Thus no odd mode vanishes. The odd periodic extension is piecewise constant; the Fourier convergence theorem gives one inside the pulse, zero outside, and $\boxed{1/2}$ at $\pi/3$ and $2\pi/3$. Both endpoints have limit zero. The jump-point sums differ from the assigned target values zero.

Step 3: Change only the assigned values. Changing finitely many points does not change any coefficient integral. The partial sums and their limits are therefore unchanged. The series records the jump averages, not arbitrary isolated assignments at the jumps.

Step 4: Establish the uniform-error obstruction. Let g be continuous and suppose $\sup|f - g| \le E$. Taking one-sided limits at the first jump gives $|g(\pi/3)| \le E$ and $|1 - g(\pi/3)| \le E$. The triangle inequality yields $1 \le 2E$, so $\boxed{E \ge 1/2}$. Every finite sine sum is continuous and obeys this lower bound. Thus pointwise convergence away from the jumps cannot become uniform convergence to this target on the whole interval. The graph shows the finite-sum transition and oscillations.



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Question 5

Let $f(x) = x(\pi - x)$ on $[0, \pi]$ and $S_N = \sum_{n=1}^{N} b_n \sin(nx)$. You may use Parseval's identity $\int_0^\pi f^2 dx = (\pi/2) \sum_{n \geq 1} b_n^2$, and the standard convergence theorem for piecewise smooth odd periodic extensions.

Tasks

1. Derive all coefficients by integration by parts, and identify the missing modes.
2. Prove a uniform tail bound $\|f - S_N\|_\infty \leq 4/(\pi N^2)$ for $N \geq 1$. State why uniform convergence includes both endpoints here.
3. Find the smallest integer N certified by that bound to give error below 10^{-3}. Distinguish a sufficient certificate from the actual smallest successful truncation.
4. Evaluate $\int_0^\pi f^2 dx$ exactly and use Parseval to derive the sum of the reciprocal sixth powers of the positive odd integers.

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Question 5 – Solution

Strategy. Use the zero endpoint values and the constant second derivative to obtain rapidly decaying coefficients.

Step 1: Integrate twice and retain the parity. Because $f(0) = f(\pi) = 0$ and $f'' = -2$, integration by parts gives $\int_0^\pi f \sin(nx)dx = 2(1-(-1)^n)/n^3$. Therefore

$$\boxed{b_n = \frac{4(1-(-1)^n)}{\pi n^3} = \begin{cases} 8/(\pi n^3), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}}$$

Step 2: Bound the entire tail uniformly. The coefficient series is absolutely summable, so it converges uniformly. The continuous odd periodic extension and the convergence theorem identify its sum with f everywhere on $[0, \pi]$. For every x,

$$|f(x) - S_N(x)| \le \frac{8}{\pi} \sum_{n>N} \frac{1}{n^3} \le \frac{8}{\pi} \int_N^\infty t^{-3} dt = \boxed{\frac{4}{\pi N^2}}.$$

Both endpoint values of the target and all partial sums are zero; there is no endpoint mismatch or jump obstructing uniform convergence.

Step 3: Choose a certified truncation. The stated bound is below 10^{-3} when $N > \sqrt{4000/\pi}$. Since this threshold lies between 35 and 36, $\boxed{N = 36}$ is the smallest integer certified by this bound. The estimate discarded the missing even modes and possible cancellation; it does not prove that every smaller N fails.

Step 4: Extract an exact numerical sum. Expanding the polynomial square yields $\int_0^\pi x^2(\pi - x)^2 dx = \pi^5/30$. Parseval gives

$$\frac{\pi^5}{30} = \frac{32}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^6}, \qquad \boxed{\sum_{j=0}^{\infty} \frac{1}{(2j+1)^6} = \frac{\pi^6}{960}}.$$

This conclusion uses the complete series and Parseval, not a finite numerical fit.

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Question 6

For $0 < x < \pi$, the sine series of $f(x) = x$ is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Its partial sums converge to f in $L^2(0, \pi)$ as well as pointwise inside. Here $\|g\|_2 = (\int_0^\pi |g|^2 dx)^{1/2}$.

Tasks

1. Form the series obtained by formally differentiating once. Test the necessary term-to-zero condition at $x = \pi/2$ and decide whether the formal differentiation is valid there.
2. Integrate each finite partial sum from zero to x. Use Cauchy–Schwarz and the given L^2 convergence to justify passing to the limit uniformly for $0 \le x \le \pi$.
3. Derive the resulting cosine identity for $x^2/2$, including both endpoints. Give a uniform tail bound for this integrated series.
4. Evaluate the identity at $x = \pi$ to sum the reciprocal squares of the positive odd integers. Explain why integration is justified here while the attempted differentiation fails.

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Question 6 – Solution

Strategy. Test the operation itself: differentiating weakens coefficient decay, while integration improves it.

Step 1: Reject the formal derivative at a specific point. Formal differentiation produces

$$2\sum_{n\geq 1}(-1)^{n+1}\cos(nx).$$

At $x=\pi/2$, the even-indexed terms have magnitude two and do not tend to zero. Thus this series diverges there and cannot be a valid ordinary termwise derivative representation of $f'=1$.

Step 2: Pass from finite integrals to the limit. For each finite sum S_N, direct integration gives

$$\int_0^x S_N(t)dt = 2\sum_{n=1}^{N}\frac{(-1)^{n+1}}{n^2}(1-\cos nx).$$

Cauchy–Schwarz yields $|\int_0^x (S_N-f)dt| \leq \sqrt{x}\,\|S_N-f\|_2 \leq \sqrt{\pi}\,\|S_N-f\|_2 \to 0$. This bound is independent of x, proving uniform convergence of the integrals.

Step 3: State the integrated identity and its error. Consequently

$$\boxed{\frac{x^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}(1-\cos nx), \qquad 0\leq x\leq \pi.}$$

The tail is bounded uniformly by $4\sum_{n>N} n^{-2} \leq 4/N$. This also proves absolute uniform convergence of the integrated series. At zero both sides vanish; the right endpoint is included by the uniform integral argument, despite the original sine mismatch.

Step 4: Evaluate at the right endpoint. At $x=\pi$ the even terms vanish and each odd term contributes $4/n^2$. Hence $\pi^2/2 = 4\sum_{j\geq 0}(2j+1)^{-2}$, giving $\boxed{\sum_{j\geq 0}(2j+1)^{-2} = \pi^2/8}$. The integration has a proved limiting argument and summable coefficients; the differentiated series fails even a necessary condition for convergence.

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Question 7

On $[0,1]$, approximate $f(x)=e^x$ using sine functions $\sin(n\pi x)$. To preserve the endpoint data, also consider

$$\ell(x)=1+(e-1)x, \qquad r(x)=e^x-\ell(x), \qquad A_N(x)=\ell(x)+\sum_{n=1}^{N} c_n \sin(n\pi x),$$

where c_n are the sine coefficients of r.

Tasks

1. Compute the sine coefficients b_n of e^x exactly. State the ordinary sine-series limits at zero and one.
2. Compute the sine coefficients of ℓ and subtract them to derive c_n. Explain the improvement in decay.
3. Prove that A_N converges uniformly to e^x on $[0,1]$, with error at most $(1+e)/(\pi^3 N^2)$. Find an integer N certified to give error below 10^{-3}.
4. Compare endpoint errors of the raw sine partial sum and A_N. Sketch their errors for $N=12$, and explain why adding the boundary interpolant changes the approximation problem rather than contradicting the endpoint limitation of pure sine sums.

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Question 7 – Solution

Strategy. Remove the nonzero boundary values before asking a sine series for uniform approximation.

Step 1: Compute the raw coefficients. For $k = n\pi$, an antiderivative is $e^x(\sin kx - k\cos kx)/(1+k^2)$. Evaluation gives

$$\boxed{b_n = \frac{2k(1-(-1)^n e)}{1+k^2}.}$$

The odd periodic extension has opposite one-sided values at each endpoint, so its jump averages are zero. Every raw sine partial sum is also zero there.

Step 2: Cancel the leading boundary contribution. Integration of the linear interpolant gives $b_n^\ell = 2(1-(-1)^n e)/k$. Therefore

$$\boxed{c_n = b_n - b_n^\ell = -\frac{2(1-(-1)^n e)}{k(1+k^2)}.}$$

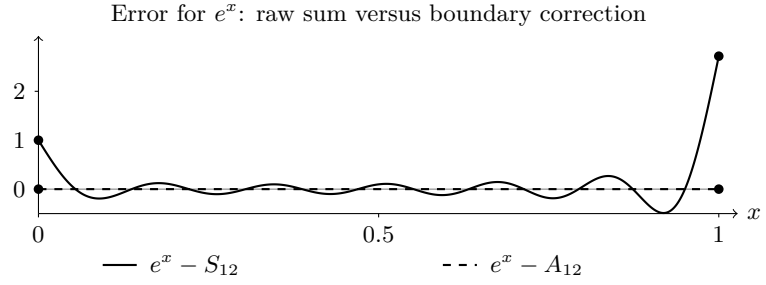
The raw coefficients are of order $1/n$, while c_n are of order $1/n^3$. The residual has zero values at both endpoints; the leading boundary term cancels.

Step 3: Establish a uniform certificate. Since $|c_n| \le 2(1+e)/(\pi^3 n^3)$, the residual series converges absolutely and uniformly. Its continuous piecewise smooth odd extension identifies the sum with r throughout the closed interval. Thus

$$\boxed{\|e^x - A_N\|_\infty \le \frac{1+e}{\pi^3 N^2}.}$$

For $N = 12$, the bound is below 10^{-3}, so twelve modes suffice.

Step 4: Interpret the endpoint comparison. For the raw sum S_N, $f - S_N$ equals one at zero and e at one. For A_N, both endpoint errors are exactly zero, and the entire error has the small bound above. The graph shows these two errors on the same scale. The corrected approximant includes the non-sine function ℓ; it is not a pure sine sum and therefore is not constrained to vanish at the endpoints.



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Question 8

Solve the boundary-value problem

$$-u'' = x, \qquad 0 \le x \le \pi, \qquad u(0) = u(\pi) = 0$$

using the sine coefficients $b_n = 2(-1)^{n+1}/n$ of the forcing x. You may use L^2 convergence of its sine series and the standard theorem allowing differentiation when the series of derivatives converges uniformly and the original series converges at one point.

Tasks

1. Derive the proposed sine coefficients of u from the action of $-d^2/dx^2$ on a sine mode. Independently solve the differential equation in closed form.
2. Verify that the closed-form solution has the proposed sine coefficients using integration by parts and the boundary values.
3. Justify one termwise differentiation of the solution series. Explain why a second differentiation cannot produce the forcing correctly at the right endpoint as an everywhere pointwise identity.
4. State precisely how the second derivatives of the finite solution sums converge in L^2. Prove a uniform error bound for the solution itself and explain the smoothing effect of solving the boundary-value problem.

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Question 8 – Solution

Strategy. Divide each forcing coefficient by its eigenvalue, then use an independent closed-form solution to verify the spectral construction.

Step 1: Obtain two descriptions of the solution. Since $-\big(\sin nx\big)'' = n^2 \sin nx$, the proposed coefficients are $\boxed{a_n = b_n/n^2 = 2(-1)^{n+1}/n^3}$. Direct integration of $u'' = -x$ and the two endpoint conditions gives

$$\boxed{u(x) = \frac{\pi^2 x - x^3}{6}.}$$

Its second derivative is $-x$ and both endpoint values vanish.

Step 2: Verify the actual coefficients. Let $\widehat{u}_n = (2/\pi)\int_0^\pi u \sin nx\, dx$. Twice integrating by parts, using zero endpoint values of u and of the sine, gives $(2/\pi)\int_0^\pi (-u'')\sin nx\, dx = n^2 \widehat{u}_n$. The left side is b_n, proving $\widehat{u}_n = a_n$. The continuous odd extension and the convergence theorem identify the series with this polynomial, including at the endpoints.

Step 3: Justify the first derivative and limit the second. The derivative coefficients have magnitude $2/n^2$, so their cosine series converges uniformly. The original series converges at zero, where it is zero; the stated differentiation theorem therefore applies once. The formal second derivative is $-\sum b_n \sin nx$. At $x = \pi$ every term is zero, whereas $u''(\pi) = -\pi$. Thus the second derivative series cannot equal u'' pointwise everywhere on the closed interval.

Step 4: State the valid convergence and smoothing. For $U_N = \sum_{n=1}^{N} a_n \sin nx$, $U_N'' = -S_N$ exactly. The supplied forcing convergence gives $\boxed{\|U_N'' + x\|_2 \to 0}$; isolated endpoint disagreement does not change this norm. Moreover

$$\boxed{\|u - U_N\|_\infty \le 2\sum_{n>N} n^{-3} \le 1/N^2.}$$

Dividing by n^2 suppresses high-frequency forcing coefficients and gives a uniformly convergent displacement with a uniformly convergent first derivative.

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Question 9

For $0 < \varepsilon < \pi/2$, define a unit-area narrow pulse centered at $\pi/2$:

$$f_\varepsilon(x) = \begin{cases} (2\varepsilon)^{-1}, & |x - \pi/2| < \varepsilon, \\ 0, & \text{otherwise}, \end{cases} \qquad 0 \le x \le \pi.$$

Use $b_n(\varepsilon) = (2/\pi) \int_0^\pi f_\varepsilon \sin nx \, dx$. You may use Parseval and Bessel's inequality for square-integrable functions.

Tasks

1. Verify the area and derive all sine coefficients. Interpret the value of $\sin z/z$ at $z = 0$ by its limit.
2. Find the limit of each fixed coefficient as $\varepsilon \downarrow 0$. Prove that the limiting coefficient sequence cannot be the sine coefficients of any $L^2(0, \pi)$ function.
3. Compute $\|f_\varepsilon\|_2^2$ and use Parseval to find $\sum_{n \ge 1} b_n(\varepsilon)^2$. Explain why a point-mass limit is not an ordinary square-integrable function.
4. Use $|\sin z/z - 1| \le z^2/6$ to bound the largest coefficient error among $1 \le n \le N$. Derive this inequality from $\sin z/z = \int_0^1 \cos(zt) dt$ and explain why the resulting estimate is not uniform over all mode numbers.

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Question 9 – Solution

Strategy. Separate a limit at each fixed frequency from a limit of the complete coefficient sequence in squared norm.

Step 1: Integrate the narrow pulse. The height times the width is one. Writing $a = \pi/2$, integration gives

$$b_n(\varepsilon) = \frac{\cos(n(a-\varepsilon)) - \cos(n(a+\varepsilon))}{\pi\varepsilon n} = \boxed{\frac{2}{\pi}\sin(n\pi/2)\frac{\sin(n\varepsilon)}{n\varepsilon}}.$$

The factor $\sin z/z$ has limiting value one at zero. Even modes vanish.

Step 2: Examine the fixed-mode limit. For each fixed n, $b_n(\varepsilon) \to \beta_n = (2/\pi)\sin(n\pi/2)$. Every odd-indexed β_n has magnitude $2/\pi$, so $\sum \beta_n^2$ diverges. Bessel's inequality would bound this sum for any L^2 function. Hence no square-integrable function has all these limiting coefficients.

Step 3: Track the diverging energy. Direct integration gives $\boxed{\|f_\varepsilon\|_2^2 = 1/(2\varepsilon)}$. Parseval therefore implies

$$\boxed{\sum_{n\geq 1} b_n(\varepsilon)^2 = \frac{1}{\pi\varepsilon}.}$$

The energies grow without bound while area stays one. Indeed, integration against a continuous test function tends to its value at $\pi/2$, by continuity and averaging over the shrinking interval. This is the point-mass limit, not an L^2 function or an ordinary pointwise sine representation of one.

Step 4: Quantify finite-band convergence. From $|1-\cos s| \leq s^2/2$ and the supplied integral representation, $|\sin z/z - 1| \leq \int_0^1 z^2t^2/2\,dt = z^2/6$. Thus

$$\boxed{\max_{1\leq n\leq N} |b_n(\varepsilon) - \beta_n| \leq \frac{N^2\varepsilon^2}{3\pi}.}$$

For each fixed N this tends to zero. It grows with N, so it does not justify uniform convergence across all frequencies or interchange with the infinite energy sum.

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Question 10

A signal on $[0, \pi]$ is

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + 2\sin 10x.$$

An approximation may retain at most three sine modes, with arbitrary real coefficients. Compare this freedom with the usual truncation to the first three frequencies. Use the ordinary squared integral error on $[0, \pi]$.

Tasks

1. Verify all sine coefficients by orthogonality and compute $\|f\|_2^2$.
2. Find the first-three-frequency partial sum and its exact squared error.
3. Find the best approximation using any three sine modes. Prove both the optimal choice of frequencies and their coefficients, and compare its error with the ordinary low-frequency truncation.
4. Suppose only the first three coefficients and the total squared norm are known, without the displayed formula for f. Determine the unseen energy and explain whether these data locate it at frequency ten. Give two distinct compatible signals.

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Question 10 – Solution

Strategy. Orthogonality makes the cost of omitting a mode explicit, so optimal selection depends on coefficient size rather than frequency alone.

Step 1: Identify the coefficient energy. The sine functions have squared norm $\pi/2$ and are mutually orthogonal. Thus $b_1 = 1, b_2 = 1/2, b_3 = 1/3, b_{10} = 2$, with all other coefficients zero. Consequently

$$\boxed{\|f\|_2^2 = \frac{\pi}{2}\left(1 + \frac{1}{4} + \frac{1}{9} + 4\right) = \frac{193\pi}{72}.}$$

Step 2: Evaluate the conventional truncation. The first-three-frequency sum is $S_3 = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$. Its residual is $2\sin 10x$, so $\boxed{\|f - S_3\|_2^2 = 2\pi}$. Frequency order alone discards the largest-amplitude component here.

Step 3: Optimize the three selected modes. For any selected frequencies, orthogonality expresses the error as $\pi/2$ times the sum of squared coefficient differences on selected modes plus the squared coefficients of omitted modes. Each retained coefficient must therefore equal the true coefficient. To minimize the omitted energy, retain the three largest magnitudes, uniquely $2, 1, 1/2$, at frequencies $10, 1, 2$. Hence

$$\boxed{g_* = 2\sin 10x + \sin x + \tfrac{1}{2}\sin 2x, \qquad \|f - g_*\|_2^2 = \frac{\pi}{18}.}$$

The squared error of S_3 is 36 times larger. Adding a zero-coefficient mode cannot improve the selection and using fewer than three omits additional energy.

Step 4: Interpret incomplete spectral information. The measured first-three energy is $(\pi/2)(1 + 1/4 + 1/9) = 49\pi/72$. Subtracting it from the known total leaves $\boxed{2\pi}$ of unseen energy. Its location is not determined: $S_3 + 2\sin 10x$ and $S_3 + 2\sin 100x$ have the same given coefficients and norm. Total energy constrains the squared size of the tail, not its frequency distribution.

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Question 1

For $f(x) = x$ on $[0, \pi]$, use the convention

$$C_N = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx, \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad (n \geq 0).$$

For piecewise smooth periodic extensions, the Fourier series converges to the average of its one-sided limits.

Tasks

1. Compute a_0 and all a_n, and identify the actual constant term. Explain the error caused by confusing a_0 with the mean.
2. Describe the even 2π-periodic extension and determine the sum at the endpoints and inside the interval.
3. Prove uniform convergence on $[0, \pi]$ and give an explicit uniform tail bound. Sketch C_{12} and f.
4. Every C_N has zero endpoint derivatives. Does the limit inherit these slopes? Give a quantitative obstruction to uniform convergence of the derivatives to $f' = 1$.

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Question 1 – Solution

Strategy. Separate the mean convention from positive-frequency normalization, and distinguish function convergence from derivative convergence.

Step 1: Compute the mean and coefficients. The zero-frequency integral gives $a_0 = \pi$, so the mean term is $\pi/2$. Integration by parts for $n \geq 1$ gives

$$\int_0^\pi x \cos nx \, dx = \frac{(-1)^n - 1}{n^2}, \qquad \boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^2}.}$$

Even modes vanish; odd coefficients are $-4/(\pi n^2)$. Using a_0 in place of $a_0/2$ adds the erroneous constant $\pi/2$.

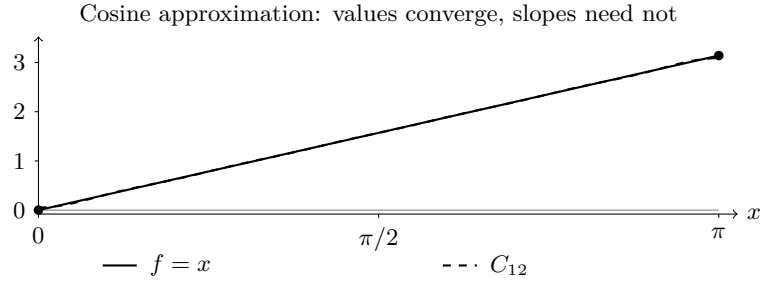
Step 2: Identify the extension and its limits. The extension is $|x|$ on $[-\pi, \pi]$, repeated with period 2π. It is continuous, with corners at multiples of π. At zero its two limits are zero; at π both are π. Therefore

$$\boxed{x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^2}, \qquad 0 \leq x \leq \pi.}$$

Step 3: Bound the uniform error. The bound $|a_n| \leq 4/(\pi n^2)$ gives absolute uniform convergence by the Weierstrass test. The Fourier convergence theorem identifies the sum with f, including the endpoints. For $N \geq 1$,

$$\boxed{\|f - C_N\|_\infty \leq \frac{4}{\pi} \sum_{n>N} n^{-2} \leq \frac{4}{\pi N}.}$$

Step 4: Reject an unjustified derivative limit. Every finite derivative is a sine sum, so $C_N'(0) = C_N'(\pi) = 0$. The target has one-sided endpoint derivatives one. Thus $\boxed{\sup_{[0,\pi]} |C_N' - 1| \geq 1}$ for every N. Uniform convergence of functions alone does not permit differentiating the limit or passing endpoint derivative conditions to it.



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Question 2

An interior cusp has unknown location $a \in (0, \pi)$:

$$f_a(x) = |x - a|, \qquad 0 \le x \le \pi.$$

Write $f_a \sim m + \sum_{n \ge 1} a_n \cos nx$, with $m = \pi^{-1} \int_0^\pi f_a$ and $a_n = (2/\pi) \int_0^\pi f_a \cos nx$.

Tasks

1. Compute the mean and all coefficients by splitting at a. Identify the endpoint-slope and interior-corner contributions separately.
2. At $a = \pi/2$, determine exactly which modes survive and find the fundamental period of the even periodic extension. Sketch the target and its first twelve-mode sum.
3. Recover a uniquely from the signed first coefficient a_1. State the admissible range of that measurement.
4. If only the mean and all coefficient magnitudes are known, is the location unique? Explain the reflection ambiguity and its exceptional symmetric case.

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Question 2 – Solution

Strategy. Retain the interior slope jump when integrating a continuous function with a corner.

Step 1: Integrate the linear pieces. The triangular areas give $\boxed{m = [a^2 + (\pi - a)^2]/(2\pi)}$. Twice integrating the pieces separately gives

$$\boxed{a_n = \frac{2}{\pi n^2}\left[1 + (-1)^n - 2\cos(na)\right], \quad n \geq 1.}$$

The endpoint slopes $-1, 1$ contribute $1 + (-1)^n$, and the interior slope jump two contributes $-2\cos(na)$. None of these terms can be dropped.

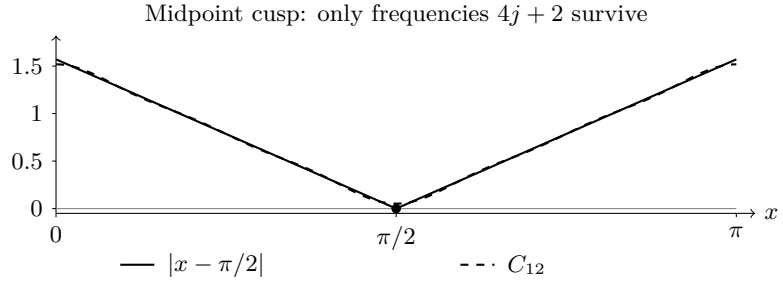
Step 2: Specialize to the midpoint cusp. At $a = \pi/2$ the mean is $\pi/4$. Odd modes vanish, and among the even modes exactly $n = 4j + 2$ survive, with coefficient $8/(\pi n^2)$. The extension has period π; its maxima occur precisely at multiples of π, so every positive period must be a positive multiple of π. Its fundamental period is therefore $\boxed{\pi}$. Absolute coefficient summability gives uniform convergence, and the continuous piecewise linear even extension identifies the sum at all points, including cusps.

Step 3: Invert the first coefficient. The general formula gives $a_1 = -4\cos a/\pi$. Strict monotonicity of cosine on $(0, \pi)$ yields

$$\boxed{a = \arccos(-\pi a_1/4), \qquad -4/\pi < a_1 < 4/\pi.}$$

The open measurement range corresponds exactly to interior cusp locations.

Step 4: Describe what magnitudes lose. Reflection gives $f_{\pi-a}(x) = f_a(\pi - x)$ and $a_n^{\text{reflected}} = (-1)^n a_n$, with unchanged mean. Thus all magnitudes coincide for a and $\pi - a$. In fact $m = \pi/4 + (a - \pi/2)^2/\pi$, so the mean leaves precisely those two possibilities unless $a = \pi/2$, when they coincide. Signed odd coefficients resolve the nonsymmetric ambiguity.



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Question 3

Consider the normalized Neumann problem

$$-u'' = x - \pi/2, \qquad u'(0) = u'(\pi) = 0, \qquad \int_0^\pi u(x)dx = 0.$$

Use the convention $u = a_0/2 + \sum_{n\geq 1} a_n \cos nx$.

Tasks

1. Derive the integral compatibility condition for a general forcing f with these derivative conditions. Explain why replacing the forcing by x makes the problem impossible.
2. Find the cosine coefficients of the stated forcing. Explain how the zero mode differs from every positive-frequency mode when solving for u.
3. Solve in closed form and as a cosine series. Verify the equation, both derivative conditions and the mean.
4. Prove uniqueness with the normalization, and state the full family without it. Explain why adding a constant cannot repair an incompatible forcing.

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Question 3 – Solution

Strategy. Check the forcing mean before dividing by eigenvalues; the constant mode has eigenvalue zero.

Step 1: Check compatibility before solving. Integration gives $\int_0^\pi f = -u'(\pi) + u'(0) = 0$. The stated forcing satisfies this condition, whereas $\int_0^\pi x = \pi^2/2 \neq 0$. The replacement is impossible regardless of the additive level of u.

Step 2: Solve the mode equations. The forcing has mean zero and coefficients $F_n = 2((-1)^n - 1)/(\pi n^2)$. Since $-(\cos nx)'' = n^2 \cos nx$,

$$\boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^4} \quad (n \geq 1).}$$

The zero-mode equation is $0 = 0$ and leaves the mean free. The normalization sets $a_0 = 0$; dividing this equation by zero would be invalid.

Step 3: Verify an independent polynomial solution. Direct integration gives

$$\boxed{u = -\frac{x^3}{6} + \frac{\pi x^2}{4} - \frac{\pi^3}{24} = -\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^4}.}$$

Its derivative is $x(\pi - x)/2$, zero at both ends, and $-u'' = x - \pi/2$. The mean of the first two terms is $\pi^3/24$, canceled by the constant. Twice integrating its coefficient integral by parts using $u' = 0$ at the endpoints gives $n^2 a_n = F_n$, verifying the coefficients. The continuous piecewise smooth even extension identifies the series sum.

Step 4: Separate the free constant from compatibility. A difference satisfies $w'' = 0$ and zero endpoint derivatives, hence is constant. Its zero mean forces it to vanish. Without normalization, all solutions are the displayed polynomial plus any real constant. Adding a constant changes neither the equation nor its derivative conditions, so it cannot fix the integrated forcing balance.

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Question 4

For $f = e^x$ on $[0, 1]$, improve a cosine approximation by matching slopes:

$$\ell = x + \frac{e-1}{2}x^2, \qquad r = e^x - \ell, \qquad A_N = \ell + m_r + \sum_{n=1}^{N} c_n \cos(n\pi x),$$

where $m_r = \int_0^1 r$ and $c_n = 2\int_0^1 r(x)\cos(n\pi x)dx$.

Tasks

1. Compute the raw mean and cosine coefficients of e^x. Find the endpoint slopes of every raw cosine partial sum.
2. Verify the slopes of ℓ, then compute m_r and all c_n. Explain the improvement from order n^{-2} to order n^{-4}.
3. Prove $\|e^x - A_N\|_\infty \le 2(1+e)/(3\pi^4 N^3)$ and give an integer N certified to make the bound less than 10^{-4}.
4. Justify one differentiation of the residual series and verify the slopes of every A_N. Sketch the derivative errors of raw and corrected approximants for $N = 8$. State what slope matching does not guarantee.

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Question 4 – Solution

Strategy. Remove the endpoint-slope contribution while retaining the residual's generally nonzero mean.

Step 1: Compute the raw data. For $k = n\pi$, direct integration gives

$$\boxed{\text{mean} = e - 1, \qquad a_n = \frac{2(e(-1)^n - 1)}{1 + k^2}.}$$

Every raw finite cosine sum has zero endpoint derivatives, disagreeing with $f'(0) = 1$ and $f'(1) = e$.

Step 2: Cancel the slope contribution. The derivative $\ell' = 1 + (e-1)x$ matches both target endpoint slopes. Two integrations by parts give its positive-frequency coefficient $2(e(-1)^n - 1)/k^2$; the constant second derivative has zero cosine integral. Subtracting yields

$$\boxed{m_r = \frac{5}{6}(e-1) - \frac{1}{2}, \qquad c_n = -\frac{2(e(-1)^n - 1)}{k^2(1+k^2)}.}$$

The residual has zero endpoint derivatives, so the leading slope term cancels and fourth-power decay remains. Its mean must still be included.

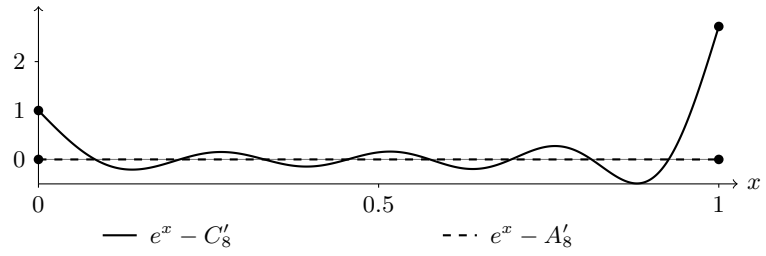
Step 3: Certify the function error. The bound $|c_n| \le 2(1+e)/(\pi^4 n^4)$ gives uniform absolute convergence. The continuous piecewise smooth even extension identifies the sum with r. Consequently

$$\boxed{\|e^x - A_N\|_\infty \le \frac{2(1+e)}{\pi^4} \int_N^\infty t^{-4} dt = \frac{2(1+e)}{3\pi^4 N^3}.}$$

The choice $\boxed{N = 7}$ makes this bound less than 10^{-4}.

Step 4: Verify the derivative behavior. Derivative coefficients are bounded by a constant times n^{-3}, so that series converges uniformly. Together with convergence at one point, this justifies differentiation. Every residual partial sum has zero endpoint slopes; adding ℓ gives $A_N'(0) = 1, A_N'(1) = e$ exactly. Slope matching does not force exact endpoint values at finite N; those are controlled by the function-error bound and become exact in the limit.

Derivative error: raw cosines versus slope correction



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Question 5

On the shifted interval $[1, 4]$, expand $f(x) = (x-1)(4-x)$ using a cosine basis adapted to its left endpoint. Put $t = x - 1 \in [0, 3]$.

Tasks

1. State the correct shifted basis and coefficient normalization. Compute the mean and all positive-frequency coefficients.
2. Use reflection about $x = 5/2$ to explain the missing odd modes. Determine the series sum at the endpoints and midpoint.
3. Someone uses $1, \cos(\pi x/3), \cos(2\pi x/3), \ldots$ with the usual orthogonal coefficient formula. Compute the inner product of the first two listed functions and diagnose the failure.
4. Derive a uniform truncation bound for the correct series. Explain why interval length alone does not specify the phase origin required by standard cosine orthogonality.

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Question 5 – Solution

Strategy. Shift the coordinate in the basis as well as in the target, so the coefficient formulas retain orthogonality.

Step 1: Compute the translated expansion. The basis is $1, \cos(n\pi(x-1)/3)$. The mean is $m = (1/3)\int_1^4 f = 3/2$, and $a_n = (2/3)\int_1^4 f\cos(n\pi(x-1)/3)$. For $f = t(3-t)$, two integrations by parts yield

$$\boxed{a_n = -\frac{18(1+(-1)^n)}{\pi^2 n^2}.}$$

The endpoint slopes are $3, -3$; the constant second derivative integrates to zero against each positive-frequency cosine.

Step 2: Use symmetry and continuity. Reflection sends t to $3-t$, multiplying mode n by $(-1)^n$ while leaving the target unchanged. Thus odd modes vanish. Coefficient summability and the continuous piecewise smooth even extension give uniform convergence to f. The sums are zero at $x = 1, 4$ and $\boxed{9/4}$ at $x = 5/2$.

Step 3: Diagnose the unshifted family. The constant and first unshifted cosine have inner product

$$\boxed{\int_1^4 \cos(\pi x/3)dx = \frac{3}{\pi}\left(\sin\frac{4\pi}{3} - \sin\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{\pi} \neq 0.}$$

They are not orthogonal. Dividing independent moments by the usual cosine norms therefore does not compute orthogonal projection coefficients here. The unshifted phase is not an innocuous substitution in that formula.

Step 4: Give a uniform certificate. For the correctly shifted partial sum,

$$\boxed{\|f - C_N\|_\infty \le \frac{36}{\pi^2}\sum_{n>N} n^{-2} \le \frac{36}{\pi^2 N}.}$$

Length sets the frequency spacing $\pi/3$; the left endpoint sets the phase. Both are used in translating the standard half-range expansion.

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Question 6

Let $f = 1$ on $[0, \pi/2)$, $f(\pi/2) = 1/2$, and $f = 0$ on $(\pi/2, \pi]$. Write $C_N = a_0/2 + \sum_{n=1}^{N} a_n \cos nx$, including $C_0 = a_0/2$, and define

$$\sigma_N = \frac{1}{N+1} \sum_{j=0}^{N} C_j.$$

Let F be the even 2π-periodic extension. The ordinary Fourier convergence theorem may be used for this piecewise constant function.

Tasks

1. Compute the mean and coefficients. Show that C_1 becomes negative although the target is nonnegative.
2. Derive the coefficient weights in σ_N. Expand $K_N(t) = (N+1)^{-1} |\sum_{j=0}^{N} e^{ijt}|^2$ and prove nonnegativity and integral 2π over one period.
3. Verify $\sigma_N(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} F(x-t) K_N(t) dt$, then prove $0 \le \sigma_N \le 1$ everywhere.
4. Determine all limits of σ_N on $[0, \pi]$ by averaging convergent sequences of partial sums. Sketch C_9, σ_9 and the target, and explain the tradeoff made by averaging.

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Question 6 – Solution

Strategy. Use an average with a nonnegative kernel and unit total mass to preserve the range of the data.

Step 1: Exhibit a negative partial sum. The mean is $1/2$ and $\boxed{a_n = 2\sin(n\pi/2)/(\pi n)}$. Thus $C_1 = 1/2 + (2/\pi)\cos x$ has minimum $\boxed{1/2 - 2/\pi < 0}$ at π, despite $0 \le f \le 1$.

Step 2: Compute the weights and kernel. Mode n appears in $N + 1 - n$ of the partial sums, giving

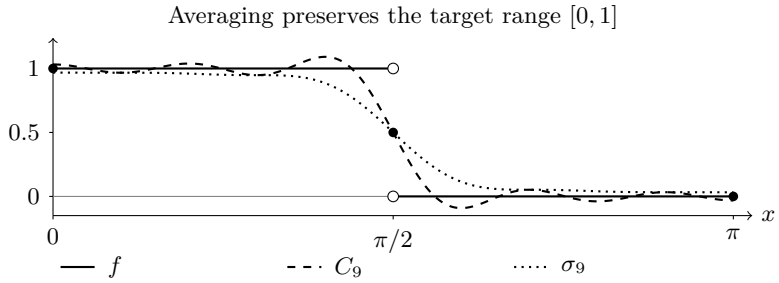
$$\sigma_N = \frac{1}{2} + \sum_{n=1}^{N} \left(1 - \frac{n}{N+1}\right) a_n \cos nx.$$

Counting pairs with index difference n in the squared modulus gives $K_N = 1 + 2\sum_{n=1}^{N}(1 - n/(N+1))\cos nt$. The squared-modulus definition proves $K_N \ge 0$, and integrating the finite expansion gives $\int_{-\pi}^{\pi} K_N = 2\pi$.

Step 3: Prove preservation of the range. Insert the finite kernel expansion into the convolution. The average integral of $F(x-t)\cos nt$ is $a_n \cos nx/2$; after shifting variables, its sine component vanishes by evenness of F. The constant term gives mean $1/2$. This proves the formula. Because $0 \le F \le 1$ and the kernel is nonnegative with mass 2π,

$$\boxed{0 \le \sigma_N(x) \le 1.}$$

Step 4: Identify the limits and tradeoff. Ordinary partial sums tend to one left of the jump, zero right of it, and $1/2$ at the jump. The extension is continuous at zero and π, with values one and zero. Arithmetic means of a convergent sequence have the same limit, so σ_N has exactly these limits. Averaging damps the higher modes and prevents overshoot, but spreads the finite transition instead of fitting the edge as sharply.



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Question 7

For $0 < \rho < 1$, define the positive profile

$$P_\rho(x) = \frac{1-\rho^2}{1-2\rho\cos x+\rho^2}, \qquad 0 \le x \le \pi.$$

Use $P_\rho = a_0/2 + \sum_{n\ge1} a_n \cos nx$ and, when needed, Parseval: $\int_0^\pi f^2 = \pi(a_0/2)^2 + (\pi/2)\sum_{n\ge1} a_n^2$.

Tasks

1. Derive the cosine expansion from a geometric series, justifying uniform convergence for fixed ρ. Identify the mean and every coefficient.
2. Compute the integral and squared norm exactly. Locate the maximum and minimum and give their values.
3. Determine the pointwise limit as $\rho \uparrow 1$. Explain why integrating it does not recover the constant mass.
4. For every continuous φ on $[0,\pi]$, prove $\int_0^\pi P_\rho\varphi \to \pi\varphi(0)$ by splitting at a small fixed $\delta > 0$. Sketch $P_{1/2}$ and $P_{4/5}$ and interpret this limit.

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Question 7 – Solution

Strategy. Sum the geometric coefficients exactly, then locate the mass accumulating near the endpoint.

Step 1: Derive the uniformly convergent expansion. The geometric identity gives $1+2\,\mathrm{Re}[\rho e^{ix}/(1-\rho e^{ix})] = P_\rho(x)$. Since $\sum \rho^n < \infty$, its cosine series converges absolutely and uniformly:

$$\boxed{P_\rho = 1 + 2\sum_{n=1}^{\infty} \rho^n \cos nx, \qquad a_0 = 2, \quad a_n = 2\rho^n.}$$

Step 2: Compute mass, energy and extrema. Uniform integration gives $\boxed{\int_0^\pi P_\rho = \pi}$. Parseval and the geometric sum give

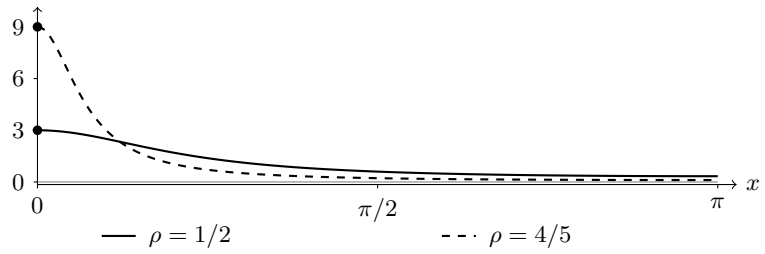
$$\boxed{\int_0^\pi P_\rho^2 = \pi \frac{1+\rho^2}{1-\rho^2}.}$$

The denominator increases with x. The maximum at zero is $(1+\rho)/(1-\rho)$; the minimum at π is $(1-\rho)/(1+\rho)$.

Step 3: Explain the apparent loss of mass. For fixed $x > 0$, the numerator tends to zero and the denominator to $2(1-\cos x) > 0$, so $P_\rho(x) \to 0$. At zero it tends to infinity. The almost-everywhere limit is zero, yet the integral stays π. Uniform convergence and interchange of the integral with this limit are not available; the exact squared norm also diverges.

Step 4: Prove endpoint concentration. Given $\eta > 0$, choose $\delta > 0$ so $|\varphi(x) - \varphi(0)| < \eta$ on $[0, \delta]$. That part of $\int P_\rho(\varphi - \varphi(0))$ is at most $\pi\eta$ in magnitude. On $[\delta, \pi]$, $P_\rho \le (1-\rho^2)/[2\rho(1-\cos\delta)] \to 0$ uniformly, so the remaining part tends to zero. Letting $\eta \downarrow 0$ proves the claim. The limit is mass π concentrated at zero, not an ordinary integrable profile.

Fixed mass π concentrates near the left endpoint



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Question 8

Initially f is known to lie in $\operatorname{span}\{1, \cos x, \cos 2x\}$ on $[0, \pi]$. Measurements give

$$\int_0^\pi f = \pi, \qquad \int_0^\pi f \cos x = \pi/4, \qquad \int_0^\pi f^2 = 5\pi/4.$$

There is also a requirement $f \geq 0$.

Tasks

1. Recover all functions in the stated span satisfying the measurements. Identify where a sign ambiguity enters.
2. Decide whether nonnegativity eliminates either candidate. Prove the answer on the entire interval using $t = \cos x \in [-1, 1]$.
3. A fourth measurement is $f(\pi) = 0$. Find the unique candidate in the stated span and verify all the data.
4. Remove the restriction to that span. Construct another nonnegative continuous function with all four measurements, and explain the limitation of the earlier uniqueness result.

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Question 8 – Solution

Strategy. Linear moments preserve coefficient signs, while a squared-energy measurement can lose them.

Step 1: Solve the coefficient equations. Write $f = A + B\cos x + C\cos 2x$. Orthogonality gives $A = 1$, $B = 1/2$ and $\pi[A^2 + (B^2 + C^2)/2] = 5\pi/4$. Thus $C^2 = 1/4$, leaving

$$\boxed{f_\pm = 1 + \tfrac{1}{2}\cos x \pm \tfrac{1}{2}\cos 2x.}$$

The energy measurement determines the magnitude, but not the sign, of C.

Step 2: Test positivity without sampling. With $t = \cos x$, $f_+ = t^2 + t/2 + 1/2 = (t + 1/4)^2 + 7/16 > 0$. The other candidate is $f_- = 3/2 + t/2 - t^2$. This concave quadratic reaches its minimum on $[-1, 1]$ at an endpoint, with values zero and one. Thus both candidates are nonnegative; positivity does not resolve the ambiguity.

Step 3: Apply the endpoint measurement. At π, $f_+(\pi) = 1$ while $f_-(\pi) = 0$. Hence the unique candidate within the specified span is $\boxed{f = f_-}$. Its mean and first moment are the prescribed ones, and its squared norm is $\pi(1 + 1/8 + 1/8) = 5\pi/4$, verifying all four data.

Step 4: Remove the model restriction. The distinct function

$$\boxed{g = 1 + \tfrac{1}{2}\cos x - \tfrac{1}{2}\cos 4x}$$

has the same mean, first cosine moment and squared norm by orthogonality. Also $g(\pi) = 0$ and $g \geq 1 - 1/2 - 1/2 = 0$ everywhere. Thus these measurements and nonnegativity do not give uniqueness among all continuous functions. The finite-span hypothesis was essential.

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Question 9

For $\alpha > 0$, consider

$$-u_\alpha'' + \alpha^2 u_\alpha = 1 + \cos x, \qquad u_\alpha'(0) = u_\alpha'(\pi) = 0.$$

The zeroth cosine mode has eigenvalue zero for $-d^2/dx^2$.

Tasks

1. Solve by cosine modes, treating the constant separately. Verify the equation and both boundary conditions.
2. Prove uniqueness for $\alpha > 0$ by an energy identity. Determine the mean and its behavior as $\alpha \downarrow 0$.
3. Subtract the mean to define $v_\alpha = u_\alpha - \pi^{-1} \int_0^\pi u_\alpha$. Find its limit and prove uniform convergence of the functions and their first two derivatives.
4. Explain why the unregularized problem with forcing $1 + \cos x$ has no solution, whereas the limit of v_α solves a different normalized problem. State that problem precisely.

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Question 9 – Solution

Strategy. Track the constant mode separately: it becomes singular as the regularization vanishes.

Step 1: Solve the forced modes. The constant is divided by α^2, while the first cosine is divided by $1+\alpha^2$. Therefore

$$\boxed{u_\alpha = \frac{1}{\alpha^2} + \frac{\cos x}{1+\alpha^2}.}$$

Its derivative is $-\sin x/(1+\alpha^2)$, zero at both endpoints. Substitution recovers both the constant forcing one and the cosine term.

Step 2: Establish uniqueness and inspect the mean. For a homogeneous difference, integration by parts gives $\int_0^\pi [(w')^2 + \alpha^2 w^2]dx = 0$, so $w = 0$. The mean is $\boxed{1/\alpha^2}$ and diverges as $\alpha \downarrow 0$; the full family cannot have a finite uniform limit.

Step 3: Control the normalized family. Subtracting the mean gives $v_\alpha = \cos x/(1+\alpha^2)$. The first two derivatives of cosine, as well as cosine itself, have supremum norm one. Thus for $j = 0, 1, 2$,

$$\boxed{\|v_\alpha^{(j)} - (\cos x)^{(j)}\|_\infty = \frac{\alpha^2}{1+\alpha^2} \longrightarrow 0.}$$

This proves function and derivative convergence on the whole interval.

Step 4: Identify the correct limiting problem. The original forcing has integral $\pi \neq 0$, violating the Neumann condition $\int f = 0$ when $\alpha = 0$. Hence the unregularized original problem is impossible. Subtracting the mean also removes the constant forcing: $-v_\alpha'' + \alpha^2 v_\alpha = \cos x$. The limit solves

$$\boxed{-v'' = \cos x, \quad v'(0) = v'(\pi) = 0, \quad \int_0^\pi v = 0, \qquad v = \cos x.}$$

It is the unique zero-mean solution of this compatible problem.

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Question 10

Samples at $x_0 = \pi/6$, $x_1 = \pi/2$, $x_2 = 5\pi/6$ give the values $(2, 0, 1)$. Initially assume $f = A+B\cos x+C\cos 2x$ and define $\langle u, v\rangle_d = \frac{1}{3}\sum_{j=0}^{2} u(x_j)v(x_j)$.

Tasks

1. List the sample vectors of $1, \cos x, \cos 2x$. Verify discrete orthogonality and compute their squared norms.
2. Recover A, B, C from the discrete moments. Verify every sample and prove uniqueness within the assumed span.
3. Drop the span restriction and exhibit infinitely many smooth functions with the same samples by adding an invisible cosine mode.
4. Do nonnegative samples imply that the reconstructed function is nonnegative throughout $[0, \pi]$? Find its exact minimum using $t = \cos x$ and explain what the samples do not certify.

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Question 10 – Solution

Strategy. Distinguish reconstruction in a finite discrete basis from claims about unmeasured modes or intermediate values.

Step 1: Compute the discrete Gram data. The vectors are

$$(1,1,1), \qquad (\sqrt{3}/2, 0, -\sqrt{3}/2), \qquad (1/2, -1, 1/2).$$

Their distinct dot products are zero. Dividing squared lengths by three gives squared norms $\boxed{1, 1/2, 1/2}$.

Step 2: Recover and verify the coefficients. The sample mean gives $A = 1$. Dividing each other discrete moment by $1/2$ gives

$$\boxed{B = 1/\sqrt{3}, \qquad C = 1, \qquad f = 1 + \frac{1}{\sqrt{3}}\cos x + \cos 2x.}$$

The values at the three nodes are $1 + 1/2 + 1/2 = 2$, $1 - 1 = 0$ and $1 - 1/2 + 1/2 = 1$. Three nonzero orthogonal vectors are independent, so the coefficient triple is unique within the specified span.

Step 3: Exhibit the invisible smooth freedom. The function $\cos 3x$ vanishes at all nodes, whose tripled arguments are $\pi/2, 3\pi/2, 5\pi/2$. Hence $\boxed{f_D = f + D\cos 3x, \quad D \in \mathbb{R}}$ has the same samples. These are distinct smooth functions; reconstruction is not unique among all smooth functions.

Step 4: Check between-node positivity exactly. Since $\cos 2x = 2t^2 - 1$, the reconstructed function is $2t^2 + t/\sqrt{3}$. Completing the square gives

$$\boxed{f = 2\left(t + \frac{1}{4\sqrt{3}}\right)^2 - \frac{1}{24}.}$$

Its minimum is $-1/24$, attained at the interior point $x = \arccos(-1/(4\sqrt{3}))$. Nonnegative samples therefore fail to certify nonnegativity between the nodes, even for this unique finite-span reconstruction.

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Question 1

Let $f(x) = x + x^2$ for $-\pi < x < \pi$, extend with period 2π, and assign $f(\pi + 2k\pi) = \pi^2$. Use

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx), \qquad (a_n, b_n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx, \sin nx)\, dx.$$

For piecewise smooth periodic functions, the Fourier series converges to the average of the one-sided limits.

Tasks

1. Separate the even and odd contributions and derive all coefficients, including a_0.
2. Determine the sum at every point, explaining why the two endpoint limits differ despite periodicity. Sketch f and S_8 on one period.
3. Find the best approximation to f in $\text{span}\{1, \cos x, \sin x\}$ for squared integral error. Prove optimality and uniqueness.
4. Compute that minimum error exactly by orthogonality and direct integration. Explain why changing the assigned endpoint value would not change this approximation.

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Question 1 – Solution

Strategy. Use parity to separate coefficients, then use orthogonal projection to measure the combined approximation.

Step 1: Separate the two symmetries. The even part is x^2 and the odd part is x. Odd integrands give zero in the cross terms. Integration by parts gives

$$\boxed{a_0 = \frac{2\pi^2}{3}, \qquad a_n = \frac{4(-1)^n}{n^2}, \qquad b_n = \frac{2(-1)^{n+1}}{n}.}$$

For example, $\int_0^\pi x \sin nx = -\pi(-1)^n/n$ and $\int_0^\pi x^2 \cos nx = 2\pi(-1)^n/n^2$.

Step 2: Identify the periodic join. Inside the base interval the sum is $x + x^2$. At the join, the left limit is $\pi^2 + \pi$ and the right limit is $\pi^2 - \pi$, so the sum is π^2. Periodicity identifies the location of the join, not its two one-sided limits. The assigned value equals this average; the target still has a jump.

Step 3: Prove the projection claim. Orthogonality gives

$$\boxed{p(x) = \pi^2/3 - 4\cos x + 2\sin x.}$$

The residual $f - p$ is orthogonal to the three basis functions. For any other $q = A + B\cos x + C\sin x$,

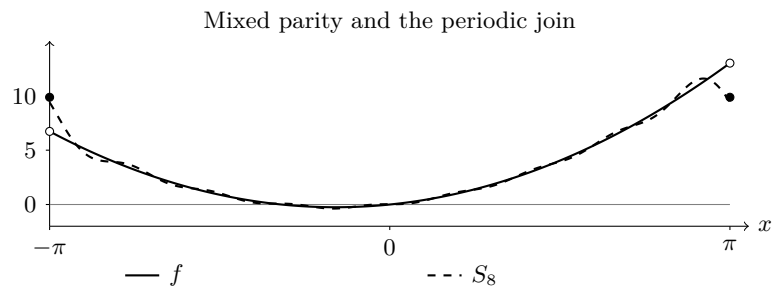
$$\|f - q\|_2^2 = \|f - p\|_2^2 + 2\pi(A - \pi^2/3)^2 + \pi(B + 4)^2 + \pi(C - 2)^2.$$

Thus p is the unique minimizer in this span.

Step 4: Compute the error independently. The odd cross term integrates to zero, giving $\|f\|_2^2 = 2\pi^5/5 + 2\pi^3/3$, while $\|p\|_2^2 = 2\pi^5/9 + 20\pi$. Consequently

$$\boxed{\min \|f - q\|_2^2 = \frac{8\pi^5}{45} + \frac{2\pi^3}{3} - 20\pi.}$$

Changing isolated endpoint values changes neither integrals nor coefficients. It may change whether the assigned values equal the series sums.



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Question 2

A 2π-periodic pulse has height one on the circular interval $c - \pi/3 < x < c + \pi/3$ (modulo 2π), height zero elsewhere, and value $1/2$ at its two edges. The unknown center c is defined modulo 2π. Use the real Fourier convention $f = a_0/2 + \sum_{n\geq 1}(a_n \cos nx + b_n \sin nx)$, with coefficients integrated over any full period. The series has the usual one-sided-average value at a jump.

Tasks

1. Derive the mean and all coefficients by centering the integration interval at c.
2. Determine exactly which entire harmonics vanish. Distinguish a missing harmonic from a zero cosine coefficient caused only by phase.
3. Recover c modulo 2π from the signed pair (a_1, b_1). State an exact consistency condition for a measured pair under this pulse model.
4. Show what is lost if only the harmonic amplitudes $\sqrt{a_n^2 + b_n^2}$ are measured. For $c = \pi/4$, sketch the pulse and its degree-12 partial sum, marking the edge values.

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Question 2 – Solution

Strategy. Centering exposes the pulse width; translating back rotates each sine/cosine coefficient pair.

Step 1: Integrate the centered pulse. The area is $2\pi/3$, so the mean is $1/3$ and $a_0 = 2/3$. Put $x = c+t$. The sine terms odd in t integrate to zero, leaving

$$\boxed{a_n = \frac{2\sin(n\pi/3)}{\pi n}\cos nc, \qquad b_n = \frac{2\sin(n\pi/3)}{\pi n}\sin nc.}$$

These formulas also apply when the pulse crosses the chosen period boundary: integrate instead over a period centered at c.

Step 2: Identify genuine missing modes. The amplitude is

$$R_n = \sqrt{a_n^2 + b_n^2} = \frac{2|\sin(n\pi/3)|}{\pi n}.$$

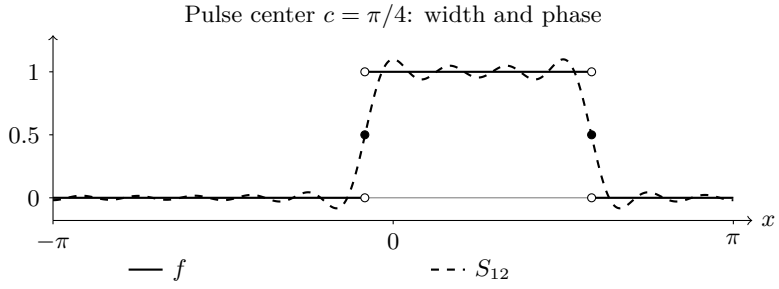
Hence the entire harmonic vanishes exactly when 3 divides n. For other n, a_n alone can vanish because $\cos nc = 0$, while $b_n \neq 0$. Discarding such a sine term would remove a real harmonic.

Step 3: Recover the circular center. For $n = 1$ the prefactor is positive:

$$(a_1, b_1) = \frac{\sqrt{3}}{\pi}(\cos c, \sin c), \qquad \boxed{c = \operatorname{atan2}(b_1, a_1) \pmod{2\pi}.}$$

Consistency is exactly $a_1^2 + b_1^2 = 3/\pi^2$ for real measured coefficients. Every pair on that circle corresponds to precisely one center modulo 2π. Using only an ordinary arctangent of b_1/a_1 loses quadrant information.

Step 4: Separate location from amplitude. Every R_n is independent of c, so amplitude measurements leave every center possible, even if all harmonics are known. At $c = \pi/4$, the pulse edges in $[-\pi, \pi]$ are $-\pi/12$ and $7\pi/12$. The sum is one between them, zero outside, and $1/2$ at each edge. The plotted partial sum uses both coefficient families; the finite oscillations do not alter these limiting edge averages.



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Question 3

Consider a half-wave rectifier with output $h(x) = \max(\sin x, 0)$, extended with period 2π. Use the mean-term convention $a_0/2$. You may use Parseval:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n\geq 1}(a_n^2 + b_n^2).$$

Tasks

1. Express h using $\sin x$ and $|\sin x|$. Derive its complete real Fourier series.
2. Explain why all sine coefficients except b_1 vanish, and why only even positive cosine frequencies occur.
3. Compute the average squared output and its division into the constant term, the fundamental harmonic, and all higher harmonics.
4. Determine the fundamental periods of h and $|\sin x|$, with a proof that the proposed periods are minimal. Sketch h and its degree-8 Fourier sum.

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Question 3 – Solution

Strategy. Rectification combines an odd fundamental with an even function of half the period.

Step 1: Derive the expansion. Since $h = (\sin x + |\sin x|)/2$, its mean is $1/\pi$. Direct integration on $(0, \pi)$ gives $b_1 = 1/2$ and $b_n = 0$ for $n \neq 1$. For $k \geq 1$, product-to-sum integration gives

$$\frac{1}{\pi}\int_0^{\pi} \sin x \cos(2kx)\, dx = -\frac{2}{\pi(4k^2-1)}.$$

Odd cosine coefficients vanish, including a_1. Thus

$$\boxed{h(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}.}$$

The coefficients are absolutely summable; the standard piecewise smooth Fourier theorem identifies the sum with this continuous target, including its corners.

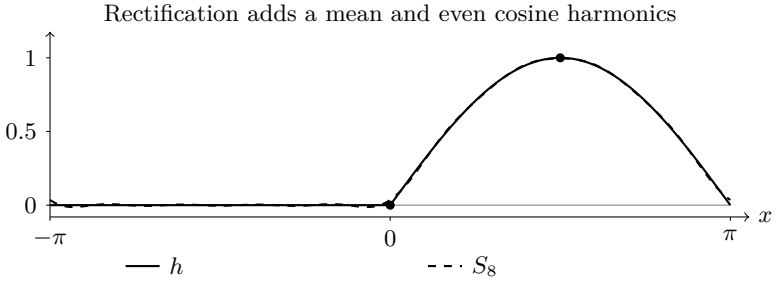
Step 2: Explain the selection rules. The function $|\sin x|$ is even, so contributes only cosines. It is also π-periodic, eliminating odd cosine frequencies in a 2π expansion. The remaining odd part is exactly $\sin x/2$, so it contributes only b_1. The full rectified waveform is neither even nor odd.

Step 3: Account for the average power. Direct integration gives $(2\pi)^{-1}\int h^2 = 1/4$. Orthogonality separates the nonnegative contributions:

$$\boxed{P_{\text{constant}} = \frac{1}{\pi^2}, \quad P_{\text{fundamental}} = \frac{1}{8}, \quad P_{\text{higher}} = \frac{1}{8} - \frac{1}{\pi^2}.}$$

They sum to $1/4$. The higher-harmonic contribution is positive; the mean squared is only the constant contribution, not the total average squared output.

Step 4: Prove the fundamental periods. The maxima of h occur exactly at $\pi/2 + 2k\pi$. Any positive period must take one maximum to another, hence be a positive multiple of 2π; 2π works. The maxima of $|\sin x|$ occur at $\pi/2 + k\pi$, so its least positive period is π. Thus the full output has fundamental period 2π despite its even-frequency cosine terms: its nonzero sine fundamental matters.



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Question 4

Let $f(x) = e^x$ for $0 < x < 2$ and extend periodically with period 2. At each join assign the average $(1+e^2)/2$. Write

$$f \sim m + \sum_{n\geq 1} \big(a_n \cos(n\pi x) + b_n \sin(n\pi x)\big), \quad m = \frac{1}{2}\int_0^2 f, \quad (a_n, b_n) = \int_0^2 f(\cos(n\pi x), \sin(n\pi x)).$$

Set $E = e^2 - 1$.

Tasks

1. Derive m, a_n, b_n. Identify the series sum at the periodic join.
2. Use integration by parts on $f' = f$ inside $(0, 2)$ to obtain two coefficient relations. Explain why omitting the boundary jump would incorrectly force every positive-frequency coefficient to zero.
3. Find a continuous 2-periodic function F with $F' = f - m$ away from the joins and $F(0) = 0$. Prove uniqueness under this normalization.
4. Compute the mean and all Fourier coefficients of F directly. Verify that differentiating its positive-frequency terms formally gives those of $f - m$, and explain why f itself cannot have a continuous periodic primitive.

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Question 4 – Solution

Strategy. Keep the endpoint jump in integration by parts, and remove the mean before seeking a periodic primitive.

Step 1: Compute the full coefficients. For $k = n\pi$, exponential-trigonometric antiderivatives at $0, 2$ give

$$\boxed{m = \frac{E}{2}, \qquad a_n = \frac{E}{1+k^2}, \qquad b_n = -\frac{kE}{1+k^2}.}$$

The Fourier sum at a join is $(1+e^2)/2$, since its two limits are e^2 and 1. The assigned join value does not enter any integral.

Step 2: Retain the boundary contribution. On the open interval, $f' = f$. Integration by parts gives

$$\int_0^2 f' \cos kx = E + k\int_0^2 f \sin kx, \qquad \int_0^2 f' \sin kx = -k\int_0^2 f \cos kx.$$

Thus $a_n = E + kb_n$ and $b_n = -ka_n$, yielding the coefficients above. Dropping E would give $(1+k^2)a_n = 0$ and then $b_n = 0$. The periodic extension has a jump, so it is not a globally smooth periodic solution of $f' = f$.

Step 3: Construct the periodic primitive. Integrating the mean-zero forcing gives, on $[0, 2]$,

$$\boxed{F(x) = e^x - 1 - \frac{E}{2}x.}$$

Both endpoint values are zero, so its periodic extension is continuous. Two such primitives differ by a constant on each open period; continuity connects those constants, and $F(0) = 0$ fixes the remaining freedom.

Step 4: Check the transformed coefficients. Direct integration gives mean $M = -1$. If A_n, B_n denote the coefficients of F, the polynomial terms have zero cosine integral and $\int_0^2 x \sin kx = -2/k$. Consequently

$$\boxed{A_n = \frac{E}{1+k^2}, \qquad B_n = \frac{E}{k(1+k^2)}.}$$

Indeed $kB_n = a_n$ and $-kA_n = b_n$. These are coefficient identities; the derivative equation is verified by the closed form on each open period. Finally, a continuous piecewise smooth periodic primitive of f would have net change $\int_0^2 f = E \neq 0$, which is impossible. Subtracting the mean is essential.

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Question 5

A change of origin replaces $f(x)$ by $f(t+c)$, where c is defined modulo 2π. Consider

$$f(x) = 2\cos x + 2\sin x + \cos 2x.$$

Tasks

1. Derive how a general coefficient pair (a_n, b_n) changes under $x = t + c$. Prove that its amplitude is unchanged.
2. Determine every origin c for which $f(t+c)$ is even, or prove that none exists.
3. Determine every origin c for which $f(t+c)$ is odd. Give the shifted expansion for one such origin.
4. Replace $\cos 2x$ by $\sin 2x$ to obtain g. Find all origins making $g(t+c)$ even and give one resulting cosine-only expansion. Explain why separately aligning each harmonic is not a valid test for a single common symmetry center.

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Question 5 – Solution

Strategy. A common origin must satisfy the phase conditions of every nonzero harmonic simultaneously.

Step 1: Derive the coefficient rotation. Angle addition gives the new coefficients

$$\boxed{a'_n = a_n \cos nc + b_n \sin nc, \qquad b'_n = -a_n \sin nc + b_n \cos nc.}$$

Squaring and adding cancels the cross terms, so $(a'_n)^2 + (b'_n)^2 = a_n^2 + b_n^2$. For a trigonometric polynomial, evenness is equivalent to every b'_n being zero; oddness requires a zero mean and every a'_n zero. Necessity follows from orthogonality, and sufficiency follows from sine/cosine parity.

Step 2: Test all evenness conditions. At frequency one, $b'_1 = 2(\cos c - \sin c) = 0$ implies $c = \pi/4 + k\pi$. At frequency two, $b'_2 = -\sin 2c$ would then equal -1. Hence

$$\boxed{\text{No origin makes } f(t+c) \text{ even.}}$$

The first frequency can be aligned, but that shift makes the second purely sine.

Step 3: Find every oddness origin. The first condition $a'_1 = 2(\cos c + \sin c) = 0$ gives $c = 3\pi/4 + k\pi$. At each of these, $a'_2 = \cos 2c = 0$, and the mean is already zero. These are all solutions. At $c = 3\pi/4$,

$$\boxed{f(t + 3\pi/4) = -2\sqrt{2} \sin t + \sin 2t.}$$

Modulo 2π the two origins are $3\pi/4$ and $7\pi/4$.

Step 4: Change the relative harmonic phase. For $g = 2\cos x + 2\sin x + \sin 2x$, the first-frequency evenness condition is again $c = \pi/4 + k\pi$. Now the second is $b'_2 = \cos 2c = 0$, which is satisfied. At $c = \pi/4$,

$$\boxed{g(t + \pi/4) = 2\sqrt{2} \cos t + \cos 2t.}$$

All evenness origins are $\pi/4 + k\pi$. Each harmonic admits its own phase alignment, but a symmetry of the full function requires the same c for all frequencies. The two examples have identical harmonic amplitudes and different answers to the evenness question.

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Question 6

Find the continuous 2π-periodic steady response of

$$y' + 2y = 1 + q(x), \qquad q(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

where q is periodically extended and assigned zero at its jumps. The differential equation is required on the open intervals between jumps. Solutions are continuous and piecewise continuously differentiable.

Tasks

1. Derive the full Fourier coefficients of q, including its mean.
2. Obtain the mean and all cosine and sine coefficients of the periodic response by mode equations.
3. Independently solve on the two intervals and enforce continuity and periodicity. Prove that this response is unique.
4. Verify its mean, extrema and behavior at the switches. Sketch the exact response and its degree-9 Fourier approximation. Explain why a sine-only forcing produces cosine terms in the response.

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Question 6 – Solution

Strategy. Solve the harmonic response and the piecewise differential equation independently.

Step 1: Expand the input. The periodic extension of q is odd. Direct integration gives zero mean and zero cosine coefficients, with

$$\boxed{b_n^{(q)} = \frac{2(1-(-1)^n)}{\pi n} = \begin{cases} 4/(\pi n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}}$$

The constant in $1+q$ is one.

Step 2: Solve the coupled mode equations. Write $y = m + \sum(A_n \cos nx + B_n \sin nx)$. Integration by parts is legitimate piecewise: continuity and periodicity cancel the internal and endpoint terms. Thus $2m = 1$, $nB_n + 2A_n = 0$, and $-nA_n + 2B_n = b_n^{(q)}$. Hence

$$\boxed{m = \frac{1}{2}, \quad A_n = -\frac{4}{\pi(n^2+4)}, \quad B_n = \frac{8}{\pi n(n^2+4)} \quad (n \text{ odd}),}$$

and both coefficients vanish for even n.

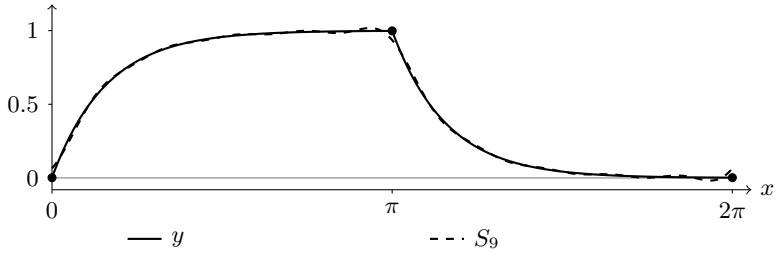
Step 3: Verify a closed-form periodic response. Put $r = e^{-2\pi}$. Solving the constant-forcing equations and matching gives

$$\boxed{y(x) = \begin{cases} 1 - \dfrac{e^{-2x}}{1+r}, & 0 \le x \le \pi, \\ \dfrac{e^{-2(x-\pi)}}{1+r}, & \pi \le x \le 2\pi. \end{cases}}$$

Both expressions give $y(\pi) = 1/(1+r)$, and $y(0) = y(2\pi) = r/(1+r)$. A difference of two responses is Ce^{-2x} across the whole period by continuity; periodicity forces $C = 0$. The verified closed form therefore has the coefficients determined above.

Step 4: Interpret the response. The first branch increases and the second decreases, so the extrema are $r/(1+r)$ and $1/(1+r)$. Integrating the equation over a period gives mean $1/2$. The response is continuous, but y' jumps by -2 at π and by 2 at 0 modulo 2π; the assigned input values do not require a derivative there. Differentiation couples sine and cosine coefficients, creating a phase shift. Absolute coefficient summability and the Fourier theorem identify the plotted series with the continuous response.

Continuous response with corners at the switches



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Question 7

For real parameters α, β, seek a twice continuously differentiable 2π-periodic solution of

$$y'' + 4y = 3 + \cos x + \alpha \cos 2x + \beta \sin 2x.$$

Tasks

1. Derive necessary conditions on α, β by multiplying by each resonant mode and integrating over one period.
2. Prove those conditions are sufficient and find all periodic solutions when they hold.
3. Under the same conditions, impose $y(0) = y'(0) = 0$. Find the resulting solution and verify all conditions directly.
4. Explain why initial conditions cannot restore periodic solvability when a resonant forcing coefficient is nonzero. Relate this to the failed division by $4 - n^2$ in the Fourier equations.

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Question 7 – Solution

Strategy. Test the forcing against the periodic homogeneous modes before dividing by a spectral factor.

Step 1: Derive the compatibility conditions. For a periodic C^2 function, both y and y' match at the period endpoints. Two integrations by parts therefore give

$$\int_0^{2\pi} (y'' + 4y)\cos 2x = 0, \qquad \int_0^{2\pi} (y'' + 4y)\sin 2x = 0.$$

Orthogonality makes the corresponding right sides $\pi\alpha$ and $\pi\beta$. Thus

$$\boxed{\alpha = \beta = 0}$$

is necessary. The constant and first harmonic do not contribute to these tests.

Step 2: Construct every compatible solution. With these parameters zero, a particular solution is $3/4 + \cos x/3$. The homogeneous equation has basis $\cos 2x, \sin 2x$, both periodic. Thus

$$\boxed{y(x) = \frac{3}{4} + \frac{1}{3}\cos x + A\cos 2x + B\sin 2x, \qquad A, B \in \mathbb{R}.}$$

This proves sufficiency as well as completeness: subtract the particular solution from any solution and solve the homogeneous equation.

Step 3: Enforce the additional normalization. The initial values give $3/4 + 1/3 + A = 0$ and $2B = 0$. Hence

$$\boxed{y(x) = \frac{3}{4} + \frac{1}{3}\cos x - \frac{13}{12}\cos 2x.}$$

It has $y(0) = 0$, $y'(0) = 0$, and is 2π-periodic together with all derivatives. Substitution gives $y'' + 4y = 3 + \cos x$ because the frequency-two term is annihilated.

Step 4: Identify the resonance obstruction. For frequency n, the mode equation multiplies each coefficient of y by $4 - n^2$. At $n = 2$ it reads $0 = \alpha$ or $0 = \beta$: it is a compatibility condition, not a formula for an unknown coefficient. When it holds, the two coefficients are free until extra conditions are supplied. When it fails, no periodic C^2 solution exists. Although an initial-value problem still has a unique solution, its resonant response cannot be periodic; altering homogeneous constants does not change either integral obstruction.

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Question 8

Let $f(x) = 1 + \cos x + 2\sin x + \cos 2x$. Its periodic autocorrelation is

$$R(\tau) = \frac{1}{2\pi}\int_0^{2\pi} f(x)f(x+\tau)\,dx.$$

Tasks

1. Derive R using orthogonality. Explain why sine terms disappear from the correlation even though f has a nonzero sine coefficient.
2. Find the exact maximum and minimum of R, including all shifts attaining them modulo 2π.
3. Construct a real trigonometric polynomial g with the same mean and the same R which is not a translate of f. Prove both claims.
4. Derive the identity relating R to the squared distance between f and its translate. Determine every shift leaving f unchanged and connect this with the fundamental period.

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Question 8 – Solution

Strategy. Correlation retains harmonic energies but discards their individual phases.

Step 1: Evaluate the correlation. Different frequencies are orthogonal. At frequency n, a pair $a_n \cos nx + b_n \sin nx$ contributes $(a_n^2 + b_n^2)\cos(n\tau)/2$; the two mixed sine terms cancel. The constant contributes the square of the mean. Thus

$$\boxed{R(\tau) = 1 + \frac{5}{2}\cos\tau + \frac{1}{2}\cos 2\tau.}$$

In particular, $R(0) = 4$ is the mean square of f, not the square of its mean. The correlation is even regardless of the parity of f.

Step 2: Optimize over all shifts. Put $t = \cos\tau \in [-1, 1]$. Then $R = t^2 + (5/2)t + 1/2$, whose derivative $2t + 5/2$ is positive throughout this interval. Hence

$$\boxed{\max R = 4 \quad (\tau = 0 \bmod 2\pi), \qquad \min R = -1 \quad (\tau = \pi \bmod 2\pi).}$$

A correlation can be negative here; the original signal is not required to be nonnegative.

Step 3: Exhibit phase ambiguity beyond translation. Take $g(x) = 1 + \sqrt{5}\cos x + \cos 2x$. Its mean is one and its harmonic squared amplitudes are 5 and 1, so it gives exactly the same R. If $g(x) = f(x + c)$, equality at frequency two requires $\cos 2c = 1$ and $\sin 2c = 0$, so $c = k\pi$. But then the first sine coefficient of $f(x+c)$ is $2(-1)^k \neq 0$, unlike g. Thus g is not a translate. Correlation does not determine relative harmonic phases.

Step 4: Recover the exact translation invariance. Translation preserves $\int f^2$. Expanding the squared difference gives

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi} [f(x+\tau) - f(x)]^2\, dx = 2\big(R(0) - R(\tau)\big).}$$

This is zero precisely at $\tau = 0$ modulo 2π. Zero integral of a continuous nonnegative square means equality at every point. Therefore the invariant shifts are exactly $2k\pi$, and the fundamental period is 2π.

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Question 9

For real a, b, a modulated waveform is

$$s_{a,b}(x) = (1 + a\cos x)\cos 3x + b\sin x \sin 3x.$$

Use a 2π Fourier expansion, and define average power as $(2\pi)^{-1}\int_0^{2\pi} s_{a,b}^2$.

Tasks

1. Derive the complete Fourier spectrum using product identities. Identify the frequencies surrounding the carrier frequency 3.
2. Recover a, b from measured cosine coefficients C_2, C_4. Determine all choices canceling the frequency-4 term while leaving a prescribed frequency-2 coefficient d.
3. For $d = 1/2$ with frequency 4 canceled, find the fundamental period. Prove minimality even though the frequency-1 coefficient is zero.
4. Compute the average power for general a, b, then for this special case. Prove a sharp global bound on the special waveform's absolute value and identify points where equality holds.

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Question 9 – Solution

Strategy. Resolve the products into actual harmonics before inferring frequency content, period or energy.

Step 1: Resolve the two products. The identities $\cos x \cos 3x = (\cos 2x + \cos 4x)/2$ and $\sin x \sin 3x = (\cos 2x - \cos 4x)/2$ give

$$\boxed{s_{a,b}(x) = \cos 3x + \frac{a+b}{2}\cos 2x + \frac{a-b}{2}\cos 4x.}$$

The mean and every sine coefficient are zero. The neighboring frequencies, often called sidebands, are 2 and 4, not frequency 1 from the modulation alone.

Step 2: Invert the coefficient measurements. The linear equations give

$$\boxed{a = C_2 + C_4, \qquad b = C_2 - C_4.}$$

Requiring $C_4 = 0$ and $C_2 = d$ fixes $a = b = d$ uniquely. The two modulation terms reinforce the lower frequency and cancel the upper frequency when their amplitudes agree.

Step 3: Establish the least period. For $d = 1/2$ the signal is $s = \cos 3x + \frac{1}{2}\cos 2x$. If $T > 0$ is a period, translating this finite expansion and using orthogonality forces $3T$ and $2T$ each to be integer multiples of 2π. Subtracting gives T an integer multiple of 2π. Conversely 2π works. Thus the fundamental period is $\boxed{2\pi}$. A nonzero first harmonic is sufficient for this period in a 2π polynomial, but is not necessary: frequencies 2 and 3 already enforce it.

Step 4: Measure power and peak amplitude. Orthogonality gives

$$\boxed{P = \frac{1}{2}\left(1 + \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4}\right) = \frac{1}{2} + \frac{a^2+b^2}{4}.}$$

At $a = b = 1/2$, $P = 5/8$. Also $|s| \le |\cos 3x| + \frac{1}{2}|\cos 2x| \le 3/2$, with equality at $x = 2k\pi$. For negative equality both cosines would have to be -1; the equations $3x = (2j+1)\pi$, $2x = (2k+1)\pi$ are incompatible by parity. Positive equality requires both to be 1, giving exactly $x = 2k\pi$.

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Question 10

Two periodic signals are $f(x) = \cos x + \cos 2x$ and $g(x) = \sin x + \cos 2x$. To align them by one delay, minimize

$$D(\delta) = \int_0^{2\pi} [f(x-\delta) - g(x)]^2 \, dx, \qquad \delta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Tasks

1. Use orthogonality to derive $D(\delta)$ without expanding a long pointwise square.
2. Determine all globally optimal delays and the exact minimum error. Prove global optimality.
3. Compare delays 0 and $\pi/2$. Explain why exactly aligning the first harmonic need not give the best alignment of the full signal.
4. Decide whether any delay makes the two functions identical. Sketch $D(\delta)/\pi$ over one period, marking every minimizer, and explain the nonuniqueness of the best delay.

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Question 10 – Solution

Strategy. One delay rotates each harmonic by a different multiple of the same angle, creating a coupled optimization.

Step 1: Compute distance through correlation. Both signals have squared norm 2π. Angle addition and orthogonality give

$$\int_0^{2\pi} f(x-\delta)g(x)\,dx = \pi(\sin\delta + \cos 2\delta).$$

Therefore

$$\boxed{\frac{D(\delta)}{\pi} = 4 - 2(\sin\delta + \cos 2\delta).}$$

The sine sign follows from $\cos(x-\delta) = \cos x\cos\delta + \sin x\sin\delta$.

Step 2: Optimize over the whole circle. Put $t = \sin\delta \in [-1,1]$. Since $\cos 2\delta = 1 - 2t^2$,

$$\boxed{\frac{D(\delta)}{\pi} = \frac{7}{4} + 4\left(t - \frac{1}{4}\right)^2.}$$

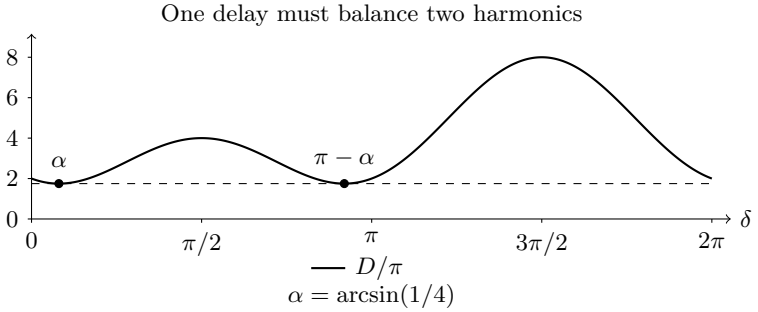
The feasible value $t = 1/4$ is the unique minimizing sine value. Consequently

$$\boxed{D_{\min} = \frac{7\pi}{4}, \qquad \delta = \arcsin(1/4) \text{ or } \pi - \arcsin(1/4) \pmod{2\pi}.}$$

Completing the square proves these are all global minimizers.

Step 3: Compare two tempting alignments. At $\delta = 0$, the second harmonics agree and $D = 2\pi$. At $\delta = \pi/2$, the first harmonic matches $\sin x$, but the second becomes $-\cos 2x$, so $D = 4\pi$. Exactly matching one harmonic can worsen the full error. The optimum balances their contributions and improves on both choices.

Step 4: Interpret the two minima. The minimum is strictly positive, so no delay makes the signals identical. The objective depends only on $\sin\delta$; the two distinct delays with sine $1/4$ give the same compromise. They do not produce the same shifted function: their first cosine coefficients are opposite and nonzero. Thus nonuniqueness of the best delay does not imply exact agreement or identical aligned waveforms. The graph shows both minima in $[0, 2\pi)$.



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Question 1

Define a 2π-periodic function by $f = -1$ on $(-\pi, 0)$, $f = 2$ on $(0, \pi)$, $f(0) = 7$ and $f(\pi) = -4$. Let S_N be its real Fourier partial sum with mean $a_0/2$. Use the piecewise smooth Fourier theorem (the sum is the average of the one-sided limits) and Parseval. Here $\|u\|_2^2 = \int_{-\pi}^{\pi} |u|^2$.

Tasks

1. Derive the mean and all coefficients. Determine the pointwise series sum everywhere, including both types of assigned exceptional values.
2. Compute $\|f - S_N\|_2^2$ as a coefficient tail and prove it tends to zero.
3. Prove that S_N does not converge uniformly to f. Also prove a lower bound of $3/2$ for the essential supremum error, which ignores sets of measure zero.
4. Decide whether redefining only the jump values can restore uniform convergence. Sketch f and S_9, displaying the assigned values separately from the open one-sided limits.

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Question 1 – Solution

Strategy. Distinguish the assigned function, its Fourier pointwise limit and the equivalence class measured by an integral norm.

Step 1: Compute coefficients and limits. Apart from isolated values, $f = 1/2 + (3/2)\operatorname{sgn}(\sin x)$. Thus $a_0 = 1$, every positive cosine coefficient is zero, and

$$\boxed{b_n = \frac{3(1-(-1)^n)}{\pi n} = \begin{cases} 6/(\pi n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}}$$

The series sum equals -1 or 2 on the open intervals and $1/2$ at every jump. It does not equal either assigned value 7 or -4 there.

Step 2: Verify mean-square convergence. Isolated values do not affect integrals. Parseval and orthogonality give

$$\boxed{\|f - S_N\|_2^2 = \frac{36}{\pi} \sum_{\substack{n>N \\ n \text{ odd}}} \frac{1}{n^2} \le \frac{36}{\pi N} \longrightarrow 0.}$$

The coefficient tail is square summable despite failure at the assigned jump values.

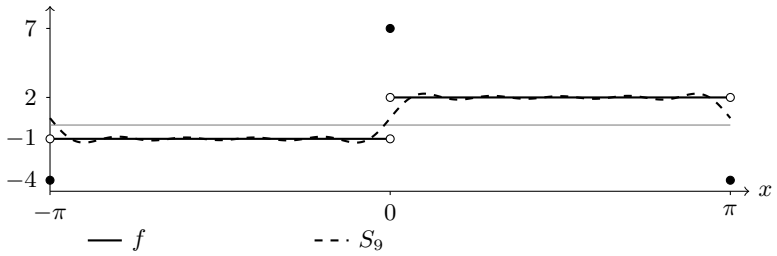
Step 3: Separate point and essential errors. Every $S_N(0) = 1/2$, so $\|f - S_N\|_\infty \ge 13/2$ for every N. Moreover, by continuity of S_N, as $x \downarrow 0$ its error against the right-hand value 2 tends to $3/2$. For any $\varepsilon > 0$, that error exceeds $3/2 - \varepsilon$ on an interval of positive length. Hence

$$\boxed{\operatorname{ess\,sup} |f - S_N| \ge 3/2.}$$

Removing isolated exceptional values cannot remove this obstruction.

Step 4: Test possible redefinitions. The one-sided limits remain -1 and 2 after any change only at the jumps. The resulting target is still discontinuous. A uniform limit of the continuous trigonometric polynomials S_N would be continuous, so no such redefinition restores uniform convergence. Mean-square convergence and the open-interval pointwise limits are unchanged. Filled points in the graph show the assigned values.

Assigned values, jump limits and a Fourier sum



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Question 2

For integers $m \geq 1$, let

$$h_m(x) = \max(1 - m|x|, 0), \qquad -\pi \leq x \leq \pi,$$

and extend periodically. Write $S_N^{(m)}$ for the degree-N Fourier partial sum of this particular function. The target changes with m; these are not partial sums of one fixed function.

Tasks

1. Compute $\|h_m\|_\infty$, $\|h_m\|_2^2$ and the pointwise limit as $m \to \infty$. Distinguish mean-square convergence to zero from pointwise convergence to zero.
2. Derive the mean and all Fourier coefficients of h_m.
3. For each fixed m, prove $S_N^{(m)} \to h_m$ uniformly as $N \to \infty$. State a bound whose dependence on m is explicit.
4. Evaluate both iterated limits at $x = 0$, first letting N tend to infinity and then reversing the order. Explain why they differ, and sketch h_1, h_3, h_9 near the origin.

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Question 2 – Solution

Strategy. A shrinking support can make integral errors small while retaining a fixed peak.

Step 1: Compare the modes of convergence. The peak is always one and direct integration gives

$$\boxed{\|h_m\|_\infty = 1, \qquad \|h_m\|_2^2 = 2\int_0^{1/m} (1-mx)^2\,dx = \frac{2}{3m}.}$$

The pointwise limit is one at 0 modulo 2π and zero elsewhere. Thus $h_m \to 0$ in mean square and almost everywhere, but not at every point and not uniformly. The pointwise limit has a single-point spike in each period.

Step 2: Compute the spectrum. The function is even, so $b_n = 0$. Its area is $1/m$, giving mean $1/(2\pi m)$. Integration over its support gives

$$\boxed{a_n = \frac{2}{\pi}\int_0^{1/m} (1-mx)\cos nx\,dx = \frac{2m}{\pi n^2}\big(1-\cos(n/m)\big).}$$

For each fixed n, $0 \le a_n \le 1/(\pi m)$ by $1-\cos u \le u^2/2$. Therefore every fixed coefficient tends to zero as $m \to \infty$.

Step 3: Keep the target fixed when taking a Fourier limit. For fixed m, $|a_n| \le 4m/(\pi n^2)$, so the series converges absolutely and uniformly. The piecewise smooth Fourier theorem identifies its sum with the continuous target h_m. In particular,

$$\boxed{\|h_m - S_N^{(m)}\|_\infty \le \frac{4m}{\pi N}.}$$

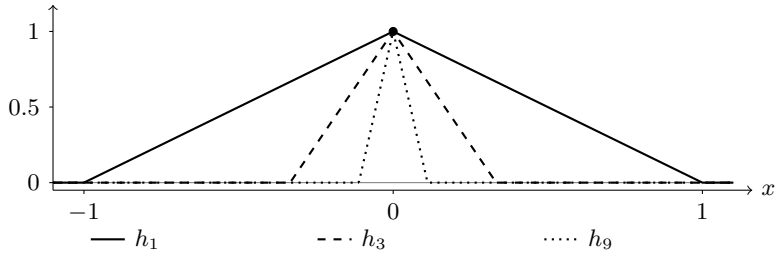
The factor m prevents using this bound uniformly over all targets.

Step 4: Compare the iterated limits. For fixed m, the inner N-limit at zero is $h_m(0) = 1$. For fixed N, the mean and finitely many coefficients all tend to zero. Thus

$$\boxed{\lim_{m\to\infty}\lim_{N\to\infty} S_N^{(m)}(0) = 1, \qquad \lim_{N\to\infty}\lim_{m\to\infty} S_N^{(m)}(0) = 0.}$$

Fixed-mode convergence does not control the collective contribution of modes whose number grows with m. There is no common uniform estimate permitting this interchange.

Shrinking support preserves a unit peak



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Question 3

You may use the Fourier identity

$$F(x) = \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \qquad -\pi \le x \le \pi,$$

extended periodically. Let $P_N = \sum_{n=1}^{N} \cos(nx)/n^2$, $N \ge 1$.

Tasks

1. Prove uniform convergence and find the exact value of $\|F - P_N\|_\infty$ as a positive numerical series.
2. Determine all points where this maximum absolute error occurs, modulo 2π.
3. Prove strict two-sided bounds using integrals and determine the smallest integer N for which the uniform error is less than $1/100$.
4. Find the exact squared mean-square error as a series and compare its decay rate with that of the uniform error. Sketch $F - P_5$ and its upper and lower uniform-error levels.

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Question 3 – Solution

Strategy. Positive coefficients make the usual triangle bound attain its value at a common phase.

Step 1: Turn the tail estimate into an equality. The Weierstrass test applies because $\sum n^{-2} < \infty$. Put $T_N = \sum_{n>N} n^{-2}$. For every x, $|F(x) - P_N(x)| \le T_N$, while at $x = 0$ every cosine equals one. Hence

$$\boxed{\|F - P_N\|_\infty = T_N.}$$

This is an exact error, not merely a sufficient upper bound.

Step 2: Identify every equality point. For the tail to equal T_N, every positive summand must have $\cos nx = 1$. Using two consecutive integers $n > N$ forces $x = 0$ modulo 2π. For the tail to equal $-T_N$, every $\cos nx$ would have to be -1. But then $\cos(2nx) = 1$ for a frequency also in the tail, a contradiction. Thus the maximum absolute error occurs exactly at $x = 0$ modulo 2π.

Step 3: Certify the minimal truncation. Strict monotonicity of t^{-2} gives

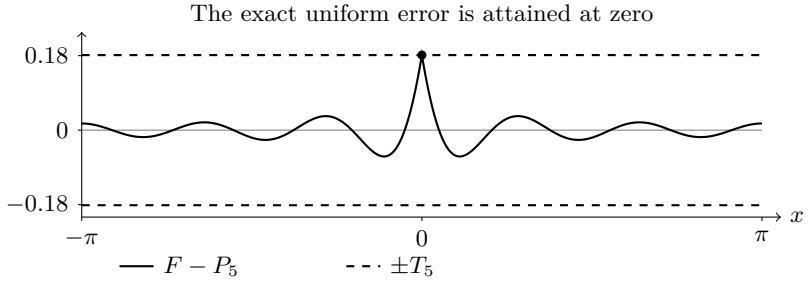
$$\boxed{\frac{1}{N+1} < T_N < \frac{1}{N}.}$$

At $N = 100$, $T_N < 1/100$. At $N = 99$, $T_N > 1/100$; monotonicity excludes every smaller N. Therefore the exact smallest choice is $\boxed{N = 100}$. The two integral bounds straddle the threshold tightly enough to decide it.

Step 4: Compare the two norms. Orthogonality gives

$$\boxed{\|F - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^4}, \qquad \frac{\pi}{3(N+1)^3} < \|F - P_N\|_2^2 < \frac{\pi}{3N^3}.}$$

The uniform error is of order N^{-1}, whereas the L^2 error is of order $N^{-3/2}$ (its square is of order N^{-3}). Distinguish the norm from its square.



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Question 4

Let $G(x) = (\pi - x)/2$ on $0 < x < 2\pi$, extended periodically with value zero at the joins. Its Fourier coefficients give

$$G(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi).$$

This identity follows from the piecewise smooth Fourier theorem. Set $P_N = \sum_{n=1}^{N} \sin(nx)/n$ and fix $0 < \delta < \pi$.

Tasks

1. Use a finite geometric sum to bound $\left|\sum_{n=p}^{q} \sin nx\right|$ uniformly in p, q for $x \in [\delta, 2\pi - \delta]$.
2. Apply summation by parts to prove $\|G - P_N\|_{\infty,[\delta,2\pi-\delta]} \le 1/((N+1)\sin(\delta/2))$.
3. Prove that convergence is not uniform even on the full open interval $(0, 2\pi)$. Explain why the preceding bound cannot be used with $\delta = 0$.
4. Compute the squared $L^2(0, 2\pi)$ error and bound it. Sketch the error for $N = 8$, marking the endpoint limits separately from the assigned endpoint errors.

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Question 4 – Solution

Strategy. Cancellation controls a tail away from a jump, but the cancellation bound degenerates near the join.

Step 1: Bound finite oscillatory blocks. For $x \notin 2\pi\mathbb{Z}$,

$$\left|\sum_{n=p}^{q} e^{inx}\right| = \left|\frac{e^{ipx}(1-e^{i(q-p+1)x})}{1-e^{ix}}\right| \le \frac{1}{|\sin(x/2)|}.$$

The same bound holds for the imaginary part. On the specified closed interval, it is at most $B = 1/\sin(\delta/2)$.

Step 2: Sum by parts with a decreasing weight. Put $A_k = \sum_{n=N+1}^{k} \sin nx$, so $|A_k| \le B$. Then

$$\sum_{n=N+1}^{M} \frac{\sin nx}{n} = \frac{A_M}{M} + \sum_{k=N+1}^{M-1} A_k \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

The absolute value is bounded by $B/(N+1)$. Passing to the convergent tail gives

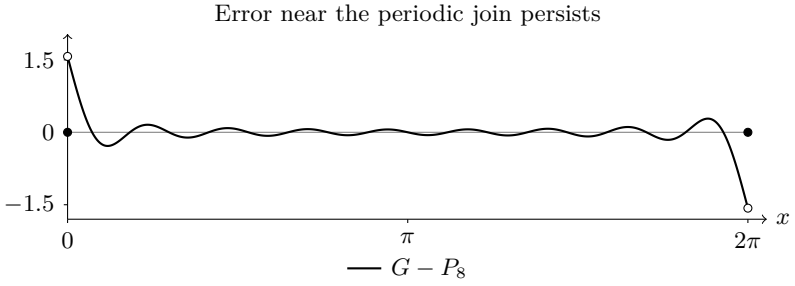
$$\boxed{\sup_{\delta \le x \le 2\pi-\delta} |G - P_N| \le \frac{1}{(N+1)\sin(\delta/2)} \longrightarrow 0.}$$

Step 3: Locate the global obstruction. For every fixed N, $P_N(x) \to 0$ as $x \downarrow 0$, while $G(x) \to \pi/2$. Therefore $\sup_{0<x<2\pi} |G - P_N| \ge \pi/2$ for every N. Excluding the endpoint itself does not exclude points arbitrarily close to it. At $\delta = 0$ the geometric-sum bound has a zero denominator and gives no finite uniform control.

Step 4: Verify convergence in the integral norm. The sine functions have squared norm π over this period. Hence

$$\boxed{\|G - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^2} < \frac{\pi}{N}.}$$

The error tends to zero in mean square despite persistent near-jump suprema. Its assigned endpoint errors are zero, whereas its interior limits are $\pi/2$ at 0 and $-\pi/2$ at 2π.



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Question 5

Consider

$$H(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

You may use the theorem: if a series of C^1 functions converges at one point and its derivative series converges uniformly, its sum is C^1 and may be differentiated term by term. Also use

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}, \qquad 0 \le x \le 2\pi.$$

Tasks

1. Verify every hypothesis needed to differentiate the series once on $[0, 2\pi]$.
2. Integrate the given derivative formula to obtain a closed form for H. Check the endpoint values and slopes of its periodic extension.
3. Determine whether the periodic extension is C^2. Identify precisely where the formal second derivative series agrees with the classical second derivative.
4. Prove that the second derivative partial sums cannot converge uniformly to the interior second derivative on $(0, 2\pi)$. Explain why the first differentiation remains valid.

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Question 5 – Solution

Strategy. Check a separate convergence condition for each differentiation; one valid differentiation does not authorize the next.

Step 1: Justify the first differentiation. Every summand is C^1. The original series converges at 0 (all terms are zero), and the derivative terms satisfy $|\cos nx/n^2| \le 1/n^2$. The Weierstrass test gives uniform convergence of the derivative series. The stated theorem therefore gives

$$\boxed{H'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.}$$

This includes one-sided endpoint derivatives on the closed interval.

Step 2: Verify the explicit function. Integrating the provided formula and using $H(0) = 0$ gives

$$\boxed{H(x) = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}, \qquad 0 \le x \le 2\pi.}$$

Both endpoint values are zero. Both endpoint slopes equal $\pi^2/6$, so the periodic extension is C^1 across the join.

Step 3: Identify the second derivative obstruction. Inside the period, $H''(x) = -\pi/2 + x/2$. Its right limit at zero is $-\pi/2$, whereas its left periodic limit is $\pi/2$. Thus the periodic function is not twice differentiable at the join and is not C^2. The formal second derivative series is $-\sum \sin(nx)/n$. The piecewise smooth Fourier theorem, or direct coefficients of the displayed linear function, gives H'' on $0 < x < 2\pi$. At a join the series equals zero, the average of the limits, where the classical periodic second derivative does not exist.

Step 4: Rule out uniform second differentiation. Every finite sum $-\sum_{n=1}^{N} \sin(nx)/n$ tends to zero as $x \downarrow 0$, but the interior target tends to $-\pi/2$. Thus its error supremum on the full open period is at least $\pi/2$ for every N. The derivative series for the first differentiation had summable bounds n^{-2}; the next series has no such global uniform convergence. There is no contradiction between a valid first differentiation and failure of the stronger second-derivative conclusion.

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Question 6

For $N \geq 1$, consider real trigonometric polynomials $p(x) = A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$ on $[-\pi, \pi]$. Use $\|p\|_2^2 = \int_{-\pi}^{\pi} p^2$ and define $D_N(x) = 1 + 2\sum_{n=1}^{N} \cos nx$.

Tasks

1. Prove $\|p\|_\infty \leq \sqrt{(2N+1)/(2\pi)}\,\|p\|_2$ by an orthonormal-basis argument.
2. Prove the constant is sharp and characterize all nonzero equality cases at a prescribed point x_0.
3. For $p_N = D_N/(2N+1)$, compute both norms exactly and determine its pointwise limit. Explain what this says about point evaluation under mean-square convergence.
4. Derive the sine-quotient form of D_N and sketch p_2, p_8 over a period. Explain why this sequence is not the sequence of Fourier partial sums of one fixed function.

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Question 6 – Solution

Strategy. Point evaluation on a finite-dimensional space has a norm that grows with the number of available modes.

Step 1: Apply Cauchy–Schwarz in coefficient space. An orthonormal basis is $1/\sqrt{2\pi}$, $\cos nx/\sqrt{\pi}$, $\sin nx/\sqrt{\pi}$. The sum of their squared values at any point is

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}(\cos^2 nx + \sin^2 nx) = \frac{2N+1}{2\pi}.$$

Cauchy–Schwarz against the orthonormal coefficient vector proves

$$\boxed{\|p\|_\infty \le \sqrt{\frac{2N+1}{2\pi}}\,\|p\|_2.}$$

Step 2: Identify equality cases. Equality at x_0 holds exactly when the coefficient vector is a real nonzero multiple of the basis-value vector at x_0. Equivalently,

$$\boxed{p(x) = CD_N(x - x_0), \qquad C \in \mathbb{R}\setminus\{0\}.}$$

This exhibits the sharp constant, not merely an upper estimate.

Step 3: Build a vanishing-energy sequence with a fixed peak. The triangle inequality and the value at zero give $\|D_N\|_\infty = 2N+1$. Orthogonality gives $\|D_N\|_2^2 = 2\pi(2N+1)$. Consequently

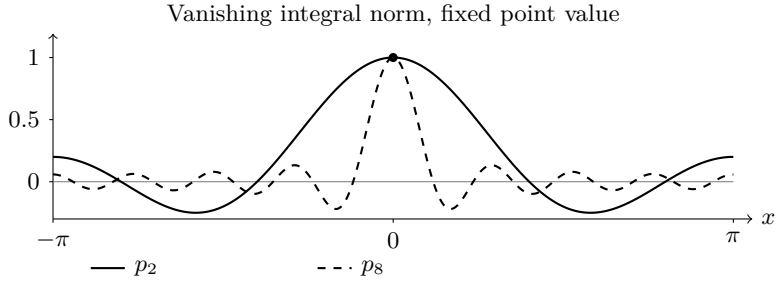
$$\boxed{\|p_N\|_\infty = 1, \qquad \|p_N\|_2^2 = \frac{2\pi}{2N+1} \to 0.}$$

At 0 modulo 2π, $p_N = 1$. At every other fixed point it tends to zero by the quotient below. Point evaluation is therefore not continuous in the L^2 norm on the union of these polynomial spaces.

Step 4: Explain the oscillatory peak. Summing $\sum_{n=-N}^{N} e^{inx}$ yields

$$\boxed{D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)} \quad (x \notin 2\pi\mathbb{Z}).}$$

The removable value at zero is $2N+1$. The normalized peak narrows with N. These p_N cannot be successive partial sums of one fixed function: their constant coefficients $1/(2N+1)$ change with N, whereas truncation preserves every already-included Fourier coefficient.



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Question 7

For real bounded piecewise continuous 2π-periodic f, its degree-one Fourier projection is

$$(S_1 f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)(1 + 2\cos t)\, dt.$$

Use the essential supremum norm, which ignores changes on sets of measure zero.

Tasks

1. Prove $\|S_1 f\|_\infty \le \Lambda \|f\|_\infty$, where $\Lambda = (2\pi)^{-1} \int_{-\pi}^{\pi} |1 + 2\cos t|\, dt$.
2. Compute Λ exactly, locating every sign change of the kernel.
3. Construct an admissible f with $\|f\|_\infty = 1$ for which equality is attained at $x = 0$. Verify its mean and first cosine coefficient directly.
4. Use the result to give a sharp worst-case amplification factor for a bounded input error under S_1. Explain why orthogonal projection being a contraction in L^2 does not imply contraction in the supremum norm.

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Question 7 – Solution

Strategy. The signed Fourier kernel controls integral projection, while its absolute integral controls worst-case pointwise error.

Step 1: Bound the projection in the supremum norm. Taking absolute values in the given integral gives

$$|S_1 f(x)| \le \frac{\|f\|_\infty}{2\pi} \int_{-\pi}^{\pi} |1 + 2\cos t|\, dt.$$

Null-set changes do not affect this estimate or the projected polynomial. Taking the supremum over x proves the claimed inequality.

Step 2: Integrate the absolute kernel. The kernel is nonnegative for $|t| \le 2\pi/3$ and negative outside. Its full signed integral is 2π. Its integral on the two negative intervals is

$$2\int_{2\pi/3}^{\pi} (1 + 2\cos t)\, dt = \frac{2\pi}{3} - 2\sqrt{3}.$$

Changing the sign of that contribution gives

$$\boxed{\Lambda = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} > 1.}$$

Step 3: Attain the bound with an explicit input. Choose $f(t) = \operatorname{sgn}(1 + 2\cos t)$, with value zero at its two zeros. It is even and piecewise constant with essential supremum one. Its positive region has length $4\pi/3$, so its mean is $1/3$. Direct integration gives

$$a_1 = \frac{2}{\pi}\left(\int_0^{2\pi/3} \cos t\, dt - \int_{2\pi/3}^{\pi} \cos t\, dt\right) = \frac{2\sqrt{3}}{\pi}, \qquad b_1 = 0.$$

Hence $S_1 f(0) = 1/3 + 2\sqrt{3}/\pi = \Lambda$, proving sharpness.

Step 4: Interpret the stability statement. For input error e with $\|e\|_\infty \le \varepsilon$, linearity gives $\|S_1(f + e) - S_1 f\|_\infty \le \Lambda\varepsilon$. The choice $e = \varepsilon \operatorname{sgn}(1 + 2\cos t)$ attains this bound. Orthogonal projection still satisfies $\|S_1 e\|_2 \le \|e\|_2$ by the Pythagorean identity. That statement uses an integral norm, and does not prevent a larger pointwise peak. Contraction is a property of a specified norm.

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Question 8

Let real coefficient sequences satisfy

$$\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \le M, \qquad M > 0.$$

Define the formal series $f = m + \sum_{n\ge1}(a_n \cos nx + b_n \sin nx)$ and let S_N be its degree-N truncation, $N \ge 1$. Write $T_N = \sum_{n>N} n^{-2}$.

Tasks

1. Prove that the series defines a continuous periodic function by establishing uniform convergence.
2. Prove $\|f - S_N\|_\infty \le \sqrt{MT_N} < \sqrt{M/N}$.
3. For a fixed N and prescribed point x_0, construct coefficients satisfying the constraint for which the bound $\sqrt{MT_N}$ is attained.
4. Prove $\|f - S_N\|_2^2 \le \pi M/(N+1)^2$ and construct an equality case for this bound. Explain why the two optimizing coefficient patterns differ.

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Question 8 – Solution

Strategy. Use the weighted coefficient budget with two different dual estimates: one for a point value and one for energy.

Step 1: Prove uniform convergence from the budget. For any finite tail from p to q, Cauchy–Schwarz gives

$$\left|\sum_{n=p}^{q}(a_n\cos nx+b_n\sin nx)\right|\le\left(\sum_{n=p}^{q}n^2(a_n^2+b_n^2)\right)^{1/2}\left(\sum_{n=p}^{q}\frac{\cos^2 nx+\sin^2 nx}{n^2}\right)^{1/2}.$$

It is at most $\sqrt{M}(\sum_{n\ge p}n^{-2})^{1/2}$, independently of x, and tends to zero as $p\to\infty$. The uniform Cauchy criterion applies. The limit of these continuous periodic partial sums is continuous and periodic.

Step 2: Bound the infinite tail. Passing to the uniform limit of the finite-tail inequality gives

$$\boxed{\|f-S_N\|_\infty\le\sqrt{MT_N}<\sqrt{M/N}\quad(N\ge1).}$$

The weighted hypothesis controls the full collection of omitted modes, not just a finite list of observed coefficients.

Step 3: Attain the exact pointwise bound. Set every coefficient with $n\le N$ to zero and, for $n>N$, choose

$$\boxed{a_n=\sqrt{\frac{M}{T_N}}\frac{\cos(nx_0)}{n^2},\qquad b_n=\sqrt{\frac{M}{T_N}}\frac{\sin(nx_0)}{n^2}.}$$

The weighted sum equals M, and at x_0 the tail equals $\sqrt{MT_N}$. This proves sharpness of the first bound over all admissible sequences. The mean m may be arbitrary because it cancels from the error.

Step 4: Optimize the integral error. Orthogonality, justified by the uniform convergence, gives

$$\|f-S_N\|_2^2=\pi\sum_{n>N}(a_n^2+b_n^2)\le\frac{\pi}{(N+1)^2}\sum_{n>N}n^2(a_n^2+b_n^2)\le\boxed{\frac{\pi M}{(N+1)^2}}.$$

Equality is attained by $a_{N+1}=\sqrt{M}/(N+1)$ and all other coefficients zero. The integral optimizer concentrates its budget in the cheapest omitted mode; the pointwise optimizer distributes it across all omitted modes with phases aligned at the chosen point.

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Question 9

For $p > 0$, study $F_{p,N}(x) = \sum_{n=1}^{N} \cos(nx)/n^p$ on $[-\pi, \pi]$. You may use completeness of L^2: a Cauchy sequence in its norm has an L^2 limit; taking an inner product with any fixed L^2 function is continuous. Also use Dirichlet's test for a scalar series with bounded partial sums multiplied by decreasing positive weights tending to zero.

Tasks

1. Classify exactly those p for which the sequence converges in L^2, proving both directions by orthogonality.
2. Determine convergence at $x = 0$ modulo 2π, and at every other fixed point.
3. Classify uniform convergence on the full period. Prove uniform convergence on each closed subinterval avoiding $2\pi\mathbb{Z}$ for every $p > 0$.
4. For $1/2 < p \leq 1$, explain rigorously how an L^2 function can have these Fourier partial sums although they diverge at zero. Can redefining the limit's value at zero change that divergence?

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Question 9 – Solution

Strategy. The thresholds for square summability, pointwise convergence at a common phase, and uniform convergence need not coincide.

Step 1: Classify mean-square convergence. For $M > N$, orthogonality gives

$$\|F_{p,M} - F_{p,N}\|_2^2 = \pi \sum_{n=N+1}^{M} n^{-2p}.$$

This tends to zero uniformly in $M > N$ exactly when $2p > 1$. Completeness proves existence of an L^2 limit in that case. For $p \le 1/2$, the norms $\pi \sum_{n=1}^{N} n^{-2p}$ are unbounded, so no L^2 limit is possible. Thus $\boxed{p > 1/2}$ is necessary and sufficient.

Step 2: Classify each pointwise series. At zero modulo 2π the series is $\sum n^{-p}$, converging exactly when $p > 1$; otherwise its partial sums tend to $+\infty$. At any other fixed x, the geometric-sum bound gives bounded partial sums of $\cos nx$. The weights n^{-p} decrease to zero for every $p > 0$, so Dirichlet's test proves convergence at every such point.

Step 3: Distinguish global and local uniform convergence. For $p > 1$, the Weierstrass test proves absolute uniform convergence globally. For $0 < p \le 1$, divergence at zero precludes uniform convergence on the full period. Away from $2\pi\mathbb{Z}$, the cosine block sums are bounded by $1/\sin(\delta/2)$. Summation by parts with weights n^{-p} gives

$$\boxed{\sup_{\delta \le x \le 2\pi - \delta} \left| \sum_{n>N} \frac{\cos nx}{n^p} \right| \le \frac{(N+1)^{-p}}{\sin(\delta/2)}.}$$

Hence local uniform convergence holds for every $p > 0$.

Step 4: Interpret the L^2 limit. For $1/2 < p \le 1$, continuity of inner products shows that the L^2 limit has mean zero, cosine coefficients n^{-p} and zero sine coefficients. Thus these really are its Fourier partial sums, not arbitrary approximations. An L^2 function is defined only up to changes on sets of measure zero; a single divergent point does not contradict convergence in the integral norm. Assigning any value to the limit at zero changes neither coefficients nor partial sums, and therefore cannot remove the divergence there.

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Question 10

Let $f \in L^2(-\pi, \pi)$ be real and periodic with mean m and Fourier coefficients a_n, b_n. For $t > 0$, define the damped expansion

$$U_t(x) = m + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n \cos nx + b_n \sin nx).$$

Use Parseval and L^2 convergence of Fourier partial sums. This problem concerns the series itself; no partial differential equation is assumed.

Tasks

1. Prove that for each fixed $t > 0$ the series, and every series obtained by differentiating in x finitely many times, converge uniformly.
2. Prove $U_t \to f$ in L^2 as $t \downarrow 0$ using a finite-head and small-tail argument, with an exact coefficient formula for the error.
3. If $\sum n^2(a_n^2 + b_n^2) \le M$, prove $\|U_t - f\|_2 \le \sqrt{\pi M t}$.
4. Show that the mean is preserved and that $\|U_t - m\|_2 \le \|f - m\|_2$. Explain why the general L^2 assumption alone cannot guarantee uniform convergence to the assigned function f as $t \downarrow 0$.

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Question 10 – Solution

Strategy. Damping controls all high frequencies for positive t; removing it requires a separate convergence argument.

Step 1: Prove smoothness for positive damping. Parseval gives $E = \sum(a_n^2 + b_n^2) < \infty$. After k derivatives, the absolute value of the nth harmonic is at most $n^k e^{-n^2 t}\sqrt{a_n^2 + b_n^2}$. Cauchy–Schwarz bounds the sum by

$$\sqrt{E}\left(\sum_{n\geq 1} n^{2k} e^{-2n^2 t}\right)^{1/2} < \infty.$$

The exponential dominates every power, so the Weierstrass test applies for each fixed $t > 0$ and each integer $k \geq 0$. Repeated use of the uniform derivative theorem justifies differentiation; U_t is smooth and periodic.

Step 2: Remove damping in the integral norm. The Fourier coefficients of the difference give

$$\boxed{\|U_t - f\|_2^2 = \pi \sum_{n\geq 1}(1 - e^{-n^2 t})^2(a_n^2 + b_n^2).}$$

Given $\varepsilon > 0$, choose K so that $\pi \sum_{n>K}(a_n^2 + b_n^2) < \varepsilon/2$. Since $0 \leq 1 - e^{-n^2 t} \leq 1$, this bounds the error tail for every t. The remaining finite sum tends to zero, hence is below $\varepsilon/2$ for sufficiently small t. This proves the limit.

Step 3: Obtain a quantitative rate under extra information. For $z \geq 0$, $1 - e^{-z} \leq \min(z, 1)$, so $(1 - e^{-z})^2 \leq z$. Applying this with $z = n^2 t$ gives

$$\boxed{\|U_t - f\|_2^2 \leq \pi t \sum n^2(a_n^2 + b_n^2) \leq \pi M t.}$$

Taking square roots yields the requested rate; this uses a stronger hypothesis than mere square summability.

Step 4: Separate smoothing from uniform recovery. Uniform integration of the damped series preserves the mean m. Parseval gives $\|U_t - m\|_2^2 = \pi \sum e^{-2n^2 t}(a_n^2 + b_n^2) \leq \|f - m\|_2^2$. Every U_t is continuous. Choose an L^2 periodic step function with a jump: it cannot be a uniform limit of these continuous functions. Even isolated redefinitions of f are invisible to the coefficients. Thus smooth approximants and mean-square recovery do not alone guarantee uniform recovery of an arbitrarily assigned representative.

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Question 1

A stationary rod has cross-sectional area $A(x) > 0$, conductivity $k(x) > 0$ and volumetric heat capacity $C(x) > 0$. Temperature $u(x, t)$ is uniform across each section; the lateral surface is insulated. A volumetric source $s(x, t)$ has units $\mathrm{W/m^3}$. Define signed heat power toward increasing x by $Q = -kAu_x$, in watts. All coefficients are independent of time.

Tasks

1. Apply energy conservation to an arbitrary interval $[a, b]$ and derive the local heat equation. State the units of every term in the balance.
2. Expand the spatial derivative and identify the diffusivity. Explain when the equation reduces to $u_t = \alpha u_{xx}$.
3. After nondimensionalization, take $0 < x < 1$, $A = e^x$, $k = C = 1$, $s = 0$ and an instantaneous profile $u = x$. Find Q, the storage rate Au_t and u_t. Sketch the first two quantities.
4. Explain why this linear temperature profile need not be steady, even though its curvature is zero. Distinguish heat flux per area from total heat power.

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Question 1 – Solution

Strategy. Balance total heat power before dividing by the local cross-sectional area.

Step 1: Derive the conservation law. The stored thermal energy relative to a fixed reference has time derivative $\int_a^b CAu_t\,dx$. Inflow minus outflow plus generation gives

$$\int_a^b CAu_t\,dx = Q(a,t) - Q(b,t) + \int_a^b As\,dx.$$

Since this holds on every interval, with Fourier's law it yields

$$\boxed{CAu_t = -Q_x + As = (kAu_x)_x + As.}$$

Each local term has units W/m: C is J/(m^{3}K), A is m^2 and u_t is K/s. Each integrated term, including $Q(a) - Q(b)$, is in watts.

Step 2: Identify the geometric contribution. Dividing by CA gives

$$\boxed{u_t = \frac{k}{C}u_{xx} + \frac{(kA)'}{CA}u_x + \frac{s}{C}.}$$

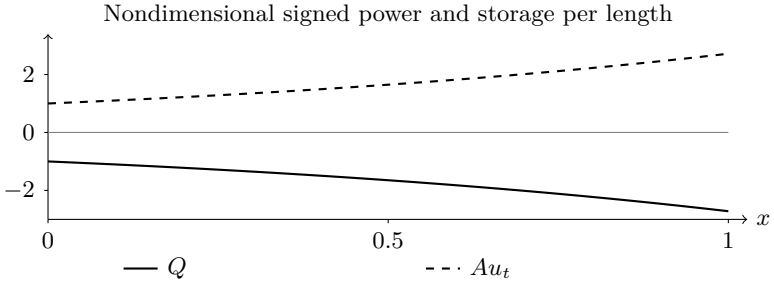
The diffusivity k/C has units m^2/s. For a homogeneous constant-area rod without a source, this is $u_t = \alpha u_{xx}$ with constant $\alpha = k/C$. More generally, the first-derivative term vanishes when kA is constant; constant diffusivity additionally requires k/C constant.

Step 3: Evaluate the nondimensional example. Here $u_x = 1$ and $u_{xx} = 0$, but

$$\boxed{Q(x) = -e^x, \qquad Au_t = -Q_x = e^x, \qquad u_t = 1.}$$

The signed power is negative because heat moves toward decreasing temperature, which here is the negative x direction. Its magnitude increases toward the larger cross section.

Step 4: Interpret the nonzero storage. Flux per unit area is $q = -ku_x = -1$, constant in this example. Total power is $Q = Aq = -e^x$, which is not constant. More heat enters a small interval through its larger right face than leaves through its smaller left face, so energy accumulates. Zero curvature alone is a steady-state test only under the coefficient and geometry assumptions that remove the extra term.



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Question 2

A homogeneous rod of length L and constant area A obeys $Cu_t = ku_{xx}$, with $C, k > 0$ and insulated sides. Both ends exchange heat with an ambient temperature T_a. The outward heat flux per area equals $h_j(u - T_a)$ at end j, where $h_0, h_L \geq 0$. Set $v = u - T_a$ and use the signed flux $q = -ku_x$ toward increasing x.

Tasks

1. Derive the two boundary conditions, explaining the different signs at the left and right ends.
2. Derive the rate of change of the total excess heat $E = CA \int_0^L v \, dx$. Must this signed quantity always decrease?
3. Derive a dissipation identity for $V = (CA/2) \int_0^L v^2 \, dx$.
4. Use the identity to classify every steady state, including $h_0 = h_L = 0$, and prove uniqueness for a prescribed initial temperature profile.

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Question 2 – Solution

Strategy. Outward flux has opposite coordinate signs at the two ends; quadratic energy provides a sign-definite quantity.

Step 1: Orient both boundary fluxes. At the left end, outward flux is $-q(0) = ku_x(0)$; at the right it is $q(L) = -ku_x(L)$. Thus

$$\boxed{kv_x(0,t) = h_0 v(0,t), \qquad -kv_x(L,t) = h_L v(L,t).}$$

Using the same coordinate sign at both ends would reverse one heat transfer.

Step 2: Balance the signed excess heat. Integrating the equation gives

$$\boxed{E' = kA[v_x(L) - v_x(0)] = -A\big(h_L v(L) + h_0 v(0)\big).}$$

It is negative when both ends are hotter than ambient and positive when both are colder. It need not always decrease, since E is a signed excess rather than a nonnegative measure of departure from ambient.

Step 3: Derive the dissipative quantity. Multiplying by Av and integrating by parts yields

$$\boxed{V' = -kA\int_0^L v_x^2\,dx - Ah_0 v(0)^2 - Ah_L v(L)^2 \le 0.}$$

The endpoint terms have the correct negative signs precisely because the boundary conditions were oriented outward.

Step 4: Classify steady states and prove uniqueness. For a steady state $V' = 0$, every nonnegative term on the right must vanish. Thus $v_x = 0$. If at least one $h_j > 0$, its boundary term forces this constant to be zero, so the sole equilibrium is $u = T_a$. If both coefficients vanish, every spatially constant temperature is steady. For uniqueness, the difference of two sufficiently smooth solutions with the same initial data obeys the same homogeneous equation and boundary conditions. Its quadratic energy starts at zero and cannot increase, so the difference vanishes everywhere by continuity. This also covers the insulated case: its possible equilibrium constants do not cause nonuniqueness for fixed initial data.

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Question 3

A rod has constant area, insulated sides, no internal source and conductivity $k(x) = k_0(1 + x/L)$, with $k_0, L > 0$. Its end temperatures are $u(0) = T_h$, $u(L) = T_c$, with $T_h > T_c$. The steady equation is $(k(x)u_x)_x = 0$, and flux per area toward the right is $q = -k(x)u_x$.

Tasks

1. Derive the exact temperature and heat flux.
2. Verify the equation and both boundary values, and locate the largest magnitude of the temperature gradient.
3. Test the straight-line interpolant between the same endpoints in the correct variable-conductivity equation. Explain its physical failure.
4. Define k_{eff} by $q = k_{\text{eff}}(T_h - T_c)/L$. Compute it and compare it with the arithmetic spatial average of k. Sketch the normalized exact and linear temperature profiles.

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Question 3 – Solution

Strategy. Constant steady heat flux requires the gradient to vary inversely with conductivity.

Step 1: Integrate the flux law. Since $(ku_x)_x = 0$, the flux q is constant. Integrating $u_x = -q/k(x)$ gives

$$T_c - T_h = -\frac{qL}{k_0}\log 2.$$

Consequently

$$\boxed{q = \frac{k_0(T_h - T_c)}{L\log 2}, \qquad u(x) = T_h - (T_h - T_c)\frac{\log(1 + x/L)}{\log 2}.}$$

The flux is positive: heat travels from the hotter left end toward the right.

Step 2: Check the profile and gradient. The logarithm is zero at 0 and $\log 2$ at L, giving the two endpoint values. Also

$$u_x = -\frac{T_h - T_c}{L\log 2\,(1 + x/L)}, \qquad k(x)u_x = -\frac{k_0(T_h - T_c)}{L\log 2}.$$

Thus the product is constant as required. The gradient magnitude is largest at $x = 0$ and decreases by a factor of two from left to right.

Step 3: Diagnose the linear candidate. For $\ell = T_h - (T_h - T_c)x/L$, one has $\ell_{xx} = 0$, but

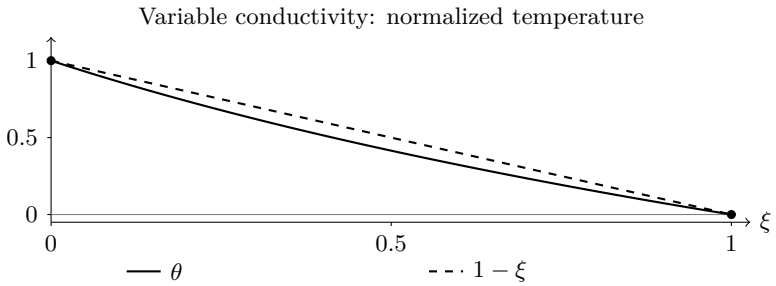
$$\boxed{(k\ell_x)_x = -\frac{k_0(T_h - T_c)}{L^2} \neq 0.}$$

Its rightward flux grows along the rod, so more heat leaves each small interval than enters. Without a source this profile would cool locally, not remain steady. Using $ku_{xx} = 0$ discards the essential term $k'u_x$.

Step 4: Interpret the effective conductivity. The definition gives $k_{\text{eff}} = k_0/\log 2$, while the arithmetic average is $3k_0/2$. In fact $\log 2 > 2/3$: the function $1/t$ is strictly convex and its average on $[1, 2]$ exceeds its midpoint value $2/3$. Hence

$$\boxed{k_{\text{eff}} = \frac{k_0}{\log 2} < \frac{3k_0}{2}.}$$

Steady conduction uses the reciprocal average $k_{\text{eff}} = L/\int_0^L k^{-1}\,dx$. The graph uses $\xi = x/L$ and $\theta = (u - T_c)/(T_h - T_c)$.



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Question 4

Two homogeneous layers in series have thicknesses L_1, L_2, conductivities $k_1, k_2 > 0$ and the same cross-sectional area. The outer temperatures are $T_h > T_c$. There are no sources. A contact resistance per area $R \geq 0$ satisfies $u_- - u_+ = Rq$, where u_-, u_+ are the left and right interface traces and q is the steady rightward heat flux. The interface stores no energy.

Tasks

1. Explain why flux is continuous at the interface while temperature may jump. State all equations determining the steady profiles.
2. Derive q, u_-, u_+ and the two linear temperature profiles.
3. In consistent nondimensional units, take $L_1 = L_2 = 1$, $k_1 = 1$, $k_2 = 2$, $R = 1/2$, $T_h = 100$ and $T_c = 0$. Compute and sketch the profiles, showing both interface traces.
4. Analyze the limits $R \to 0$ and $R \to \infty$. Explain why imposing temperature continuity when $R > 0$ would generally contradict the model.

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Question 4 – Solution

Strategy. Add the two layer resistances and the contact resistance, retaining the interface temperature drop.

Step 1: Apply the interface balance. No interfacial storage or source means incoming flux equals outgoing flux. Thus the same q satisfies $u_x = -q/k_1$ in the first layer and $u_x = -q/k_2$ in the second. The boundary and contact relations are

$$u(0) = T_h, \quad u(L_1 + L_2) = T_c, \quad u_- - u_+ = Rq.$$

Temperature continuity is a special case of zero resistance, not the general interface law.

Step 2: Add all temperature drops. Writing $\mathcal{R} = L_1/k_1 + R + L_2/k_2$ gives

$$\boxed{q = \frac{T_h - T_c}{\mathcal{R}}, \quad u_- = T_h - \frac{qL_1}{k_1}, \quad u_+ = T_c + \frac{qL_2}{k_2}.}$$

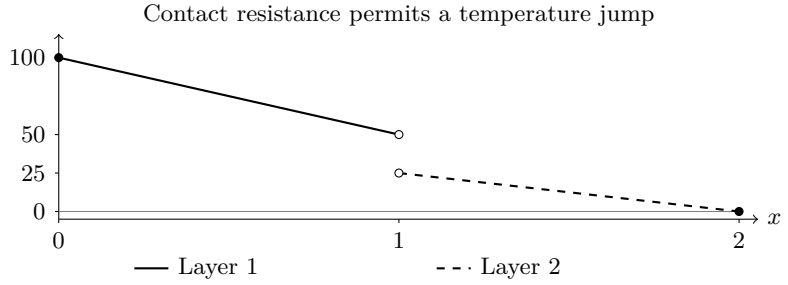
The profiles are $T_h - qx/k_1$ on the first layer and $T_c + q(L_1 + L_2 - x)/k_2$ on the second. Their traces differ by Rq.

Step 3: Verify the numerical example. Here $\mathcal{R} = 1 + 1/2 + 1/2 = 2$, so

$$\boxed{q = 50, \quad u_- = 50, \quad u_+ = 25, \qquad u(x) = \begin{cases} 100 - 50x, & 0 \le x < 1, \\ 50 - 25x, & 1 < x \le 2. \end{cases}}$$

The flux is 50 on both sides, although the slopes differ. The temperature jump is $25 = Rq$. Open circles show separate traces, not a single assigned temperature at an idealized zero-thickness contact.

Step 4: Take the resistance limits. As $R \to 0$, the flux tends to $(T_h - T_c)/(L_1/k_1 + L_2/k_2)$ and the traces meet. As $R \to \infty$, $q \to 0$, each layer becomes nearly constant at its own outer temperature, and $Rq \to T_h - T_c$. A large resistance can therefore sustain a finite temperature jump while the flux vanishes. For finite $R > 0$, forcing $u_- = u_+$ would give $q = 0$, contradicting $T_h > T_c$ and the finite total resistance.



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Question 5

A homogeneous rod with insulated ends is observed to have temperature

$$u(x,t) = T_a + Be^{-\lambda t}\cos(\pi x/L), \qquad 0 \le x \le L, \quad B, \lambda, L > 0.$$

The unknown constant conductivity and volumetric heat capacity satisfy $Cu_t = ku_{xx}$ and $k, C > 0$. Assume B, L, λ are known.

Tasks

1. Verify the boundary conditions and infer the diffusivity $\alpha = k/C$ from the observed field.
2. Prove that temperature observations of this field alone do not identify k and C separately. Describe the full family of positive parameter pairs.
3. An independent measurement gives rightward heat flux per area $q_m > 0$ at $x = L/2, t = 0$. Recover k and C uniquely and check their units.
4. Explain why the midpoint is useful for this flux measurement but useless for measuring the temperature decay. Identify which temperature sensor locations can recover λ from two distinct measurement times.

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Question 5 – Solution

Strategy. The temperature equation determines a ratio of material parameters; a flux measurement provides an independent scale.

Step 1: Verify the field and recover diffusivity. Differentiation gives

$$u_t = -\lambda B e^{-\lambda t}\cos(\pi x/L), \qquad u_{xx} = -\frac{\pi^2}{L^2} B e^{-\lambda t}\cos(\pi x/L).$$

Also $u_x = -(\pi B/L)e^{-\lambda t}\sin(\pi x/L)$, so $u_x = 0$ at both ends. Matching the equation at points where the cosine is nonzero gives

$$\boxed{\alpha = \frac{k}{C} = \frac{\lambda L^2}{\pi^2}.}$$

The same equality then verifies the equation everywhere, including the node.

Step 2: Identify the nonuniqueness. For every choice $C_* > 0$, the pair

$$\boxed{C = C_*, \qquad k = \frac{\lambda L^2}{\pi^2} C_*}$$

produces exactly the observed field. Multiplying both parameters by the same positive factor leaves the equation and insulated boundary conditions unchanged. More exact observations of this same temperature field cannot resolve that scaling ambiguity.

Step 3: Use the independent flux data. At the midpoint initially, $q = -ku_x = kB\pi/L$. Therefore

$$\boxed{k = \frac{q_m L}{B\pi}, \qquad C = \frac{k}{\alpha} = \frac{q_m \pi}{B\lambda L}.}$$

With q_m in $\mathrm{W/m^2}$, B in kelvins and λ in $\mathrm{s^{-1}}$, these have units $\mathrm{W/(mK)}$ and $\mathrm{J/(m^3K)}$, respectively. Positivity of the measured quantities ensures physically positive parameters.

Step 4: Choose informative sensor locations. At $x = L/2$, $u = T_a$ for all time: both u_t and u_{xx} vanish, so a ratio of these derivatives would be $0/0$. But $|u_x|$ is largest there, giving a strong flux signal. At any fixed $x \neq L/2$ in $[0, L]$, the excess temperature is nonzero and

$$\boxed{\lambda = \frac{1}{t_2 - t_1}\log\left|\frac{u(x,t_1) - T_a}{u(x,t_2) - T_a}\right|, \quad t_2 > t_1.}$$

The absolute values allow sensors on either side of the node; the ratio eliminates the unknown local amplitude.

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Question 6

On an infinite homogeneous rod, consider

$$u(x,t) = T_a + \frac{B}{\sqrt{t+\tau}} \exp\left(-\frac{x^2}{4\alpha(t+\tau)}\right), \quad B, \tau, \alpha > 0, \quad t \geq 0.$$

Use $C > 0$ for volumetric heat capacity and interpret integrated heat per unit cross-sectional area. You may use $\int_{\mathbb{R}} e^{-z^2}\, dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} z^2 e^{-z^2}\, dz = \sqrt{\pi}/2$.

Tasks

1. Verify $u_t = \alpha u_{xx}$ directly, retaining the derivative of the time-dependent prefactor.
2. Compute the total excess heat per area and show that it is constant.
3. Compute the variance of the excess-temperature profile about zero and describe its peak and width as time increases.
4. Determine where the temperature is increasing at a given time, and the time of maximum temperature at a fixed point x. Sketch the profiles for $\alpha = B = \tau = 1$ at $t = 0, 3$, marking the inflection points.

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Question 6 – Solution

Strategy. Differentiate the amplitude and width together; local warming can coexist with a falling global peak and conserved heat.

Step 1: Verify the differential equation. Put $v = u - T_a$ and $\sigma = t + \tau > 0$. Logarithmic differentiation gives

$$v_t = v\left(-\frac{1}{2\sigma} + \frac{x^2}{4\alpha\sigma^2}\right), \qquad v_{xx} = v\left(-\frac{1}{2\alpha\sigma} + \frac{x^2}{4\alpha^2\sigma^2}\right).$$

Thus $v_t = \alpha v_{xx}$. Omitting the derivative of $\sigma^{-1/2}$ would remove the first term and invalidate the equation.

Step 2: Integrate the excess heat. With $z = x/\sqrt{4\alpha\sigma}$,

$$\boxed{\mathcal{H} = C\int_{\mathbb{R}} v\,dx = 2CB\sqrt{\pi\alpha}.}$$

The factor from the increasing width cancels the decreasing prefactor. The Gaussian flux tends to zero at both infinities, consistent with no net heat loss from the infinite rod.

Step 3: Measure spreading. Symmetry gives center zero. The provided Gaussian integrals yield

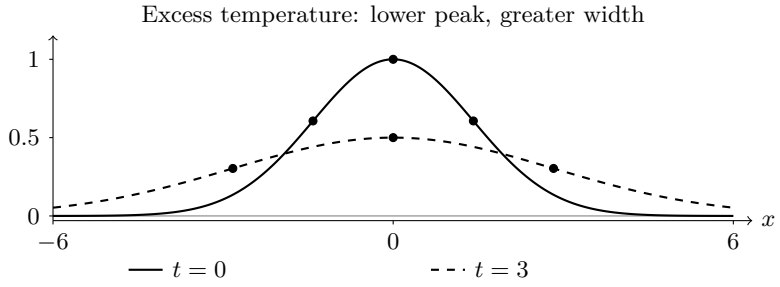
$$\boxed{\frac{\int_{\mathbb{R}} x^2 v\,dx}{\int_{\mathbb{R}} v\,dx} = 2\alpha\sigma, \qquad \max_x v = \frac{B}{\sqrt{\sigma}}.}$$

Thus the root-mean-square width grows as $\sqrt{2\alpha(t+\tau)}$, while the peak decreases as $(t+\tau)^{-1/2}$. The inflection points occur at $x = \pm\sqrt{2\alpha\sigma}$.

Step 4: Locate local warming. Since $v > 0$, $u_t > 0$ precisely when $x^2 > 2\alpha(t+\tau)$. A fixed point reaches its maximum on $t \geq 0$ at

$$\boxed{t_* = \max\left(0, \frac{x^2}{2\alpha} - \tau\right).}$$

Outside the initial inflection points it first warms and then cools. Inside them it cools from the start; equality gives an initially zero derivative. Therefore diffusion does not force temperature to decrease at every location. The graph uses excess temperature v, so the ambient offset is removed.



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Question 7

A constant-area rod with constant $k, C > 0$ is prescribed the candidate field

$$u(x,t) = T_a + at + b\,x(L-x), \qquad a, b, L > 0, \quad 0 \le x \le L.$$

The model permits a volumetric source s and obeys $Cu_t = ku_{xx} + s$. Its lateral surface is insulated; end conditions must be determined consistently.

Tasks

1. Determine the unique source field that makes the candidate satisfy the equation.
2. State the initial and Dirichlet boundary data realized by the candidate. Decide whether it also satisfies insulated-end conditions.
3. Compute both outward end heat fluxes and verify the integrated energy balance, using cross-sectional area A.
4. Suppose the source is removed while the same formula is retained but a is allowed to be real. Find the required a and explain why changing only the boundary values cannot fix a nonzero interior residual.

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Question 7 – Solution

Strategy. A manufactured temperature field determines its source and boundary data; those data cannot be prescribed independently afterward.

Step 1: Compute the interior residual. The derivatives are $u_t = a$, $u_x = b(L - 2x)$ and $u_{xx} = -2b$. Substitution requires

$$\boxed{s = Ca + 2kb.}$$

This is a positive spatially and temporally constant source, in $\mathrm{W/m^3}$. It supplies both the rising stored energy and heat lost at the ends.

Step 2: State the compatible data. At $t = 0$, $u(x, 0) = T_a + bx(L - x)$. Both endpoint temperatures equal $T_a + at$. The endpoint derivatives are bL and $-bL$, neither zero, so this field does not satisfy insulated-end conditions when $b > 0$. Prescribing insulation in addition to the realized end temperatures would contradict the candidate rather than add useful information.

Step 3: Verify total energy accounting. The outward fluxes are $ku_x(0) = kbL$ and $-ku_x(L) = kbL$. The stored heat relative to T_a is

$$E = CA \int_0^L (u - T_a)\, dx = CA \left(atL + \frac{bL^3}{6} \right), \qquad E' = CAaL.$$

Total source power minus total outward power is

$$\boxed{AL(Ca + 2kb) - 2AkbL = CAaL = E'.}$$

This independently checks the source sign and both boundary signs.

Step 4: Remove the source consistently. For $s = 0$, the formula can satisfy the PDE only if

$$\boxed{a = -\frac{2kb}{C}.}$$

Its initially warm curved profile then decreases at a constant rate while both endpoint temperatures follow the same decreasing law. If a retains a different value, the residual $Ca+2kb$ is nonzero at every interior point. Boundary values alone cannot repair that local equation while keeping the same candidate field. Changing boundary data generally requires changing the field itself.

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Question 8

Let u be continuous on $[0, L] \times [0, T]$, sufficiently differentiable in the interior, and satisfy $u_t = \alpha u_{xx}$ with $\alpha > 0$. Assume $0 \leq u(x, 0) \leq M$ and $0 \leq u(0, t), u(L, t) \leq M$, where $M > 0$.

Tasks

1. Prove $u \leq M$ by applying a maximum argument to $w = u - M - \varepsilon t$ and then letting $\varepsilon \downarrow 0$. Address a maximum occurring at the final observation time.
2. Prove $u \geq 0$ and explain why these bounds concern the global range rather than monotonic cooling at every fixed position.
3. Test $v(x, t) = Me^{\alpha(\pi/L)^2 t} \sin(\pi x/L)$ against the source-free heat equation and the range bound.
4. Find the source needed to realize this growing candidate in $Cv_t = kv_{xx} + s$, with $k/C = \alpha$. Give and verify a source-free decaying candidate with the same initial and endpoint data.

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Question 8 – Solution

Strategy. A small time-dependent perturbation makes the maximum argument strict, while a growing test field exposes a missing source.

Step 1: Prove the upper bound. For $\varepsilon > 0$, w is nonpositive initially and at the spatial endpoints. If it has a positive maximum on the space-time rectangle, that maximum occurs at an interior spatial point and a time $t_* > 0$. There $w_{xx} \le 0$ and $w_t \ge 0$: at the latest time, use the derivative from earlier times; at an interior time it is zero. Yet

$$w_t - \alpha w_{xx} = -\varepsilon < 0,$$

a contradiction. One can first apply the argument on any shorter time rectangle and pass to T by continuity if needed. Thus $u \le M + \varepsilon t$, and letting $\varepsilon \downarrow 0$ gives $u \le M$.

Step 2: Prove the lower bound. Apply the same argument to $-u$ with upper bound zero to obtain $-u \le 0$. Hence $\boxed{0 \le u \le M}$ throughout the rectangle. The theorem does not say $u_t \le 0$ at every point. At a local temperature minimum with positive curvature, the equation gives $u_t = \alpha u_{xx} > 0$. Redistribution can warm a cooler location without exceeding the allowed global range.

Step 3: Diagnose the growing sine. Write $\mu = (\pi/L)^2$. Then $v_t = \alpha\mu v$ and $v_{xx} = -\mu v$, so

$$\boxed{v_t - \alpha v_{xx} = 2\alpha\mu v \neq 0}$$

in the spatial interior. It satisfies zero endpoint values and the allowed initial profile, but for $t > 0$ its midpoint exceeds M. This violates the conclusion because it fails the source-free PDE hypothesis.

Step 4: Restore the missing physics or correct the time factor. The required source is $s = C(v_t - \alpha v_{xx}) = 2k\mu v > 0$ in the interior. With that source, the earlier source-free maximum argument no longer applies. Alternatively,

$$\boxed{\widetilde{v}(x,t) = Me^{-\alpha\mu t}\sin(\pi x/L)}$$

has $\widetilde{v}_t = -\alpha\mu\widetilde{v} = \alpha\widetilde{v}_{xx}$, the same initial profile and zero endpoints. It remains between zero and M.

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Question 9

Two homogeneous rods have lengths L and $2L$, the same $k, C > 0$, insulated sides and no sources. At both ends each rod exchanges heat with the same ambient T_a by an outward convection law. The first rod has coefficient $h > 0$. The two initial profiles have the same shape as functions of relative position. Set $\alpha = k/C$ and choose a nonzero temperature scale ΔT.

Tasks

1. Nondimensionalize the first problem using $\xi = x/L$, $\eta = \alpha t/L^2$ and $\theta = (u - T_a)/\Delta T$. Identify the dimensionless boundary parameter.
2. Verify the PDE and initial shape for the proposed rescaling $u_2(x, t) = u_1(x/2, t/4)$ on the longer rod.
3. Derive exactly which convection coefficient h_2 makes this rescaling satisfy both boundary laws.
4. Assess the claim that doubling length always quadruples every thermal response time when the same h is used. State when the proposed scaling is valid, including the perfectly insulated case.

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Question 9 – Solution

Strategy. Match the dimensionless boundary conditions as well as the interior diffusion time scale.

Step 1: Identify the dimensionless problem. The chain rule gives $\theta_\eta = \theta_{\xi\xi}$. The outward convection laws become

$$\boxed{\theta_\xi(0,\eta) = \mathrm{Bi}\,\theta(0,\eta), \qquad -\theta_\xi(1,\eta) = \mathrm{Bi}\,\theta(1,\eta), \quad \mathrm{Bi} = \frac{hL}{k}.}$$

This dimensionless coefficient is the Biot number for the chosen length scale. The interior time scale is L^2/α, but the boundary parameter also affects the dimensionless response.

Step 2: Verify the rescaled interior field. For $u_2(x,t) = u_1(x/2, t/4)$,

$$(u_2)_t = \tfrac{1}{4}(u_1)_t, \qquad (u_2)_{xx} = \tfrac{1}{4}(u_1)_{xx}.$$

Thus the same diffusivity equation holds. At $t = 0$, relative positions agree: $x/(2L) = (x/2)/L$, so the stated initial-shape requirement is met.

Step 3: Match both convection conditions. At the left endpoint,

$$k(u_2)_x(0,t) = \tfrac{1}{2}k(u_1)_x(0,t/4) = \frac{h}{2}\big(u_2(0,t) - T_a\big).$$

The right outward derivative scales by the same factor. Therefore the coefficient making this a rescaling of the full problem is

$$\boxed{h_2 = h/2, \qquad h_2(2L)/k = hL/k.}$$

Both rods then have the same dimensionless boundary problem.

Step 4: Qualify the length-squared claim. If $h_2 = h$, the longer rod instead has Biot number $2hL/k$. The proposed field generally fails the boundary law; the full response is not obtained merely by multiplying time by four. A special zero-excess equilibrium cannot establish a general scaling rule. The factor-four rescaling is valid when convection is also adjusted to $h/2$ and the initial shapes match. If both rods are perfectly insulated ($h = 0$), the zero-flux conditions are preserved automatically. Interior dimensional analysis alone does not justify ignoring changed boundary parameters.

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Question 10

Divide an insulated homogeneous rod into three equal cells of width Δx. Let U_i^n be cell temperatures and $r = \alpha\Delta t/(\Delta x)^2 \geq 0$. Approximate each internal rightward flux by $-k(U_{i+1}^n - U_i^n)/\Delta x$, set outer fluxes to zero, and use a forward Euler time step for the cell energy balances.

Tasks

1. Derive the three-cell update matrix and prove that the sum of temperatures is conserved.
2. Find the exact range of r for which every update preserves the range of every input vector. Prove necessity as well as sufficiency.
3. Using the orthogonal vectors $(1, 1, 1)$, $(1, 0, -1)$ and $(1, -2, 1)$, determine exactly when $\sum_i (U_i^n - \overline{U})^2$ cannot increase in one step.
4. Apply one step with $r = 3/5$ to $(0, 1, 0)$. Check heat conservation and squared norm, and sketch the old and new cell values. Explain why energy stability need not imply preservation of nonnegative temperatures.

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Question 10 – Solution

Strategy. Conservation, preservation of the temperature range and decay of a quadratic norm impose different conditions on a numerical scheme.

Step 1: Balance each cell. For the first cell, inflow minus outflow gives $U_1^{n+1} = U_1^n + r(U_2^n - U_1^n)$. Applying the same balance to the others yields

$$\boxed{U^{n+1} = \begin{pmatrix} 1-r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-r \end{pmatrix} U^n.}$$

Every column sums to one, so $\sum_i U_i$ is conserved. Equal cell volumes and constant capacity make this proportional to total heat. Every row also sums to one, so constant profiles are fixed.

Step 2: Characterize range preservation. All entries are nonnegative exactly when $0 \le r \le 1/2$. Each new value is then a convex combination of old values, preserving their range. For any $r > 1/2$, input $(0, 1, 0)$ gives a middle value $1 - 2r < 0$, outside $[0, 1]$. Thus the exact range is $\boxed{0 \le r \le 1/2}$.

Step 3: Test the quadratic energy on each mode. The stated mutually orthogonal vectors have update factors 1, $1-r$ and $1-3r$, respectively. The mean is fixed; the remaining squared mode components do not grow exactly when $|1-r| \le 1$ and $|1-3r| \le 1$. For $r \ge 0$, this is

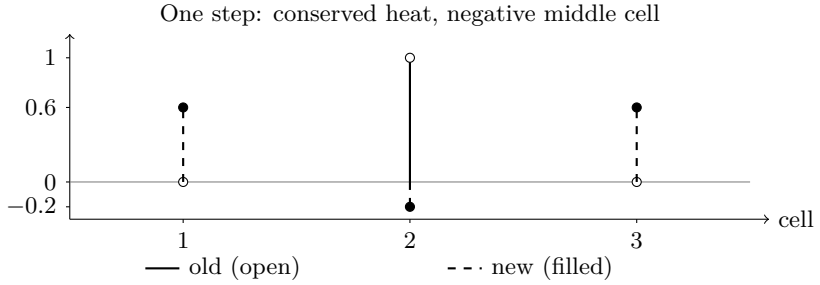
$$\boxed{0 \le r \le 2/3.}$$

Necessity follows by choosing the offending eigenvector as input. This threshold belongs to this particular three-cell insulated matrix.

Step 4: Exhibit the distinction numerically. At $r = 3/5$,

$$\boxed{(0, 1, 0) \longmapsto (3/5, -1/5, 3/5).}$$

Both sums equal one. The uncentered squared norm drops from 1 to $9/25 + 1/25 + 9/25 = 19/25$; since the mean is unchanged, the centered squared norm also drops. Nevertheless the middle temperature becomes negative. The scheme is quadratically stable here but fails the discrete range principle. The stem plot shows individual cell values, without suggesting a smooth interpolated temperature profile between cell centers.



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Question 1

An ideal string has constant tension $\mathcal{T} > 0$ and mass per unit length $\mu > 0$. Its small transverse displacement is $u(x, t)$. Assume $|u_x| \ll 1$, negligible longitudinal motion and bending stiffness, and a transverse applied force $p(x, t)$ per unit length. A linear drag force per unit length is $-bu_t$, with $b \geq 0$.

Tasks

1. Apply transverse momentum balance on an arbitrary interval and derive the linear wave model, explaining the small-slope approximation.
2. State the units of $\mu, \mathcal{T}, p, b$ and check each term. Identify the undamped wave speed and the effect of quadrupling μ at fixed tension.
3. Explain why both initial displacement and initial velocity are needed. Give two different undamped, source-free solutions on the whole line with the same zero initial displacement.
4. For $p = b = 0$, test $u = F(x - vt)$ with $F \in C^2$. Determine when this represents a traveling wave of the string and identify the exceptional profiles that do not determine v.

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Question 1 – Solution

Strategy. Transverse acceleration is driven by the difference of vertical tension components, not by a first time derivative.

Step 1: Derive the momentum equation. The vertical tension component is $\mathcal{T}\sin\theta \approx \mathcal{T}u_x$ under the small-slope approximation. For an interval $[a, d]$,

$$\int_a^d \mu u_{tt}\,dx = \mathcal{T}u_x(d,t) - \mathcal{T}u_x(a,t) + \int_a^d (p - bu_t)\,dx.$$

Localization gives

$$\boxed{\mu u_{tt} + bu_t = \mathcal{T}u_{xx} + p.}$$

The linearization neglects changes of length and tension of higher order in the slope.

Step 2: Check dimensions and speed. The units are μ : kg/m, $\mathcal{T}$: N, p : N/m and b : Ns/m^2. Since u is a length, each term in the local equation has units N/m. Without drag or forcing,

$$\boxed{u_{tt} = c^2 u_{xx}, \qquad c = \sqrt{\mathcal{T}/\mu}.}$$

The speed is in m/s. Quadrupling μ at fixed tension halves c.

Step 3: Explain the two initial data. Acceleration determines how velocity changes, so displacement alone cannot specify the motion. For example, on the whole line, both $u = 0$ and $u = Vt$, with constant $V \neq 0$, solve the source-free undamped equation and have $u(x, 0) = 0$. Their initial velocities differ. A prescribed pair $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ removes this particular ambiguity.

Step 4: Verify a moving profile. For $u = F(x - vt)$, $u_{tt} = v^2F''$ and $u_{xx} = F''$. Hence

$$\boxed{(v^2 - c^2)F'' = 0.}$$

If F'' is not identically zero, the admissible speeds are $v = \pm c$, corresponding to opposite directions. If F is affine, both second derivatives vanish for every v; a translating straight line cannot by itself identify a propagation speed. A non-affine smooth profile is needed to draw that inference from the PDE.

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Question 2

For the whole-line wave equation $u_{tt} = c^2 u_{xx}$, $c > 0$, you may use

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Take $f = 0$ and $g(s) = (1-s^2)^2$ for $|s| < 1$, $g(s) = 0$ for $|s| \geq 1$, in consistent nondimensional units. Here g is continuously differentiable.

Tasks

1. Differentiate the integral formula to verify the equation and both initial conditions for these data.
2. Find the exact center displacement $u(0,t)$ for all $t \geq 0$.
3. Determine where the solution is zero and when a fixed point reaches the constant plateau behind the departing waves. Explain why compact initial velocity can leave nonzero displacement behind.
4. Sketch the profiles for $c = 1$ at $t = 1/2, 3/2, 3$. Check that the plateau height does not grow without bound and distinguish zero displacement from zero velocity.

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Question 2 – Solution

Strategy. Initial velocity contributes an integral over a growing interval; it is not simply two displaced copies of the initial displacement.

Step 1: Verify the integral solution. Leibniz differentiation gives

$$u_t = \frac{g(x+ct)+g(x-ct)}{2}, \quad u_{tt} = \frac{c}{2}\big(g'(x+ct)-g'(x-ct)\big), \quad u_{xx} = \frac{1}{2c}\big(g'(x+ct)-g'(x-ct)\big).$$

Thus $u_{tt} = c^2 u_{xx}$. At $t = 0$, the integral is zero and $u_t(x,0) = g(x)$. The C^1 regularity of g makes these second derivatives continuous.

Step 2: Integrate at the center. Symmetry gives $u(0,t) = c^{-1}\int_0^{\min(ct,1)} (1-s^2)^2\,ds$. Consequently

$$\boxed{u(0,t) = \begin{cases} t - \dfrac{2c^2t^3}{3} + \dfrac{c^4t^5}{5}, & 0 \le ct \le 1, \\ \dfrac{8}{15c}, & ct \ge 1. \end{cases}}$$

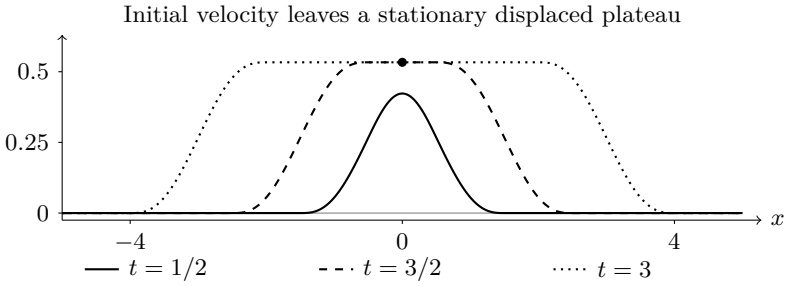
The two expressions agree at $ct = 1$.

Step 3: Locate the moving fronts and plateau. For $t > 0$, the integral is positive exactly when $[x-ct, x+ct]$ overlaps $(-1,1)$ in positive length. It is therefore zero for $|x| \ge 1+ct$. When $ct \ge |x|+1$, the integration interval contains the entire support of g, so

$$\boxed{u(x,t) = \frac{1}{2c}\int_{-1}^{1} g(s)\,ds = \frac{8}{15c}.}$$

The same value holds at equality because the endpoints of g contribute nothing. Behind the fronts, the string is displaced but locally stationary and flat. Displacement itself need not return to zero after a velocity pulse.

Step 4: Interpret the plotted profiles. Since $g \ge 0$, every interval integral lies between zero and the full integral. Thus $0 \le u \le 8/(15c)$ for every x, t. For $c = 1$, the fronts are at $x = \pm(1+t)$; a plateau appears once $t \ge 1$ and occupies $|x| \le t-1$. Its height stays $8/15$ as its width grows. Zero local velocity on that plateau does not mean zero local displacement.



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Question 3

Use the whole-line wave formula

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, \qquad c>0.$$

At an observation time $t_0 > 0$, compare two solutions with initial data (f, g) and $(\widetilde{f}, \widetilde{g})$.

Tasks

1. Identify exactly which initial positions can influence an observation (x_0, t_0) in this formula.
2. If $|f - \widetilde{f}| \le \varepsilon$ and $|g - \widetilde{g}| \le \eta$ on that interval, prove $|u(x_0,t_0) - \widetilde{u}(x_0,t_0)| \le \varepsilon + t_0\eta$. Show the two constants are sharp.
3. Take $c = 1$, $g = 0$ and $f(x) = (x-2)^3(3-x)^3$ on $2 < x < 3$, zero elsewhere. Find the entire time trace at $x = 0$, including its first and last nonzero times and maximum.
4. Sketch the normalized trace $128u(0,t)$ for $0 \le t \le 4$. Explain why changes to initial data outside the backward interval cannot affect a given observation, however large those changes are.

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Question 3 – Solution

Strategy. Read causality directly from the interval sampled by the solution formula.

Step 1: Identify the backward interval. Only the initial positions in

$$\boxed{[x_0 - ct_0, x_0 + ct_0]}$$

enter: f is sampled at its two endpoints, while g is integrated across it. This interval is a sufficient domain of dependence for both types of data; interior values of f alone do not enter this one-dimensional formula.

Step 2: Bound and attain the data error. The endpoint contribution is at most $(\varepsilon+\varepsilon)/2$. The integral contribution is at most $(2ct_0)\eta/(2c)$. Hence

$$\boxed{|u(x_0,t_0) - \widetilde{u}(x_0,t_0)| \le \varepsilon + t_0\eta.}$$

Constant differences $f - \widetilde{f} = \varepsilon$ and $g - \widetilde{g} = \eta$ attain equality. Taking either difference zero proves the corresponding coefficient cannot be improved.

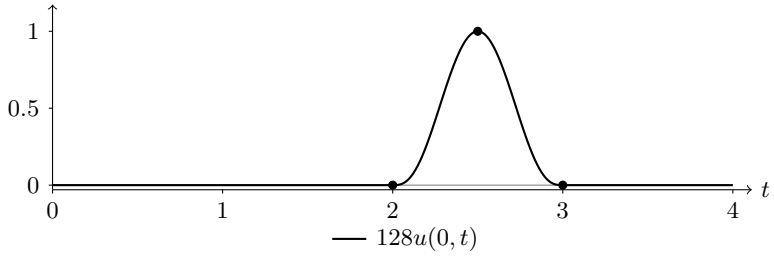
Step 3: Compute the sensor trace. At $x = 0$, $u(0,t) = [f(-t) + f(t)]/2$. For $t \ge 0$, the first term is always zero. Thus

$$\boxed{u(0,t) = \begin{cases} \frac{1}{2}(t-2)^3(3-t)^3, & 2 < t < 3, \\ 0, & 0 \le t \le 2 \text{ or } t \ge 3. \end{cases}}$$

The trace starts to become nonzero immediately after $t = 2$ and returns to zero at $t = 3$. Since $(t-2)(3-t) = 1/4 - (t-5/2)^2$, its unique maximum is $1/128$ at $t = 5/2$. The initial profile is C^2 even at its support edges.

Step 4: Interpret finite propagation. At times before 2, the backward interval does not reach the initial pulse, so the sensor cannot respond. No alteration outside the sampled interval changes either endpoint value or the integral in the formula. This conclusion places no bound on the magnitude of the remote alteration. It is a statement about finite propagation and dependence on initial data, not about small remote effects that have merely been neglected.

Finite arrival and departure at a remote sensor



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Question 4

Two semi-infinite strings meet at $x = 0$, share tension $\mathcal{T} > 0$ and have densities $\mu_1, \mu_2 > 0$. Set $c_j = \sqrt{\mathcal{T}/\mu_j}$ and $Z_j = \mathcal{T}/c_j$. The massless junction has continuous displacement and continuous transverse force $\mathcal{T}u_x$. For a smooth incident pulse with $0 < \int_{\mathbb{R}} (F')^2 < \infty$, write

$$u_1 = F(t - x/c_1) + RF(t + x/c_1) \quad (x < 0), \qquad u_2 = DF(t - x/c_2) \quad (x > 0).$$

Tasks

1. Verify each traveling term solves its local wave equation and derive the two junction equations for R, D.
2. Solve for R, D and classify the sign of the reflected displacement when the second string is denser or lighter.
3. Using rightward power $\mathcal{P} = -\mathcal{T}u_t u_x$, derive the reflected and transmitted energy fractions and prove their sum is one.
4. Evaluate the case $\mu_2 = 9\mu_1$. Analyze the limits $Z_2/Z_1 \to 0$ and $\to \infty$, and explain why a transmitted displacement amplitude above one need not violate energy conservation.

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Question 4 – Solution

Strategy. Displacement amplitudes and energy fractions differ because the strings have different wave impedances.

Step 1: Apply the interface conditions. Every term of the form $F(t \mp x/c_j)$ has $u_{tt} = c_j^2 u_{xx}$. At the junction, displacement continuity gives $1 + R = D$. The slope condition gives

$$\frac{\mathcal{T}}{c_1}(R-1)F'(t) = -\frac{\mathcal{T}}{c_2}DF'(t), \qquad \boxed{Z_1(1-R) = Z_2 D.}$$

Since F is nonconstant, these amplitude equations follow.

Step 2: Solve and interpret the amplitudes. Solving the two equations gives

$$\boxed{R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \qquad D = \frac{2Z_1}{Z_1 + Z_2}.}$$

Here $Z_j = \sqrt{\mathcal{T}\mu_j}$. A denser second string gives $R < 0$, an inverted reflected displacement. A lighter string gives $R > 0$. Equal impedances give no reflection and $D = 1$.

Step 3: Account for energy flux. The incident rightward power is $Z_1(F')^2$, the reflected leftward power has magnitude $Z_1 R^2 (F')^2$, and transmitted power is $Z_2 D^2 (F')^2$. The cross terms in $-\mathcal{T} u_t u_x$ on the incident side cancel. For a pulse with finite nonzero $\int (F')^2$, integration in time therefore gives

$$\boxed{\mathcal{R} = R^2, \qquad \mathcal{D} = \frac{Z_2}{Z_1} D^2, \qquad \mathcal{R} + \mathcal{D} = \frac{(Z_1 - Z_2)^2 + 4Z_1 Z_2}{(Z_1 + Z_2)^2} = 1.}$$

Step 4: Test examples and extreme impedances. If $\mu_2 = 9\mu_1$, then $Z_2 = 3Z_1$, so $R = -1/2$, $D = 1/2$, $\mathcal{R} = 1/4$ and $\mathcal{D} = 3/4$. As $Z_2/Z_1 \to 0$, $R \to 1$ and $D \to 2$, but $\mathcal{D} \to 0$. As $Z_2/Z_1 \to \infty$, $R \to -1$, $D \to 0$ and again $\mathcal{D} \to 0$. These are the free-end and fixed-end reflection limits, respectively. Energy depends on impedance as well as squared displacement amplitude.

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Question 5

On $x \geq 0$, use wave speed one and the even profile $F(z) = e^{-z^2}$. For $a > 0$, compare

$$u_D(x,t) = F(x+t-a) - F(x-t+a), \qquad u_N(x,t) = F(x+t-a) + F(x-t+a).$$

These formulas include the reflected field from the start; the Gaussian has noncompact tails.

Tasks

1. Verify the wave equation for both fields and show which one satisfies a fixed end and which one satisfies a free end at $x = 0$.
2. Compute their initial displacements and velocities explicitly. Explain why keeping only the incident Gaussian would violate the boundary condition.
3. At $t = a$, compute each field and its velocity. Decide whether zero displacement everywhere implies the string is at rest.
4. For $a = 3$, sketch u_D at $t = 1, 3, 5$ on $0 \leq x \leq 6$. Interpret the sign of the reflected pulse and compare it with the free-end reflection.

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Question 5 – Solution

Strategy. An odd or even spatial reflection enforces the corresponding boundary condition.

Step 1: Verify the two boundary models. Each translated profile solves $u_{tt} = u_{xx}$. Since F is even and F' is odd,

$$\boxed{u_D(0,t) = F(t-a) - F(a-t) = 0, \qquad (u_N)_x(0,t) = F'(t-a) + F'(a-t) = 0.}$$

Thus u_D satisfies the fixed-end condition and u_N the free-end condition.

Step 2: Recover the actual initial data. Differentiating the time arguments carefully gives

$$u_D(x,0) = F(x-a) - F(x+a), \quad (u_D)_t(x,0) = F'(x-a) + F'(x+a),$$
$$u_N(x,0) = F(x-a) + F(x+a), \quad (u_N)_t(x,0) = F'(x-a) - F'(x+a),$$

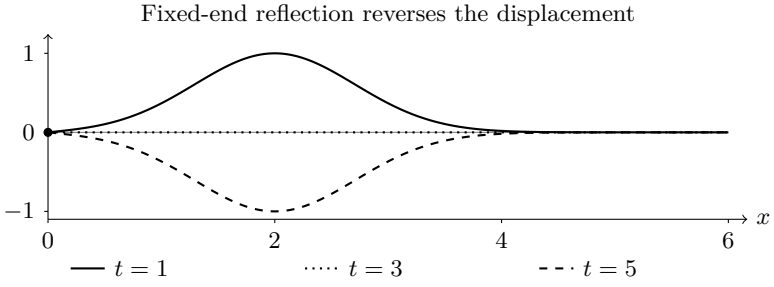
where $F'(z) = -2ze^{-z^2}$. The incident term alone generally has neither zero boundary displacement nor zero boundary slope. Its reflected partner supplies the exact cancellation; small Gaussian tails are not identically zero.

Step 3: Inspect the collision instant. At $t = a$, the two profile arguments both equal x. Therefore

$$\boxed{u_D(x,a) = 0, \quad (u_D)_t(x,a) = 2F'(x), \qquad u_N(x,a) = 2F(x), \quad (u_N)_t(x,a) = 0.}$$

The fixed-end string has zero displacement but nonzero velocity for $x > 0$: it is not at rest. The free-end field has a doubled displacement profile at that instant and zero instantaneous velocity. Displacement and velocity must both be known to specify the state.

Step 4: Interpret the reflected sign. For $a = 3$, the incoming peak at $t = 1$ is near $x = 2$. At $t = 3$ the fixed-end displacement is exactly zero everywhere. At $t = 5$ the reflected pulse is near $x = 2$ with opposite sign. Indeed $u_D(x,5) = -u_D(x,1)$. A fixed end reverses displacement upon reflection; a free end preserves its sign. The plotted zero snapshot is an exact cancellation, not disappearance of the motion or its energy.



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Question 6

A string on $0 \le x \le L$ has fixed ends, constant $\mu, \mathcal{T} > 0$ and equation $\mu u_{tt} = \mathcal{T} u_{xx} + p(x,t)$. Assume sufficient smoothness and define

$$E(t) = \frac{1}{2} \int_0^L \left(\mu u_t^2 + \mathcal{T} u_x^2 \right) dx.$$

Tasks

1. Derive the energy identity, first retaining boundary power and then using the fixed-end conditions.
2. Prove uniqueness for prescribed initial displacement and velocity when the applied force is prescribed.
3. For $p = 0$ and $A \neq 0$, verify $u = A \sin(\pi x/L) \cos(\omega t)$ with $\omega = (\pi/L)\sqrt{\mathcal{T}/\mu}$. Compute its kinetic, elastic and total energies.
4. Sketch the two energy fractions over $0 \le \omega t \le 2\pi$. Explain why a zero-displacement snapshot can have positive energy, and why the energy identity does not claim that displacement at each point is monotone.

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Question 6 – Solution

Strategy. Integrate the PDE against velocity so that acceleration and curvature become time derivatives of kinetic and elastic energies.

Step 1: Derive the work balance. Multiplying by u_t and integrating by parts gives

$$\boxed{E'(t) = \mathcal{T}[u_t u_x]_0^L + \int_0^L p\, u_t\, dx.}$$

At a fixed endpoint, $u = 0$ for all time, hence $u_t = 0$ there. Thus $E' = \int p\, u_t$. With no applied force, total energy is constant. The sign of the work rate depends on the force and velocity together.

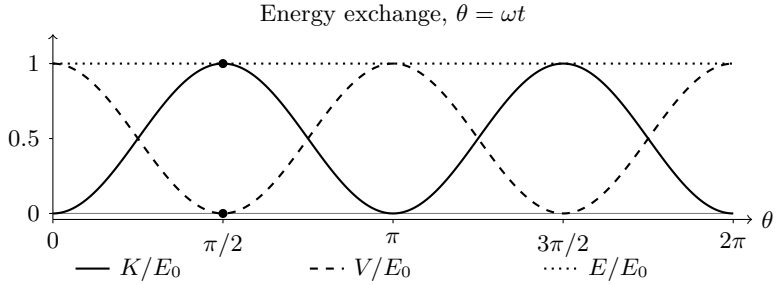
Step 2: Prove uniqueness for the full initial state. The difference w of two solutions with the same force, endpoints and initial data obeys the homogeneous equation and has zero initial energy. Its energy stays zero, so $w_t = w_x = 0$ everywhere by continuity. The fixed endpoints then force the spatial constant to be zero. Initial displacement alone would not make the kinetic part of the initial difference energy vanish; both initial data are essential.

Step 3: Compute the standing-wave energy. Put $k = \pi/L$, so $\omega^2 = (\mathcal{T}/\mu)k^2$. Then $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$, and both endpoints are zero. Using $\int_0^L \sin^2 kx = \int_0^L \cos^2 kx = L/2$ gives

$$\boxed{K(t) = E_0 \sin^2(\omega t), \qquad V(t) = E_0 \cos^2(\omega t), \qquad E_0 = \frac{\mathcal{T} A^2 k^2 L}{4}.}$$

Their sum is the constant E_0.

Step 4: Interpret the exchange of energy. For $A \neq 0$, at $\omega t = \pi/2$ the displacement and elastic energy are zero, but the velocity is nonzero and all energy is kinetic. At $\omega t = 0$ the velocity is zero and all energy is elastic. The field exchanges these forms while oscillating; total-energy conservation does not require pointwise monotonic motion. The graph shows K/E_0, V/E_0 and their constant sum.



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Question 7

A fixed-end string is known to have displacement

$$u(x,t) = A\sin(\pi x/L)\cos(c\pi t/L), \qquad A, L > 0,$$

with unknown speed $c > 0$. A snapshot at a known time $\Delta > 0$ shows $u(x, \Delta) = 0$ for every x. Initially the shape and zero velocity are known.

Tasks

1. Find every positive speed consistent with the zero snapshot and prove the list is complete.
2. Determine the speed if the independent prior bound $0 < c < L/\Delta$ is available. Explain why the snapshot alone is insufficient without it.
3. Derive the midpoint velocity at time Δ for each admissible speed. Give a consistency test for an exact signed velocity measurement and show that it identifies the speed uniquely when consistent.
4. If the measured velocity is $-5\pi A/(2\Delta)$, find c and predict the midpoint displacement at time $\Delta/2$. Verify the signs rather than using only the velocity magnitude.

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Question 7 – Solution

Strategy. A snapshot can hide how many oscillations have occurred; a velocity measurement retains both rate and direction.

Step 1: List all compatible speeds. Since the spatial sine is not identically zero, the snapshot requires $\cos(c\pi\Delta/L) = 0$. For positive c,

$$\boxed{c_n = \frac{L}{\Delta}\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \ldots.}$$

Every listed speed gives the same zero snapshot, as well as the specified initial displacement and zero velocity. The positive zeros of cosine prove completeness.

Step 2: Use an actual prior restriction. The bound $c < L/\Delta$ allows only $n = 0$, so $c = L/(2\Delta)$. Without this independent information, arbitrarily many half-oscillations may have occurred before the snapshot. A zero snapshot does not imply zero speed or determine which crossing of equilibrium was observed.

Step 3: Recover speed from signed velocity. At the midpoint,

$$\boxed{v_n = u_t(L/2, \Delta) = (-1)^{n+1}\frac{\pi A}{\Delta}\left(n + \frac{1}{2}\right).}$$

A measured v is consistent exactly when $n = |v|\Delta/(\pi A) - 1/2$ is a nonnegative integer and its sign is $(-1)^{n+1}$. The strictly increasing magnitudes then select one n uniquely. In particular, a zero velocity measurement is incompatible with the stated nonzero-amplitude zero snapshot.

Step 4: Verify the numerical multiple. For $v = -5\pi A/(2\Delta)$, the magnitude gives $n = 2$ and the predicted sign $(-1)^3$ is negative, as measured. Thus

$$\boxed{c = \frac{5L}{2\Delta}, \qquad u(L/2, \Delta/2) = A\cos(5\pi/4) = -\frac{A}{\sqrt{2}}.}$$

The signed result distinguishes this phase from other times at which the displacement vanishes. Inferring the lowest possible speed without a prior bound would produce the wrong prediction for the intermediate snapshot.

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Question 8

Compare the ideal wave equation with the given dispersive extension

$$u_{tt} = c^2 u_{xx} - \beta u_{xxxx}, \qquad c > 0, \quad \beta \geq 0.$$

For a sinusoidal wave $u = \cos(kx - \omega t)$ with $k > 0, \omega > 0$, define phase speed $v_p = \omega/k$ and group speed $v_g = d\omega/dk$. No general theory of wave packets is required.

Tasks

1. Substitute the sinusoid and derive the frequency relation. Determine the units of β.
2. Compute v_p, v_g and compare them for $\beta > 0$. Find their small-k limits and their values when $\beta = 0$.
3. For $\beta > 0$, consider a sum of two nonzero sinusoidal components with distinct positive wave numbers. Prove that the sum cannot translate rigidly with one fixed speed while both components follow the derived positive-frequency relation.
4. Sketch v_p, v_g for $c = 1, \beta = 1/4$, $0 < k \leq 3$, showing the limiting value at zero separately. Explain which feature of the ordinary wave equation is lost.

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Question 8 – Solution

Strategy. Test sinusoidal components before claiming that an arbitrary composite profile moves without changing shape.

Step 1: Derive the frequency relation. The derivatives give $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$ and $u_{xxxx} = k^4 u$. Therefore

$$\boxed{\omega^2 = c^2 k^2 + \beta k^4, \qquad \omega = k\sqrt{c^2 + \beta k^2}.}$$

Since k has units m^{-1} and ω units s^{-1}, β has units $\mathrm{m}^4/\mathrm{s}^2$.

Step 2: Compare the two speeds. Differentiating the positive-frequency branch gives

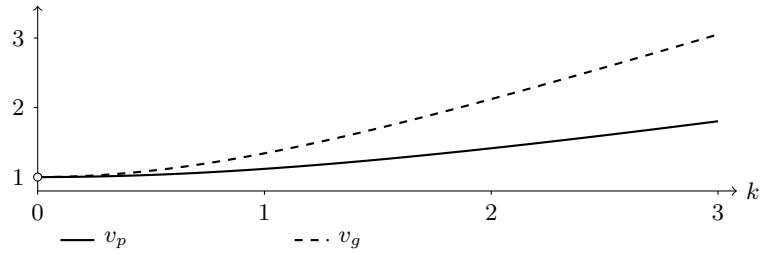
$$\boxed{v_p = \sqrt{c^2 + \beta k^2}, \qquad v_g = \frac{c^2 + 2\beta k^2}{\sqrt{c^2 + \beta k^2}}.}$$

Their difference is $\beta k^2/\sqrt{c^2 + \beta k^2}$, strictly positive if $\beta > 0$. Both tend to c as $k \downarrow 0$ and both equal c for every k when $\beta = 0$.

Step 3: Test rigid translation of two frequencies. A fixed profile $\cos(k_1 x + \phi_1) + a\cos(k_2 x + \phi_2)$ with $a \neq 0$ translated by speed v gives temporal frequencies vk_1, vk_2. Independence of the two spatial frequencies requires $v = \omega(k_1)/k_1 = \omega(k_2)/k_2$ for agreement at every x, t. For $\beta > 0$, $v_p(k)$ is strictly increasing for $k > 0$, so distinct wave numbers cannot share this value. Relative phases change in a way that cannot be represented by one common translation.

Step 4: Interpret the comparison. In the ideal equation, all right-moving sinusoidal components travel with the same speed c, so a smooth profile $F(x-ct)$ retains its shape. The given extension makes phase speed depend on frequency; a multi-frequency profile generally changes shape. The graph shows this dependence and the strict separation of the two defined speeds for positive k. The open point at $k = 0$ records their common limit, since the definitions above were made for $k > 0$.

Frequency-dependent speeds in the given extension



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Question 9

Let a sufficiently smooth displacement be periodic in x with period 2π and satisfy $u_{tt} = c^2 u_{xx}$, $c > 0$. Define its mean $m(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} u(x,t)\,dx$ and momentum $P(t) = \mu \int_{-\pi}^{\pi} u_t\,dx$, with $\mathcal{T} = \mu c^2$.

Tasks

1. Derive the evolution of the mean and prove momentum conservation.
2. Derive conservation of $E = \frac{1}{2} \int_{-\pi}^{\pi} (\mu u_t^2 + \mathcal{T} u_x^2)\,dx$. Does bounded energy force bounded displacement for all time?
3. Test the explicit family $u(x,t) = Vt + A\cos x \cos(ct)$. Compute its mean, momentum and energy exactly and use it to answer the preceding question.
4. If the mean initial velocity is zero, prove that a uniform displacement bound does follow from energy and the initial mean. You may use continuity, the intermediate value theorem and Cauchy–Schwarz.

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Question 9 – Solution

Strategy. Wave energy controls velocity and slope, but a spatially uniform motion can carry the mean displacement indefinitely.

Step 1: Track the constant spatial mode. Integrating the PDE and using matching periodic slopes gives

$$m''(t) = \frac{c^2}{2\pi}[u_x]_{-\pi}^{\pi} = 0.$$

Hence

$$\boxed{m(t) = m(0) + m'(0)t, \qquad P(t) = 2\pi\mu m'(0).}$$

The mean velocity, rather than the mean displacement, is conserved.

Step 2: Derive the periodic energy balance. Multiplication by u_t and integration by parts gives $E' = \mathcal{T}[u_t u_x]_{-\pi}^{\pi} = 0$ by periodicity. The energy contains u_t and u_x, but not u itself. A nonzero constant mean velocity is therefore compatible with a linearly growing mean displacement and a fixed total energy.

Step 3: Verify a drifting example. For $u = Vt + A\cos x\cos(ct)$, the term Vt and the oscillatory term each solve the equation. Orthogonality gives

$$\boxed{m(t) = Vt, \quad P = 2\pi\mu V, \quad E = \pi\mu V^2 + \frac{\pi\mathcal{T}A^2}{2}.}$$

The cosine's kinetic and elastic contributions sum to the second constant. For $V \neq 0$, displacement is unbounded in time even though the energy is finite and constant.

Step 4: Bound displacement when the mean cannot drift. If $m'(0) = 0$, then $m(t) = m(0)$. At each time the continuous function $w = u - m(0)$ has zero spatial mean, so it has a zero at some x_* (or is identically zero). Integrate its derivative from that point over a path of length at most 2π. Cauchy–Schwarz gives

$$|w(x,t)| \le \sqrt{2\pi}\,\|u_x(\cdot,t)\|_2 \le \sqrt{\frac{4\pi E(0)}{\mathcal{T}}}.$$

Thus $\boxed{|u(x,t)| \le |m(0)| + \sqrt{4\pi E(0)/\mathcal{T}}}$ for all time. The estimate need not be sharp; it identifies the missing mean-velocity condition.

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Question 10

A fixed-end string is driven by

$$\mu u_{tt} = \mathcal{T} u_{xx} + F \sin(kx)\cos(\omega t), \qquad k = \pi/L, \quad \omega = k\sqrt{\mathcal{T}/\mu}, \quad F > 0.$$

Both initial displacement and velocity are zero. Test solutions of the supplied form $u(x,t) = q(t)\sin(kx)$; no general separation procedure is required.

Tasks

1. Derive the scalar equation for q and verify $q(t) = Ft\sin(\omega t)/(2\mu\omega)$.
2. Check both initial conditions and both fixed-end conditions. Explain why the factor t does not alter the propagation-speed parameter of the PDE.
3. Compute the energy and verify its rate equals the applied work rate. Evaluate the energy at $t_j = j\pi/\omega$ and describe its growth.
4. Replace the driving frequency by $\Omega > 0$, $\Omega \neq \omega$. Verify the zero-initial-data response $q_\Omega = F(\cos\Omega t - \cos\omega t)/(\mu(\omega^2 - \Omega^2))$ and recover the resonant formula as $\Omega \to \omega$ for fixed t.

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Question 10 – Solution

Strategy. A force at the natural frequency produces growing amplitude while the same local wave equation retains its original speed.

Step 1: Verify the resonant amplitude equation. Substitution gives $q'' + \omega^2 q = (F/\mu)\cos\omega t$. If $D = F/(2\mu\omega)$ and $q = Dt\sin\omega t$, then

$$q' = D(\sin\omega t + \omega t\cos\omega t), \qquad q'' + \omega^2 q = 2D\omega\cos\omega t.$$

Thus the proposed response satisfies the forced equation.

Step 2: Check the full initial-boundary data. One has $q(0) = q'(0) = 0$. The spatial sine vanishes at $0, L$, giving both fixed ends for all time. The growing factor is a resonant response to sustained external forcing. It changes the amplitude history, not the coefficient $c^2 = \mathcal{T}/\mu$ that sets the source-free propagation speed.

Step 3: Measure energy and work. Spatial integration gives

$$\boxed{E(t) = \frac{\mu L}{4}(q'^2 + \omega^2 q^2) = \frac{F^2 L}{16\mu\omega^2}\left[(\sin\omega t + \omega t\cos\omega t)^2 + (\omega t\sin\omega t)^2\right].}$$

Differentiating the first expression and using the amplitude equation yields $E' = (FL/2)q'\cos\omega t$, exactly $\int_0^L F\sin(kx)\cos(\omega t)\,u_t\,dx$. At $t_j = j\pi/\omega$,

$$\boxed{E(t_j) = \frac{F^2 L}{16\mu}t_j^2.}$$

Energy is unbounded along these times. Its instantaneous rate can have either sign; sustained resonant input produces the growing envelope.

Step 4: Take the frequency limit carefully. The supplied q_Ω has zero value and derivative at zero. Direct differentiation gives $q_\Omega'' + \omega^2 q_\Omega = (F/\mu)\cos\Omega t$. For fixed t, differentiating numerator and denominator with respect to Ω at ω gives

$$\boxed{\lim_{\Omega\to\omega} q_\Omega(t) = \frac{F}{\mu}\frac{-t\sin\omega t}{-2\omega} = \frac{Ft\sin\omega t}{2\mu\omega}.}$$

Each fixed nonresonant response is bounded in time, but those bounds are not uniform as its frequency approaches resonance.

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Question 1

Let $u = u(x, t)$ be an unknown scalar field on $\mathbb{R}^2$, and let a, b, c be real constants. Consider the family

$$au_{tt} + u_t - (1 + x^2)u_{xx} + buu_x + c(u_x)^2 = \sin t.$$

The order of a PDE is the highest derivative order actually present. A linear PDE has coefficients depending only on the independent variables, and the unknown and all its derivatives occur linearly. For a linear PDE, homogeneous means that its prescribed right-hand side is zero.

Tasks

1. Identify the independent and dependent variables and determine the order for every a, including $a = 0$. Explain why second order need not mean second order in time.
2. Classify exactly which pairs (b, c) make the equation linear. In that case, decide whether it has constant coefficients and whether it is homogeneous.
3. The nonlinear term uu_x vanishes on every field independent of x. Explain why testing only those fields cannot establish linearity. Give a scaling test that detects each nonzero nonlinear coefficient.
4. Replace the right-hand side by zero. Explain what changes in the linear case, and why a zero right-hand side alone cannot make a nonlinear equation linear.

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Question 1 – Solution

Strategy. Classify the differential expression on arbitrary admissible fields, rather than on a special solution.

Step 1: Count the derivatives that remain. The independent variables are x, t and the dependent variable is u. Since $1 + x^2 > 0$, the term u_{xx} never disappears. The order is therefore $\boxed{2}$ for every a. When $a = 0$, the equation is first order in time but still second order as a PDE. When $a \neq 0$, it is also second order in time.

Step 2: Separate coefficients from unknowns. The expression is linear exactly when $\boxed{b = c = 0}$. Then $1 + x^2$ is an allowed variable coefficient, so the PDE is linear with variable coefficients. It is nonhomogeneous because $\sin t$ is not identically zero on the domain. A forcing term may vanish at some times without making the equation homogeneous.

Step 3: Test scaling on informative fields. Write the nonlinear part as $N[u] = buu_x + c(u_x)^2$. It satisfies $N[2u] - 2N[u] = 2N[u]$, whereas a linear operator would give zero. For the smooth test field $u = x$, this difference is

$$2(bx + c).$$

It vanishes for every x only when $b = c = 0$. In particular, if $b \neq 0$ it is not the zero function; if $b = 0$ and $c \neq 0$ it is the nonzero constant $2c$. The restricted tests $u = h(t)$ have $u_x = 0$ and conceal both nonlinear terms. Linearity must hold on all fields in the operator's domain.

Step 4: Classify the equation after changing the forcing. With zero right-hand side and $b = c = 0$, the equation is linear homogeneous. With $(b, c) \neq (0, 0)$, the same scaling obstruction remains, regardless of the right-hand side. Under the stated convention, "homogeneous linear PDE" is not a label for an arbitrary nonlinear equation written with zero on one side.

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Question 2

On $\mathbb{R} \times [0, \infty)$ define $L[u] = u_t - u_{xx}$ and consider $L[u] = 2$. Set

$$u_1(x,t) = 2t, \qquad u_2(x,t) = 2t + e^{-t}\cos x.$$

No initial or boundary conditions are imposed unless explicitly stated.

Tasks

1. Verify that u_1, u_2 solve the PDE and find the exact condition on real α, β for $\alpha u_1 + \beta u_2$ to solve the same PDE.
2. Determine the equations satisfied by $u_2 - u_1$ and $u_1 + u_2$. Explain why a linear equation need not have a vector space of solutions.
3. Prove that the complete solution set is $u_1 + \ker L$, where $\ker L = \{w : L[w] = 0\}$ in the chosen smooth function class. Explain the word affine using this representation.
4. Now impose $u(x, 0) = \cos x$. Decide which displayed solution satisfies the full problem. Explain why this check alone is neither a construction of all solutions nor a uniqueness proof.

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Question 2 – Solution

Strategy. Apply the operator to a combination before applying a superposition rule.

Step 1: Verify the source and its coefficient. For u_1, $u_{1,t} = 2$ and $u_{1,xx} = 0$. For u_2, $u_{2,t} = 2 - e^{-t}\cos x$ and $u_{2,xx} = -e^{-t}\cos x$. Thus $L[u_j] = 2$. Linearity gives

$$L[\alpha u_1 + \beta u_2] = 2(\alpha + \beta).$$

Consequently the combination solves the same equation exactly when $\boxed{\alpha + \beta = 1}$, not for all coefficients.

Step 2: Distinguish differences from sums. The difference has source $2 - 2 = 0$, so $u_2 - u_1 \in \ker L$. The sum has source 4 and is not a solution of $L[u] = 2$. The zero field is not a solution either. Thus this solution set is not a vector space, although its defining operator is linear. For $L[w] = 0$, arbitrary linear combinations do remain solutions.

Step 3: Describe every solution by a translation. If $L[u] = 2$, then $L[u - u_1] = 0$, so $u = u_1 + w$ with $w \in \ker L$. Conversely $L[u_1 + w] = 2 + 0 = 2$ for every such w. Hence

$$\boxed{\{u : L[u] = 2\} = u_1 + \ker L.}$$

This is an affine space: a translate of a vector space. Combinations whose coefficients sum to one stay in it. The particular solution provides an origin for the translation, but is not the zero vector of the ambient function space.

Step 4: Check the additional data separately. At $t = 0$, $u_1 = 0$ and $u_2 = \cos x$, so only u_2 among these two fields satisfies the specified initial condition. This proves existence in the smooth class because u_2 is an explicit solution. It does not prove uniqueness among all fields; that requires a theorem or argument in a stated function class, including any necessary behavior at spatial infinity. Comparing two candidates cannot eliminate all other candidates.

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Question 3

Let $u(x,t)$ solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $0 < t < T$, where $\kappa, L, T > 0$. For this problem, initial data are prescribed at $t = 0$; spatial boundary data are prescribed at $x = 0, L$. The outward normal derivative is $\partial_n u = n_x u_x$, with $n_x = -1$ on the left and $n_x = 1$ on the right. Consider

$$u(x,0) = f(x), \qquad u(0,t) = A(t), \qquad \kappa \partial_n u(L,t) + h u(L,t) = r(t), \quad h > 0.$$

Dirichlet data prescribe u, Neumann data prescribe $\partial_n u$, and Robin data prescribe a linear combination with nonzero coefficients of both.

Tasks

1. Classify each condition, and identify the portions of the space–time rectangle on which they are prescribed. Is $t = T$ a spatial boundary?
2. Rewrite the right condition with u_x. Write the corresponding coordinate form of $\kappa \partial_n u + hu = r$ if it were imposed at the left endpoint instead.
3. Classify the spatial boundary operator when $h = 0$, and decide exactly when the displayed spatial data are homogeneous. Distinguish this from homogeneity of the PDE and of the initial data.
4. Verify the manufactured field $u_*(x,t) = x^2 + 2\kappa t$. Find f, A, r that make it solve the full displayed problem, and illustrate where each datum acts.

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Question 3 – Solution

Strategy. Identify both the operator being prescribed and the geometric location of the data.

Step 1: Locate and name the data. The bottom edge $t = 0$ carries initial displacement/temperature data $f(x)$. The left edge carries Dirichlet data $A(t)$; the right carries Robin data $r(t)$. The top $t = T$ is a final-time slice, not a spatial boundary. No data are prescribed there in this initial-boundary-value problem.

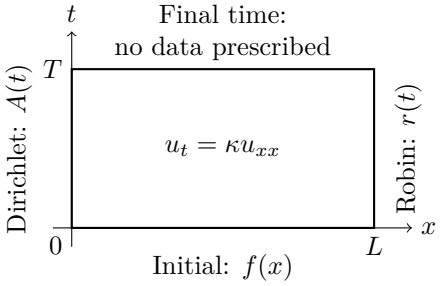
Step 2: Keep the outward signs. At the right, the condition is $\kappa u_x(L,t) + hu(L,t) = r(t)$. At the left, the analogous outward-normal condition would be $-\kappa u_x(0,t) + hu(0,t) = r(t)$. Changing the endpoint changes the sign of the normal derivative, not the sign of the prescribed value by convention.

Step 3: Classify homogeneity at each level. When $h = 0$, the right condition is Neumann, since $\kappa > 0$. The spatial data are homogeneous exactly when $A \equiv 0$ and $r \equiv 0$. The PDE is already linear homogeneous; initial data are homogeneous exactly when $f \equiv 0$. These are three separate assertions. A nonzero Robin coefficient h does not itself make the condition nonhomogeneous.

Step 4: Verify all parts of a full problem. Here $u_{*,t} = 2\kappa$ and $u_{*,xx} = 2$, so the PDE holds. Evaluation gives

$$\boxed{f(x) = x^2, \quad A(t) = 2\kappa t, \quad r(t) = 2\kappa L + h(L^2 + 2\kappa t).}$$

These data also agree at their shared corners. The diagram locates the data; it is a space–time domain, not a graph of the solution value.



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Question 4

Consider $u_t = u_{xx}$ for $0 < x < 1$, $0 < t \le T$, with

$$u(x,0) = f(x) = x(1-x), \qquad u(0,t) = A(t), \quad u(1,t) = B(t).$$

For this question a solution smooth up to the corners has continuous u, u_t, u_x, u_{xx} on the closed rectangle, satisfies the PDE there by continuity, and has $u_t(x,0)$ and $u_{xx}(x,0)$ obtained by differentiating its traces. Assume A, B are continuously differentiable.

Tasks

1. Derive the necessary value compatibility conditions at the two initial corners.
2. Derive the next necessary compatibility conditions involving $A'(0), B'(0)$ and f''.
3. Set $A = B = 0$. Decide whether value compatibility holds and whether a solution in the stated corner-smooth class can exist. Explain why this does not rule out solutions smooth only for positive time.
4. Keep f fixed and construct linear-in-time boundary data for which a polynomial solution exists. Verify the PDE and every datum explicitly; distinguish this sufficiency proof from checking necessary conditions alone.

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Question 4 – Solution

Strategy. At a corner, two different traces must describe the same value and the same time derivative.

Step 1: Match the values. Continuity at $(0,0)$ and $(1,0)$ requires

$$\boxed{A(0) = f(0) = 0, \qquad B(0) = f(1) = 0.}$$

If these equalities fail, even a continuous solution on the entire closed rectangle is impossible. Interior smoothness alone does not imply these equalities.

Step 2: Match the PDE to the time traces. The left boundary trace gives $u_t(0,0) = A'(0)$ and the initial trace gives $u_{xx}(0,0) = f''(0)$. The PDE therefore requires $A'(0) = f''(0)$. The right corner gives the analogous identity. Since $f'' = -2$,

$$\boxed{A'(0) = B'(0) = -2.}$$

These conditions use the stated corner regularity; they cannot be imposed silently on a class that has no continuous u_{xx} at the corners.

Step 3: Diagnose homogeneous endpoint data. For $A = B = 0$, both value conditions hold, but $A'(0) = B'(0) = 0 \neq -2$. Thus no solution exists in the corner-smooth class specified here. This is a regularity obstruction at $t = 0$, not a proof of nonexistence in a weaker class. A solution may be continuous up to the initial edge and smooth for $t > 0$ without its second spatial derivative extending continuously through the initial corners. No such weaker solution is needed for the contradiction.

Step 4: Construct a compatible full example. Take $A(t) = B(t) = -2t$ and

$$\boxed{u(x,t) = x(1-x) - 2t.}$$

Then $u_t = u_{xx} = -2$, $u(x,0) = f(x)$ and $u(0,t) = u(1,t) = -2t$. This polynomial has all required corner derivatives and proves existence for the modified problem. Matching a finite list of necessary compatibility conditions would not alone prove existence for arbitrary data; the explicit field supplies the sufficiency argument in this case.

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Question 5

For the first-order PDE $u_t + u_x = 0$ on $\mathbb{R}^2$, consider C^1 solutions. A characteristic curve is a curve $(x(t), t)$ along which the PDE forces u to be constant. Suppose data are prescribed on the line $x = at$:

$$u(at, t) = h(t), \qquad t \in \mathbb{R},$$

where $a \in \mathbb{R}$ and $h \in C^1(\mathbb{R})$.

Tasks

1. Use the chain rule to identify the characteristic lines, and prove that every C^1 solution has the form $u(x, t) = F(x - t)$.
2. For $a \neq 1$, determine the unique F and verify the resulting full solution for arbitrary h.
3. For $a = 1$, classify exactly which h permit a solution and whether that solution is unique. Give two distinct solutions when $h \equiv H$ is constant.
4. Compare $a = 0, h(t) = \sin t$ with $a = 1, h(t) = \sin t$. Explain geometrically why data on an entire line can determine a unique field, no field, or many fields.

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Question 5 – Solution

Strategy. Check whether the data line intersects all characteristic lines or follows just one of them.

Step 1: Recover the characteristic form. Along $x = t+s$, the chain rule gives $\frac{d}{dt}u(t+s,t) = u_x+u_t = 0$. Thus $u(t+s,t) = u(s,0)$. Defining $F(s) = u(s,0)$ proves $\boxed{u(x,t) = F(x-t)}$ for every solution. Conversely this form has $u_t = -F'$ and $u_x = F'$, so it solves the PDE.

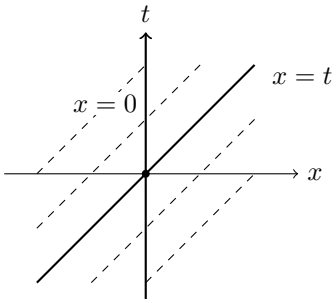
Step 2: Use a noncharacteristic data line. The condition becomes $F((a-1)t) = h(t)$. If $a \neq 1$, the argument ranges over all real numbers exactly once. Consequently

$$\boxed{F(s) = h\left(\frac{s}{a-1}\right), \qquad u(x,t) = h\left(\frac{x-t}{a-1}\right).}$$

This is C^1, solves the PDE and takes the prescribed trace. The characteristic representation proves uniqueness in the whole stated C^1 class.

Step 3: Diagnose characteristic data. If $a = 1$, the condition reduces to $F(0) = h(t)$ for every t. A solution exists exactly when h is constant, say H. Then any C^1 function with $F(0) = H$ works. For example, $u = H$ and $u = H + x - t$ are distinct solutions. Thus compatible characteristic data leave infinitely many fields undetermined.

Step 4: Interpret the two sine traces. For $a = 0$, $u(x,t) = \sin(t-x)$ is the unique solution. For $a = 1$, the varying trace $\sin t$ contradicts constancy along that characteristic, so no solution exists. The diagram shows the unique intersection with $x = 0$ and the characteristic line $x = t$; the latter supplies information on just one member of the family.



Characteristic family: $x - t = s$

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Question 6

For $u_t + u_x = 0$ on $\mathbb{R}^2$, a classical solution here means a C^1 field satisfying the PDE at every point. Compare

$$u(x,t) = |x - t|, \qquad u_\varepsilon(x,t) = \sqrt{(x-t)^2 + \varepsilon^2}, \quad \varepsilon > 0.$$

For this question only, an integral solution is a locally integrable field satisfying

$$\int_{\mathbb{R}^2} u(x,t)\big(\phi_t + \phi_x\big)(x,t)\,dx\,dt = 0$$

for every smooth compactly supported test function ϕ.

Tasks

1. Check the PDE for u off $x = t$, and decide whether u is a classical solution on the whole plane.
2. Verify that each u_ε is classical and prove a uniform error bound tending to zero as $\varepsilon \downarrow 0$.
3. Explain precisely why uniform convergence of these classical solutions does not establish that u is classical. Inspect $u_{\varepsilon,x}$ near the line $x = t$.
4. Verify that u is an integral solution using $s = x - t$, $r = t$. State which conclusions about u have and have not been proved.

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Question 6 – Solution

Strategy. A solution label includes a regularity requirement; checking a formula away from a singular line is not enough.

Step 1: Locate the failure of classical regularity. For $x > t$, $u_x = 1, u_t = -1$; for $x < t$, $u_x = -1, u_t = 1$. Thus the PDE holds on both open regions. At $x = t$, the two one-sided spatial derivatives disagree. The field is not differentiable there, hence is $\boxed{\text{not a classical solution on } \mathbb{R}^2}$.

Step 2: Verify the smooth approximations. Writing $s = x - t$, one has

$$u_{\varepsilon,x} = \frac{s}{\sqrt{s^2+\varepsilon^2}}, \qquad u_{\varepsilon,t} = -\frac{s}{\sqrt{s^2+\varepsilon^2}}.$$

These are smooth and sum to zero. Also

$$0 \le \sqrt{s^2+\varepsilon^2} - |s| = \frac{\varepsilon^2}{\sqrt{s^2+\varepsilon^2}+|s|} \le \varepsilon.$$

Equality occurs at $s = 0$, so the uniform error on the whole plane is exactly ε.

Step 3: Separate convergence of values from derivatives. For $s \neq 0$, $u_{\varepsilon,x} \to \operatorname{sgn}(s)$, while its value at $s = 0$ is always zero. This pointwise limit is discontinuous at zero. It cannot be a uniform limit on any closed interval about zero of these continuous derivatives. Uniform convergence of function values does not preserve differentiability, so it does not prove classical solvability.

Step 4: Check the stated integral formulation. Set $\psi(s, r) = \phi(s + r, r)$. The change of variables has Jacobian one, and $\psi_r = \phi_x + \phi_t$. Compact support and local integrability allow integration in either order. Therefore

$$\int_{\mathbb{R}^2} |x-t|(\phi_t + \phi_x)\,dx\,dt = \int_{\mathbb{R}} |s| \left(\int_{\mathbb{R}} \psi_r(s, r)\,dr \right) ds = 0.$$

The inner integral vanishes by compact support. Thus u is an integral solution under the supplied definition, as well as a classical solution off the line. It remains nonclassical on the whole plane; no uniqueness assertion follows from this verification alone.

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Question 7

For a real second-order linear PDE in two spatial variables, use the convention

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower-order terms} = 0, \qquad D = B^2 - AC.$$

At points where the second-order part is nonzero, call the equation hyperbolic if $D > 0$, elliptic if $D < 0$, and parabolic if $D = 0$. Consider

$$u_{xx} + 2yu_{xy} + xu_{yy} + 3u_x - u = 0.$$

Tasks

1. Identify A, B, C and classify the equation at every point of the plane, including the separating curve. Explain why the coefficient of u_{xy} is not B itself.
2. Classify the points $(0, 1), (2, 1), (1, 1)$, and sketch the type regions with correct axes and boundary curve.
3. At a fixed point, rewrite the principal quadratic form $Q(\xi, \eta) = \xi^2 + 2y\xi\eta + x\eta^2$ as a completed square. Explain its signs and rank in all three regions.
4. Replace $3u_x - u$ by $e^{x+y}u_x + 7u$ and then multiply the entire PDE by a smooth nowhere-zero function $m(x, y)$. Decide which changes can alter the type and justify your answer.

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Question 7 – Solution

Strategy. Classify the principal second-order part point by point, keeping the mixed-derivative convention fixed.

Step 1: Read the discriminant correctly. Here $A = 1$, $B = y$ and $C = x$, since the mixed coefficient is $2B = 2y$. Thus $D = y^2 - x$, giving

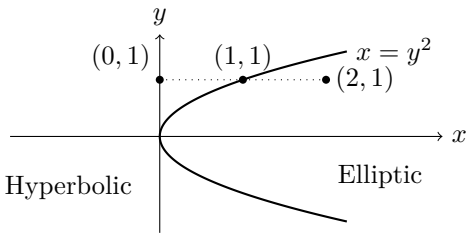
$$\boxed{\begin{array}{ll} x < y^2: & \text{hyperbolic,} \\ x = y^2: & \text{parabolic,} \\ x > y^2: & \text{elliptic.} \end{array}}$$

Since $A = 1$, the principal part never vanishes completely. The curve is therefore parabolic under the stated convention, not an order-zero exception.

Step 2: Check the points and the geometry. The three discriminants are respectively $1, -1, 0$, so $(0, 1)$ is hyperbolic, $(2, 1)$ elliptic and $(1, 1)$ parabolic. With x horizontal and y vertical, $x = y^2$ is a right-opening parabola; ellipticity lies to its right.

Step 3: Interpret the principal form. Completing the square gives $Q = (\xi + y\eta)^2 + (x - y^2)\eta^2$. For $x > y^2$, it is positive definite, with rank two. For $x < y^2$, the choices $(\xi, \eta) = (1, 0)$ and $(-y, 1)$ give opposite signs; it is indefinite with rank two. On $x = y^2$, it is a nonnegative square of rank one, vanishing on the nonzero direction $(-y, 1)$. This explains the type boundary algebraically.

Step 4: Identify changes that preserve type. Changing lower-order terms leaves A, B, C unchanged. Multiplying by m gives $\widetilde{D} = (mB)^2 - (mA)(mC) = m^2 D$, with the same sign because $m \neq 0$. Neither operation changes type. Allowing $m = 0$ could erase the principal part at those points, which is why the nowhere-zero restriction matters.



On the curve: parabolic

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Question 8

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and

$$u(0,t) = A(t), \qquad u(L,t) = B(t), \qquad u(x,0) = f(x).$$

Assume all data are smooth and $f(0) = A(0), f(L) = B(0)$. Define a boundary lifting

$$\ell(x,t) = \left(1 - \frac{x}{L}\right) A(t) + \frac{x}{L} B(t), \qquad v = u - \ell.$$

Tasks

1. Determine the PDE, endpoint data and initial data for v. Classify the homogeneity of its PDE and its spatial boundary conditions separately.
2. Find the exact condition on A, B for this affine lifting to leave the transformed PDE homogeneous for every x, t.
3. For $A(t) = t$, $B(t) = 2t$, $f = 0$, compute the transformed source and initial data. Does making the endpoint conditions homogeneous remove the time-dependent forcing from the full problem?
4. Show that a lifting is not unique. Replace ℓ by $\widetilde{\ell} = \ell + q$, where $q(0,t) = q(L,t) = 0$, and derive the new source and initial data. Explain why this is a change of unknown, not a different physical field u.

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Question 8 – Solution

Strategy. Subtract the lifting from every part of the problem, including the differential equation and the initial condition.

Step 1: Compute the transformed full problem. Since $\ell_{xx} = 0$,

$$\boxed{v_t - \kappa v_{xx} = -\ell_t = -\left(1 - \frac{x}{L}\right) A'(t) - \frac{x}{L} B'(t).}$$

The endpoints are $v(0,t) = v(L,t) = 0$, and $v(x,0) = f(x) - \ell(x,0)$. The endpoint conditions are homogeneous Dirichlet. The transformed PDE is linear, but generally nonhomogeneous. Its initial condition is homogeneous only if $f = \ell(\cdot, 0)$.

Step 2: Determine when the source vanishes. For each t, the displayed affine function of x vanishes throughout $(0, L)$ if and only if both its endpoint limits vanish. Thus $A'(t) = B'(t) = 0$ for all t, or equivalently $\boxed{A, B \text{ are constant in time}}$. Nonzero constant boundary values can therefore produce a homogeneous transformed PDE, although the original boundary data were nonhomogeneous.

Step 3: Inspect time-dependent endpoint values. For $A = t, B = 2t$, $\ell = t(1 + x/L)$, so

$$v_t - \kappa v_{xx} = -(1 + x/L), \qquad v(0,t) = v(L,t) = 0, \qquad v(x,0) = 0.$$

The boundary forcing has moved into the interior equation; it has not been removed from the full problem. In particular, $v = 0$ would fail this PDE. The stated value compatibility holds, but no corner-smooth existence is claimed for these data; higher compatibility would require additional checks.

Step 4: Track the freedom in the lifting. Let $\widetilde{v} = u - \widetilde{\ell} = v - q$. Direct differentiation gives

$$\widetilde{v}_t - \kappa \widetilde{v}_{xx} = -\ell_t - q_t + \kappa q_{xx}, \qquad \widetilde{v}(x,0) = f(x) - \ell(x,0) - q(x,0).$$

The endpoints still vanish. For example, $q = x(L - x)e^{-t}$ is a nonzero smooth choice with zero endpoints. Although the source and transformed initial data change, $u = v + \ell = \widetilde{v} + \widetilde{\ell}$ is identical. Each lifting gives an equivalent formulation when every datum is transformed consistently.

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Question 9

A problem is well posed in specified function spaces if solutions exist, are unique, and depend continuously on the prescribed data. To examine the third requirement, consider the Cauchy data problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi,\ 0 < y \le 1; \qquad u(x,0) = f(x), \quad u_y(x,0) = g(x),$$

with $u(0,y) = u(\pi,y) = 0$. Measure the bottom data by $\|f\|_\infty + \|g\|_\infty$ on $[0,\pi]$ and the observed solution by $\|u(\cdot,1)\|_\infty$. For integers $n \ge 1$, set

$$u_n(x,y) = \frac{e^{-n}}{n}\sin(nx)\sinh(ny).$$

Tasks

1. Verify the PDE and side conditions and find the exact bottom data f_n, g_n.
2. Compute the bottom data norm and top solution norm. Determine their limits as $n \to \infty$.
3. Prove that the amplification ratio is unbounded and explain why the calculation rules out a continuous solution map at zero in the stated norms if the problem is uniquely solvable on a linear class containing these fields.
4. Explain why convergence of $u_n(\cdot,1)$ to zero does not rescue continuity. Rescale the fields to obtain data converging to zero while the top norm stays exactly one, and sketch the vertical amplitude profiles for three n.

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Question 9 – Solution

Strategy. Small absolute outputs are not enough for stability; compare their size with the much smaller input, then normalize.

Step 1: Verify the supplied solutions. Two x derivatives multiply u_n by $-n^2$; two y derivatives multiply it by n^2. Their sum is zero, and $\sin(nx)$ vanishes on both sides. At the bottom, $\boxed{f_n = 0, \quad g_n = e^{-n}\sin(nx)}$. These are smooth fields with compatible side and bottom values.

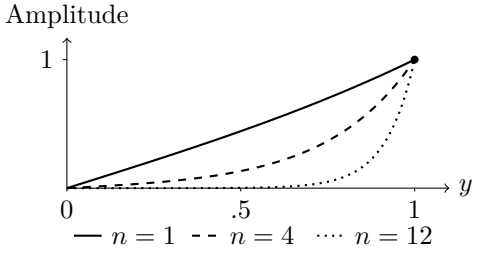
Step 2: Measure the input and output. For each integer n, $\sup_{[0,\pi]} |\sin(nx)| = 1$. Hence the bottom norm is e^{-n} and the top norm is

$$M_n = \frac{e^{-n}}{n}\sinh n = \frac{1 - e^{-2n}}{2n}.$$

Both tend to zero. Thus this sequence by itself does not contradict continuity; the relative amplification is the information needed for the next step.

Step 3: Expose unbounded amplification. The ratio is $M_n/e^{-n} = \sinh n/n \to \infty$. If uniqueness holds in a linear solution class containing these fields, solving the homogeneous linear PDE with data is a linear map. A continuous linear map between the stated normed spaces must obey a uniform bound $\|u(\cdot, 1)\|_\infty \le C(\|f\|_\infty + \|g\|_\infty)$ on its domain. The unbounded ratios rule out that bound. No existence or uniqueness theorem for arbitrary Cauchy data has been assumed.

Step 4: Give a direct discontinuity sequence. Set $w_n = u_n/M_n = \sin(nx)\sinh(ny)/\sinh n$. Its top norm is one, whereas its bottom norm is $n/\sinh n \to 0$. The zero data have the zero solution; uniqueness would force the solution map to select these w_n, contradicting continuity at zero. The vertical envelopes are $\sinh(ny)/\sinh n$, not solution traces at one fixed x for every n.



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Question 10

On the unit square $\Omega = (0,1)^2$, consider a C^2 field smooth up to each edge, satisfying

$$\Delta u = s, \qquad \partial_n u = g \text{ on } \partial\Omega,$$

where s is a constant and g is constant on each open edge. Denote these four constants by g_L, g_R, g_B, g_T. At vertices, interpret boundary data by one-sided edge traces. You may use the divergence theorem and Green's identity.

Tasks

1. Identify the boundary type and derive a necessary relation between s and the four edge values, retaining the outward-normal signs.
2. For arbitrary edge constants satisfying that relation, construct a polynomial solution of the form $u = ax^2 + by^2 + cx + dy + C$. Verify that the compatibility condition is also sufficient for this data family.
3. Prove that any two solutions for the same compatible data differ by a constant. Explain why this is uniqueness up to a constant rather than uniqueness of the original problem.
4. Take $(g_L, g_R, g_B, g_T) = (0, 2, 0, 4)$. Find s, impose the normalization $\int_\Omega u \, dA = 0$, and give the resulting unique solution. Would the same edge data be compatible with $s = 0$?

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Question 10 – Solution

Strategy. Separate existence compatibility from the constant ambiguity left by Neumann data.

Step 1: Integrate the equation before solving it. These are Neumann conditions. Since the square has area one and each edge has length one, the divergence theorem gives

$$\boxed{s = \int_\Omega \Delta u \, dA = \int_{\partial\Omega} \partial_n u \, ds = g_L + g_R + g_B + g_T.}$$

In coordinates the derivatives are $-u_x$ on the left, u_x on the right, $-u_y$ on the bottom and u_y on the top. Replacing all by positive coordinate derivatives would give the wrong compatibility relation.

Step 2: Construct every compatible polynomial. For the proposed polynomial the four traces give

$$c = -g_L, \quad 2a + c = g_R, \quad d = -g_B, \quad 2b + d = g_T.$$

Thus

$$\boxed{u = \frac{g_R + g_L}{2} x^2 + \frac{g_T + g_B}{2} y^2 - g_L x - g_B y + C.}$$

Its Laplacian is the sum of the four edge values, exactly s when compatible. Every boundary derivative has the prescribed value. This explicit construction proves sufficiency for constant source and constant edge data on this square; it is not a general sufficiency theorem for arbitrary domains and functions.

Step 3: Prove the exact remaining ambiguity. The difference w of two solutions obeys $\Delta w = 0$, $\partial_n w = 0$. Green's identity yields

$$\int_\Omega |\nabla w|^2 \, dA = \int_{\partial\Omega} w \partial_n w \, ds - \int_\Omega w \Delta w \, dA = 0.$$

Continuity gives $\nabla w = 0$ everywhere, and connectedness of the square makes w constant. Conversely any added constant leaves all equations and data unchanged. A value or mean normalization is needed for actual uniqueness.

Step 4: Normalize the concrete field. Here $s = 6$ and $u = x^2 + 2y^2 + C$. Its integral is $1/3 + 2/3 + C = 1 + C$. The zero-mean condition fixes $C = -1$, so $\boxed{u = x^2 + 2y^2 - 1}$ is unique among solutions with that normalization. For $s = 0$, the integrated compatibility condition would require $0 = 6$, so no solution exists with these edge data.

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Question 1

Let I, J be open intervals, $\kappa > 0$, $r \in \mathbb{R}$, and consider

$$u_t = \kappa u_{xx} + ru \qquad ((x,t) \in I \times J).$$

A separated field is $u(x,t) = X(x)T(t)$, with $X \in C^2(I)$ and $T \in C^1(J)$. Assume neither factor is identically zero. Do not assume either factor is nonzero at every point.

Tasks

1. Substitute the product into the PDE. Choose one time where T is nonzero and use it to derive a spatial ODE on all of I, without dividing by X.
2. Show that a single real constant λ gives $X'' + \lambda X = 0$ and $T' = (r - \kappa\lambda)T$ everywhere on their respective intervals. Prove the converse as well.
3. Determine whether a nontrivial T can vanish at a time in J. Explain why zeros of X do not invalidate the separated field and what happens if a factor is identically zero.
4. Verify $X(x) = \sin(2x)$ when $I = \mathbb{R}$. Find all corresponding time factors, determine the growth threshold in r, and explain the harmless freedom to rescale the two factors reciprocally.

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Question 1 – Solution

Strategy. Evaluate at a nonzero anchor value of one factor before deriving equations valid even at zeros of the other.

Step 1: Derive a global spatial identity. Substitution yields $XT' = \kappa X''T + rXT$. Choose $t_0 \in J$ with $T(t_0) \neq 0$. Evaluating there gives, for every $x \in I$,

$$\kappa X''(x) = \left(\frac{T'(t_0)}{T(t_0)} - r \right) X(x).$$

Set $\lambda = (r - T'(t_0)/T(t_0))/\kappa$. Then $\boxed{X'' + \lambda X = 0}$ on all of I, including any zeros of X.

Step 2: Recover the time equation and the converse. Substituting $X'' = -\lambda X$ into the product identity gives $X[T' - (r - \kappa\lambda)T] = 0$. Choose x_0 with $X(x_0) \neq 0$ to obtain

$$\boxed{T' = (r - \kappa\lambda)T \quad \text{on } J.}$$

The constant is independent of both variables; a different anchor must give the same value because X is not identically zero. Conversely, these two ODEs imply the original product identity directly, without any divisions. Thus every such pair of factors produces a PDE solution.

Step 3: Treat zeros and the excluded trivial field. The time equation gives $T(t) = Ce^{(r-\kappa\lambda)(t-t_0)}$. For a nontrivial factor $C \neq 0$, it never vanishes. Spatial zeros are allowed; the identities above remain valid there. A factor identically zero produces $u \equiv 0$, which solves the PDE but determines no separation constant from its chosen factors. The trivial field must be retained separately, not mistakenly assigned restrictions obtained by division.

Step 4: Verify a concrete mode and its normalization. For $X = \sin(2x)$, $X'' = -4X$, so $\lambda = 4$ and

$$u(x,t) = C\sin(2x)e^{(r-4\kappa)t}.$$

Its nonzero amplitude grows if $r > 4\kappa$, is constant if $r = 4\kappa$, and decays if $r < 4\kappa$. Direct differentiation verifies the PDE, including at $x = n\pi/2$. Replacing (X, T) by $(aX, T/a)$ for $a \neq 0$ leaves u unchanged; the two factors are not individually unique until a normalization is chosen.

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Question 2

Seek nonzero separated solutions of $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and mixed endpoint conditions

$$u(0,t) = 0, \qquad u_x(L,t) = 0.$$

Use the convention $X'' + \lambda X = 0$, $T' = -\kappa\lambda T$.

Tasks

1. Derive the endpoint conditions on X. Use integration by parts to prove that every nonzero spatial factor has $\lambda > 0$; address equality explicitly.
2. Solve the spatial problem and list all eigenvalues and eigenfunctions, using an index $n = 0, 1, 2, \ldots$. Explain why checking only a sine condition at both ends would give the wrong spectrum.
3. Find the time factor and compare the amplitude half-lives of the first two modes. Distinguish amplitude half-life from the half-life of squared amplitude.
4. Sketch the first three spatial factors normalized to have sine coefficient one, using $\xi = x/L$. Locate their interior zeros and verify their right-end slopes.

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Question 2 – Solution

Strategy. Let the actual boundary operators select the admissible separation constants.

Step 1: Prove positivity before solving the ODE. At a time where $T \neq 0$, the endpoint conditions give $X(0) = 0$, $X'(L) = 0$. Multiplication by X and integration yield

$$\lambda \int_0^L X^2\,dx = \int_0^L (X')^2\,dx - [XX']_0^L = \int_0^L (X')^2\,dx.$$

If $\lambda = 0$, then X' vanishes identically; $X(0) = 0$ forces $X = 0$. Thus a nonzero factor has $\boxed{\lambda > 0}$; negative values are impossible.

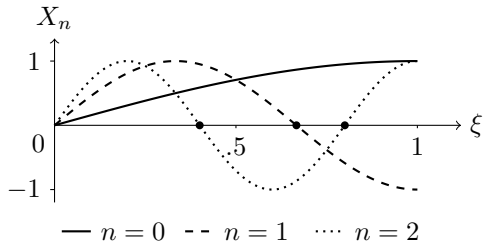
Step 2: Apply the two different endpoint conditions. Put $k = \sqrt{\lambda}$. The general solution is $A\cos(kx) + B\sin(kx)$. The left condition gives $A = 0$, and the right gives $Bk\cos(kL) = 0$. For $B \neq 0$,

$$\boxed{k_n = \frac{(n + 1/2)\pi}{L}, \quad \lambda_n = k_n^2, \quad X_n(x) = \sin(k_n x), \quad n = 0, 1, \ldots.}$$

The right condition is on the derivative, so it selects zeros of cosine. Using $\sin(kL) = 0$ would impose an unrequested Dirichlet condition there.

Step 3: Compare decay measures. The modes are $CX_n(x)e^{-\kappa\lambda_n t}$. Their amplitude half-lives are $\tau_n = \log 2/(\kappa\lambda_n)$, so $\tau_1 = \tau_0/9$. Squared amplitude decays as $e^{-2\kappa\lambda_n t}$ and has half-life $\tau_n/2$. Confusing those two quantities introduces a factor of two.

Step 4: Check the plotted nodes and slopes. For $n = 0, 1, 2$, the normalized factors are $\sin((n + 1/2)\pi\xi)$. There are respectively zero, one and two interior zeros: none; $2/3$; and $2/5, 4/5$. Every right slope is zero because $\cos((n + 1/2)\pi) = 0$. The right values alternate $1, -1, 1$ and need not vanish.



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Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, take $\kappa, L > 0$ and insulated endpoints $u_x(0,t) = u_x(L,t) = 0$. Use $X'' + \lambda X = 0$ and $T' = -\kappa\lambda T$. All fields are smooth enough for the stated derivatives and integrals.

Tasks

1. Solve the spatial eigenvalue problem, including $\lambda = 0$, and exclude $\lambda < 0$ by an integral identity.
2. Show directly from the PDE that the spatial mean is constant in time. Prove that each positive-eigenvalue spatial factor has zero mean, without evaluating its trigonometric integral.
3. Construct a finite separated-mode sum satisfying $u(x,0) = 2 + 3\cos(\pi x/L)$. Verify the PDE, both endpoints and the initial data, and identify the limiting field as $t \to \infty$.
4. Explain precisely what is lost if one divides the eigenvalue calculation by λ and discards the zero case. Can any finite sum of positive-eigenvalue modes reproduce these initial data?

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Question 3 – Solution

Strategy. The zero eigenvalue carries a conserved quantity and must be analyzed before any division by the separation constant.

Step 1: Retain the constant spatial mode. Integration by parts gives $\lambda \int X^2 = \int (X')^2 \geq 0$ because both endpoint derivatives vanish. Thus negative eigenvalues are excluded. For $\lambda = 0$, $X = ax + b$ and the endpoints force $a = 0$; nonzero constants are allowed. For $\lambda = k^2 > 0$, the left derivative condition removes the sine term and the right requires $\sin(kL) = 0$. Consequently

$$\boxed{\lambda_0 = 0,\ X_0 = 1; \qquad \lambda_n = (n\pi/L)^2,\ X_n = \cos(n\pi x/L),\ n \geq 1.}$$

Step 2: Track the conserved mean. For $m(t) = L^{-1} \int_0^L u(x,t)\, dx$, $m'(t) = \kappa [u_x]_0^L / L = 0$. Integrating $X'' + \lambda X = 0$ gives $\lambda \int_0^L X\, dx = -[X']_0^L = 0$. Thus every $\lambda > 0$ factor has zero mean. The zero mode has a constant time factor and can carry a nonzero mean.

Step 3: Verify the two-mode construction. Let $\omega = \kappa(\pi/L)^2$. Then

$$\boxed{u(x,t) = 2 + 3e^{-\omega t} \cos(\pi x/L).}$$

Its time derivative is $-3\omega e^{-\omega t} \cos(\pi x/L)$, equal to κu_{xx}. Its endpoint derivatives vanish, and its initial trace is the prescribed function. Its mean is two for every time. Since the decaying term has absolute value at most $3e^{-\omega t}$, it tends uniformly to $u = 2$. No infinite-series argument is required for this finite construction.

Step 4: Diagnose the missing mode. Dividing by λ loses the legitimate constant eigenfunction and the corresponding stationary PDE solutions. Every finite linear combination of positive-eigenvalue modes has zero spatial mean at each time, while the specified initial data have mean two. Such a combination cannot reproduce these data. This obstruction follows from the integral, not from an inability to find enough trigonometric coefficients.

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Question 4

A product field $u(x,t) = X(x)T(t)$ has proportional spatial snapshots. Consider the two-mode heat solution on $0 \le x \le \pi$, $t \ge 0$,

$$u(x,t) = e^{-t}\sin x + e^{-4t}\sin(2x).$$

For any two positions and times define

$$D = u(x_1,t_1)u(x_2,t_2) - u(x_1,t_2)u(x_2,t_1).$$

Tasks

1. Prove that a product field has $D = 0$. Conversely, if every such determinant vanishes and $u(x_0,t_0) \neq 0$ somewhere, construct factors X, T representing u everywhere.
2. Verify the PDE $u_t = u_{xx}$ and homogeneous Dirichlet endpoints for the displayed sum. Compute D at $x_1 = \pi/2, x_2 = \pi/6, t_1 = 0, t_2 = \log 2$ and decide whether the sum is a single product.
3. Find every interior spatial zero of $u(\cdot,t)$, the time at which it leaves the interior, and the sign of the field after that time.
4. Sketch the normalized snapshots $v(x,t) = e^{t}u(x,t)$ at $t = 0$, $(\log 2)/6$ and $(\log 2)/3$. Explain why division by a nonzero time factor cannot make a nonseparable field separable.

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Question 4 – Solution

Strategy. Test separation with an invariant determinant, then interpret the changing spatial shape.

Step 1: Establish an exact product test. For a product, both terms of D equal $X(x_1)X(x_2)T(t_1)T(t_2)$. Conversely, use the nonzero anchor to write $u(x,t)u(x_0,t_0) = u(x,t_0)u(x_0,t)$. Hence

$$\boxed{X(x) = u(x,t_0), \qquad T(t) = u(x_0,t)/u(x_0,t_0)}$$

gives a product everywhere, including zeros. The nonzero-anchor restriction excludes only the identically zero field, which is already a trivial product.

Step 2: Verify the solution and reject one-product form. Each term has time derivative equal to its second spatial derivative; the sum solves the PDE and vanishes at $x = 0, \pi$. At the stated points,

$$D = \frac{\sqrt{3}}{2}\left(\frac{1}{16} - \frac{1}{2}\right) = \boxed{-\frac{7\sqrt{3}}{32} \neq 0}.$$

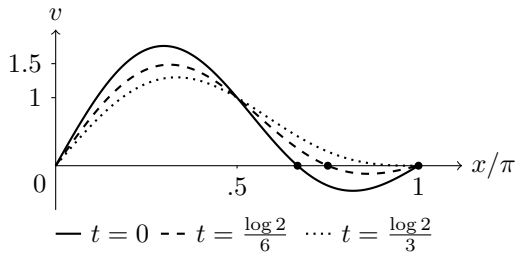
Thus superposition preserves the linear PDE but does not preserve the class of single products. The two spatial modes decay at different rates.

Step 3: Track the moving interior node. Factoring gives $u = e^{-t}\sin x[1 + 2e^{-3t}\cos x]$. On $0 < x < \pi$, $\sin x > 0$, so the only possible interior zero is

$$x_*(t) = \arccos(-e^{3t}/2), \qquad 0 \le t < t_* := \frac{\log 2}{3}.$$

It moves from $2\pi/3$ toward π. At $t = t_*$ it is at the endpoint, not an interior zero. For $t \ge t_*$ the field is strictly positive in the interior. Before then it is positive to the left of the node and negative to its right.

Step 4: Interpret the normalized plots. The three normalized fields are $\sin x + \sin 2x$, $\sin x + 2^{-1/2}\sin 2x$, and $\sin x + \frac{1}{2}\sin 2x$. Their changing zeros reveal changing shape. If $e^t u$ were a product, multiplying its time factor by e^{-t} would make u a product too, contradicting the determinant. Normalization changes amplitudes but cannot repair this failure of separation.



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Question 5

For $u_{tt} = c^2 u_{xx}$ on $0 < x < \pi$, $t \in \mathbb{R}$, with $c > 0$ and $u(0,t) = u(\pi,t) = 0$, seek a nonzero product $u = X(x)T(t)$ with both factors C^2. Use the convention $X'' + \lambda X = 0$.

Tasks

1. Derive the two factor equations using nonzero anchor values instead of assuming T has no zeros. Determine the allowed spatial factors and time factors.
2. Prove the precise restriction a single product places on its two initial functions $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$. Include the cases where one of these functions vanishes identically.
3. Determine whether $f = \sin(2x), g = \sin(3x)$ can come from one separated mode. Construct a sum of two modes that satisfies both initial functions and verify it directly.
4. Use $u(x,t) = \sin(2x)\sin(2ct)$ to explain why a zero-displacement snapshot need not mean zero solution or zero velocity. Explain the error in dividing by T at that time.

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Question 5 – Solution

Strategy. The time equation is second order and may cross zero; derive it globally and match both initial traces.

Step 1: Separate without excluding oscillation zeros. The product identity is $XT'' = c^2X''T$. A time t_0 with $T(t_0) \neq 0$ gives $X'' = -\lambda X$ everywhere, with $\lambda = -T''(t_0)/(c^2T(t_0))$. A position with $X \neq 0$ then gives $T'' + c^2\lambda T = 0$ for all times. Dirichlet endpoints and $\lambda \int X^2 = \int (X')^2$ exclude $\lambda \leq 0$. Solving the endpoint ODE gives

$$\boxed{X = \sin(nx), \quad \lambda = n^2, \quad T = A\cos(cnt) + B\sin(cnt), \quad n = 1, 2, \ldots,}$$

with overall spatial scale absorbed into A, B.

Step 2: Match both initial functions to one spatial shape. A single mode necessarily has $f = A\sin(nx)$ and $g = cnB\sin(nx)$ for the same integer n. Conversely, any such pair gives the displayed mode. Either coefficient may vanish: zero initial displacement permits a pure sine time factor, and zero initial velocity permits a pure cosine time factor. If both vanish, uniqueness for the scalar time ODE gives only the zero field. Thus proportionality alone is insufficient unless the nonzero shape is also an admissible spatial eigenfunction.

Step 3: Construct a field beyond one product. The functions $\sin 2x$ and $\sin 3x$ are not proportional, so one mode cannot provide the prescribed pair. Instead use

$$\boxed{u = \sin(2x)\cos(2ct) + \frac{1}{3c}\sin(3x)\sin(3ct).}$$

Each term satisfies the wave equation and both endpoint values. At $t = 0$, the displacement is $\sin 2x$; differentiating first gives initial velocity $\sin 3x$. These checks establish this finite construction directly.

Step 4: Preserve the solution at a zero time factor. For $u = \sin 2x \sin 2ct$, the initial displacement vanishes everywhere, while $u_t(x, 0) = 2c\sin 2x$ is not identically zero. The time factor crosses zero at $t = 0$ and other times, but its ODE and the PDE remain valid. Dividing by $T(0)$ is undefined; inferring that this mode is forbidden would discard a legitimate oscillatory solution. The anchor argument requires a nonzero value somewhere, not at every time.

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Question 6

Consider a single product $u(x, y) = X(x)Y(y)$ satisfying $u_{xx}+u_{yy} = 0$ in the rectangle $0 < x < \pi$, $0 < y < H$, where $H > 0$, with

$$u(0, y) = u(\pi, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, H) = \sin(2x).$$

Restrict attention to nonzero smooth separated fields; a general Fourier solution for arbitrary top data is not requested.

Tasks

1. Derive $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$ and determine the allowable λ from the two vertical sides.
2. Use the top data to identify the spatial mode and find Y. Verify all four boundary conditions and the PDE directly.
3. Explain why replacing the hyperbolic sine in Y by an ordinary sine generally fails. Compute the PDE residual for the proposed replacement $\sin(2x)\sin(2y)$.
4. Prove that the solution's vertical amplitude increases from zero to one, yet its absolute value never exceeds one in the rectangle. Compare the signs of the remaining factor equations for heat, wave and Laplace separation with the same positive λ.

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Question 6 – Solution

Strategy. The boundary conditions select the spatial eigenvalue, while the PDE determines the sign in the other factor equation.

Step 1: Separate with the correct opposite signs. Substitution yields $X''Y + XY'' = 0$. Nonzero anchors, as in the previous separation arguments, give $X'' = -\lambda X$, $Y'' = \lambda Y$ globally. The vertical sides impose $X(0) = X(\pi) = 0$. Integration by parts excludes $\lambda \leq 0$ for nonzero X; solving the positive case gives $\lambda = n^2$ and $X = \sin(nx)$, with the spatial constant absorbed into Y.

Step 2: Match the one-mode top trace. The top data are nonzero, so $Y(H) \neq 0$ and X must be proportional to $\sin 2x$. Thus $n = 2$. The remaining conditions are $Y(0) = 0, Y(H) = 1$, which determine

$$\boxed{u(x, y) = \sin(2x)\frac{\sinh(2y)}{\sinh(2H)}.}$$

Its second x derivative is $-4u$ and its second y derivative is $4u$. The sum is zero. The sine factor vanishes at the sides, $\sinh 0 = 0$ gives the bottom, and the ratio equals one at the top. The denominator is nonzero since $H > 0$. This uniquely determines the field within the stipulated product class.

Step 3: Reject a plausible sign error. For $\widetilde{u} = \sin 2x \sin 2y$, both second derivatives equal $-4\widetilde{u}$, so $\Delta\widetilde{u} = -8\widetilde{u}$, not identically zero. Rescaling a nonzero field to repair top values would not remove this interior residual. The required equation $Y'' - 4Y = 0$ has hyperbolic, rather than sinusoidal, solutions because the Laplace equation sums the second derivatives.

Step 4: Interpret growth in a bounded spatial direction. For $a(y) = \sinh(2y)/\sinh(2H)$, one has $a(0) = 0$, $a(H) = 1$ and $a'(y) = 2\cosh(2y)/\sinh(2H) > 0$. Hence $0 \leq a \leq 1$ and $|u| \leq 1$. For the same $X'' = -\lambda X$ with $\lambda > 0$, the remaining equations are

$$\begin{array}{ll} \text{heat:} & T' + \kappa\lambda T = 0, \\ \text{wave:} & T'' + c^2\lambda T = 0, \\ \text{Laplace:} & Y'' - \lambda Y = 0. \end{array}$$

Spatial hyperbolic growth in the last equation is compatible with bounded boundary data on a finite rectangle; it is not a statement about time instability.

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Question 7

On $1 < x < e$, $t \geq 0$, consider the variable-coefficient equation

$$\frac{1}{x}u_t = (xu_x)_x, \qquad u(1,t) = u(e,t) = 0.$$

For a nonzero product $u = X(x)T(t)$, use the separation convention $(xX')' + \lambda X/x = 0$, $T' = -\lambda T$. Introduce $s = \log x$ and $Z(s) = X(e^s)$.

Tasks

1. Derive the factor equations from the PDE and verify, by the chain rule, the transformed spatial equation and its endpoint conditions in s.
2. Determine every eigenvalue and spatial factor and verify the resulting separated fields in the original x equation.
3. Derive an integral identity that proves eigenvalue positivity. Identify the weight in the squared spatial norm; explain why an unweighted denominator would not follow from this equation.
4. Prove weighted orthogonality of distinct spatial modes by subtracting their ODE identities. Evaluate their squared weighted norms and locate the interior zeros of the third mode in the original coordinate.

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Question 7 – Solution

Strategy. A change of spatial coordinate can simplify separation, but the original equation determines the correct integration weight.

Step 1: Transform the spatial equation carefully. Substitution gives $XT'/x = (xX')'T$. Nonzero anchors yield $(xX')' = -\lambda X/x$ and $T' = -\lambda T$. Since $X' = Z_s/x$, one has $xX' = Z_s$ and $(xX')' = Z_{ss}/x$. Multiplying the spatial equation by x gives

$$\boxed{Z_{ss} + \lambda Z = 0, \qquad Z(0) = Z(1) = 0.}$$

The physical endpoints $1, e$ become $0, 1$, not an interval of length $e - 1$.

Step 2: Recover the modes in the original variable. The transformed Dirichlet problem has $\lambda_n = n^2\pi^2$ and $Z_n(s) = \sin(n\pi s)$ for $n \geq 1$. Thus

$$\boxed{u_n(x,t) = C\sin(n\pi \log x)e^{-n^2\pi^2 t}.}$$

Indeed $xX_n' = n\pi\cos(n\pi\log x)$ and $(xX_n')' = -n^2\pi^2 X_n/x$. Multiplication by the time factor gives exactly u_t/x. At $x = 1, e$ the sine values vanish. Zero and negative eigenvalues produce only the trivial Dirichlet factor.

Step 3: Derive the weighted energy quotient. Multiplication of $-(xX')' = \lambda X/x$ by X and integration give

$$\boxed{\lambda \int_1^e \frac{X^2}{x}\,dx = \int_1^e x(X')^2\,dx.}$$

The endpoint term vanishes. For a nonzero factor the denominator is positive; zero numerator would force a constant and then the zero field. Hence $\lambda > 0$. The weight $1/x$ comes from the coefficient of u_t; dropping it would change the identity and the associated eigenvalue quotient.

Step 4: Verify orthogonality, norms and nodes. For distinct modes m, n, subtracting the two integrated spatial identities cancels the derivative terms and yields $(\lambda_n - \lambda_m)\int_1^e X_n X_m\,dx/x = 0$. Thus their weighted inner product vanishes. With $s = \log x$,

$$\int_1^e \frac{X_n^2}{x}\,dx = \int_0^1 \sin^2(n\pi s)\,ds = \frac{1}{2}.$$

For $n = 3$, the interior zeros occur at $s = 1/3, 2/3$, hence at $x = e^{1/3}, e^{2/3}$. Equal spacing in s is not equal spacing in x.

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Question 8

For $u_t = u_{xx}$ on $0 < x < L$, $t \geq 0$, with $L > 0$, impose

$$u(0,t) = 0, \qquad u_x(L,t) + hu(L,t) = 0, \quad h \in \mathbb{R}.$$

A separated mode obeys $X'' + \lambda X = 0$, $T' = -\lambda T$. Define the dimensionless parameter $q = hL$. No sign restriction is imposed on h; a negative value need not model passive heat loss.

Tasks

1. Derive the identity $\lambda \int_0^L X^2 = \int_0^L (X')^2 + hX(L)^2$. Show that $h \geq 0$ guarantees positive eigenvalues, and determine exactly when $\lambda = 0$ has a nonzero factor.
2. For $\lambda > 0$, derive an eigenvalue equation using $z = L\sqrt{\lambda}$. For $\lambda < 0$, use $\rho = L\sqrt{-\lambda}$ to derive $\rho \coth \rho = -q$.
3. Prove that the negative-eigenvalue equation has exactly one root if $q < -1$ and none if $q \geq -1$. Determine the time behavior of this mode and of the zero mode.
4. For $q = -2$, bracket the positive root ρ to within 0.001, state the corresponding eigenvalue in terms of L, and sketch the intersection defining it. Mark the limiting value at $\rho = 0$ as a limit, not a negative eigenvalue.

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Question 8 – Solution

Strategy. A Robin boundary term can change the sign of the eigenvalue; analyze zero before treating the oscillatory and hyperbolic cases.

Step 1: Retain the boundary contribution. Integration by parts gives $\lambda \int X^2 = \int (X')^2 - [XX']_0^L = \int (X')^2 + hX(L)^2$. For $h \geq 0$, equality at zero would force $X' = 0$ and then $X = 0$; nonzero factors therefore have $\lambda > 0$. At $\lambda = 0$, $X = Ax$ and the right condition is $A(1 + hL) = 0$. Thus a nonzero zero mode exists exactly when $\boxed{q = -1}$.

Step 2: Separate the positive and negative cases. For $\lambda > 0$, $X = A\sin(\sqrt{\lambda}x)$ and

$$\boxed{z\cos z + q\sin z = 0, \quad z > 0.}$$

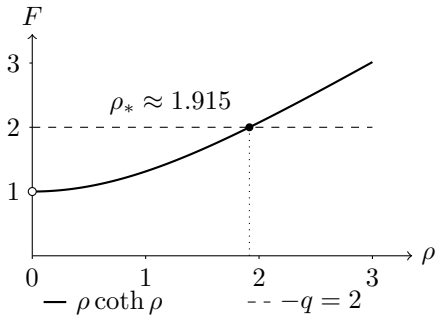
For $\lambda = -\mu^2 < 0$, $X = A\sinh(\mu x)$ and the right condition becomes $\mu\cosh(\mu L) + h\sinh(\mu L) = 0$. Since $\rho = \mu L > 0$,

$$\boxed{\rho\coth\rho = -q, \qquad \lambda = -\rho^2/L^2.}$$

Dividing by $\sinh\rho$ here is valid only in this strictly negative case.

Step 3: Prove the threshold and uniqueness. Let $F(\rho) = \rho\coth\rho$. It has limit 1 as $\rho \downarrow 0$ and $F(\rho) \to \infty$. Its derivative is $F' = (\sinh\rho\cosh\rho - \rho)/\sinh^2\rho > 0$: the numerator is zero at zero and has derivative $2\sinh^2\rho > 0$ for $\rho > 0$. Therefore its range is $(1, \infty)$, giving exactly one negative eigenvalue iff $q < -1$. That mode grows as $e^{\rho^2 t/L^2}$. At $q = -1$, the zero mode $X = x$ is stationary; it is not a negative eigenmode.

Step 4: Quantify the concrete crossing. For $q = -2$, numerical substitution gives $F(1.915) < 2 < F(1.916)$. Strict monotonicity certifies one root in this interval, approximately $\rho = 1.915008$. Thus $\lambda \approx -3.667256/L^2$. The open point at $(0, 1)$ records the excluded endpoint limit of F.



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Question 9

Consider smooth product candidates $u(x,t) = X(x)T(t)$, neither factor identically zero, on an open rectangle $I \times J$, where each interval has positive length. Compare

$$\text{(A)} \quad u_t = u_{xx} + xtu, \qquad \text{(B)} \quad u_t = u_{xx} + [a(x) + b(t)]u.$$

In (B), the given coefficient functions are smooth. No boundary data are imposed.

Tasks

1. Show that a nonzero product solving (A) would imply $T'/T - X''/X = xt$ on some smaller open rectangle where both factors are nonzero.
2. Prove that this identity is impossible by comparing two positions and two times. Explain why zeros of the factors cannot avoid the contradiction.
3. Derive separated ODEs for (B), using $X'' + a(x)X = -\lambda X$. Explain how the additive coefficient structure permits separation even though the PDE has variable coefficients.
4. Take $a(x) = -x^2$, $b(t) = t$. Construct a nonzero product with $X = e^{-x^2/2}$, determine λ and T, and verify the resulting field directly in (B).

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Question 9 – Solution

Strategy. Test the coefficient's dependence on both variables before assuming that the product ansatz can succeed.

Step 1: Divide only on a legitimate local rectangle. A nonzero value of each continuous factor has a neighborhood where that factor is nonzero. On the product of those neighborhoods, substitution into (A) gives

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} = xt.$$

The left side is a function of time minus a function of position. This local identity is already enough to test whether any nonzero product can exist.

Step 2: Use a mixed difference, not extra derivatives. Choose distinct x_1, x_2 and distinct t_1, t_2 in that rectangle. Subtract the four identities in the pattern $(x_1, t_1) - (x_1, t_2) - (x_2, t_1) + (x_2, t_2)$. All terms on the left cancel, whereas the right gives

$$\boxed{0 = (x_1 - x_2)(t_1 - t_2),}$$

which is false. Thus (A) has no nonzero product solution on an open rectangle. Factor zeros elsewhere cannot undo this contradiction in a nonzero neighborhood. The zero solution still exists; the conclusion does not assert that all solutions of (A) vanish.

Step 3: Separate an additive coefficient instead. For (B), substitution rearranges to $X[T' - b(t)T] = [X'' + a(x)X]T$. Nonzero anchors give, with one constant λ,

$$\boxed{X'' + [a(x) + \lambda]X = 0, \qquad T' = [b(t) - \lambda]T.}$$

The remaining coefficients each depend on only the appropriate variable. Conversely these ODEs imply the product PDE everywhere. Variable coefficients do not alone prevent separation; their dependence on the variables matters.

Step 4: Verify a nonconstant-coefficient example. For $X = e^{-x^2/2}$, $X'' = (x^2 - 1)X$, so $X'' - x^2X = -X$ and $\lambda = 1$. The time equation $T' = (t - 1)T$ gives $T = Ce^{t^2/2 - t}$. Hence

$$\boxed{u = Ce^{-x^2/2 + t^2/2 - t}, \qquad C \neq 0.}$$

Directly $u_t = (t - 1)u$ and $u_{xx} = (x^2 - 1)u$; adding $(-x^2 + t)u$ to the second expression gives the first. This is an explicit product solution of (B), not a solution of (A).

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Question 10

For $u_t = u_{xx} + ru$ on $0 < x < \pi$, $t \geq 0$, impose homogeneous Dirichlet endpoints. The separated spatial modes are $\sin(nx)$ with eigenvalues n^2, $n = 1, 2, \ldots$. Consider $1 < r < 4$ and the finite-mode initial field

$$u(x, 0) = \sin(2x) + \varepsilon \sin x, \qquad 0 < \varepsilon < 1.$$

Only this finite-mode construction is requested; no arbitrary-data expansion or infinite-series convergence theorem is needed.

Tasks

1. Derive the time equation for each mode and construct a solution with the given initial field. Verify its PDE and endpoints.
2. Compare the behavior when $\varepsilon = 0$ with every fixed $\varepsilon > 0$. Prove that in the latter case $e^{-(r-1)t}u$ converges uniformly to $\varepsilon \sin x$.
3. Using $\|v\|_2^2 = \int_0^\pi v^2\, dx$, compute the squared norm of each modal component and determine the time when the growing component overtakes the decaying component in this norm.
4. For $r = 2$, $\varepsilon = 1/8$, sketch the two positive modal amplitudes and mark their crossing. Explain why finding a decaying separated solution alone does not show that all solutions decay, and state the first-mode threshold in r.

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Question 10 – Solution

Strategy. Separation reveals individual growth rates; superposition shows how a small unstable component can dominate a decaying one.

Step 1: Construct and verify the finite sum. For $X_n = \sin nx$, $X_n'' = -n^2 X_n$, so the factor equation is $T_n' = (r - n^2)T_n$. Consequently

$$\boxed{u = e^{(r-4)t}\sin 2x + \varepsilon e^{(r-1)t}\sin x.}$$

Each time derivative has coefficient $r - n^2$, matching $u_{xx} + ru$. Both sine factors vanish at $0, \pi$, and their initial coefficients are $1, \varepsilon$. Linearity justifies adding these two verified modes.

Step 2: Compare exact cancellation with small contamination. For $\varepsilon = 0$, the solution decays uniformly because $r - 4 < 0$. For fixed $\varepsilon > 0$, the first mode grows because $r - 1 > 0$. Moreover

$$e^{-(r-1)t}u = \varepsilon \sin x + e^{-3t}\sin 2x, \qquad \|e^{-(r-1)t}u - \varepsilon \sin x\|_\infty = e^{-3t}.$$

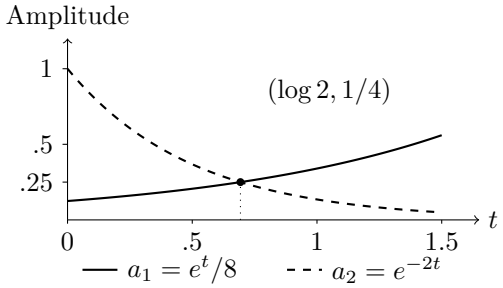
This proves uniform convergence of the normalized shape. The unnormalized solution grows without bound in sup norm, as is also seen at $x = \pi/2$.

Step 3: Compare modal norms exactly. Since $\int_0^\pi \sin^2(nx)\,dx = \pi/2$, the squared component norms are

$$N_2^2 = \frac{\pi}{2}e^{2(r-4)t}, \qquad N_1^2 = \frac{\pi}{2}\varepsilon^2 e^{2(r-1)t}.$$

Their equality is $\varepsilon^2 e^{6t} = 1$, giving $\boxed{t_c = \tfrac{1}{3}\log(1/\varepsilon)}$. The growing component is larger for $t > t_c$. The crossing time is independent of r in the stated range because the difference of modal rates is always three. Orthogonality also gives $\|u\|_2^2 = N_1^2 + N_2^2$.

Step 4: Interpret the crossing and the threshold. For $r = 2$, $\varepsilon = 1/8$, the amplitudes are $a_1 = e^t/8$ and $a_2 = e^{-2t}$. They cross at $t_c = \log 2$, with common value $1/4$. A decaying mode only describes its own initial shape. A nonzero first-mode component decays for $r < 1$, is stationary for $r = 1$, and grows for $r > 1$. Thus one decaying example cannot establish decay for every admissible field.



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Question 1

Solve $u_t = u_{xx}$ for $0 < x < \pi$, $t > 0$, with

$$u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \sin x + \tfrac{1}{2}\sin(2x).$$

The temperature is measured relative to the zero endpoint temperature.

Tasks

1. Construct the solution from the initial modes and verify the PDE, both endpoint conditions and the initial trace.
2. Prove directly from the formula that the initial temperature and every positive-time temperature are nonnegative. Does this force the temperature at every fixed interior point to decrease?
3. At $x = 5\pi/6$, determine the initial rate of change and the exact time of the unique temperature maximum. Justify the sign change of the time derivative.
4. Compute the total heat $H(t) = \int_0^\pi u(x,t)\,dx$ and its boundary-flux balance. Sketch profiles at $t = 0, 1/5, 1$ and explain how local warming is consistent with net heat loss.

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Question 1 – Solution

Strategy. Different spatial modes decay at different rates; local temperature and total heat need not have the same monotonicity.

Step 1: Evolve each initial mode. Since $(\sin nx)'' = -n^2 \sin nx$, the solution is

$$\boxed{u = e^{-t}\sin x + \tfrac{1}{2}e^{-4t}\sin 2x.}$$

Termwise differentiation of this finite sum gives $u_t = u_{xx}$. Both sines vanish at the endpoints, and setting $t = 0$ gives the prescribed field. The formula is smooth on the closed space interval for all $t \geq 0$.

Step 2: Prove nonnegativity without a plot. Factor the solution as $u = e^{-t}\sin x[1 + e^{-3t}\cos x]$. For $0 \leq x \leq \pi$ and $t \geq 0$, $\sin x \geq 0$ and the bracket is nonnegative. At positive time the bracket is strictly positive; hence the interior temperature is positive. This sign statement concerns values, not their time derivatives. Diffusion can transport heat into a cooler interior region.

Step 3: Find the local warming interval. At $x = 5\pi/6$,

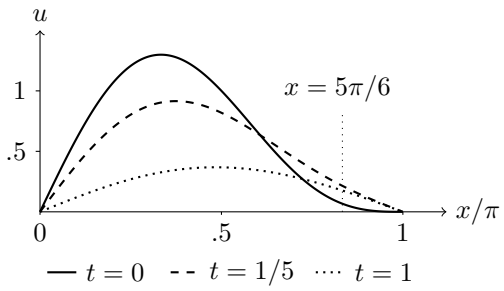
$$u = \tfrac{1}{2}e^{-t} - \tfrac{\sqrt{3}}{4}e^{-4t}, \qquad u_t = -\tfrac{1}{2}e^{-t} + \sqrt{3}e^{-4t}.$$

Thus $u_t(x, 0) = \sqrt{3} - 1/2 > 0$. Factoring out e^{-4t} shows that the derivative changes sign exactly when $e^{3t} = 2\sqrt{3}$. The unique maximum occurs at

$$\boxed{t_* = \tfrac{1}{3}\log(2\sqrt{3}).}$$

Before t_* the derivative is positive; afterward it is negative. This gives a concrete counterexample to pointwise monotone cooling.

Step 4: Compare with the global heat balance. Integration gives $H(t) = 2e^{-t}$, since the second sine has zero integral. Also $u_x = e^{-t}\cos x + e^{-4t}\cos 2x$, so $H'(t) = [u_x]_0^\pi = -2e^{-t} < 0$. The right endpoint derivative minus the left is the signed flux balance obtained by integrating the PDE. Total heat falls even while a neighborhood of $5\pi/6$ initially warms.



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Question 2

Let $\kappa, L > 0$ and solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with zero endpoint values and initial temperature $f(x) = x(L - x)$. You may use Fourier sine convergence for piecewise smooth odd extensions.

Tasks

1. Compute every coefficient $b_n = (2/L) \int_0^L f(x) \sin(n\pi x/L)\, dx$ and construct the solution series.
2. Justify uniform convergence to the initial trace as $t \downarrow 0$ and termwise verification of the PDE for $t > 0$. Do not infer corner smoothness from uniform convergence of the values.
3. Explain the absence of the even modes from reflection symmetry. Determine the total heat $H(t)$ as a convergent series and evaluate $H(0)$ directly.
4. Differentiate H for positive time and verify $H' = \kappa[u_x]_0^L$. Explain why H strictly decreases and why the solution is not a single separated product.

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Question 2 – Solution

Strategy. Exact coefficients control the initial convergence, while Gaussian mode decay justifies later differentiation.

Step 1: Compute the sine coefficients. Put $k = n\pi/L$. Twice integrating by parts, using $f(0) = f(L) = 0$ and $f'' = -2$, gives $\int_0^L f \sin(kx)\,dx = 2(1-(-1)^n)/k^3$. Thus $b_n = 4L^2(1-(-1)^n)/(n^3\pi^3)$, and

$$\boxed{u(x,t) = \frac{8L^2}{\pi^3} \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x/L)}{n^3} e^{-\kappa(n\pi/L)^2 t}.}$$

Step 2: Verify the full initial-value solution. The coefficients are absolutely summable, so the series at $t = 0$ converges uniformly. Fourier convergence identifies its sum as f on $[0, L]$, including the zero endpoints. Dominated convergence of the coefficient sum then gives $\|u(\cdot,t) - f\|_\infty \to 0$. For $t \geq \tau > 0$, any fixed number of spatial or time derivatives only adds powers of n; the factor $e^{-\kappa(n\pi/L)^2\tau}$ makes these series uniformly summable. Hence $u_t = \kappa u_{xx}$ termwise for positive time. At an initial corner, $f'' = -2$ conflicts with the zero boundary time derivative; continuity of all second spatial derivatives through those corners is not claimed.

Step 3: Use symmetry and integrate the solution. Since $f(L-x) = f(x)$ and $\sin(n\pi(L-x)/L) = (-1)^{n+1}\sin(n\pi x/L)$, even coefficients vanish. Integrating the uniformly convergent series gives

$$H(t) = \frac{16L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^4}, \qquad H(0) = \int_0^L x(L-x)\,dx = \frac{L^3}{6}.$$

Step 4: Check heat loss and the mode content. For $t > 0$, differentiation gives

$$H'(t) = -\frac{16\kappa L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^2} = \kappa[u_x]_0^L.$$

The equality on the right follows by differentiating the spatial sine series and using the odd-mode cosine endpoint values. Every summand is positive, so $H' < 0$. Infinitely many nonzero coefficients decay at distinct rates; in particular the ratio of the third coefficient to the first changes with time. A single product would have fixed ratios wherever its time factor is nonzero.

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Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$ with $\kappa, L > 0$, zero endpoint values, and $0 < a < L$, take the continuous tent-shaped initial temperature

$$f(x) = \begin{cases} x/a, & 0 \le x \le a, \\ (L-x)/(L-a), & a \le x \le L. \end{cases}$$

Its height is one. You may use Fourier convergence for continuous piecewise smooth functions with zero endpoint values.

Tasks

1. Derive all sine coefficients by splitting the integral at a, retaining the change in slope, and write the heat solution.
2. Justify the initial trace and positive-time smoothing even though f' jumps at a.
3. Show how the location a can be recovered from the ratio b_2/b_1. State its allowed range and explain why the recovery is unique when $0 < a < L$.
4. For $L = \kappa = 1$, $a = 1/3$, simplify the coefficients, identify the missing modes, and sketch the initial tent and the profiles at $t = 0.01, 0.05, 0.15$.

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Question 3 – Solution

Strategy. The corner in the initial profile appears as a slope jump in the coefficient calculation, not as a failure of the positive-time heat solution.

Step 1: Compute the contribution of the slope jump. With $k = n\pi/L$, integration by parts once gives $\int f\sin(kx) = k^{-1}\int f'\cos(kx)$ since f vanishes at both endpoints. Using $f' = 1/a$ to the left and $-1/(L-a)$ to the right gives

$$\int_0^L f\sin(kx)\,dx = \frac{L\sin(ka)}{a(L-a)k^2}.$$

Therefore

$$\boxed{b_n = \frac{2L^2\sin(n\pi a/L)}{\pi^2 n^2 a(L-a)}, \qquad u = \sum_{n=1}^{\infty} b_n e^{-\kappa(n\pi/L)^2 t}\sin(n\pi x/L).}$$

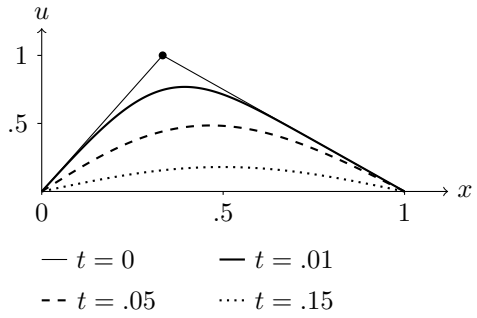
Step 2: Separate initial regularity from later regularity. Since $|b_n| \le C/n^2$, the sine series converges absolutely and uniformly, and Fourier convergence identifies its initial sum as the continuous tent. Summable domination gives uniform convergence to f as $t \downarrow 0$. For every $\tau > 0$, polynomial factors from any finite number of derivatives are dominated by Gaussian decay for $t \ge \tau$. Thus the series is smooth for positive time, satisfies the PDE termwise and retains zero endpoints. Its initial derivative jump does not persist as a corner in a positive-time profile.

Step 3: Recover the peak location from two modes. Put $\theta = \pi a/L \in (0,\pi)$. Then $b_1 > 0$ and

$$\frac{b_2}{b_1} = \frac{\sin 2\theta}{4\sin\theta} = \tfrac{1}{2}\cos\theta, \qquad \boxed{a = \frac{L}{\pi}\arccos(2b_2/b_1).}$$

The ratio must lie strictly between $-1/2$ and $1/2$. Cosine is strictly decreasing on $(0,\pi)$, so this reconstruction is unique in the stated family.

Step 4: Specialize and interpret the profiles. For $L = \kappa = 1$, $a = 1/3$, $b_n = 9\sin(n\pi/3)/(\pi^2 n^2)$. Exactly the multiples of three vanish, and $b_2/b_1 = 1/4$ recovers $a = 1/3$. The figure uses the exact piecewise initial trace and rapidly convergent positive-time series. The peak rounds as the profile diffuses toward zero.



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Question 4

Solve $u_t = u_{xx}$ on $0 < x < \pi$, $t > 0$, with $u(0,t) = u(\pi,t) = 0$ and initial data in the L^2 sense

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 1/2, & x = \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

The L^2 norm is $\|v\|_2^2 = \int_0^\pi |v|^2\,dx$. You may use sine-series convergence at continuity points and to the average at a jump, and Parseval's identity.

Tasks

1. Compute the sine coefficients and construct the heat solution.
2. Prove that the solution is smooth and solves the PDE for positive time, and that it approaches f in L^2 as $t \downarrow 0$.
3. State the pointwise limit at the interior jump and at each boundary when boundary values are retained as zero. Explain why convergence cannot be uniform on $[0,\pi]$, even if the endpoints of f are defined to be zero.
4. Prove uniqueness among solutions smooth up to the spatial boundaries for $t > 0$ and continuous into L^2 at $t = 0$, with these zero endpoint values. Use an energy estimate starting at $t = \delta > 0$ and then let $\delta \downarrow 0$.

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Question 4 – Solution

Strategy. Interpret the initial condition in its stated norm; incompatible corners and jumps do not invalidate a positive-time series solution.

Step 1: Compute all modal amplitudes. Integration over the nonzero half of the interval gives

$$\boxed{b_n = \frac{2}{\pi n}\big(1-\cos(n\pi/2)\big), \qquad u(x,t) = \sum_{n\geq 1} b_n e^{-n^2 t}\sin(nx).}$$

The point value assigned at $\pi/2$ does not affect the integral. Modes divisible by four vanish; the other coefficients must not all be replaced by the odd-only coefficients for a constant temperature on the entire interval.

Step 2: Verify positive-time smoothness and the norm trace. For $t \geq \tau > 0$, $|b_n| \leq 4/(\pi n)$ and the exponential makes all finitely differentiated series uniformly convergent. Thus the PDE and endpoints hold. Parseval and the sine expansion of f give

$$\|u(\cdot,t) - f\|_2^2 = \frac{\pi}{2}\sum_{n\geq 1} b_n^2 (1-e^{-n^2 t})^2 \longrightarrow 0,$$

by summable domination, since $\sum b_n^2 < \infty$. This is precisely the specified initial condition.

Step 3: State the pointwise limits correctly. The limits are 1 on $(0,\pi/2)$, 0 on $(\pi/2,\pi)$ and $1/2$ at the jump. To justify Gaussian damping of a pointwise convergent Fourier series, apply summation by parts: the decreasing weights $e^{-n^2 t}$ average its partial sums, have total weight tending to one, and move past each fixed index as $t \downarrow 0$. Thus they preserve the Fourier limit. At both endpoints u is identically zero for $t > 0$, so the limit is zero. A uniform limit of continuous functions on $[0,\pi]$ would be continuous; the jump and the left corner mismatch preclude uniform convergence to these data.

Step 4: Prove uniqueness in the stated class. Let w be the difference of two such solutions. For $t \geq \delta > 0$,

$$\frac{d}{dt}\|w\|_2^2 = 2\int_0^\pi w w_{xx}\,dx = -2\int_0^\pi w_x^2\,dx \leq 0.$$

The boundary term vanishes. Hence $\|w(\cdot,t)\|_2 \leq \|w(\cdot,\delta)\|_2$. Both solutions have the same L^2 initial trace, so the right side tends to zero as $\delta \downarrow 0$. Positive-time continuity then makes $w = 0$ pointwise. No smoothness at the discontinuous initial trace was assumed.

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Question 5

Let $\kappa, L > 0$. Solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with insulated ends $u_x(0,t) = u_x(L,t) = 0$ and initial temperature $f(x) = x$. Use the convention $f = a_0 + \sum_{n\ge 1} a_n \cos(n\pi x/L)$, where $a_0 = L^{-1}\int_0^L f$ and $a_n = (2/L)\int_0^L f\cos(n\pi x/L)$. You may use cosine-series convergence and Parseval's identity.

Tasks

1. Compute the coefficients and construct the solution, retaining the zero mode.
2. Prove conservation of the mean and the reflection identity $u(L-x,t) = L - u(x,t)$. Determine the limiting equilibrium.
3. Justify uniform recovery of the initial values, and explain why the spatial derivative cannot extend continuously through the initial boundary corners.
4. Compute $V(t) = \int_0^L (u - L/2)^2\,dx$ as a series, find $V(0)$ and establish $V(t) \le e^{-2\kappa(\pi/L)^2 t}V(0)$. Sketch profiles for $L = \kappa = 1$ at $t = 0, 0.03, 0.15$.

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Question 5 – Solution

Strategy. Insulation preserves the constant component while all nonconstant cosine modes decay.

Step 1: Compute and evolve the cosine coefficients. Integration gives $a_0 = L/2$ and $a_n = 2L((-1)^n - 1)/(\pi^2 n^2)$. Therefore

$$\boxed{u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \cos(n\pi x/L)}{n^2}.}$$

Gaussian decay justifies the PDE and derivative boundary conditions termwise for $t > 0$. The constant mode remains stationary.

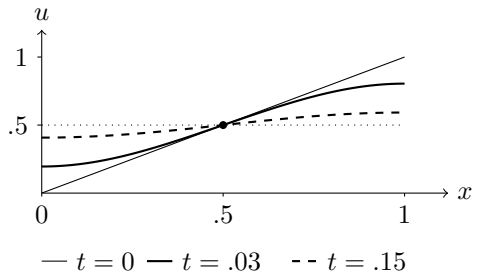
Step 2: Identify the conserved and symmetric parts. Integrating the PDE yields $(\int u)' = \kappa[u_x]_0^L = 0$, so the mean is $L/2$. For odd n, $\cos(n\pi(L-x)/L) = -\cos(n\pi x/L)$. The series therefore gives $u(L-x,t) = L - u(x,t)$, in particular $u(L/2,t) = L/2$ at all times. The absolutely summable coefficients and positive modal decay imply uniform convergence to the equilibrium $L/2$ as $t \to \infty$.

Step 3: Check values and derivatives separately. The n^{-2} coefficients give absolute uniform convergence at $t = 0$; the cosine convergence theorem identifies the sum as x, including both endpoints. Summable domination proves uniform recovery of those values as $t \downarrow 0$. But the initial trace has derivative one, whereas $u_x = 0$ at the endpoints for every positive time. A continuously extending spatial derivative would have to take both values at each initial corner, which is impossible. Uniform convergence of temperatures does not imply convergence of derivatives.

Step 4: Quantify equilibration. Orthogonality gives

$$V(t) = \frac{8L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-2\kappa(n\pi/L)^2 t}}{n^4}, \qquad V(0) = \int_0^L (x - L/2)^2 \, dx = \frac{L^3}{12}.$$

Each exponential is at most $e^{-2\kappa(\pi/L)^2 t}$, proving the required bound. The profiles flatten toward their conserved mean, rather than toward zero. Only the initial profile has the incompatible endpoint slopes.



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Question 6

Solve $u_t = \kappa u_{xx}$ for $0 < x < L$, $t > 0$, with $\kappa, L > 0$,

$$u(0,t) = 0, \qquad u_x(L,t) = 0, \qquad u(x,0) = x(2L - x).$$

You may use the complete orthogonal family $\sin(k_n x)$, $k_n = (n + 1/2)\pi/L$, $n = 0, 1, \ldots$, with squared norm $L/2$, and its Fourier convergence for the odd-at-zero, even-at-L extension of these data.

Tasks

1. Compute the expansion coefficients using the actual mixed conditions and construct the solution.
2. Verify the initial trace, PDE and two different endpoint conditions, justifying the series operations.
3. Compute total heat and its rate of loss. Explain why only the left endpoint contributes to the heat flux.
4. Determine the limiting shape of $e^{\kappa k_0^2 t} u$ uniformly on $[0, L]$. Compare its slowest exponential decay rate with that of a rod of the same length and diffusivity with both endpoints fixed at zero.

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Question 6 – Solution

Strategy. Use half-integer modes for the mixed boundary problem; integer sine modes would solve a different problem.

Step 1: Evaluate the coefficient integral. For $f = x(2L - x)$, $f(0) = 0$, $f'(L) = 0$ and $f'' = -2$. Two integrations by parts, using $\cos(k_n L) = 0$, give

$$\int_0^L f \sin(k_n x)\, dx = \frac{2}{k_n^3}, \qquad \boxed{u(x,t) = \frac{4}{L} \sum_{n=0}^{\infty} \frac{e^{-\kappa k_n^2 t} \sin(k_n x)}{k_n^3}.}$$

The vanishing boundary terms use the derivative condition at L, not a zero value of $f(L)$, which is actually L^2.

Step 2: Verify the complete solution. The coefficients are summable, so the initial series is uniformly convergent; the supplied Fourier theorem identifies it as f. Summable domination gives uniform convergence to f as $t \downarrow 0$. For $t \geq \tau > 0$, differentiated series converge uniformly by exponential decay and verify the PDE. At zero, all sine factors vanish; at L, all derivative cosine factors vanish. The initial function also satisfies these value/first-derivative data, although higher corner compatibility is a separate issue.

Step 3: Compute heat loss through the open thermal boundary. Since $\int_0^L \sin(k_n x)\, dx = 1/k_n$,

$$H(t) = \frac{4}{L} \sum_{n \geq 0} \frac{e^{-\kappa k_n^2 t}}{k_n^4}, \qquad H(0) = \frac{2L^3}{3}.$$

For $t > 0$, $H' = -(4\kappa/L) \sum e^{-\kappa k_n^2 t}/k_n^2 = -\kappa u_x(0,t)$. This is $\kappa[u_x]_0^L$ because the right derivative is zero. Every term in the loss sum is positive; insulation blocks flux at the right endpoint only.

Step 4: Extract the slowest mode rigorously. Multiplying the series by $e^{\kappa k_0^2 t}$ leaves the first term and a remainder bounded by $e^{-\kappa(k_1^2 - k_0^2)t} \sum_{n \geq 1} 4/(L k_n^3)$, which tends to zero. Hence uniformly,

$$\boxed{e^{\kappa k_0^2 t} u \longrightarrow \frac{32L^2}{\pi^3} \sin(\pi x/(2L)).}$$

The slowest rate is $\kappa\pi^2/(4L^2)$, one quarter of the Dirichlet–Dirichlet rate $\kappa\pi^2/L^2$. Equal material properties and length do not imply equal decay rates when the boundary conditions differ.

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Question 7

For the zero-endpoint heat problem $u_t = u_{xx}$ on $(0, \pi)$, let $f \in L^2(0, \pi)$ with $\|f\|_2 \le M$, where $M > 0$. Its solution for $t > 0$ is

$$u = \sum_{n \ge 1} b_n e^{-n^2 t} \sin(nx), \qquad u_N = \sum_{n=1}^{N} b_n e^{-n^2 t} \sin(nx), \quad N \ge 1.$$

You may use Parseval: $\sum b_n^2 = 2\|f\|_2^2/\pi$, and Cauchy–Schwarz.

Tasks

1. Prove $\|u - u_N\|_2 \le M e^{-(N+1)^2 t}$. Show that the bound is sharp over the stated data class for each N, t.
2. Derive a uniform-in-x error bound using Cauchy–Schwarz and the tail sum $\sum_{n>N} e^{-2n^2 t}$.
3. Bound that sum by a geometric series using $n = N + 1 + j$, $j \ge 0$, and obtain an explicit bound requiring no infinite summation.
4. For $M = 1, t \ge 0.1$, certify that six modes give a uniform error below 0.01. Explain why the available bound does not certify five modes, and why this does not prove that five modes fail for every particular initial field.

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Question 7 – Solution

Strategy. A truncation certificate needs a bound on the omitted coefficients, not just a graph of the retained modes.

Step 1: Prove the sharp norm estimate. Orthogonality gives

$$\|u-u_N\|_2^2=\frac{\pi}{2}\sum_{n>N}b_n^2e^{-2n^2t}\le e^{-2(N+1)^2t}\|f\|_2^2.$$

Taking square roots proves the bound. Equality is attained by $f=M\sqrt{2/\pi}\sin((N+1)x)$: all its energy lies in the first omitted mode. Thus the rate and constant are sharp for the whole L^2 ball.

Step 2: Control pointwise errors uniformly. Using $|\sin nx|\le 1$ and Cauchy–Schwarz,

$$|u-u_N|\le\left(\sum_{n>N}b_n^2\right)^{1/2}\left(\sum_{n>N}e^{-2n^2t}\right)^{1/2}\le\sqrt{\frac{2}{\pi}}M\left(\sum_{n>N}e^{-2n^2t}\right)^{1/2}.$$

This bound is independent of x and finite for $t>0$. It also justifies uniform convergence of the positive-time solution for arbitrary L^2 data.

Step 3: Replace the tail by a geometric bound. For the integer $j\ge 0$, $(N+1+j)^2\ge(N+1)^2+(2N+3)j$, since $j^2\ge j$. Therefore

$$\sum_{n>N}e^{-2n^2t}\le\frac{e^{-2(N+1)^2t}}{1-e^{-2(2N+3)t}},\qquad\boxed{\|u-u_N\|_\infty\le\sqrt{\frac{2}{\pi}}\,\frac{Me^{-(N+1)^2t}}{\sqrt{1-e^{-2(2N+3)t}}}.}$$

The denominator is positive because $t>0$. Each factor makes this certificate decrease as time increases.

Step 4: Make a numerical guarantee with the correct scope. At $t=0.1$, the bound is less than 0.00611 for $N=6$ and greater than 0.0226 for $N=5$. Hence six modes certify the requested 0.01 uniform error for every $t\ge 0.1$ and every datum with $\|f\|_2\le 1$. The five-mode certificate is too large; it does not prove failure for each field. For example, a datum supported entirely in the first five modes has zero truncation error with five modes. The bound is a sufficient test, not an exact error formula for unknown data.

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Question 8

For $\alpha > 0$, solve the forced heat problem

$$u_t = u_{xx} + e^{-\alpha t}\sin x, \qquad 0 < x < \pi,\ t > 0, \qquad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = 0.$$

Use a field of the form $u(x,t) = q_\alpha(t)\sin x$. This is interior forcing; the boundary temperatures remain zero.

Tasks

1. Derive and solve the scalar initial-value problem for q_α, treating $\alpha = 1$ separately.
2. Verify the PDE and all data, and prove that the constructed amplitude is positive for $t > 0$. Explain why uniqueness holds in the class of smooth solutions continuous into L^2 initially.
3. Determine the unique time of maximum amplitude for $\alpha \neq 1$ and its limit as $\alpha \to 1$. Check the maximum directly for $\alpha = 1$.
4. Derive the total-heat balance and compare matching temporal decay rates here with unbounded oscillatory resonance. Does the factor t in the exceptional solution cause unbounded temperature?

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Question 8 – Solution

Strategy. The source has one spatial eigenfunction, so the full problem reduces to a forced scalar decay equation.

Step 1: Solve the amplitude equation including the exceptional rate. Substitution gives $q'_\alpha + q_\alpha = e^{-\alpha t}$, $q_\alpha(0) = 0$. Multiplying by e^t and integrating yields

$$\boxed{q_\alpha(t) = \begin{cases} (e^{-\alpha t} - e^{-t})/(1-\alpha), & \alpha \neq 1, \\ te^{-t}, & \alpha = 1. \end{cases}}$$

The exceptional formula follows from the same integrating-factor integral; it is not obtained by substituting into a zero denominator.

Step 2: Verify existence, sign and uniqueness. The scalar ODE verifies the PDE, while the sine factor and $q_\alpha(0) = 0$ verify endpoints and initial data. The integral representation

$$q_\alpha(t) = \int_0^t e^{-(t-s)} e^{-\alpha s}\, ds$$

is strictly positive for $t > 0$. The difference of two solutions with the same source solves the homogeneous zero-data problem. Its squared L^2 norm is nonincreasing by integration by parts; starting at a positive time and using the zero initial norm proves uniqueness in the stated class.

Step 3: Locate the unique maximum. For $\alpha \neq 1$, $q'_\alpha = (e^{-t} - \alpha e^{-\alpha t})/(1-\alpha)$. The equation $q'_\alpha = 0$ is $e^{(\alpha-1)t} = \alpha$, giving

$$\boxed{t_\alpha = \frac{\log \alpha}{\alpha - 1} > 0.}$$

There is exactly one root; $q'_\alpha(0) = 1 > 0$ and the derivative is negative for sufficiently large time, proving it is the unique maximum. As $\alpha \to 1$, $t_\alpha \to 1$. Directly $(te^{-t})' = e^{-t}(1-t)$, with maximum $1/e$ at $t = 1$.

Step 4: Interpret the forced heat balance. The total heat is $H = 2q_\alpha$. Since $[u_x]_0^\pi = -2q_\alpha$ and the source integral is $2e^{-\alpha t}$,

$$H' = -2q_\alpha + 2e^{-\alpha t},$$

which agrees with the scalar ODE. At $\alpha = 1$, matching the source decay rate to the homogeneous decay produces te^{-t}. This field is bounded and tends to zero; its time factor does not imply unbounded oscillatory resonance. For every fixed $\alpha > 0$, the source and the solution ultimately decay.

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Question 9

For $u_t = u_{xx}$ on $(0, \pi)$ with zero endpoint values, let $\phi_n(x) = \sqrt{2/\pi}\sin(nx)$ be the orthonormal spatial modes. Write $u(x, 0) = \sum c_n \phi_n(x)$ and $u(x, T) = \sum d_n \phi_n(x)$, where $T > 0$. All norms in this question are $L^2(0, \pi)$ norms.

Tasks

1. Relate c_n and d_n. Prove that there is at most one L^2 initial field for given exact final data, and state the precise coefficient condition for such an initial field to exist.
2. Use initial fields ϕ_n to show that uniqueness of backward reconstruction does not imply continuous dependence on final data in these norms.
3. Suppose measurement noise in the final field has norm at most δ. Reconstruct only modes $1, \ldots, N$ and prove that the reconstructed noise has norm at most $e^{N^2 T}\delta$. Show that this bound is sharp.
4. For $T = 0.1$, $\delta = 10^{-6}$, choose the largest integer N certified to keep reconstructed noise below 0.01. Plot $\log_{10}$ of modal amplification against integer mode number, and explain why controlling noise alone does not bound the omitted true initial modes.

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Question 9 – Solution

Strategy. Each forward-decaying mode becomes an exponentially amplified mode when time is reversed.

Step 1: Characterize exact inversion. Forward evolution gives $d_n = c_n e^{-n^2 T}$, so the only possible coefficients are $\boxed{c_n = e^{n^2 T} d_n}$. Completeness implies at most one initial field. It exists in L^2 exactly when

$$\boxed{\sum_{n\geq 1} e^{2n^2 T}|d_n|^2 < \infty.}$$

Indeed this condition constructs an L^2 sine expansion whose forward solution has the given coefficients. Arbitrary L^2 final data need not satisfy this stronger condition.

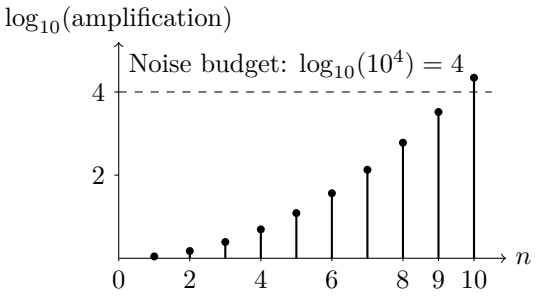
Step 2: Disprove continuity despite uniqueness. Take $f_n = \phi_n$, whose initial norm is one. Its final field has norm $e^{-n^2 T} \to 0$. Thus final data tend to zero while their uniquely determined initial fields do not. The backward map on its attainable-data domain is not continuous at zero in the stated norms. No contradiction arises with the stable forward problem, which damps these same modes.

Step 3: Bound noise after truncating the inverse. Let η_n be the noise coefficients, so $\sum |\eta_n|^2 \leq \delta^2$. The retained reconstructed noise satisfies

$$\left\| \sum_{n=1}^{N} e^{n^2 T} \eta_n \phi_n \right\|_2^2 = \sum_{n=1}^{N} e^{2n^2 T}|\eta_n|^2 \leq e^{2N^2 T}\delta^2.$$

Taking square roots gives the bound, attained by noise $\delta\phi_N$.

Step 4: Choose a noise cutoff and state its limitation. The requirement is $e^{0.1N^2}10^{-6} < 10^{-2}$, or $N^2 < 40\log 10$. Thus $\boxed{N = 9}$ is the largest certified integer: the bounds for $9, 10$ are approximately 0.003295 and 0.022027. The plot uses discrete stems, since n is an integer; their heights are $0.1n^2/\log 10$. The omitted initial tail has norm $(\sum_{n>N} |c_n|^2)^{1/2}$, for which no numerical bound was supplied. A noise cutoff controls amplification but requires additional prior information to control that approximation error.



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Question 10

Let $u_t = \kappa u_{xx}$ on $(0, L)$ with $\kappa, L > 0$, zero endpoint values, and initial field $f \in L^2(0, L)$. Let b_n be its sine coefficients and $E(t) = \frac{1}{2}\int_0^L u(x,t)^2\,dx$ its quadratic energy. You may use Parseval's identity, positive-time termwise differentiation and sine completeness.

Tasks

1. Derive the energy identity and prove $E(t) \le e^{-2\kappa(\pi/L)^2 t}E(0)$. Characterize every nonzero datum attaining equality at a fixed positive time.
2. Suppose $f(L - x) = -f(x)$ almost everywhere. Prove that only even modes remain and obtain the improved bound with rate $8\kappa\pi^2/L^2$. Identify when that bound is attained.
3. Show that a nonnegative, nonzero datum cannot have $b_1 = 0$. Explain why the enhanced asymptotic decay caused by removing the first mode requires sign changes in nonzero real initial data.
4. Prove the long-time L^2 asymptotic formula $e^{\kappa(\pi/L)^2 t}u \to b_1 \sin(\pi x/L)$, with a quantitative remainder bound. State what the formula says when $b_1 = 0$.

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Question 10 – Solution

Strategy. Orthogonality turns energy decay and symmetry restrictions into precise statements about which coefficients are present.

Step 1: Prove the sharp basic energy decay. For $t > 0$, integration by parts gives $E' = \kappa \int u u_{xx} = -\kappa \int u_x^2$ because $u = 0$ at both ends. With $\lambda_1 = (\pi/L)^2$, Parseval yields

$$E(t) = \frac{L}{4} \sum_{n \geq 1} b_n^2 e^{-2\kappa n^2 \lambda_1 t} \leq e^{-2\kappa\lambda_1 t} E(0).$$

For fixed $t > 0$, every term with $n \geq 2$ has strictly smaller decay factor. Equality for nonzero data holds exactly when $f = b_1 \sin(\pi x/L)$ with $b_1 \neq 0$. This also proves sharpness of the uniform energy rate.

Step 2: Use antisymmetry to improve the rate. Changing variable $x \mapsto L - x$ in the coefficient integral and using $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$ gives $b_n = (-1)^n b_n$. Thus odd coefficients vanish. The smallest remaining index is two, giving

$$\boxed{E(t) \leq e^{-8\kappa\pi^2 t/L^2} E(0).}$$

Nonzero equality data are precisely scalar multiples of $\sin(2\pi x/L)$. The symmetry is preserved by the evolved even-mode series.

Step 3: Identify the sign constraint. If $f \geq 0$ almost everywhere and is nonzero in L^2, then it is positive on a set of positive measure. The first sine is strictly positive on $(0, L)$; therefore $b_1 = (2/L) \int f \sin(\pi x/L) > 0$. Similarly a nonpositive nonzero datum has $b_1 < 0$. Thus any nonzero real datum with $b_1 = 0$ must have both positive and negative parts of positive measure. Removing the first mode is possible, but not for a nontrivial nonnegative initial temperature relative to the zero boundaries.

Step 4: Prove the normalized asymptotic statement. After removing the first term, Parseval gives

$$\left\| e^{\kappa\lambda_1 t} u - b_1 \sin(\pi x/L) \right\|_2^2 = \frac{L}{2} \sum_{n \geq 2} b_n^2 e^{-2\kappa(n^2-1)\lambda_1 t} \leq e^{-6\kappa\lambda_1 t} \|f\|_2^2.$$

The remainder norm is at most $e^{-3\kappa\lambda_1 t}\|f\|_2$ and tends to zero. If $b_1 = 0$, the normalized limit is zero and the actual field decays at least at the second-mode norm rate $e^{-4\kappa\lambda_1 t}$. The next nonzero coefficient, if present, can give an even faster decay rate.

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Question 1

Let $\kappa, L > 0$ and $A, B, C \in \mathbb{R}$. Solve

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0,t) = A, \quad u(L,t) = B,$$

with $u(x,0) = A + (B-A)x/L + C\sin(\pi x/L)$. Define the rightward diffusive flux as $j = -\kappa u_x$; the outward flux at the right endpoint is $j(L,t)$.

Tasks

1. Determine the unique stationary temperature and explain why it need not have zero flux.
2. Subtract that stationary field, solve the transformed initial-boundary problem, and verify the reconstructed solution against every original datum.
3. Take $L = \kappa = 1$, $A = 1$, $B = 3$, $C = 2$. Determine when the right reservoir receives heat from the rod and when it supplies heat to the rod. Find the exact switching time.
4. Compute total heat and its flux balance in this example. Sketch profiles at $t = 0, 0.08, 0.3$, together with the stationary field, and explain why the hotter endpoint can initially receive heat.

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Question 1 – Solution

Strategy. Separate the maintained stationary gradient from the transient temperature excess.

Step 1: Find the stationary conduction state. Stationarity requires $S'' = 0$, and the two boundary values give

$$\boxed{S(x) = A + (B - A)x/L, \qquad j_S = -\kappa(B - A)/L.}$$

This state is unique because a linear function is determined by two endpoint values. Unless $A = B$, it carries a constant nonzero flux. Stationary temperature means zero local storage rate, not an absence of heat transport.

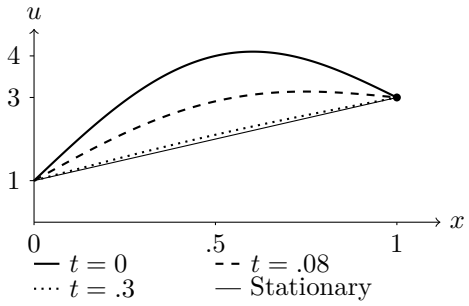
Step 2: Solve the homogeneous-boundary transient. With $v = u - S$, $v_t = \kappa v_{xx}$, $v(0,t) = v(L,t) = 0$ and $v(x,0) = C\sin(\pi x/L)$. Thus, writing $\lambda = \kappa\pi^2/L^2$,

$$\boxed{u(x,t) = S(x) + Ce^{-\lambda t}\sin(\pi x/L).}$$

Since $S'' = 0$, differentiation verifies the PDE. The sine vanishes at both ends, and at $t = 0$ the original initial field is recovered exactly.

Step 3: Determine the direction of endpoint heat flow. For the given numbers, $u_x(1,t) = 2 - 2\pi e^{-\pi^2 t}$, so $j(1,t) = -2 + 2\pi e^{-\pi^2 t}$. The right reservoir receives heat when $j > 0$, that is $0 \le t < t_* := \log \pi/\pi^2$. At t_* the right flux vanishes; for $t > t_*$ it supplies heat to the rod. The eventual rightward flux is -2.

Step 4: Verify the global balance and the profiles. Here $H(t) = \int_0^1 u\,dx = 2 + (4/\pi)e^{-\pi^2 t}$, so $H' = -4\pi e^{-\pi^2 t}$. Directly $j(0,t) = -2 - 2\pi e^{-\pi^2 t}$ and $j(1,t) = -2 + 2\pi e^{-\pi^2 t}$, giving $H' = j(0,t) - j(1,t)$. Initially the interior exceeds the right endpoint temperature, so heat can leave through that endpoint despite its being hotter than the left one.



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Question 2

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ and is initially at zero temperature in $L^2(0,1)$. At $t = 0$ the left endpoint is raised to one and held there, while the right remains zero: $u(0,t) = 1$, $u(1,t) = 0$ for $t > 0$. You may use Fourier sine convergence, Parseval, and $\int_0^\infty e^{-\pi^2 s^2}\, ds = 1/(2\sqrt{\pi})$.

Tasks

1. Subtract the stationary field, compute the sine coefficients of the transformed initial data and construct the full solution.
2. Justify the PDE for positive time and the L^2 initial trace. State the initial limit at interior points and at the left endpoint; explain why uniform convergence to the zero initial field is impossible.
3. Derive a convergent total-heat formula and prove that total heat increases for every positive time, although the source-free PDE has no interior forcing.
4. Find the rightward flux entering at $x = 0$ and its leading behavior as $t \downarrow 0$. Use a Riemann-sum limit to justify the singular rate rather than evaluating a divergent series at $t = 0$.

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Question 2 – Solution

Strategy. The sudden boundary change creates an initial corner mismatch; subtracting the stationary profile isolates its decaying sine expansion.

Step 1: Construct the boundary step response. The stationary field is $S = 1 - x$. For $v = u - S$, the endpoints are zero and $v(x, 0) = x - 1$. Integration gives $2\int_0^1 (x-1)\sin(n\pi x)\,dx = -2/(n\pi)$. Hence

$$\boxed{u(x,t) = 1 - x - \frac{2}{\pi}\sum_{n\geq 1} \frac{e^{-n^2\pi^2 t}\sin(n\pi x)}{n}.}$$

Step 2: Verify the appropriate initial trace. For $t \geq \tau > 0$, Gaussian decay makes every finitely differentiated transient series uniformly convergent. It satisfies the heat equation and zero endpoint values; adding S gives the required boundary data. The square-summable coefficients and Parseval prove $v \to x - 1$ in L^2, hence $u \to 0$ in L^2. At each interior point, the Fourier limit is zero after reconstruction; Gaussian damping preserves that limit by summation by parts. At $x = 0$, the positive-time value is always one, so its limit is one. Uniform convergence to zero on the closed interval is therefore impossible. No corner-continuous solution is claimed.

Step 3: Integrate the boundary-driven heating. For positive time,

$$H(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\ \text{odd}} \frac{e^{-n^2\pi^2 t}}{n^2}, \qquad H'(t) = 4\sum_{n\ \text{odd}} e^{-n^2\pi^2 t} > 0.$$

The initial norm trace gives $H(0+) = 0$ and the final heat is $1/2$. Differentiating the spatial series gives $H' = [u_x]_0^1 = j(0,t) - j(1,t)$, where $j = -u_x$. The endpoint supplies heat; no interior source is needed.

Step 4: Quantify the singular initial input. At the left endpoint,

$$j(0,t) = 1 + 2\sum_{n\geq 1} e^{-n^2\pi^2 t}.$$

With $h = \sqrt{t}$, the decreasing, integrable Gaussian gives $h\sum_{n\geq 1} e^{-\pi^2(nh)^2} \to \int_0^\infty e^{-\pi^2 s^2}\,ds$ by upper/lower integral bounds. Thus $\sqrt{t}\,j(0,t) \to 1/\sqrt{\pi}$, or $\boxed{j(0,t) \sim 1/\sqrt{\pi t}}$. The singular rate is integrable in time and is consistent with vanishing initial heat. Substituting $t = 0$ into the differentiated series would be invalid.

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Question 3

For $\kappa, L, Q > 0$, consider uniform internal heating with unequal fixed endpoints:

$$u_t = \kappa u_{xx} + Q, \quad u(0,t) = A, \quad u(L,t) = B, \qquad u(x,0) = \ell(x) := A + (B-A)x/L.$$

You may use the sine expansion of a smooth zero-endpoint function.

Tasks

1. Determine the stationary field and find exactly when its maximum lies strictly inside the rod. Locate that maximum.
2. Subtract the stationary field, compute the transient coefficients and construct the solution satisfying the original linear initial profile.
3. Verify the reconstructed PDE and data, explain its initial and positive-time regularity, and check the stationary heat-flux balance against the source integral.
4. For $L = \kappa = 1$, $A = 1$, $B = 2$, $Q = 4$, determine the stationary maximum and sketch the initial field, the profiles at $t = 0.02, 0.1$, and the stationary field. Explain why the maximum may exceed both boundary values.

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Question 3 – Solution

Strategy. Subtract a stationary solution of the forced equation, not just the line joining its boundary values.

Step 1: Solve the stationary forced problem. The equation $\kappa S'' + Q = 0$ and the endpoint data give

$$\boxed{S = \ell + \frac{Q}{2\kappa} x(L - x).}$$

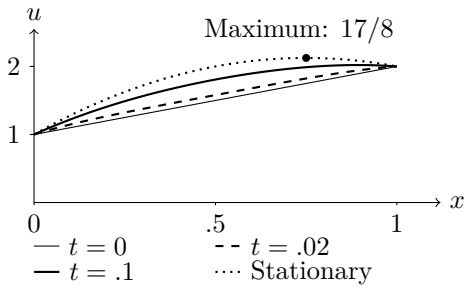
It is strictly concave, with critical point $x_* = L/2 + \kappa(B - A)/(QL)$. This is strictly interior exactly when $|B - A| < QL^2/(2\kappa)$. If equality holds, the maximum is at an endpoint; otherwise the maximum is at the endpoint in the direction of the larger value.

Step 2: Construct the transient from the actual initial difference. For $v = u - S$, the PDE and boundaries are homogeneous and $v(x, 0) = -Qx(L - x)/(2\kappa)$. Two integrations by parts give the nonzero coefficients $-4QL^2/(\kappa\pi^3 n^3)$ for odd n. Therefore

$$\boxed{u = S - \frac{4QL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.}$$

Step 3: Check the PDE, trace and maintained fluxes. The transient solves the homogeneous heat equation; $\kappa S'' + Q = 0$ then verifies the forced equation for u. The transient vanishes at the ends. Summable coefficients and Fourier convergence recover $u(\cdot, 0) = \ell$ uniformly. Gaussian decay permits differentiation for $t > 0$. The initial linear profile has $\kappa\ell'' + Q = Q$, inconsistent with a zero endpoint time derivative at an initial corner, so full corner smoothness is not asserted. For $j_S = -\kappa S'$, one has $j_S(L) - j_S(0) = QL$. Hence the steady balance $0 = j_S(0) - j_S(L) + QL$ accounts for all generated heat.

Step 4: Interpret the internally heated profile. In the example, $S = 1 + x + 2x(1 - x)$, $x_* = 3/4$ and $S(x_*) = 17/8$. The final maximum exceeds both boundary values because heat is generated inside the rod. The source-free maximum principle cannot be applied unchanged to this forced equation. The transient decays uniformly toward the plotted stationary field.



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Question 4

On $0 < x < 1$, let the storage and conductivity vary so that

$$\frac{u_t}{1+x} = \big((1+x)u_x\big)_x, \qquad u(0,t) = 1, \quad u(1,t) = 3.$$

Define $j = -(1+x)u_x$, $\ell = \log 2$, and let the initial field be

$$f(x) = 1 + \frac{2\log(1+x)}{\ell} + \sin\!\left(\frac{\pi \log(1+x)}{\ell}\right).$$

Tasks

1. Derive the stationary temperature and stationary flux. Explain why the straight line $1 + 2x$ is not stationary for this equation.
2. Use $s = \log(1+x)$ to transform the entire PDE and its endpoints. Construct and verify the full solution for the given initial field.
3. Compute the stored quantity $H(t) = \int_0^1 u(x,t)/(1+x)\,dx$ and verify $H' = j(0,t) - j(1,t)$.
4. For the error $v = u - S$, derive the weighted energy identity. Evaluate its weighted squared norm for this solution and explain why the weight cannot simply be omitted from the identity.

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Question 4 – Solution

Strategy. The stationary gradient and the natural storage norm both follow from the variable coefficients.

Step 1: Integrate constant stationary flux. The stationary equation is $((1+x)S')' = 0$, so $(1+x)S' = C$. The endpoint difference gives $C = 2/\ell$, hence

$$\boxed{S = 1 + 2\log(1+x)/\ell, \qquad j_S = -2/\ell.}$$

For $1 + 2x$, the quantity $((1+x)u_x)' = 2$ is nonzero, so it is not stationary. Equal endpoint values do not make different interior profiles solve the same PDE.

Step 2: Transform and solve the full problem. Write $U(s,t) = u(e^s - 1, t)$ on $0 < s < \ell$. Then $u_x = U_s/(1+x)$ and $((1+x)u_x)_x = U_{ss}/(1+x)$. The transformed equation is $U_t = U_{ss}$, with endpoints one and three. Subtracting $1 + 2s/\ell$ gives one sine mode. With $\lambda = \pi^2/\ell^2$,

$$\boxed{u = S + e^{-\lambda t}\sin(\pi\log(1+x)/\ell).}$$

The chain-rule identities verify the original PDE, and the sine gives zero endpoint correction and exactly the prescribed initial field.

Step 3: Check storage against boundary flux. Since $dx/(1+x) = ds$,

$$H(t) = 2\ell + \frac{2\ell}{\pi}e^{-\lambda t}, \qquad H' = -\frac{2\pi}{\ell}e^{-\lambda t}.$$

Also $j = -U_s = -2/\ell - (\pi/\ell)e^{-\lambda t}\cos(\pi s/\ell)$. Its left value minus its right value equals the displayed derivative. The unweighted integral of u is not the stored quantity in this model.

Step 4: Derive the weighted error dissipation. Multiplying $v_t/(1+x) = ((1+x)v_x)_x$ by v and integrating, with zero error at both ends, gives

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{v^2}{1+x}\,dx = -\int_0^1 (1+x)v_x^2\,dx.$$

For this mode the squared norm is $(\ell/2)e^{-2\lambda t}$ and the right-hand integral is $(\pi^2/(2\ell))e^{-2\lambda t}$, agreeing with the identity. The weight is the storage coefficient; dropping it changes the energy calculation.

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Question 5

Let both endpoint temperatures rise at a constant rate $r > 0$:

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \qquad u(0,t) = A + rt, \quad u(L,t) = B + rt,$$

where $\kappa, L > 0$ and $u(x,0) = \ell(x) := A + (B - A)x/L$. You may use the sine expansion of $x(L - x)$.

Tasks

1. Subtract the instantaneous boundary line $\ell(x)+rt$ and derive the PDE and initial data for the remaining field. Explain why that line alone is not a solution.
2. Find a time-independent zero-endpoint correction $P(x)$ so that $\ell + rt + P$ solves the PDE. Add the transient needed to recover the original initial field.
3. Verify the reconstructed solution, justify the initial trace and positive-time differentiations, and determine its limiting offset relative to the moving boundary line.
4. Compute the limiting midpoint lag, the limiting mean lag and the eventual total-heat growth rate. Explain why there is no stationary equilibrium even though the offset approaches a fixed shape.

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Question 5 – Solution

Strategy. A boundary lifting with nonzero time derivative creates a source in the transformed equation; a persistent lag balances that source.

Step 1: Keep the source created by the lifting. For $w = u - \ell - rt$,

$$w_t = \kappa w_{xx} - r, \qquad w(0,t) = w(L,t) = 0, \qquad w(x,0) = 0.$$

The line $\ell + rt$ has time derivative r and zero second spatial derivative, so it fails the original source-free PDE when $r > 0$.

Step 2: Find the lag profile and the initial correction. A fixed P must satisfy $\kappa P'' = r$, $P(0) = P(L) = 0$, giving $P = -rx(L-x)/(2\kappa)$. The remaining transient has initial data $-P$. Its odd sine coefficients are $4rL^2/(\kappa\pi^3 n^3)$. Thus

$$\boxed{u = \ell + rt + P + \frac{4rL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.}$$

Step 3: Verify all data and the moving-frame limit. The field $\ell + rt + P$ satisfies $u_t = \kappa u_{xx} = r$; its added transient satisfies the homogeneous heat equation. All correction terms vanish at the endpoints. At $t = 0$, the summable sine expansion equals $-P$ uniformly and recovers ℓ. For $t \geq \tau > 0$, the exponential transient permits all termwise derivatives. Initial corner smoothness is not implied: the boundary time derivative is r, whereas $\kappa\ell'' = 0$. As $t \to \infty$, the transient tends uniformly to zero, so $u - \ell - rt \to P$ uniformly.

Step 4: Quantify the persistent lag and heat input. The midpoint offset is $P(L/2) = -rL^2/(8\kappa)$, and the mean offset is $L^{-1}\int P = -rL^2/(12\kappa)$. Their negatives are the positive lags behind the boundary line. Since $P'(L) - P'(0) = rL/\kappa$, the eventual flux balance is $H' = \kappa[u_x]_0^L \to rL$, also obtained by differentiating the integrated solution. Boundary values grow indefinitely; it is the offset, not the absolute temperature, that approaches a fixed state.

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Question 6

A semi-infinite rod has $u_t = \kappa u_{xx}$ on $x > 0$, with $\kappa, \omega > 0$ and a maintained oscillating boundary $u(0,t) = A + B\cos(\omega t)$, where $A > |B| > 0$. Seek its periodic response of the form

$$u(x,t) = A + Be^{-\alpha x}\cos(\omega t - \alpha x), \qquad \alpha > 0,$$

with $u(x,t) \to A$ as $x \to \infty$. The initial field is to be the trace of this periodic response, so no additional startup transient is requested. Use the rightward flux convention $j = -\kappa u_x$.

Tasks

1. Determine α by direct substitution, and state the initial field that makes the response a solution of the full initial-boundary problem.
2. Derive the amplitude attenuation and phase delay at distance x. Find the depth where the oscillation amplitude falls to $1/e$ of its boundary value, and express it in terms of the period $P = 2\pi/\omega$.
3. Compute the boundary flux and its phase relative to the imposed temperature. Explain why constant far-field temperature does not imply zero instantaneous boundary flux.
4. Prove positivity of the temperature and approach to the far-field value as $x \to \infty$, uniformly in time. For $A = 2, B = 1, \kappa = 1, \omega = 2$, sketch profiles at phases $\omega t = 0, \pi/2, \pi$.

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Question 6 – Solution

Strategy. A periodic boundary signal diffuses inward with both amplitude attenuation and a spatial phase shift.

Step 1: Match the PDE and the initial trace. Put $\theta = \omega t - \alpha x$. Then $u_t = -\omega B e^{-\alpha x}\sin\theta$ and $u_{xx} = -2\alpha^2 B e^{-\alpha x}\sin\theta$. Thus $\boxed{\alpha = \sqrt{\omega/(2\kappa)}}$. The boundary trace is the imposed cosine, and the compatible initial field is $f(x) = A + Be^{-\alpha x}\cos(\alpha x)$. It is generally not the constant A; a constant initial field would require an additional transient.

Step 2: Quantify attenuation and delay. The local oscillation amplitude is $|B|e^{-\alpha x}$ and its phase lag is αx. Relative to the boundary waveform, the time delay is $\alpha x/\omega$, understood modulo the period. The penetration depth is

$$\boxed{d = 1/\alpha = \sqrt{2\kappa/\omega} = \sqrt{\kappa P/\pi}.}$$

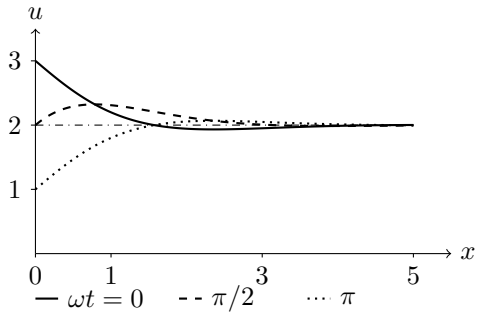
Longer periods penetrate farther, with depth proportional to the square root of the period rather than the period itself.

Step 3: Determine the phase of the boundary flux. Differentiation gives $u_x = \alpha B e^{-\alpha x}(-\cos\theta + \sin\theta)$. Hence

$$\boxed{j(0,t) = B\sqrt{\kappa\omega}\cos(\omega t + \pi/4).}$$

The flux leads the boundary temperature oscillation by $\pi/4$ in phase and has zero average over a full period. Instantaneous storage and release near the driven boundary are compatible with a constant far-field limit.

Step 4: Verify bounds and interpret the profiles. For all $x \geq 0, t$, $|u - A| \leq |B|e^{-\alpha x} \leq |B|$, so $u \geq A - |B| > 0$. Also $\sup_t |u(x,t) - A| \leq |B|e^{-\alpha x} \to 0$ as $x \to \infty$. This is a spatial far-field limit uniform in time, not decay of the maintained oscillation at a fixed point as time increases. For the plotted parameters, $\alpha = 1$ and the penetration depth is one.



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Question 7

On $0 < x < 1$, let $u_t = u_{xx}$ and $u(x, 0) = A$. The right endpoint stays at A. The left is held at $A + B$ for $0 < t < \tau$ and returns to A for $t \geq \tau$, where $B, \tau > 0$. You may use the unit boundary-step response

$$S(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n\pi x)}{n}, \qquad t > 0,$$

which has zero initial interior trace, left boundary one and right boundary zero. Set $S(x, 0) = 0$ for $0 < x < 1$ when discussing the switching time.

Tasks

1. Construct the solution before and after the switch by superposing boundary step responses. Verify the boundary traces on both open time intervals.
2. Derive a homogeneous-boundary sine series for $t > \tau$, with coefficients at time τ. Explain continuity at the switch for interior points and why continuity at the switching boundary corner is impossible.
3. Compute the total excess heat after the switch and prove that it strictly decreases. Determine the long-time equilibrium.
4. For $A = B = 1$, $\tau = 0.1$, sketch profiles at $t = 0.03, 0.08, 0.13, 0.3$. Explain why the rod does not return instantly to its initial field when the boundary pulse ends.

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Question 7 – Solution

Strategy. Switching off a boundary step is a delayed negative step; its instantaneous boundary effect does not erase the stored interior heat.

Step 1: Superpose the two boundary events. Linearity gives

$$\boxed{u = A + BS(x,t) \quad (0 < t < \tau), \qquad u = A + B[S(x,t) - S(x,t-\tau)] \quad (t > \tau).}$$

Each term solves the heat equation in its time interval. Before the switch, the left trace is $A + B$; afterward the two unit left traces cancel and leave A. Both right traces are always A. The initial interior trace is A.

Step 2: Identify the post-switch initial data. The stationary lines cancel after the switch, leaving

$$u - A = \frac{2B}{\pi} \sum_{n \geq 1} \frac{1 - e^{-n^2\pi^2\tau}}{n} e^{-n^2\pi^2(t-\tau)} \sin(n\pi x).$$

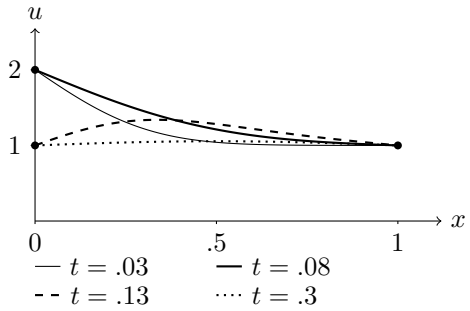
These are the sine coefficients of $BS(x,\tau)$ on the open interval. For every interior x, $S(x, t-\tau) \to 0$ as $t \downarrow \tau$, so the interior trace agrees with the pre-switch value. The left boundary itself jumps from $A + B$ to A; no single continuous trace at that corner can match both. For times strictly after the switch, all differentiated series converge uniformly by Gaussian decay. The switching trace is also recovered in L^2.

Step 3: Track the retained heat. For $t > \tau$, integration yields

$$H_{\text{ex}}(t) = \frac{4B}{\pi^2} \sum_{n \text{ odd}} \frac{1 - e^{-n^2\pi^2\tau}}{n^2} e^{-n^2\pi^2(t-\tau)}.$$

Its derivative is $-4B \sum_{n \text{ odd}} (1 - e^{-n^2\pi^2\tau}) e^{-n^2\pi^2(t-\tau)} < 0$. This equals $[u_x]_0^1$ after the switch. At any later fixed starting time the exponentially damped coefficients are summable, which proves uniform convergence to the equilibrium $u = A$ as $t \to \infty$.

Step 4: Interpret the finite pulse. The profiles before the switch have left value two; those after it have left value one. The right value is one throughout. At the switch the excess heat is $B \int_0^1 S(x,\tau)\, dx > 0$, as follows from the positive derivative of step-response heat and its zero initial value. It must subsequently diffuse out. A changed boundary value is not an instantaneous reset of the interior.



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Question 8

Consider $u_t = u_{xx}$ on $0 < x < 1$, with

$$u(0,t) = e^t, \qquad u(1,t) = e^t \cosh 1, \qquad u(x,0) = \cosh x.$$

Compare the affine boundary lifting $\ell(x,t) = e^t[1+(\cosh 1-1)x]$ with the nonaffine lifting $P(x,t) = e^t \cosh x$. You may use the sine expansion of a smooth function vanishing at both endpoints.

Tasks

1. Verify a closed-form solution of the original problem using P, including the PDE and every datum.
2. For $v = u - \ell$, derive the homogeneous-boundary forced PDE and its initial data. Explain why setting $v = 0$ would give the wrong physical field.
3. Set $k_n = n\pi$ and $D_n = 1 - (-1)^n \cosh 1$. Compute the sine coefficients of the initial difference and the transformed source. Solve the resulting modal initial-value equations.
4. Reconstruct u from the modal solution and prove that it agrees with the closed form. Explain why different liftings can produce different source terms without producing different solutions of the original problem.

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Question 8 – Solution

Strategy. A convenient lifting can solve the PDE as well as its boundary traces; a less convenient lifting is still valid if every transformed term is retained.

Step 1: Verify the exact nonaffine solution. The field $\boxed{u = e^t \cosh x}$ has $u_t = u_{xx} = e^t \cosh x$. Its two endpoint values and its initial trace are exactly the prescribed ones. With lifting P, the remaining field is zero, with zero source and initial data.

Step 2: Transform all parts under the affine lifting. Write $p(x) = 1 + (\cosh 1 - 1)x$, so $\ell = e^t p$. Since $\ell_{xx} = 0$,

$$v_t - v_{xx} = -e^t p(x), \quad v(0,t) = v(1,t) = 0, \quad v(x,0) = \cosh x - p(x).$$

The zero field fails both the interior source equation and the initial trace. The affine lifting satisfies the boundary values but is not itself a heat solution.

Step 3: Solve the transformed modal equations. Direct integration gives

$$2\int_0^1 p(x)\sin(k_n x)\,dx = \frac{2D_n}{k_n}, \qquad 2\int_0^1 \cosh x \sin(k_n x)\,dx = \frac{2k_n D_n}{1+k_n^2}.$$

Thus the initial coefficient is $c_n = -2D_n/[k_n(1+k_n^2)]$. The modal equation is $v_n' + k_n^2 v_n = -(2D_n/k_n)e^t$, $v_n(0) = c_n$. Since $(1+k_n^2)c_n = -2D_n/k_n$, its solution is $\boxed{v_n(t) = c_n e^t}$. No additional decaying term is needed for these data.

Step 4: Reconstruct the same physical temperature. The coefficients $c_n = O(n^{-3})$ are summable. Their sine series equals the smooth zero-endpoint function $\cosh x - p(x)$ uniformly on $[0,1]$. Consequently

$$u = \ell + e^t \sum_{n\geq 1} c_n \sin(k_n x) = e^t p + e^t(\cosh x - p) = e^t \cosh x.$$

This identity verifies the reconstructed PDE directly without claiming absolute uniform convergence of every differentiated sine series at the boundary. Different liftings change the auxiliary source and initial data together; retaining both transformations preserves the original solution.

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Question 9

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ with unknown constant boundary temperatures A, B. It is known that its full field has the form

$$u(x,t) = A + (B - A)x + Ce^{-\pi^2 t}\sin(\pi x).$$

Sensors at $x = 1/3, 2/3$ record $y_L(t), y_R(t)$. Set $m(t) = (y_L(t) + y_R(t))/2$ and let $\Delta > 0$ be known.

Tasks

1. Derive which combination of the readings identifies the stationary gradient immediately, even before the transient has decayed.
2. Use readings at times $0, \Delta$ to recover A, B, C uniquely. State the consistency condition on the differences of the two pairs of readings.
3. For $\Delta = (\log 2)/\pi^2$, use readings $(8/3, 10/3)$ at zero and $(13/6, 17/6)$ at Δ to find the complete field and verify the readings and boundary values.
4. Explain why one pair of simultaneous readings cannot identify all three parameters. For small Δ, quantify amplification of errors in the two averaged readings when estimating the mean of the boundary temperatures.

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Question 9 – Solution

Strategy. Equal sine values at the two sensors let their difference cancel the transient; a second observation time separates the remaining steady and decaying parts.

Step 1: Cancel the equal transient values. At both sensor positions, $\sin(\pi x) = \sqrt{3}/2$. Thus

$$d := B - A = 3(y_R - y_L), \qquad m(t) = S + Ke^{-\pi^2 t}, \quad S = (A+B)/2, \quad K = \sqrt{3}C/2.$$

The difference determines the stationary gradient at any measurement time. It must be unchanged between the two observations if the assumed model holds.

Step 2: Invert the two-time averages. Put $r = e^{-\pi^2 \Delta} \in (0,1)$. From $m_0 = S + K$, $m_\Delta = S + rK$,

$$\boxed{K = \frac{m_0 - m_\Delta}{1-r}, \quad S = \frac{m_\Delta - r m_0}{1-r}, \quad A = S - d/2,\ B = S + d/2,\ C = 2K/\sqrt{3}.}$$

The denominator is nonzero because $\Delta > 0$. Equality of the observed differences is the only additional consistency condition; with it these parameters reconstruct all four readings under the stated model.

Step 3: Reconstruct the numerical example. Here $r = 1/2$, $d = 2$, $m_0 = 3$ and $m_\Delta = 5/2$. Thus $S = 2$, $K = 1$ and

$$\boxed{A = 1, \quad B = 3, \quad C = 2/\sqrt{3}, \qquad u = 1 + 2x + \frac{2}{\sqrt{3}} e^{-\pi^2 t} \sin(\pi x).}$$

The endpoint values are one and three. The sensor transients are one at zero and one half at Δ, giving exactly the four supplied readings. The linear stationary part and the decaying sine verify $u_t = u_{xx}$.

Step 4: State the identifiability and conditioning limits. A single pair gives d and $S + K$, leaving infinitely many choices of S, K. For two times, errors e_0, e_Δ in the averages cause $\delta S = (e_\Delta - r e_0)/(1-r)$. If $|e_0|, |e_\Delta| \le \eta$,

$$|\delta S| \le \frac{1+r}{1-r}\eta \sim \frac{2\eta}{\pi^2 \Delta} \quad (\Delta \downarrow 0).$$

Opposite-signed errors attain the bound. Very close observations are therefore poorly conditioned even though the noiseless parameters are unique.

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Question 10

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, with fixed endpoint temperatures A, B, $\kappa, L > 0$, and stationary field $S = A + (B - A)x/L$. For smooth compatible data define the error energy $E_v = \frac{1}{2}\int_0^L (u - S)^2\, dx$ and the raw quadratic quantity $E_u = \frac{1}{2}\int_0^L u^2\, dx$. You may use the zero-endpoint Poincare inequality $\int v_x^2 \geq (\pi/L)^2 \int v^2$ and the parabolic maximum principle.

Tasks

1. Derive the energy identities for E_v and E_u, retaining all boundary terms. Prove the sharp bound $E_v(t) \leq e^{-2\lambda t} E_v(0)$, where $\lambda = \kappa\pi^2/L^2$.
2. For $A = B = 1$ and $u(x, 0) = 1 - \varepsilon \sin(\pi x/L)$, $0 < \varepsilon \leq 1$, construct the solution. Prove that its raw quadratic quantity increases while its error energy decreases.
3. If $|u(x, 0) - S(x)| \leq C \sin(\pi x/L)$, $C > 0$, use comparison to prove a pointwise error bound for all later times. Give a sufficient time for uniform temperature error at most δ, where $0 < \delta < C$.
4. Give a sufficient time for reducing error energy to a fraction $\eta \in (0, 1)$ of its initial value, and explain why neither total heat nor raw quadratic energy alone measures distance from a maintained nonzero equilibrium.

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Question 10 – Solution

Strategy. Dissipation applies naturally to deviations from the maintained equilibrium; the boundary reservoirs can increase the unshifted quantities.

Step 1: Identify which energy has no boundary work. For $v = u - S$, the endpoints vanish and $v_t = \kappa v_{xx}$. Therefore

$$E_v' = -\kappa \int_0^L v_x^2 \, dx \le -2\lambda E_v, \qquad E_v(t) \le e^{-2\lambda t} E_v(0).$$

A first-sine error attains equality, so the rate is sharp. For the raw quantity,

$$\boxed{E_u' = \kappa [u u_x]_0^L - \kappa \int_0^L u_x^2 \, dx.}$$

The first term need not vanish with nonzero maintained boundary values.

Step 2: Exhibit simultaneous raw growth and error decay. The stated initial data give $u = 1 - \varepsilon e^{-\lambda t} \sin(\pi x/L)$. Then

$$E_u = \frac{L}{2} - \frac{2\varepsilon L}{\pi} e^{-\lambda t} + \frac{\varepsilon^2 L}{4} e^{-2\lambda t}, \qquad E_v = \frac{\varepsilon^2 L}{4} e^{-2\lambda t}.$$

Writing $z = \varepsilon e^{-\lambda t} \in (0, 1]$ gives $E_u' = \lambda L z(2/\pi - z/2) > 0$, since $2/\pi > 1/2$. Meanwhile $E_v' = -2\lambda E_v < 0$. Heating by the reservoirs moves this field closer to equilibrium while increasing its raw quadratic quantity.

Step 3: Establish a pointwise stopping criterion. The barriers $\pm C e^{-\lambda t} \sin(\pi x/L)$ solve the homogeneous zero-endpoint heat equation and bound the initial error. Applying comparison to their differences with v on any finite time interval gives

$$\boxed{|u - S| \le C e^{-\lambda t} \sin(\pi x/L) \le C e^{-\lambda t}.}$$

Thus $t \ge \lambda^{-1} \log(C/\delta)$ suffices for uniform error at most δ. This conclusion uses the supplied pointwise initial bound, not merely an initial L^2 norm.

Step 4: Distinguish the two stopping criteria. The energy inequality gives $E_v(t)/E_v(0) \le \eta$ once $t \ge (2\lambda)^{-1} \log(1/\eta)$ for nonzero initial error; zero error is already at equilibrium. The factor of two comes from squaring a decaying amplitude. Total heat can conceal spatial differences, and raw energy includes the maintained background and boundary work. Neither is a substitute for an explicit norm of $u - S$ when measuring equilibration.

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Question 1

A square plate occupies $0 < x < 1$, $0 < y < 1$. Its steady temperature satisfies

$$u_{xx} + u_{yy} = 0, \qquad u(0,y) = u(1,y) = u(x,0) = 0, \qquad u(x,1) = \sin(\pi x) + \tfrac{1}{2}\sin(3\pi x).$$

The conductivity is one, so the outward heat flux is $-\partial_n u$.

Tasks

1. Construct the harmonic temperature by separation of variables and verify all four boundary values.
2. At height $y = 1/2$, determine the attenuation of each boundary mode and their amplitude ratio. Explain which spatial feature penetrates farther into the plate.
3. Compute the total outward heat flux on each edge and verify global balance. Use outward normals consistently.
4. Sketch temperature profiles at $y = 1/4, 1/2, 1$. Prove that the temperature is strictly positive in the interior despite the sign changes of the third sine mode.

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Question 1 – Solution

Strategy. Separate the two imposed spatial frequencies and track their normal derivatives as well as their temperatures.

Step 1: Extend each boundary mode. The zero side values give $\sin(n\pi x)$; the corresponding vertical equation is $Y'' - (n\pi)^2 Y = 0$. The bottom value selects $\sinh(n\pi y)$ and the top value normalizes it:

$$\boxed{u = \frac{\sinh(\pi y)}{\sinh \pi} \sin(\pi x) + \frac{1}{2} \frac{\sinh(3\pi y)}{\sinh(3\pi)} \sin(3\pi x).}$$

Each term has zero Laplacian. Substitution gives all four traces, including compatible corner values. The maximum principle gives uniqueness among continuous harmonic extensions.

Step 2: Compare penetration of the modes. Since $\sinh(a/2)/\sinh a = 1/(2\cosh(a/2))$, the first and third modes are attenuated by $1/(2\cosh(\pi/2))$ and $1/(2\cosh(3\pi/2))$. Their third-to-first amplitude ratio is

$$\boxed{\frac{\cosh(\pi/2)}{2\cosh(3\pi/2)}.}$$

It is smaller than the boundary ratio $1/2$: the higher spatial frequency loses amplitude faster away from the driven edge.

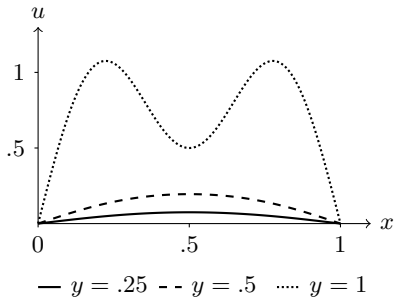
Step 3: Balance the four edge fluxes. Integrating $-\partial_n u$ along each unit edge gives

$$F_{\text{top}} = -2\coth \pi - \coth(3\pi), \qquad F_{\text{bottom}} = 2\ \operatorname{csch} \pi + \operatorname{csch}(3\pi),$$

$$F_{\text{left}} = F_{\text{right}} = \tanh(\pi/2) + \tfrac{1}{2}\tanh(3\pi/2).$$

For example, at the bottom the outward normal is $(0, -1)$, so the outward heat flux is u_y, not $-u_y$. The identity $\coth a - \operatorname{csch} a = \tanh(a/2)$ proves that the four totals sum to zero.

Step 4: Interpret the profiles and their sign. Writing $s = \sin(\pi x) \in [0, 1]$, the top value is $s + \frac{1}{2}(3s - 4s^3) = s(5/2 - 2s^2) \geq 0$, and it is positive for $0 < x < 1$. The weak maximum principle gives $u \geq 0$; the strong maximum principle gives $u > 0$ inside because the boundary data are not constant. Individual modes need not be positive.



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Question 2

Let u be harmonic on the unit square, with zero values on the left, right and bottom edges and

$$u(x, 1) = x(1 - x).$$

You may use Fourier sine convergence for a smooth function vanishing at both endpoints and Green's first identity.

Tasks

1. Derive the sine coefficients of the top boundary function and construct the complete harmonic extension.
2. Justify continuity up to every edge and harmonicity in the interior. Explain why boundary convergence and interior differentiation require different estimates.
3. Prove the pointwise bounds $0 \leq u(x, y) \leq y/4$. Determine whether equality can occur at an interior point.
4. Compute the Dirichlet energy $\mathcal{E} = \int_0^1 \int_0^1 |\nabla u|^2 \, dx \, dy$ as an explicitly convergent positive series. Justify the limiting operation.

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Question 2 – Solution

Strategy. Use the boundary Fourier coefficients for construction, a harmonic barrier for bounds, and orthogonality for energy.

Step 1: Compute the boundary expansion. Two integrations by parts give

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{4(1-(-1)^n)}{(n\pi)^3}.$$

Thus $b_n = 8/(n\pi)^3$ for odd n, and $b_n = 0$ for even n. The solution is

$$\boxed{u(x,y) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{8}{(n\pi)^3}\frac{\sinh(n\pi y)}{\sinh(n\pi)}\sin(n\pi x).}$$

Step 2: Establish the appropriate convergence. The hyperbolic ratio is between zero and one for $0 \leq y \leq 1$. Since $\sum |b_n| < \infty$, the series converges uniformly on the closed square. Its top trace is the sine series of $x(1-x)$, and its other traces vanish. On any strip $y \leq 1-\epsilon$, derivatives of every fixed order are bounded by a polynomial in n times $e^{-n\pi\epsilon}$. They converge uniformly there, so the Laplacian vanishes term by term. Uniform convergence of the original series alone would not justify taking two derivatives.

Step 3: Compare with a harmonic barrier. The boundary values are nonnegative. The function $h(x,y) = y/4$ is harmonic, is nonnegative on the sides, vanishes on the bottom, and dominates $x(1-x) \leq 1/4$ on the top. Applying the maximum principle to u and $u-h$ gives

$$\boxed{0 \leq u(x,y) \leq y/4.}$$

Both inequalities are strict in the interior: either equality there would make the corresponding harmonic difference constant, contrary to its boundary values.

Step 4: Evaluate energy without losing boundary terms. For a finite partial sum, Green's identity leaves only the top contribution $\int_0^1 u(x,1)u_y(x,1)\,dx$. Sine orthogonality gives

$$\mathcal{E}_N = \frac{1}{2}\sum_{\substack{n\leq N\\ n \text{ odd}}} b_n^2\, n\pi\coth(n\pi).$$

The same identity applied to a difference of partial sums proves their gradients are Cauchy in L^2, since the summands are $O(n^{-5})$. The uniform limit has that weak gradient and is the harmonic field already constructed. Therefore

$$\boxed{\mathcal{E} = \frac{32}{\pi^5}\sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{\coth(n\pi)}{n^5}.}$$

Every term is positive; truncating this energy series gives a lower bound.

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Question 3

Use centered coordinates $X = x - \frac{1}{2}$, $Y = y - \frac{1}{2}$ on the unit square. A harmonic temperature has the following boundary data:

$$u(x,0) = X^2 - X - \tfrac{1}{4}, \quad u(x,1) = X^2 + X - \tfrac{1}{4},$$
$$u(0,y) = \tfrac{1}{4} - Y - Y^2, \quad u(1,y) = \tfrac{1}{4} + Y - Y^2.$$

Seek a harmonic polynomial of degree at most two.

Tasks

1. Recover the polynomial from the four traces. Check their corner compatibility and prove uniqueness among all continuous harmonic solutions.
2. Find and classify every interior critical point. Explain why its existence does not contradict the maximum principle.
3. Find the exact global maximum and minimum on the closed square and all points where they occur.
4. Draw the zero-temperature lines and label the signs of the regions they separate. Verify that the total outward conductive flux, with conductivity one, is zero.

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Question 3 – Solution

Strategy. Boundary polynomials can identify an exact harmonic field, but its interior stationary point must be classified rather than assumed to be an extremum.

Step 1: Reconstruct and verify the polynomial. A general quadratic $aX^2+bXY+cY^2+dX+eY+f$ is harmonic precisely when $c=-a$. The top and bottom traces force $a=1$, $b=2$, $d=e=0$, and $f=0$. The side traces then agree:

$$\boxed{u = X^2 + 2XY - Y^2.}$$

At the corners its values are $1/2, -1/2, -1/2, 1/2$, in the order $(0,0),(1,0),(0,1),(1,1)$, whether computed from horizontal or vertical edges. The difference of two continuous harmonic solutions has zero boundary data; its maximum and minimum are zero.

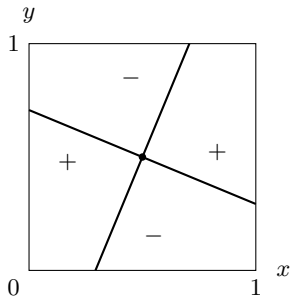
Step 2: Classify the critical point. The gradient is $(2X+2Y, 2X-2Y)$, so its sole zero is $(X,Y)=(0,0)$. The Hessian has eigenvalues $2\sqrt{2}$ and $-2\sqrt{2}$, hence the center is a saddle. In particular $u(X,0)=X^2$ while $u(0,Y)=-Y^2$. The maximum principle prohibits a nonconstant harmonic field's interior maximum or minimum, not a saddle.

Step 3: Locate the actual extrema. On the bottom, $X^2-X-1/4$ decreases on $[-1/2,1/2]$, with range $[-1/2,1/2]$; the top has the same range. On the left, $1/4-Y-Y^2$ decreases over the allowed interval, also with that range, and the right behaves symmetrically. Therefore

$$\boxed{\max u = \tfrac{1}{2} \text{ at } (0,0),(1,1), \qquad \min u = -\tfrac{1}{2} \text{ at } (1,0),(0,1).}$$

No interior point attains either value by the strong maximum principle.

Step 4: Draw the zero set and check flux. Solving $X^2+2XY-Y^2=0$ gives the two lines $Y=(1\pm\sqrt{2})X$, clipped to the square. The horizontal regions are positive and the vertical ones negative. Direct integration of $-\partial_n u$ gives flux -1 on each vertical edge and $+1$ on each horizontal edge; the sum is zero, also following from $\Delta u=0$.



$u=0$ on solid lines; dot: saddle.

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Question 4

Inside the unit disk, seek the bounded harmonic extension of the boundary function

$$g(\theta) = \begin{cases} 1, & -\pi/2 < \theta < \pi/2, \\ 0, & \pi/2 < \theta < 3\pi/2. \end{cases}$$

Angles are interpreted modulo 2π. Values assigned at the two jump points do not affect the Poisson integral

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}\, g(\phi)\, d\phi, \qquad 0 \le r < 1.$$

You may use that this kernel is positive, integrates to 2π, and has Fourier expansion $1 + 2\sum_{n \ge 1} r^n \cos(n\alpha)$.

Tasks

1. Compute the boundary Fourier coefficients and the resulting harmonic series.
2. Prove $0 < u < 1$ in the disk and find the center value. Explain the uniqueness class for these discontinuous data.
3. Determine the radial boundary limits on both open semicircles and at the two jump points. Explain why no continuous extension to the entire closed disk exists.
4. Find a closed expression for $u(x, 0)$ along the horizontal diameter and sketch it, including its limiting endpoint values.

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Question 4 – Solution

Strategy. The Poisson kernel supplies a bounded harmonic extension without imposing impossible continuity at jumps.

Step 1: Compute and damp the Fourier modes. The average is $1/2$. Symmetry makes every sine coefficient zero, while

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(n\phi)\, d\phi = \frac{2\sin(n\pi/2)}{n\pi}.$$

The kernel expansion therefore gives

$$\boxed{u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{k\geq 0} \frac{(-1)^k r^{2k+1}\cos((2k+1)\theta)}{2k+1}.}$$

On every smaller closed disk all derivatives converge uniformly by geometric decay. Each polar harmonic is the real part of $(x+iy)^{2k+1}$.

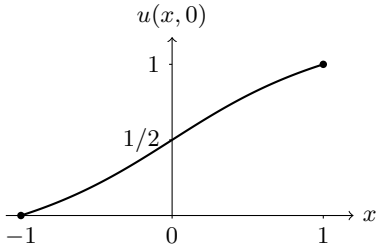
Step 2: Use positivity and specify uniqueness. Both semicircles have positive length, and the kernel is strictly positive inside. Thus their harmonic weights lie strictly between zero and one, giving $0 < u < 1$ and $u(0) = 1/2$. Uniqueness holds among bounded harmonic functions with these radial traces almost everywhere. Indeed, for the difference w, the Poisson formula on a circle of radius $R < 1$ expresses a fixed interior value through $w(R,\phi)$. As $R \uparrow 1$, boundedness and the a.e. zero trace let dominated convergence give $w = 0$. No common continuous trace at the jumps is required.

Step 3: Distinguish the boundary limits. At a point strictly within either semicircle the kernel is an approximate identity: away from any fixed angular neighborhood it tends uniformly to zero, while its total mass is fixed. The radial limit is therefore the local constant, one or zero. At $\theta = \pm\pi/2$ every displayed cosine term vanishes, so $u(r, \pm\pi/2) = 1/2$ for all $r < 1$. Neighboring boundary limits disagree; assigning $1/2$ at the jumps cannot make the closed-disk extension continuous.

Step 4: Sum the diameter profile. For $-1 < x < 1$, integrate the geometric series for $1/(1+x^2)$ from zero to x to obtain $\arctan x = \sum_{k\geq 0}(-1)^k x^{2k+1}/(2k+1)$. Hence

$$\boxed{u(x,0) = \tfrac{1}{2} + \tfrac{2}{\pi}\arctan x.}$$

Its derivative is $2/[\pi(1+x^2)] > 0$; its limits at $x = -1, 1$ are zero and one. These endpoints lie on the open constant-data arcs, not at the jumps.



Endpoint dots show boundary limits.

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Question 5

A disk of radius one has prescribed boundary temperature

$$g(\theta) = 2 + \cos\theta + \tfrac{1}{2}\cos(2\theta).$$

Let u be its continuous harmonic extension. For $r > 0$, the polar Laplacian is $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$.

Tasks

1. Construct u in polar and Cartesian coordinates. Explain why excluding singular radial terms is essential at the center.
2. Determine the exact boundary range and the locations of its extrema. Deduce sharp strict bounds for interior temperature.
3. Calculate the center temperature and the angular mean on every concentric circle. Explain their relation to the boundary average.
4. Find $\partial_n u$ on the boundary, verify zero total outward heat flux for conductivity one, and compute $\int_{\text{disk}} |\nabla u|^2 \, dA$.

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Question 5 – Solution

Strategy. Regular disk modes preserve the mean, damp angular oscillations and diagonalize the boundary energy calculation.

Step 1: Retain only regular radial factors. For mode $n \geq 1$, separation gives r^n and r^{-n}; the mean mode gives a constant and $\log r$. The singular factors cannot occur in a harmonic function on the full disk. Thus

$$\boxed{u = 2 + r\cos\theta + \tfrac{1}{2}r^2\cos(2\theta) = 2 + x + \tfrac{1}{2}(x^2 - y^2).}$$

The Cartesian polynomial verifies $\Delta u = 1 - 1 = 0$ even at the center, and setting $r = 1$ gives the prescribed data.

Step 2: Find the sharp range. Put $c = \cos\theta$. Then $g = 3/2 + c + c^2 = (c + 1/2)^2 + 5/4$, with $-1 \leq c \leq 1$. The minimum $5/4$ occurs at $\theta = 2\pi/3, 4\pi/3$; the maximum $7/2$ occurs at $\theta = 0$ modulo 2π. Consequently

$$\boxed{\tfrac{5}{4} < u(x, y) < \tfrac{7}{2} \quad \text{inside the disk.}}$$

The inequalities are strict by the strong maximum principle. The constants cannot be improved over the whole interior, since approaching the stated boundary points approaches those values.

Step 3: Identify the preserved mean. Both cosine terms have zero angular integral, so

$$\frac{1}{2\pi}\int_0^{2\pi} u(r, \theta)\, d\theta = 2 = u(0) \qquad (0 < r \leq 1).$$

This explicitly realizes the mean-value property. The center equals the boundary average, although most boundary points and most interior points have different temperatures.

Step 4: Compute flux and energy. At $r = 1$, the outward derivative is $u_r = \cos\theta + \cos(2\theta)$. Its integral is zero, hence so is the total conductive flux $-\int u_r\, d\theta$. Green's identity and trigonometric orthogonality give

$$\int_{\text{disk}} |\nabla u|^2\, dA = \int_0^{2\pi} g(\theta) u_r(1, \theta)\, d\theta = \pi + \tfrac{1}{2}\pi = \boxed{\tfrac{3}{2}\pi}.$$

Zero net heat flux does not imply zero gradient or zero energy: heat enters through some boundary portions and leaves through others.

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Question 6

A conducting annulus $a < r < R$ has constant inner and outer temperatures T_i, T_o, where $0 < a < R$ and $T_i \neq T_o$. Conductivity is one. Seek a radial harmonic field. Total outward heat flux means the integral of $-\partial_n u$ over the indicated boundary circle.

Tasks

1. Derive the temperature and prove it is the unique continuous harmonic solution of these boundary data, including among nonradial competitors.
2. Compute both total boundary fluxes with the annulus's outward normals. Identify its conductance, defined as the magnitude of transmitted total heat flux divided by $|T_i - T_o|$.
3. Compute the Dirichlet energy and describe its relation to the temperature difference and conductance.
4. Hold R, T_i, T_o fixed and let $a \downarrow 0$. Determine the local limit of temperature away from the origin, the energy limit and the largest gradient. Explain the failure of uniform convergence to T_o. Sketch the profiles for $R = 1, T_i = 1, T_o = 0$ and $a = 0.1, 0.3, 0.5$.

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Question 6 – Solution

Strategy. The logarithmic radial mode is admissible on an annulus, but its small-hole limit is not uniform near the shrinking inner boundary.

Step 1: Solve the radial equation. The equation $(ru_r)' = 0$ gives $u = A\log r + B$. Writing $D = \log(R/a) > 0$ and $\Delta T = T_i - T_o$, the boundary data give

$$\boxed{u(r) = T_o + \Delta T\,\frac{\log(R/r)}{D}.}$$

Its radial Laplacian is zero and its two traces are correct. Any other continuous harmonic solution differs by a harmonic function with zero boundary data; the maximum principle on the annulus proves uniqueness.

Step 2: Respect the inner normal. Here $u_r = -\Delta T/(rD)$. At $r = R$ the outward normal points toward increasing r, whereas at $r = a$ it points toward decreasing r. Thus

$$\boxed{F_{\text{outer}} = \frac{2\pi\Delta T}{D}, \qquad F_{\text{inner}} = -\frac{2\pi\Delta T}{D}, \qquad G = \frac{2\pi}{D}.}$$

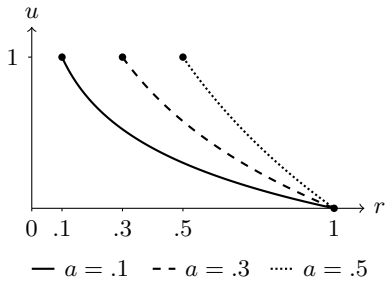
The sum is zero. When $\Delta T > 0$, heat enters the annulus at the inner boundary and leaves at the outer boundary.

Step 3: Integrate the energy. Direct radial integration yields

$$\mathcal{E} = 2\pi\int_a^R \frac{(\Delta T)^2}{r^2D^2}\,r\,dr = \boxed{\frac{2\pi(\Delta T)^2}{D} = G(\Delta T)^2}.$$

Equivalently, Green's identity gives $\mathcal{E} = -T_iF_{\text{inner}} - T_oF_{\text{outer}}$, with the same sign and value.

Step 4: Separate local and uniform limits. As $a \downarrow 0$, $D \to \infty$. On each fixed $r_0 \le r \le R$ the field tends uniformly to T_o, and $\mathcal{E} \to 0$. However $\sup_{a\le r\le R}|u - T_o| = |\Delta T|$ for every a. Also $\max|\nabla u| = |\Delta T|/[a\log(R/a)] \to \infty$ at the inner circle. The concentration near a shrinking boundary is compatible with vanishing integrated energy.



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Question 7

On the unit disk consider the Neumann problem

$$\Delta u = 0, \qquad \partial_n u = 1 + 3\cos\theta + 4\sin(2\theta) \quad \text{on } r = 1.$$

Here $\partial_n u$ is the outward normal derivative, not the outward conductive heat flux.

Tasks

1. Decide whether a classical solution exists. Derive the necessary compatibility condition and evaluate it for these data.
2. Subtract a constant c from the prescribed derivative to make the data compatible. Find c and construct a regular harmonic solution whose area average on the disk is zero.
3. Prove uniqueness under that normalization, and state precisely the nonuniqueness without it.
4. For the corrected solution, compute its boundary trace, Dirichlet energy and net conductive heat flux. Check the energy independently by integrating the Cartesian gradient.

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Question 7 – Solution

Strategy. Test flux compatibility before separation; then divide each nonconstant disk mode by its normal-derivative multiplier.

Step 1: Test the solvability condition. The divergence theorem gives

$$0 = \int_{\text{disk}} \Delta u \, dA = \int_{\partial\text{disk}} \partial_n u \, ds.$$

The prescribed derivative integrates to 2π, since both oscillatory terms integrate to zero. It violates a necessary condition, so the original problem has no classical solution.

Step 2: Correct the mean and construct the field. The corrected integral is $2\pi(1-c)$, so the only constant correction is $c = 1$. Since $\partial_r(r^n) = n$ at $r = 1$, the regular solution is

$$u = 3r\cos\theta + 2r^2\sin(2\theta) + C = 3x + 4xy + C.$$

The nonconstant terms have zero area average by symmetry. Thus normalization gives $C = 0$, and

$$\boxed{u(x,y) = 3x + 4xy.}$$

Its Cartesian Laplacian vanishes, and its outward derivative is exactly $3\cos\theta + 4\sin(2\theta)$.

Step 3: Establish uniqueness up to a constant. For two solutions, their difference w has $\Delta w = 0$ and $\partial_n w = 0$. Green's identity gives $\int |\nabla w|^2 = \int_{\partial\text{disk}} w\partial_n w = 0$. Hence $\nabla w = 0$, so w is constant on the connected disk. The zero-area-mean condition forces that constant to vanish. Without a normalization, every constant translate is another solution and these are all possibilities.

Step 4: Check energy in two coordinate systems. The boundary trace is $3\cos\theta + 2\sin(2\theta)$. Orthogonality gives

$$\mathcal{E} = \int_0^{2\pi} [3\cos\theta + 2\sin(2\theta)][3\cos\theta + 4\sin(2\theta)]\, d\theta = \boxed{17\pi}.$$

Independently, $|\nabla u|^2 = (3+4y)^2 + 16x^2 = 9 + 24y + 16r^2$. Its disk integral is $9\pi + 0 + 16(\pi/2) = 17\pi$. The corrected normal derivative integrates to zero, so its negative, the conductive flux, has zero total as required.

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Question 8

A unit square has insulated vertical sides and prescribed horizontal temperatures:

$$\Delta u = 0, \qquad u_x(0,y) = u_x(1,y) = 0, \qquad u(x,0) = 0, \quad u(x,1) = 1 + \cos(2\pi x).$$

Seek a solution smooth up to the boundary. Conductivity is one.

Tasks

1. Construct the solution by separation, treating the zero spatial eigenvalue explicitly.
2. Verify every datum and prove uniqueness for this mixed boundary problem using Green's identity.
3. Compute the horizontal average at each height and the integrated outward heat flux on each edge. Explain why zero side flux does not force constant temperature.
4. Calculate the Dirichlet energy. Prove $0 < u < 2$ at every interior point, justifying how the insulated sides are handled.

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Question 8 – Solution

Strategy. Neumann side conditions select cosine modes, including a zero mode whose vertical factor is linear rather than hyperbolic.

Step 1: Keep the zero mode. The side eigenfunctions are $1, \cos(n\pi x)$ with eigenvalues $0, (n\pi)^2$. The mean mode satisfies $Y'' = 0$ with bottom value zero and top value one, so it is y. The second cosine mode gives

$$\boxed{u(x,y) = y + \frac{\sinh(2\pi y)}{\sinh(2\pi)} \cos(2\pi x).}$$

Dropping the zero mode would lose the entire prescribed mean temperature.

Step 2: Verify and prove mixed uniqueness. The displayed terms are harmonic. The sine factor in u_x vanishes at both vertical sides; the bottom and top traces are exactly zero and $1 + \cos(2\pi x)$. For the difference w of two solutions, Green's identity yields $\int |\nabla w|^2 = \int_{\partial\Omega} w \partial_n w = 0$: $w = 0$ on the horizontal edges and $\partial_n w = 0$ on the vertical ones. Thus w is constant, and its bottom value forces $w = 0$.

Step 3: Track the mean and edge fluxes. The cosine has zero horizontal integral, so $\overline{u}(y) = \int_0^1 u(x,y)\,dx = y$. The integrated outward conductive fluxes are

$$\boxed{F_{\text{top}} = -1, \qquad F_{\text{bottom}} = 1, \qquad F_{\text{left}} = F_{\text{right}} = 0.}$$

They balance. Insulation forbids heat crossing the vertical sides; it does not forbid a vertical gradient or heat transfer between the horizontal edges.

Step 4: Compute energy and bound the field. Only the top contributes to $\int_{\partial\Omega} u \partial_n u$. At the top, $u_y = 1 + 2\pi \coth(2\pi) \cos(2\pi x)$, so orthogonality gives

$$\boxed{\mathcal{E} = 1 + \pi \coth(2\pi).}$$

For $0 < y < 1$, strict convexity of $\sinh(2\pi y)$ gives $0 < \sinh(2\pi y)/\sinh(2\pi) < y$. Therefore $0 < y - \sinh(2\pi y)/\sinh(2\pi) \le u \le y + \sinh(2\pi y)/\sinh(2\pi) < 2y < 2$. This direct estimate avoids incorrectly treating the unspecified side temperatures as Dirichlet data.

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Question 9

Let u be regular and harmonic on the unit disk. Its boundary temperature is known to have the form

$$g(\theta) = A + B\cos(N\theta), \qquad N \geq 1,$$

where N is known and A, B are unknown. The temperature is measured on the circle $r = \rho$, where $0 < \rho < 1$.

Tasks

1. Derive exact recovery formulas for A, B from the measured angular mean and cosine coefficient. Specify the consistency conditions on the other measured Fourier coefficients.
2. For $\rho = 1/2$, $N = 6$, and measured trace $2 + 0.03\cos(6\theta)$, recover the boundary data and the full field.
3. If the measured Nth cosine coefficient has error at most η, find the sharp error bound for the recovered B. Derive the largest allowable mode number when the desired per-coefficient error is at most $\varepsilon > \eta$. Evaluate it for $\rho = 1/2, \eta = 10^{-4}, \varepsilon = 10^{-2}$.
4. Contrast forward boundary stability with inverse instability when arbitrarily high modes are allowed. Give an explicit sequence showing why small interior-circle errors need not imply small boundary errors.

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Question 9 – Solution

Strategy. Harmonic continuation damps a mode by ρ^N; inversion must divide by exactly that small factor.

Step 1: Recover the two parameters. The regular harmonic extension is $u = A + Br^N \cos(N\theta)$. For the measured trace $h(\theta) = u(\rho, \theta)$, define

$$m = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta, \qquad c_N = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos(N\theta)\,d\theta.$$

Orthogonality gives

$$\boxed{A = m, \qquad B = \rho^{-N} c_N.}$$

Consistency with the stated model requires every sine coefficient and every nonconstant cosine coefficient except c_N to vanish. Since $\rho > 0$, the two parameters are unique.

Step 2: Reconstruct the given field. Here $\rho^N = 1/64$, $m = 2$, and $c_6 = 0.03 = 3/100$, so $B = 48/25 = 1.92$. Thus

$$\boxed{g(\theta) = 2 + \frac{48}{25}\cos(6\theta), \qquad u(r,\theta) = 2 + \frac{48}{25} r^6 \cos(6\theta).}$$

Substitution at $r = 1/2$ reproduces the measured coefficient $3/100$. The polar mode is a Cartesian harmonic polynomial and is regular at the center.

Step 3: Quantify inverse amplification. A coefficient error e, $|e| \le \eta$, produces error $\rho^{-N} e$ in B. Consequently $|\delta B| \le \eta \rho^{-N}$, with equality when $|e| = \eta$. The desired bound is met exactly when

$$N \le \frac{\log(\varepsilon/\eta)}{\log(1/\rho)}, \qquad \boxed{N_{\max} = \left\lfloor \frac{\log(\varepsilon/\eta)}{\log(1/\rho)} \right\rfloor.}$$

If this integer is zero, no positive mode qualifies. The given numbers give $N_{\max} = 6$: mode six has error bound 0.0064, whereas mode seven has 0.0128. This is a per-coefficient criterion, not a bound on a sum of many mode errors.

Step 4: Separate forward and inverse stability. For continuous boundary perturbations, the maximum principle gives $\|u_1 - u_2\|_{\infty,\text{disk}} \le \|g_1 - g_2\|_{\infty,\partial\text{disk}}$. In contrast, take $g_n(\theta) = \cos(n\theta)$ and $u_n = r^n \cos(n\theta)$. Then $\|g_n\|_\infty = 1$ but $\|u_n(\rho,\cdot)\|_\infty = \rho^n \to 0$. Thus no fixed bound can control boundary sup-norm errors by measured-circle sup-norm errors across all modes, although each exact finite-mode inversion is unique.

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Question 10

Let u be harmonic on the unit square Ω, continuous on its closure, with prescribed boundary values g. A proposed approximation v is twice continuously differentiable on the closed square and satisfies

$$|v - g| \le \varepsilon \quad \text{on } \partial\Omega, \qquad |\Delta v| \le \delta \quad \text{in } \Omega, \qquad \varepsilon, \delta \ge 0.$$

You may use the weak maximum principle for subharmonic functions: if $\Delta w \ge 0$, then $\max_{\overline{\Omega}} w \le \max_{\partial\Omega} w$.

Tasks

1. Construct a quadratic nonnegative barrier ψ with $-\Delta\psi = 1$ and maximum $1/8$ on the square.
2. Prove $|u - v| \le \varepsilon + \delta\psi \le \varepsilon + \delta/8$. Check the signs of both comparison functions.
3. Test the estimate for $g = x + y$ and $v = x + y + a\,x(1-x)y(1-y)$, where $a \ne 0$. Determine the exact boundary error, residual bound and actual maximum error, and compare with the certificate.
4. Explain why checking only the PDE residual cannot certify the answer. Show that the coefficient of ε in the estimate is sharp, and state whether this example proves sharpness of $1/8$.

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Question 10 – Solution

Strategy. Convert a bounded residual into an explicit comparison barrier, while retaining a separate bound for incorrect boundary values.

Step 1: Construct the barrier. Set

$$\boxed{\psi(x,y) = \frac{x(1-x) + y(1-y)}{4}.}$$

It is nonnegative on the closed square and satisfies $\Delta\psi = -1$. Each quadratic is at most $1/4$, so $\max \psi = 1/8$ at $(1/2, 1/2)$. The barrier need not vanish on every edge; its nonnegative boundary values are sufficient for the comparison.

Step 2: Compare the error in both directions. Put $e = u - v$, so $\Delta e = -\Delta v$. For $w_+ = e - \varepsilon - \delta\psi$ and $w_- = -e - \varepsilon - \delta\psi$,

$$\Delta w_+ = -\Delta v + \delta \geq 0, \qquad \Delta w_- = \Delta v + \delta \geq 0.$$

On the boundary, $|e| = |g - v| \leq \varepsilon$ and $\psi \geq 0$, so both $w_\pm \leq 0$. The stated maximum principle yields

$$\boxed{|u - v| \leq \varepsilon + \delta\psi(x,y) \leq \varepsilon + \delta/8.}$$

This is an analytic bound over the entire square, not an inference from sampled residuals.

Step 3: Compare the certificate with the exact error. The exact harmonic solution for $g = x + y$ is $u = x + y$. The added factor vanishes on every edge, so $\varepsilon = 0$. Direct differentiation gives

$$\Delta v = -2a[x(1-x) + y(1-y)], \qquad \|\Delta v\|_\infty = |a|.$$

Thus the certificate gives $\|u - v\|_\infty \leq |a|/8$. The exact error is $|a|x(1-x)y(1-y)$, with maximum $|a|/16$ at the center. The certificate is valid and overestimates this example's maximum by a factor of two.

Step 4: Identify what must be checked and what is sharp. The field $v = u + M$ has zero residual for every constant M, yet its error is $|M|$ everywhere. Without a boundary-error check, an arbitrarily inaccurate temperature can pass the PDE test. Taking $|M| = \varepsilon$ and $\delta = 0$ attains the bound $\|u - v\|_\infty = \varepsilon$, proving that coefficient one is sharp. The polynomial example does not attain the residual coefficient $1/8$ and does not prove it sharp. The barrier supplies a rigorous convenient certificate without a claim of optimality.

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Question 1

A string on $0 < x < 1$ has unit tension and unit mass per length, fixed endpoints, and initial data

$$u_{tt} = u_{xx}, \qquad u(0,t) = u(1,t) = 0, \qquad u(x,0) = \sin(\pi x) + \tfrac{1}{2}\sin(2\pi x), \quad u_t(x,0) = 2\pi \sin(3\pi x).$$

Its energy is $E(t) = \frac{1}{2}\int_0^1 (u_t^2 + u_x^2)\,dx$.

Tasks

1. Construct the complete modal solution, including the correct coefficient of the velocity-generated mode.
2. Verify the PDE, both endpoint conditions and both initial data. Determine whether the entire string can ever have zero displacement.
3. Compute the energy carried by each active mode and the total energy. Explain why cross terms do not contribute.
4. Find the smallest positive period of the full state (u, u_t). Prove minimality rather than merely giving a return time.

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Question 1 – Solution

Strategy. Initial displacement and initial velocity set different temporal coefficients; spatial orthogonality separates both energy and state recurrence.

Step 1: Normalize the three temporal modes. Fixed ends select $\sin(n\pi x)$ with frequencies $\omega_n = n\pi$. A modal initial velocity g_n gives coefficient g_n/ω_n multiplying $\sin(\omega_n t)$. Therefore

$$\boxed{u = \sin(\pi x)\cos(\pi t) + \tfrac{1}{2}\sin(2\pi x)\cos(2\pi t) + \tfrac{2}{3}\sin(3\pi x)\sin(3\pi t).}$$

The last coefficient is $2/3$, not 2π.

Step 2: Verify the state and test for a flat snapshot. Each term has equal second derivatives in x and t; each spatial sine vanishes at both endpoints. At $t = 0$ the first two terms give the stated displacement. Differentiation gives $2\pi\sin(3\pi x)$ as the initial velocity. If $u(\cdot, t)$ were identically zero, orthogonality would force both $\cos(\pi t) = 0$ and $\cos(2\pi t) = 0$. But the first condition gives $t = k + 1/2$, at which $\cos(2\pi t) = -1$. Thus the string is never entirely flat.

Step 3: Add the orthogonal modal energies. For $q_n(t)\sin(n\pi x)$, the energy is $E_n = \frac{1}{4}[q_n'^2 + (n\pi)^2 q_n^2]$. Distinct sines are orthogonal in the kinetic term, and distinct cosines in the elastic term, so no cross terms remain. Evaluating at $t = 0$ gives

$$\boxed{E_1 = \frac{\pi^2}{4}, \qquad E_2 = \frac{\pi^2}{4}, \qquad E_3 = \pi^2, \qquad E = \frac{3\pi^2}{2}.}$$

Each modal energy is constant because $q_n'' + (n\pi)^2 q_n = 0$.

Step 4: Find the fundamental state period. All three temporal modes return with their derivatives after time 2. Conversely, a return of the full state forces its nonzero first mode to satisfy $\cos(\pi P) = 1$ and $\sin(\pi P) = 0$. Thus P must be a positive even integer. The smallest is

$$\boxed{P = 2.}$$

A state recurrence requires displacement and velocity together; isolated zeros of one modal displacement do not establish a period.

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Question 2

A uniform string of length L, tension $T > 0$ and density $\mu > 0$ has fixed endpoints and wave speed $c = \sqrt{T/\mu}$. It is plucked to height $h > 0$ at $x = a$, where $0 < a < L$, then released from rest:

$$f(x) = \begin{cases} hx/a, & 0 \le x \le a, \\ h(L-x)/(L-a), & a \le x \le L, \end{cases} \qquad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Use the finite-energy solution of $u_{tt} = c^2 u_{xx}$.

Tasks

1. Derive the sine coefficients and the full solution. State in what sense the initial data and PDE are satisfied near the moving corners.
2. Set $a = L/3$. Identify precisely which modes are absent and compute the initial, hence conserved, energy.
3. Prove $u(x, L/c) = -f(L-x)$ and $u(x, 2L/c) = f(x)$. Include the velocity state at these times.
4. Let F be the odd, $2L$-periodic extension of f. Use it to obtain exact piecewise-linear profiles at $t = 0, L/(2c), L/c$ for $L = c = h = 1$, $a = 1/3$, and sketch them without truncating a Fourier series.

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Question 2 – Solution

Strategy. Fourier coefficients reveal the missing harmonics, while reflected traveling profiles preserve the corners exactly.

Step 1: Compute the pluck coefficients. Integrating separately on $[0, a]$ and $[a, L]$ gives

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2hL^2}{a(L-a)n^2\pi^2}\sin\frac{n\pi a}{L}.$$

Thus

$$\boxed{u(x,t) = \sum_{n\geq 1} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.}$$

The $O(n^{-2})$ coefficients give uniform convergence of displacement. The initial velocity is zero in L^2, and the solution has finite conserved energy. It solves the PDE weakly and classically away from the propagating slope corners; no globally C^2 solution is claimed.

Step 2: Identify the spectrum and energy. For $a = L/3$, $b_n = 9h\sin(n\pi/3)/(n^2\pi^2)$, so exactly the positive multiples of three are absent. Initially all energy is elastic:

$$E = \frac{T}{2}\int_0^L f'^2\,dx = \frac{Th^2L}{2a(L-a)} = \boxed{\frac{9Th^2}{4L} \quad (a = L/3).}$$

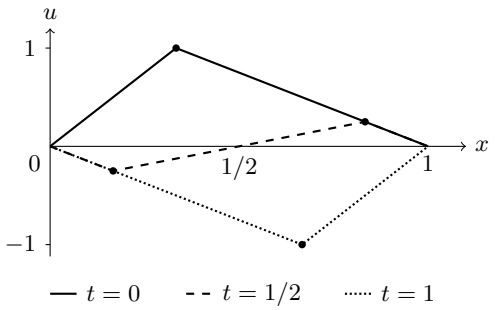
The finite-energy wave evolution conserves this value.

Step 3: Verify the reflected and full returns. At $t = L/c$, each cosine is $(-1)^n$, and $\sin(n\pi(L-x)/L) = (-1)^{n+1}\sin(n\pi x/L)$. Hence $u(x, L/c) = -f(L-x)$. At $2L/c$ every cosine equals one, so $u = f$. The modal velocities vanish at both times, giving zero velocity in L^2.

Step 4: Construct exact corner-preserving snapshots. The reflected d'Alembert form is $u(x,t) = \frac{1}{2}[F(x-ct) + F(x+ct)]$. Oddness and periodicity enforce both endpoints and recover both initial data. For the stated numbers, the half-time profile is

$$u(x, \tfrac{1}{2}) = \begin{cases} -3x/2, & 0 \leq x \leq 1/6, \\ 3x/4 - 3/8, & 1/6 \leq x \leq 5/6, \\ 3(1-x)/2, & 5/6 \leq x \leq 1. \end{cases}$$

The other two profiles are $f(x)$ and $-f(1-x)$; their corners are plotted at their exact positions.



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Question 3

A unit string has fixed endpoints and unit tension and density. It starts straight and is struck so that its initial velocity is

$$u(x,0)=0, \qquad g(x)=u_t(x,0)=\begin{cases} V, & 1/3<x<2/3, \\ 0, & 0<x<1/3 \text{ or } 2/3<x<1, \end{cases} \qquad V>0.$$

Values assigned at the two jump points do not change the finite-energy solution.

Tasks

1. Derive the velocity sine coefficients and construct the complete fixed-end solution of $u_{tt}=u_{xx}$.
2. Explain the symmetry and identify exactly which modes vanish. State the appropriate initial trace and regularity claims.
3. Find an exact formula for the midpoint displacement for $0 \le t \le 1/2$. Identify when its velocity first changes and explain why the displacement then stays positive.
4. Compute the conserved energy directly from the initial data. Derive the corresponding modal Parseval identity, including every normalization factor.

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Question 3 – Solution

Strategy. A velocity profile must be integrated in time; its jumps do not permit a globally classical interpretation of the initial state.

Step 1: Convert the strike to temporal sine amplitudes. The velocity coefficients are

$$g_n = 2V \int_{1/3}^{2/3} \sin(n\pi x)\, dx = \frac{2V}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right).$$

Because the initial displacement is zero,

$$\boxed{u(x,t) = \sum_{n\geq 1} \frac{g_n}{n\pi} \sin(n\pi t)\sin(n\pi x).}$$

The additional division by $n\pi$ is essential to recover $u_t(\cdot,0) = g$.

Step 2: Interpret the symmetry and traces. The strike satisfies $g(1-x) = g(x)$, so only the symmetric, odd-indexed sine modes can occur. The displayed coefficient vanishes for every even n and is nonzero for every odd n: the cosine difference for odd n is either 1 or -2. Displacement converges uniformly since its coefficients are $O(n^{-2})$. Velocity recovers g in L^2; its initial sine series takes the average $V/2$ at either jump. The PDE holds weakly everywhere and classically away from the reflected characteristic fronts.

Step 3: Compute the midpoint before reflected arrivals. For $0 \leq t \leq 1/2$, the interval $[1/2-t, 1/2+t]$ remains within $[0,1]$. The velocity-only d'Alembert formula therefore gives

$$u(\tfrac{1}{2},t) = \frac{1}{2}\int_{1/2-t}^{1/2+t} g(s)\, ds = \boxed{\begin{cases} Vt, & 0 \leq t \leq 1/6, \\ V/6, & 1/6 \leq t \leq 1/2. \end{cases}}$$

At $t = 1/6$, both edges of the moving integration interval pass the strike edges. The one-sided midpoint velocities change from V to zero. A zero subsequent velocity does not undo the displacement already accumulated.

Step 4: Match physical and modal energies. Initially $u_x = 0$, so

$$\boxed{E = \frac{1}{2}\int_0^1 g^2\, dx = \frac{V^2}{6}.}$$

Parseval with $\int_0^1 \sin^2(n\pi x)\, dx = 1/2$ gives $\int g^2 = \frac{1}{2}\sum g_n^2$, hence $E = \frac{1}{4}\sum g_n^2$. Equivalently,

$$\sum_{n\geq 1} \frac{\left[\cos(n\pi/3) - \cos(2n\pi/3)\right]^2}{n^2} = \frac{\pi^2}{6}.$$

Each mode trades kinetic and elastic energy while preserving its contribution $g_n^2/4$.

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Question 4

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. Its left end is fixed and its right end is free for $t > 0$:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u_x(L,t) = 0.$$

At release its displacement is $f(x) = hx/L$, $h > 0$, and its velocity is zero. Interpret this as a finite-energy problem, since the initial slope need not satisfy the newly released end condition.

Tasks

1. Derive the mixed-end eigenfunctions, frequencies and coefficients of f. Construct the solution.
2. Explain the initial corner incompatibility. Prove that the full state reverses sign after $2L/c$ and has smallest positive period $4L/c$.
3. Show that at $t = L/c$ the string is flat but is not at rest. Find its velocity in the open interval and verify the conserved energy.
4. For $L = c = h = 1$, sketch exact profiles at $t = 0, 1/2, 1$. Use the odd reflection at the fixed end and even reflection at the free end, and mark the moving corner.

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Question 4 – Solution

Strategy. A free end selects half-integer modes; the release launches a corner rather than making the initial slope compatible retroactively.

Step 1: Construct the mixed-mode solution. The conditions $X(0) = 0$, $X'(L) = 0$ give $k_n = (2n+1)\pi/(2L)$, $n \geq 0$, and $\omega_n = ck_n$. Since $\int_0^L \sin^2(k_n x)\,dx = L/2$,

$$b_n = \frac{2}{L}\int_0^L \frac{hx}{L}\sin(k_n x)\,dx = \frac{8h(-1)^n}{(2n+1)^2\pi^2}, \qquad \boxed{u = \sum_{n\geq 0} b_n \sin(k_n x)\cos(\omega_n t).}$$

Step 2: State the trace and recurrence precisely. Initially $f'(L) = h/L \neq 0$, so a solution with continuous u_x at the release corner cannot satisfy the new free-end condition there. The displayed finite-energy solution recovers f uniformly and zero initial velocity in L^2; its boundary condition has the usual weak/natural meaning at wavefront events. Every active frequency is an odd multiple of $\pi c/(2L)$. Thus both u and u_t change sign after $2L/c$ and return after $4L/c$. The nonzero lowest mode requires that full period, proving minimality.

Step 3: Resolve the flat but moving state. At $t = L/c$ every temporal cosine vanishes. For $0 < x < L$,

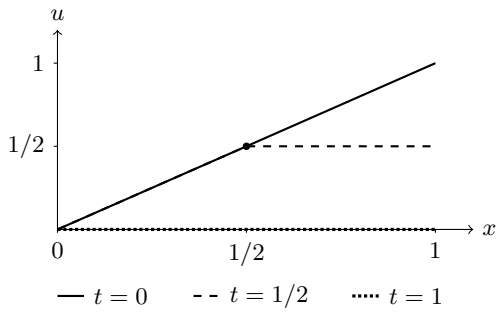
$$u_t(x, L/c) = -\frac{4hc}{L\pi}\sum_{n\geq 0}\frac{\sin((2n+1)\pi x/(2L))}{2n+1} = \boxed{-hc/L}.$$

Here $\sum_{n\geq 0}\sin((2n+1)z)/(2n+1) = \pi/4$ for $0 < z < \pi$, the sine series of a constant on that interval. The fixed-end velocity remains zero; the interior velocity trace is an L^2 statement, not corner continuity. Initially $E = Th^2/(2L)$. At this flat snapshot, the kinetic energy is $\mu L(hc/L)^2/2 = Th^2/(2L)$, so the energy is unchanged.

Step 4: Follow the released-end corner. Extend f oddly across zero and evenly across L, with period $4L$, and take the average of its two translates. For the stated units and $0 \leq t \leq 1$ this gives

$$\boxed{u(x,t) = \min(x, 1-t), \qquad 0 \leq x \leq 1.}$$

For $0 < t < 1$, the corner is at $x = 1 - t$: the left part retains slope one and the right part is flat and moving downward. The snapshots below use exact line segments.



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Question 5

A unit string with fixed ends, unit density and unit tension experiences uniform viscous drag:

$$u_{tt} + 2\beta u_t = u_{xx}, \qquad \beta = 2\pi, \qquad u(x,0) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x), \quad u_t(x,0) = 0.$$

Tasks

1. Derive the modal equations and classify each of the three active modes as over-, critically or underdamped.
2. Construct the full solution, satisfying both initial conditions for every mode.
3. Derive the energy dissipation law for $E = \frac{1}{2}\int_0^1 (u_t^2 + u_x^2)\,dx$. Explain why energy is nonincreasing even when a modal displacement oscillates.
4. Find the leading long-time displacement profile and its exact exponential decay rate. More generally, state when all positive string modes are underdamped.

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Question 5 – Solution

Strategy. The same drag rate competes with different natural frequencies, so one string can exhibit all three damping regimes at once.

Step 1: Classify the modal roots. With $u = \sum q_n(t)\sin(n\pi x)$, each coefficient satisfies $q_n'' + 2\beta q_n' + (n\pi)^2 q_n = 0$. For $\beta = 2\pi$, the discriminant sign is the sign of $4 - n^2$. Thus mode one is overdamped, mode two is critical, and mode three is underdamped.

Step 2: Impose both initial conditions. The required coefficient functions, all with $q_n(0) = 1$ and $q_n'(0) = 0$, are

$$q_1 = e^{-2\pi t}\left[\cosh(\sqrt{3}\pi t) + \frac{2}{\sqrt{3}}\sinh(\sqrt{3}\pi t)\right],$$
$$q_2 = e^{-2\pi t}(1 + 2\pi t),$$
$$q_3 = e^{-2\pi t}\left[\cos(\sqrt{5}\pi t) + \frac{2}{\sqrt{5}}\sin(\sqrt{5}\pi t)\right].$$

Then $\boxed{u = \sum_{n=1}^{3} q_n(t)\sin(n\pi x)}$ satisfies the damped PDE and both endpoints. The added sinh, linear and sine terms are necessary to make the initial velocities zero.

Step 3: Account for dissipated energy. Differentiate E and integrate $u_x u_{xt}$ by parts:

$$E' = [u_x u_t]_0^1 + \int_0^1 u_t(u_{tt} - u_{xx})\,dx = \boxed{-2\beta\int_0^1 u_t^2\,dx \le 0}.$$

The boundary term is zero because fixed ends have zero velocity. Oscillation changes how energy is partitioned; drag still removes energy whenever the velocity is not identically zero.

Step 4: Identify the slow surviving mode. Expanding q_1 into two exponentials, its slower coefficient is $C = (1 + 2/\sqrt{3})/2$ and its slower decay rate is $\gamma = (2 - \sqrt{3})\pi$. The other two modes decay as $e^{-2\pi t}$ times bounded or linear factors. Therefore

$$\boxed{e^{\gamma t}u(x,t) \longrightarrow C\sin(\pi x) \quad \text{uniformly for } 0 \le x \le 1.}$$

For a nonnegative general drag parameter, mode n is underdamped when $\beta < n\pi$; all positive modes are underdamped exactly when $0 \le \beta < \pi$. The threshold compares drag with the lowest frequency, not with the highest one.

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Question 6

A unit string with unit tension and density has fixed ends and is initially at rest. A spatially sinusoidal force is held constant for a chosen duration $\tau > 0$:

$$u_{tt} - u_{xx} = F_0 \sin(\pi x)\, \chi_{(0,\tau)}(t), \qquad u(x,0) = u_t(x,0) = 0, \qquad F_0 > 0.$$

The indicator is one during the pulse and zero otherwise; isolated switch-time values do not affect the motion.

Tasks

1. Derive the modal displacement during and after the pulse, imposing continuity of displacement and velocity at switch-off.
2. Determine every positive duration that leaves the string exactly at rest after switch-off. Prove that both state variables vanish for those durations.
3. Compute the residual energy as a function of τ and verify it equals the net work of the applied force.
4. Find the durations that maximize residual energy. Plot its normalized dependence on pulse duration and explain how a force of one sign can do zero net work over a nontrivial pulse.

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Question 6 – Solution

Strategy. Switch-off is a delayed negative step in forcing; controlling pulse duration can cancel the entire excited state.

Step 1: Superpose the two forcing steps. Only the first spatial mode is driven. Writing $u = q(t)\sin(\pi x)$ gives $q'' + \pi^2 q = F_0\chi_{(0,\tau)}$ with zero initial data. Hence

$$q(t) = \frac{F_0}{\pi^2}\begin{cases} 1-\cos(\pi t), & 0 \le t \le \tau, \\ \cos(\pi(t-\tau)) - \cos(\pi t), & t \ge \tau. \end{cases}$$

The two expressions and their first derivatives agree at $t = \tau$; only acceleration can jump. They verify the forcing on each open time interval.

Step 2: Find exact cancellation durations. For $t \ge \tau$, the cosine difference becomes

$$\boxed{q(t) = \frac{2F_0}{\pi^2}\sin\frac{\pi\tau}{2}\sin\big(\pi(t-\tau/2)\big).}$$

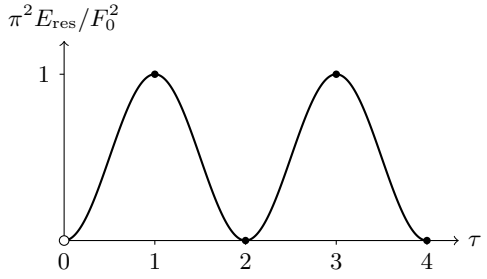
Its displacement and velocity vanish identically precisely when $\sin(\pi\tau/2) = 0$, giving $\boxed{\tau = 2m,\ m = 1, 2, \dots}$. Equivalently, the switch-off state $q(\tau) = F_0(1-\cos\pi\tau)/\pi^2$, $q'(\tau) = F_0\sin\pi\tau/\pi$ is zero exactly for those durations.

Step 3: Evaluate residual energy and work. After switch-off,

$$\boxed{E_{\text{res}} = \frac{1}{4}(q'^2 + \pi^2 q^2) = \frac{F_0^2}{\pi^2}\sin^2\frac{\pi\tau}{2}.}$$

During forcing, the power is $\int_0^1 F_0\sin(\pi x)u_t\,dx = F_0 q'/2$. Its integral from zero to τ is $F_0 q(\tau)/2 = F_0^2(1-\cos\pi\tau)/(2\pi^2)$, the same energy.

Step 4: Interpret the pulse-duration curve. The maximum is F_0^2/π^2, attained for $\tau = 2m+1$, $m \ge 0$. For an even-integer duration, positive work during motion in the force direction is canceled by negative work while the string moves against the force. A positive force need not supply positive power at every instant.



Open dot: the excluded zero-duration limit.

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Question 7

A unit fixed-end string with speed one is known to involve only modes $1, \dots, N$:

$$u(x,t) = \sum_{n=1}^{N} \left[f_n \cos(n\pi t) + \frac{g_n}{n\pi} \sin(n\pi t) \right] \sin(n\pi x).$$

A displacement sensor at $x = s \in (0,1)$ records $h_s(t) = u(s,t)$ for $0 \le t \le 2$.

Tasks

1. Derive recovery formulas for f_n, g_n from the trace whenever the nth mode is visible.
2. For a rational sensor position $s = p/q$ in lowest terms, identify exactly the invisible modes. Explain why continuous-time sampling cannot recover those modes.
3. Use sensors at $s = 1/2$ and $s = 1/3$. Determine their common invisible modes and the largest N for which the entire stated finite-mode state is uniquely recoverable.
4. Prove that an irrational sensor sees every individual mode but can be arbitrarily poorly conditioned as N grows. You may use the pigeonhole fact that for every integer $Q \ge 1$ there are integers $1 \le n \le Q, k$ with $|ns - k| \le 1/Q$.

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Question 7 – Solution

Strategy. Time orthogonality separates frequencies, but it cannot restore a spatial mode whose node lies at the sensor.

Step 1: Project the temporal record. On $[0, 2]$, distinct integer-frequency sines and cosines are orthogonal and each squared norm is one. Consequently

$$\boxed{f_n = \frac{\int_0^2 h_s(t)\cos(n\pi t)\,dt}{\sin(n\pi s)}, \qquad g_n = \frac{n\pi\int_0^2 h_s(t)\sin(n\pi t)\,dt}{\sin(n\pi s)}}$$

whenever $\sin(n\pi s) \neq 0$. These recover both initial displacement and velocity coefficients.

Step 2: Identify rational-position blindness. For $s = p/q$ in lowest terms, $\sin(n\pi s) = 0$ exactly when q divides n. Every such mode contributes zero to $h_s(t)$ for every time and for every choice of its two coefficients. Thus even a perfect continuous trace cannot distinguish states that differ only in those modes.

Step 3: Combine the two sensors. The midpoint misses the even modes; the one-third sensor misses multiples of three. Their common invisible modes are exactly the multiples of six. For $N \leq 5$, each mode has a nonzero factor at at least one sensor, so the formulas recover the entire state. For $N \geq 6$, arbitrary sixth-mode displacement or velocity can be added without changing either record. The largest fully identifiable cutoff is therefore

$$\boxed{N = 5.}$$

Step 4: Distinguish visibility from stability. If s is irrational, no ns is an integer, so each denominator is nonzero. But the supplied approximation gives $|\sin(n\pi s)| = |\sin(\pi(ns - k))| \leq \pi/Q$. As $Q \to \infty$, the selected indices cannot stay in a finite set: each such index has a fixed positive distance from the integers. Hence arbitrarily large modes have arbitrarily small sensor factors. An error e_n in the cosine projection becomes $e_n/\sin(n\pi s)$ in f_n; the sine projection has the additional factor $n\pi$ for g_n. Exact visibility at all modes does not provide a uniform noise bound.

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Question 8

A uniform string of length L, tension $T > 0$ and density $\mu > 0$ is fixed at both ends and carries a point mass $M > 0$ at its midpoint. Away from that mass, $u_{tt} = c^2 u_{xx}$ with $c = \sqrt{T/\mu}$. Displacement is continuous at $x = L/2$, and the mass obeys

$$M u_{tt}(L/2, t) = T\big[u_x(L/2+, t) - u_x(L/2-, t)\big].$$

Tasks

1. For a normal mode $u = \phi(x)\cos(\omega t)$, derive the interface condition and classify antisymmetric modes about the midpoint.
2. Derive the frequency equation for symmetric modes using $z = \omega L/(2c)$ and $\alpha = M/(\mu L)$. Prove there is exactly one positive symmetric root in each interval $(j\pi, j\pi + \pi/2)$, $j = 0, 1, \ldots$.
3. Find the limiting lowest symmetric frequency as $M \downarrow 0$ and its leading behavior as $M \to \infty$. Interpret the latter as an effective spring-mass system.
4. Write the total conserved energy, including the attached mass, and prove conservation by checking the interface terms.

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Question 8 – Solution

Strategy. The midpoint mass changes the dynamic matching condition, but modes with a node there do not move the added mass.

Step 1: Separate and use symmetry. On each half, $\phi'' + k^2\phi = 0$, $k = \omega/c$, and the mass equation becomes

$$T(\phi'_+ - \phi'_-) + M\omega^2\phi(L/2) = 0.$$

Antisymmetry gives $\phi(L/2) = 0$ and matching slopes. The fixed endpoints then yield

$$\boxed{\phi(x) = A\sin(2m\pi x/L), \qquad \omega = 2m\pi c/L, \quad m = 1, 2, \ldots.}$$

These frequencies are independent of M.

Step 2: Derive and locate the symmetric frequencies. A symmetric shape has $\phi = A\sin(kx)$ on the left and $\phi = A\sin(k(L-x))$ on the right. Its slope jump is $-2Ak\cos z$ and midpoint value is $A\sin z$. Substitution gives

$$2Tk\cos z = Mc^2k^2\sin z, \qquad \boxed{\cot z = \alpha z, \quad \alpha = \frac{M}{\mu L}.}$$

On $(j\pi, j\pi + \pi/2)$, $\cot z - \alpha z$ decreases strictly from positive infinity to a negative value. It has exactly one root. On the intervening intervals the cotangent is negative, so no positive root occurs there. Values $\sin z = 0$ do not solve the original matching equation. There is no nonzero static mode with both endpoints fixed.

Step 3: Interpret the lowest root. For the root $z_0 \in (0, \pi/2)$, $\alpha \downarrow 0$ gives $z_0 \to \pi/2$, recovering $\omega_0 \to \pi c/L$. As $\alpha \to \infty$, $z_0 \to 0$ and $z\cot z \to 1$ gives $\alpha z_0^2 \to 1$. Thus

$$\boxed{\omega_0 \sim 2\sqrt{\frac{T}{ML}} \quad (M \to \infty).}$$

A small midpoint deflection d gives each half-string slope magnitude $2d/L$; their restoring forces add to $4Td/L$. The effective stiffness is $4T/L$, consistent with the displayed mass-spring frequency.

Step 4: Include the concentrated kinetic energy. For piecewise smooth motion, the energy is

$$E = \frac{1}{2}\int_0^L (\mu u_t^2 + Tu_x^2)\,dx + \frac{M}{2}u_t(L/2, t)^2.$$

Integration by parts on each half leaves $Tu_t(L/2)(u_x^- - u_x^+)$; the fixed-end contributions vanish. The point-mass term differentiates to $Mu_t(L/2)u_{tt}(L/2) = Tu_t(L/2)(u_x^+ - u_x^-)$. They cancel, so $\boxed{E' = 0}$. Omitting the mass term would falsely suggest loss or gain of energy at the interface.

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Question 9

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. It is initially straight and at rest in the open interval. For $t > 0$, its left end stays fixed while its right end moves upward at constant speed $V > 0$:

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = 0, \quad u(L,t) = Vt.$$

Only times $0 \le t \le 2L/c$ are considered. Define $r_+ = \max(r, 0)$.

Tasks

1. Construct the incident wave before it reaches the left end. Locate its front and verify the boundary and initial data in their appropriate senses.
2. Add the fixed-end reflected wave and obtain one formula valid through $t = 2L/c$. Verify both endpoint displacements and explain the time restriction.
3. Compute the total string energy on each side of the first reflection time and verify the boundary power balance.
4. Explain the release-corner and wavefront regularity. For $L = c = V = 1$, sketch exact profiles at $t = 1/2, 1, 3/2, 2$ and describe the interior state at the last time.

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Question 9 – Solution

Strategy. A prescribed end motion launches a ramp front; reflection reverses its displacement sign and transfers the injected energy into a growing slope.

Step 1: Launch the incoming ramp. For $0 \le t \le L/c$,

$$u_{\text{in}}(x,t) = V\big(t-(L-x)/c\big)_+.$$

It vanishes ahead of the front $x = L - ct$ and equals Vt at $x = L$. Every function of $t + x/c$ satisfies the wave equation distributionally; here it is classical away from the front. At $t = 0$, displacement is zero and velocity is zero at every interior point.

Step 2: Reflect at the fixed left end. The reflected ramp must cancel the incoming displacement at $x = 0$:

$$\boxed{u = V\big(t-(L-x)/c\big)_+ - V\big(t-(L+x)/c\big)_+.}$$

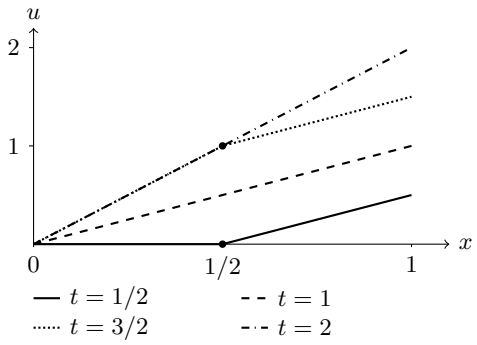
The left trace is identically zero. At $x = L$, the second term is zero through $t = 2L/c$, giving $u(L,t) = Vt$. After that time it would change the right trace, so additional reflections would be needed. The formula is not asserted for later times.

Step 3: Balance energy and injected power. Before the first reflection, the moving portion has length ct, with $u_t = V$ and $u_x = V/c$. Since $T/c^2 = \mu$, its energy is $E = \mu c V^2 t$. For $L/c < t < 2L/c$, the reflected left portion has length $ct - L$, velocity zero and slope $2V/c$; the right portion has length $2L - ct$, velocity V and slope V/c. Hence again

$$E = 2\mu V^2(ct - L) + \mu V^2(2L - ct) = \boxed{\mu c V^2 t}.$$

For $0 < t < 2L/c$, the right-end slope is V/c, so the input power is $Tu_x(L,t)u_t(L,t) = TV^2/c = \mu c V^2 = E'$.

Step 4: Interpret the corner and the final snapshot. The initial interior velocity is zero, but the driven endpoint velocity jumps to V: no continuous velocity at that release corner is possible. Displacement stays continuous across each front, while first derivatives jump; the weak wave equation remains valid. At $t = 2L/c$, $u = 2Vx/c$ and the open-interval velocity is zero; energy is entirely elastic, $2\mu LV^2$. The endpoint still moves, so this is not a smooth state of rest that persists afterward.



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Question 10

A unit fixed-end string with unit tension and density is released from rest with $u(x,0) = x(1-x)$. Let u_N retain only sine modes with index $n \le N$, where $N \ge 1$ is an integer, from the exact finite-energy solution. Define the error energy

$$E_{\mathrm{err}}(t) = \frac{1}{2}\int_0^1 \left[(u_t - u_{N,t})^2 + (u_x - u_{N,x})^2\right] dx.$$

Tasks

1. Derive the Fourier solution and its total energy. State the regularity caveat at the initial boundary corners.
2. Derive an exact series for E_{err} and prove the all-time estimate $E_{\mathrm{err}} \le 16/(3\pi^4 N^3)$.
3. Prove the all-time displacement estimate $\|u - u_N\|_\infty \le 4/(\pi^3 N^2)$. Find a sufficient integer N for error at most 10^{-3} everywhere.
4. Find a sufficient integer N for relative energy error at most 10^{-4}. Explain why neither estimate depends on elapsed time and why wave evolution supplies no heat-like smoothing factor.

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Question 10 – Solution

Strategy. Orthogonality gives a conserved tail energy, while absolute Fourier tails give a different uniform displacement certificate.

Step 1: Construct the solution and normalize energy. Two integrations by parts give

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx = \frac{4(1-(-1)^n)}{(n\pi)^3}.$$

Thus

$$\boxed{u(x,t) = \frac{8}{\pi^3}\sum_{\substack{n\geq 1\\ n\ \text{odd}}} \frac{\sin(n\pi x)\cos(n\pi t)}{n^3}, \qquad E = \frac{1}{2}\int_0^1 (1-2x)^2\,dx = \frac{1}{6}.}$$

Displacement and first-derivative series converge uniformly, and the PDE has a finite-energy interpretation. A globally C^2 corner solution is impossible: fixed endpoints would have $u_{tt} = 0$ there, while the initial second spatial derivative is -2.

Step 2: Bound the conserved energy tail. Each omitted mode contributes $(n\pi)^2 b_n^2/4$, independent of time. Orthogonality therefore gives

$$E_{\text{err}} = \frac{16}{\pi^4}\sum_{\substack{n>N\\ n\ \text{odd}}} \frac{1}{n^4} \leq \frac{16}{\pi^4}\sum_{n>N}\frac{1}{n^4} \leq \boxed{\frac{16}{3\pi^4 N^3}}.$$

The last inequality is the decreasing-function integral bound $\sum_{n>N} n^{-4} \leq \int_N^\infty s^{-4}\,ds$.

Step 3: Certify displacement independently. Using $|\sin|, |\cos| \leq 1$ and the same tail comparison,

$$\boxed{\|u - u_N\|_\infty \leq \frac{8}{\pi^3}\sum_{n>N}\frac{1}{n^3} \leq \frac{4}{\pi^3 N^2}.}$$

This holds simultaneously for every x and every t. For tolerance $\delta = 10^{-3}$, it suffices to take $N \geq \sqrt{4/(\pi^3\delta)} \approx 11.36$, so $\boxed{N = 12}$ suffices. The bound includes even indices for convenience; it is sufficient, not a claim of the smallest possible retained set.

Step 4: Choose an energy-based cutoff. Since $E = 1/6$, $E_{\text{err}}/E \leq 32/(\pi^4 N^3)$. For tolerance $\eta = 10^{-4}$, it suffices that

$$N \geq \left(\frac{32}{\pi^4\eta}\right)^{1/3} \approx 14.87, \qquad \boxed{N = 15}.$$

Each wave mode preserves its energy and merely oscillates. There is no factor decaying with $n^2 t$ as in diffusion, so waiting does not reduce the truncation's energy error. These two cutoffs measure different errors and need not agree.

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Question 1

Three problems use the interval $0 < x < \pi$ with zero values at $x = 0, \pi$:

$$H_t = \tfrac{1}{2}H_{xx}, \quad H(x,0) = \sin(2x); \qquad W_{tt} = 9W_{xx}, \quad W(x,0) = \sin(2x), \quad W_t(x,0) = 3\sin(2x).$$

The third is $P_{xx} + P_{yy} = 0$ on $0 < x < \pi$, $0 < y < 1$, with $P(x,0) = 0$ and $P(x,1) = \sin(2x)$, as well as the zero side values.

Tasks

1. Derive the common spatial eigenproblem using $-X'' = \lambda X$. Explain why neither $\lambda = 0$ nor $\lambda < 0$ gives a nonzero Dirichlet eigenfunction.
2. Derive the remaining ordinary differential equation for each PDE and solve all three specified problems.
3. Verify every original datum, including the wave velocity. Explain why the three remaining factors require different kinds or numbers of conditions.
4. A proposed rule says a positive spatial eigenvalue always produces an exponential decay factor. Diagnose the rule and state the correct behavior for these three problems.

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Question 1 – Solution

Strategy. Keep one spatial sign convention, then let the original PDE determine whether the remaining equation is first-order decay, oscillation or a spatial boundary problem.

Step 1: Fix the spatial spectrum. The conditions are $X(0) = X(\pi) = 0$. At $\lambda = 0$, the affine solution must vanish. For $\lambda = -\sigma^2 < 0$, $X = A\sinh(\sigma x)$ after the left condition, and the right condition forces $A = 0$. For $\lambda > 0$, the nonzero possibilities are

$$\boxed{\lambda_n = n^2, \quad X_n(x) = \sin(nx), \quad n = 1, 2, \dots.}$$

All supplied data occupy the $n = 2$ mode.

Step 2: Derive the three remaining equations. Substitution gives $a' = -\lambda a/2$ for heat, $b'' = -9\lambda b$ for the wave, and $d'' = \lambda d$ for Laplace's equation. At $\lambda = 4$, the specified solutions are

$$\boxed{H = e^{-2t}\sin(2x), \quad W = \big(\cos(6t) + \tfrac{1}{2}\sin(6t)\big)\sin(2x), \quad P = \frac{\sinh(2y)}{\sinh 2}\sin(2x).}$$

Step 3: Check the original conditions. Each spatial sine vanishes at both ends. At $t = 0$, the two time-dependent fields have displacement $\sin(2x)$, while $W_t(x, 0) = 6(\frac{1}{2})\sin(2x) = 3\sin(2x)$. The harmonic field is zero at $y = 0$ and equals $\sin(2x)$ at $y = 1$. Direct second derivatives verify all three PDEs. The heat factor needs one initial value; the wave factor needs position and velocity; the harmonic factor uses two values on different spatial edges.

Step 4: Correct the proposed rule. A positive eigenvalue produces heat decay $e^{-\lambda t/2}$, wave frequency $3\sqrt{\lambda}$, and hyperbolic spatial factors for the harmonic extension. The last factor increases with y because it connects a zero bottom trace to a nonzero top trace; y is not time. The sign of the spatial eigenvalue alone does not identify the remaining factor without the PDE and its data.

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Question 2

Impose insulated side conditions: $H_x = W_x = P_x = 0$ at $x = 0, \pi$. Consider

$$H_t = H_{xx}, \quad H(x,0) = 2 + \cos x; \qquad W_{tt} = W_{xx}, \quad W(x,0) = 2 + \cos x, \quad W_t(x,0) = 3 + 2\cos x.$$

On $0 < x < \pi$, $0 < y < 1$, let P be harmonic with $P(x,0) = 0$ and $P(x,1) = 2 + \cos x$.

Tasks

1. Derive the Neumann eigenfunctions, including the zero eigenvalue. State the norm of the constant eigenfunction and of each positive cosine mode.
2. Construct all three solutions, treating their zero modes explicitly.
3. Compute the horizontal mean in each problem and explain exactly which data would be lost if the zero mode were omitted.
4. Compute the wave energy $\frac{1}{2}\int_0^\pi (W_t^2 + W_x^2)\,dx$. Explain why a constant energy does not imply bounded wave displacement in this setting.

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Question 2 – Solution

Strategy. The spatial constant is an actual eigenfunction for insulated sides; its remaining equation changes with the type of PDE.

Step 1: Retain the zero eigenvalue. The problem $-X'' = \lambda X$, $X'(0) = X'(\pi) = 0$ has $X_0 = 1$, $\lambda_0 = 0$, and $X_n = \cos(nx)$, $\lambda_n = n^2$ for $n \geq 1$. Their squared $L^2(0, \pi)$ norms are π and $\pi/2$, respectively. The mean coefficient therefore uses $1/\pi$, not the positive-mode factor $2/\pi$.

Step 2: Solve the zero and first-mode equations. For $\lambda = 0$, the remaining equations are $a' = 0$, $b'' = 0$ and $d'' = 0$. Using all the supplied data gives

$$\boxed{\begin{aligned} H(x,t) &= 2 + e^{-t}\cos x, \\ W(x,t) &= 2 + 3t + (\cos t + 2\sin t)\cos x, \\ P(x,y) &= 2y + \frac{\sinh y}{\sinh 1}\cos x. \end{aligned}}$$

The cosine derivatives vanish at the side boundaries. The scalar factors verify the PDEs, both wave initial data and the two harmonic edge traces.

Step 3: Compare the three means. Since $\int_0^\pi \cos x\, dx = 0$, their horizontal means are

$$\boxed{\overline{H} = 2, \qquad \overline{W} = 2 + 3t, \qquad \overline{P} = 2y.}$$

Omitting the constant spatial eigenfunction would lose the heat initial mean, both the wave initial mean and mean velocity, and the harmonic top mean. The zero mode preserves a heat mean, permits a freely drifting wave mean, and linearly joins two spatial mean values in the harmonic problem.

Step 4: Separate drift energy from modal energy. Put $q = \cos t + 2\sin t$. Orthogonality gives

$$E = \frac{1}{2}\left[9\pi + \frac{\pi}{2}(q'^2 + q^2)\right] = \frac{9\pi}{2} + \frac{5\pi}{4} = \boxed{\frac{23\pi}{4}}.$$

The spatially constant displacement $2 + 3t$ contributes no elastic energy, although its velocity contributes a fixed kinetic energy. Thus W becomes unbounded through its mean while its energy remains constant. A displacement bound from this energy alone would require control of that mean.

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Question 3

Consider the separated spatial problem

$$-X'' = \lambda X, \qquad 0 < x < L, \quad X(0) = 0, \quad X'(L) + hX(L) = 0,$$

where $L > 0$ and $h \in \mathbb{R}$. The same boundary conditions are imposed on the heat equation $u_t = \kappa u_{xx}$ and the wave equation $v_{tt} = c^2 v_{xx}$, with $\kappa, c > 0$.

Tasks

1. Derive the energy identity for λ. Use $X(L)^2 \le L \int_0^L X'^2\,dx$ to identify when every eigenvalue is positive and when a zero eigenvalue occurs.
2. For $h < -1/L$, derive the negative-eigenvalue equation and prove there is exactly one negative eigenvalue.
3. Choose $L = 1$ and $h = -\coth 1$. Construct a heat solution and a wave solution with initial profile $\sinh x$, taking zero initial wave velocity.
4. Explain why these solutions invalidate a rule that separated heat modes always decay and separated wave modes always oscillate. Identify the boundary term responsible.

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Question 3 – Solution

Strategy. The admissible spatial spectrum depends on the boundary operator, including the sign of a Robin coefficient.

Step 1: Include the Robin boundary term. Multiplication by X and integration by parts give

$$\boxed{\lambda \int_0^L X^2\,dx = \int_0^L X'^2\,dx + hX(L)^2.}$$

For $h \geq 0$ the right side is positive for nonzero X with $X(0) = 0$. For $-1/L < h < 0$, the supplied trace inequality bounds it below by $(1 + hL)\int X'^2 > 0$. At $h = -1/L$ it is nonnegative, with equality precisely for an affine function through zero; $X = x$ is the zero mode. Conversely, solving $X'' = 0$ shows a nonzero zero mode occurs only at $h = -1/L$.

Step 2: Find the negative mode. For $\lambda = -\sigma^2$, $\sigma > 0$, the left condition gives $X = A\sinh(\sigma x)$. The right condition is

$$\boxed{\sigma \coth(\sigma L) = -h.}$$

The left side increases strictly from $1/L$ to infinity: with $z = \sigma L$, the derivative of $z\coth z$ is $(\sinh z\cosh z - z)/\sinh^2 z > 0$. Thus there is exactly one positive σ when $-h > 1/L$, and none otherwise. This counts all negative eigenvalues because every such eigenfunction must have the displayed hyperbolic form.

Step 3: Build the two growing examples. For the specified L, h, $\sigma = 1$ and $\lambda = -1$. Hence

$$\boxed{u(x,t) = e^{\kappa t}\sinh x, \qquad v(x,t) = \cosh(ct)\sinh x.}$$

Their spatial second derivatives equal the fields; their time derivatives satisfy the respective PDEs. The left traces are zero, and at $x = 1$ the Robin expression is proportional to $\cosh 1 - \coth 1 \sinh 1 = 0$. The initial profiles are $\sinh x$ and the initial wave velocity is zero.

Step 4: Locate the failure of the rule. Heat factors satisfy $a' = -\kappa\lambda a$; wave factors satisfy $b'' + c^2\lambda b = 0$. A negative eigenvalue therefore gives growth rather than decay or pure oscillation. In the spatial energy identity, $hX(L)^2$ is negative here. The usual nonnegative energy form for dissipative or fixed boundaries cannot be assumed for this Robin condition. The counterexample changes the spectrum, not the separation calculation.

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Question 4

On $0 < x < 1$, compare $H_t = H_{xx}$ and $W_{tt} = W_{xx}$ with the moving boundary values

$$H(0,t) = W(0,t) = t, \qquad H(1,t) = W(1,t) = 2t.$$

Take $H(x,0) = W(x,0) = 0$ and $W_t(x,0) = 1 + x$.

Tasks

1. For general boundary values $A(t), B(t)$, derive the transformed PDE and initial data after subtracting $\ell(x,t) = (1-x)A(t) + xB(t)$ in each problem.
2. For the specified data, decide whether the affine lifting itself solves each original PDE.
3. Construct the complete heat solution using a stationary correction to its transformed forcing and a sine expansion. Derive the required coefficients.
4. Verify the reconstructed original data and find the long-time heat offset relative to the moving line. Contrast it with the wave field and state the heat release-corner regularity caveat.

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Question 4 – Solution

Strategy. Homogenizing boundary values creates a source involving the first or second time derivative of the lifting, depending on the PDE.

Step 1: Transform the PDE and every initial datum. Since $\ell_{xx} = 0$, the zero-endpoint fields $v = H - \ell$ and $z = W - \ell$ obey

$$v_t - v_{xx} = -\ell_t, \qquad z_{tt} - z_{xx} = -\ell_{tt}.$$

Their initial displacements are the respective original profiles minus $\ell(x, 0)$; the wave initial velocity is the original velocity minus $\ell_t(x, 0)$. Transforming only the endpoints would omit essential source and initial terms.

Step 2: Test the actual moving line. Here $\ell = t(1 + x)$, so $\ell_t = 1 + x$ but $\ell_{tt} = \ell_{xx} = 0$. It fails the heat PDE but solves the wave PDE. The wave's transformed displacement and velocity are both zero, giving

$$\boxed{W(x, t) = t(1 + x).}$$

The heat correction has $v_t - v_{xx} = -(1 + x)$ and $v(x, 0) = 0$.

Step 3: Solve the forced heat correction. Seek $P'' = 1 + x$ with $P(0) = P(1) = 0$. Integration gives

$$P(x) = \frac{x^2}{2} + \frac{x^3}{6} - \frac{2x}{3} = \frac{x(x - 1)(x + 4)}{6}.$$

The sine coefficients of $1 + x$ are $a_n = 2[1 - 2(-1)^n]/(n\pi)$. Twice integrating P'' against a sine gives $p_n = -a_n/(n\pi)^2$. Thus

$$\boxed{H = t(1 + x) + P(x) + \sum_{n \geq 1} \frac{2[1 - 2(-1)^n]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x).}$$

Step 4: Verify and interpret the offset. At both ends, P and every sine vanish. At $t = 0$ the summable sine coefficients reconstruct $-P$ uniformly, giving zero initial temperature. For $t > 0$, Gaussian damping permits differentiated convergence; substitution verifies $H_t = H_{xx}$. The initial flat field has $H_{xx} = 0$ while the boundary time derivatives are one and two, so joint classical corner smoothness is not asserted. Finally $H - \ell \to P < 0$ on $0 < x < 1$, uniformly on the closed interval, whereas $W - \ell = 0$ at all times. An identical boundary lifting can have different physical consequences in the two PDEs.

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Question 5

On the square $0 < x, y < \pi$, let

$$U_t = U_{xx} + U_{yy}, \qquad U = 0 \text{ on every edge}, \qquad U(x, y, 0) = \sin x \sin(2y) + 2\sin(2x)\sin y.$$

Tasks

1. Derive the product eigenfunctions and explain why the two active products have the same decay rate. Construct U.
2. Determine whether U is separable as $T(t)F(x, y)$ and whether its nonzero spatial profile can be written as $X(x)Y(y)$. Prove both conclusions.
3. Find the interior zero-temperature curve, its limiting edge endpoints and the signs on either side. Sketch that geometry.
4. Replace the initial coefficients by unknown A, B. Explain why one fixed-point time trace cannot distinguish them, but traces at $(\pi/2, \pi/4)$ and $(\pi/4, \pi/2)$ can. Give exact recovery formulas.

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Question 5 – Solution

Strategy. A repeated eigenvalue allows a common time factor without forcing the spatial combination to be a single coordinate product.

Step 1: Identify the repeated eigenvalue. Zero edges give $\sin(mx)\sin(ny)$ with eigenvalues m^2+n^2. The pairs $(1,2)$ and $(2,1)$ both give five. Therefore

$$\boxed{U = e^{-5t}\big[\sin x \sin(2y) + 2\sin(2x)\sin y\big].}$$

Every edge and the initial field are correct, and applying the Laplacian multiplies the bracket by -5.

Step 2: Distinguish two meanings of separation. The displayed formula is exactly one product $T(t)F(x,y)$. It is not one product $T(t)X(x)Y(y)$. Indeed, the spatial slice at $y=\pi/2$ is $2\sin(2x)$, whereas at $y=\pi/4$ it is $\sin x + \sqrt{2}\sin(2x)$. These two nonzero slices are not proportional by sine orthogonality, as they would have to be for $X(x)Y(y)$.

Step 3: Determine the nodal geometry. Factor the bracket as $2\sin x \sin y\,(\cos y + 2\cos x)$. The first two factors are positive inside the square, so the interior zero set is

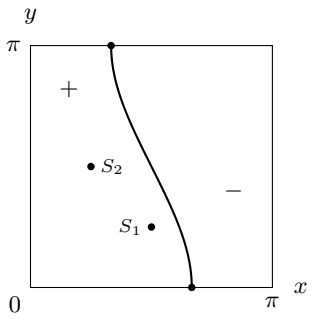
$$\boxed{\cos y = -2\cos x, \qquad \pi/3 < x < 2\pi/3.}$$

Its limiting endpoints are $(\pi/3,\pi)$ and $(2\pi/3,0)$. At fixed y, the field is positive for $x < \arccos(-\cos y/2)$ and negative to its right. The zero curve is unchanged by the positive decay factor.

Step 4: Resolve the equal-rate inverse problem. At any one sensor, the entire trace has the form $e^{-5t}(A\alpha + B\beta)$ with fixed spatial factors α,β; it supplies only one scalar equation. At the two specified sensors the unwanted product vanishes and the wanted product equals one. Hence

$$\boxed{A = e^{5t}U(\pi/2,\pi/4,t), \qquad B = e^{5t}U(\pi/4,\pi/2,t).}$$

Distinct time samples at one point cannot separate equal decay rates; spatially independent observations can.



Curve: interior $U = 0$

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Question 6

Consider diffusion with variable spatial coefficient:

$$u_t = (x^2 u_x)_x, \quad 1 < x < e, \qquad u(1,t) = u(e,t) = 0,$$

$$u(x,0) = x^{-1/2}\big[\sin(\pi \log x) + 2\sin(2\pi \log x)\big].$$

Use $s = \log x$ and $U(s,t) = u(e^s, t)$.

Tasks

1. Derive the transformed PDE and find a substitution $U = e^{-s/2}V$ that removes its first spatial derivative.
2. Derive the original spatial eigenfunctions, eigenvalues and their $L^2(1, e; dx)$ norms. Construct the complete solution.
3. Derive the coefficient formula for general initial data in these eigenfunctions. Explain why ordinary unweighted sine projection in s of $u(e^s, 0)$ is incorrect.
4. Verify the original PDE and data and derive the energy identity $\frac{1}{2}(d/dt)\int_1^e u^2\,dx = -\int_1^e x^2 u_x^2\,dx$. Check it on each separated eigenfunction.

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Question 6 – Solution

Strategy. Changing coordinates changes the differential operator and the integration measure; a further amplitude factor restores ordinary sine orthogonality.

Step 1: Transform the operator, not just the variable. The chain rule gives $u_x = U_s/x$ and $(x^2u_x)_x = U_{ss} + U_s$. With $U = e^{-s/2}V$, direct differentiation yields

$$\boxed{V_t = V_{ss} - \tfrac{1}{4}V, \qquad 0 < s < 1, \quad V(0,t) = V(1,t) = 0.}$$

Ignoring U_s would shift every eigenvalue incorrectly.

Step 2: Recover the spectrum and field. The sine modes give

$$X_n(x) = x^{-1/2}\sin(n\pi \log x), \qquad \lambda_n = (n\pi)^2 + \tfrac{1}{4}.$$

Since $dx = e^s ds$, the two $x^{-1/2}$ factors cancel the measure in a product:

$$\int_1^e X_m X_n \, dx = \int_0^1 \sin(m\pi s)\sin(n\pi s)\, ds.$$

Thus the squared norms are $1/2$, with zero cross products. The solution is

$$\boxed{u = e^{-\lambda_1 t}X_1 + 2e^{-\lambda_2 t}X_2.}$$

Step 3: Use the correct projection. For initial data f,

$$\boxed{b_n = 2\int_1^e f(x)x^{-1/2}\sin(n\pi \log x)\, dx = 2\int_0^1 e^{s/2}f(e^s)\sin(n\pi s)\, ds.}$$

It is the transformed field $V(s,0) = e^{s/2}f(e^s)$ that has ordinary sine coefficients. For example, the ungauged sines are not orthogonal in dx:

$$\int_1^e \sin(\pi \log x)\sin(2\pi \log x)\, dx = -\frac{4\pi^2(e+1)}{(1+\pi^2)(1+9\pi^2)} \neq 0.$$

Step 4: Verify diffusion and dissipation. Each X_n satisfies $-(x^2X_n')' = \lambda_n X_n$ and vanishes at both endpoints. The temporal factors therefore verify the original PDE and exactly recover the supplied initial coefficients. Integration by parts gives $\frac{1}{2}(d/dt)\int u^2 = [ux^2u_x]_1^e - \int x^2u_x^2 = -\int x^2u_x^2$. On a mode $ae^{-\lambda_n t}X_n$, both sides equal $-\lambda_n a^2 e^{-2\lambda_n t}/2$, since $\int x^2X_n'^2 = \lambda_n \int X_n^2 = \lambda_n/2$.

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Question 7

On $0 < x < \pi$, compare the heat equation $H_t = H_{xx}$ with the wave equation $W_{tt} = W_{xx}$. Both have zero endpoint displacement for positive time and the initial interior profile $f(x) = 1$; the wave has zero initial velocity. Use L^2 convergence for the initial displacement. For the wave, use the reflected bounded function obtained from the odd, 2π-periodic extension F of f, not a presumed finite-energy solution.

Tasks

1. Derive the common sine coefficients and the heat and wave series, stating the appropriate interpretation of each.
2. Use $W(x,t) = \frac{1}{2}[F(x-t) + F(x+t)]$ to determine the wave profile for $0 < t < \pi/2$, including values at its two fronts.
3. Explain the contrast with heat smoothing, and why neither field can converge uniformly to 1 over the whole open interval as $t \downarrow 0$. Sketch both profiles at $t = \pi/4$.
4. Compute the energy of the first N wave modes and show why this problem has no finite-energy wave solution with those Dirichlet initial data. Explain what remains valid in the reflected interpretation.

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Question 7 – Solution

Strategy. The same initial Fourier coefficients can lead to a smooth heat field and a discontinuous wave field; convergence alone does not guarantee finite energy.

Step 1: Compute and evolve the coefficients. The coefficients are $b_n = 2(1-(-1)^n)/(n\pi)$, so

$$H = \frac{4}{\pi}\sum_{n\ \text{odd}} \frac{e^{-n^2 t}\sin(nx)}{n}, \qquad W = \frac{4}{\pi}\sum_{n\ \text{odd}} \frac{\cos(nt)\sin(nx)}{n}.$$

Both recover f in L^2 as $t \downarrow 0$. Heat derivatives converge uniformly for every $t \geq \tau > 0$. The wave series represents the reflected function in L^2 and takes average values at jumps; its zero initial time derivative is understood in distributions.

Step 2: Track the two reflected fronts. The extension is $F(s) = \operatorname{sgn}(\sin s)$ away from multiples of π, with value zero at those multiples. For $0 < t < \pi/2$,

$$\boxed{W(x,t) = \begin{cases} 0, & 0 < x < t \text{ or } \pi - t < x < \pi, \\ 1, & t < x < \pi - t, \\ 1/2, & x = t \text{ or } x = \pi - t. \end{cases}}$$

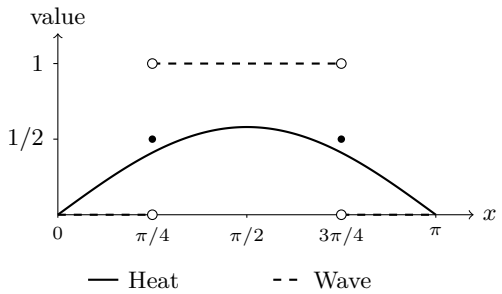
Each endpoint trace is zero. Reflection has produced two moving jumps, not a gradually smoothing profile.

Step 3: Compare smoothing and initial convergence. For $t > 0$, the heat field is smooth and $0 < H < 1$ in the interior, by diffusion and the maximum principle. The wave instead has the step profile above. For any positive time approaching zero, the wave's zero strips give sup-norm error one; the heat's continuous zero endpoint trace makes its supremum error also one over the open interval. Thus L^2 initial convergence does not imply uniform convergence across the incompatible endpoint layer.

Step 4: Test energy rather than assume it. The partial wave energy is

$$E_N = \frac{\pi}{4}\sum_{n \leq N} n^2 b_n^2 = \boxed{\frac{4}{\pi}\#\{n \leq N : n \text{ odd}\} \longrightarrow \infty.}$$

No finite-energy Dirichlet wave has this initial profile; it is not in the zero-trace energy space. The reflected bounded field still solves the wave equation distributionally, with the stated initial displacement and zero initial velocity in the corresponding weak sense. At a moving jump, its derivatives need not be ordinary square-integrable functions.



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Question 8

On $0 < x < \pi$ with zero endpoint values, consider the stationary problem

$$-u'' - 4u = f(x), \qquad f(x) = \sin x + a\sin(2x) + \sin(3x), \quad a \in \mathbb{R},$$

and the associated evolution $v_t = v_{xx} + 4v + f$, initially $v(x, 0) = 0$.

Tasks

1. Derive the necessary stationary solvability condition by testing against the kernel. Show that it is sufficient for these data.
2. Find every stationary solution when one exists, and identify the one orthogonal to the kernel.
3. Solve the evolution from rest. Distinguish growth caused by a kernel forcing from growth caused by a negative eigenvalue of the stationary operator.
4. Replace the zero initial state by $b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x)$. Determine exactly when the evolution stays bounded and give its limiting stationary field.

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Question 8 – Solution

Strategy. A separated coefficient equation can be singular or unstable; stationarity, uniqueness and attraction are separate questions.

Step 1: Test the stationary kernel. For $\mathcal{L} = -d^2/dx^2 - 4$, the eigenvalues on $\sin(nx)$ are $n^2 - 4$. The kernel is spanned by $\sin(2x)$. Integration by parts, using both zero endpoints, gives $0 = \int_0^\pi (\mathcal{L}u)\sin(2x)\,dx = \int_0^\pi f\sin(2x)\,dx = a\pi/2$. Thus $\boxed{a = 0}$ is necessary.

Step 2: Solve the compatible stationary equations. When $a = 0$, divide the first and third coefficients by $1 - 4 = -3$ and $9 - 4 = 5$:

$$\boxed{u = -\tfrac{1}{3}\sin x + \tfrac{1}{5}\sin(3x) + C\sin(2x), \qquad C \in \mathbb{R}.}$$

Direct substitution proves sufficiency. Orthogonality to the kernel forces $C = 0$, and without it the stationary field is not unique.

Step 3: Evolve from rest. The modal equations are $q_1' = 3q_1 + 1$, $q_2' = a$, $q_3' = -5q_3 + 1$. Their zero-initial-value solutions give

$$\boxed{v = \frac{e^{3t} - 1}{3}\sin x + at\sin(2x) + \frac{1 - e^{-5t}}{5}\sin(3x).}$$

Nonzero a drives linear growth in the kernel mode. Separately, the first mode grows exponentially because $\mathcal{L}$ has eigenvalue -3. Therefore even compatible forcing does not make the stationary family an attractor from rest.

Step 4: Identify the bounded initial states. For the prescribed three-mode initial state,

$$q_1 = -\tfrac{1}{3} + (b_1 + \tfrac{1}{3})e^{3t}, \qquad q_2 = b_2 + at, \qquad q_3 = \tfrac{1}{5} + (b_3 - \tfrac{1}{5})e^{-5t}.$$

Orthogonality prevents cancellation between spatial modes. Boundedness holds exactly when

$$\boxed{a = 0, \qquad b_1 = -1/3.}$$

The coefficients b_2, b_3 are unrestricted. Then the uniform limit is $-\frac{1}{3}\sin x + b_2\sin(2x) + \frac{1}{5}\sin(3x)$. Eliminating the unstable component and satisfying the kernel compatibility are both necessary.

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Question 9

Let $u_t = u_{xx}$ on $0 < x < \pi$, with zero endpoint values and $u(x, 0) = \sin x + \sin(2x)$. A proposed solution strategy assumes that the complete answer is one product $X(x)T(t)$ because the PDE is separable.

Tasks

1. Derive the exact solution and explain why separability of the PDE does not force every solution to be a single product.
2. Prove that a nonzero product solution must use a spatial eigenfunction. Explain how the argument handles zeros of X or T.
3. Test the exact solution with the two-by-two sample determinant at $x_1 = \pi/3$, $x_2 = 2\pi/3$, $t_1 = 0$, $t_2 = \log 2$. Compute it and use it to rule out one-product factorization.
4. Find the exact uniform error when $e^t u(x, t)$ is approximated by $\sin x$, and a sufficient time for that error to be at most $\varepsilon \in (0, 1)$. Sketch normalized profiles for second-mode ratios $1, 1/2, 1/10$ and zero.

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Question 9 – Solution

Strategy. An eigenfunction expansion can become nearly one mode without ever being exactly one product at a finite time.

Step 1: Superpose the two decays. The eigenvalues are one and four, so

$$\boxed{u = e^{-t}\sin x + e^{-4t}\sin(2x).}$$

The PDE and every datum follow term by term. Separation constructs building blocks; linearity allows their sums, whose distinct temporal factors need not combine into one product.

Step 2: Derive the restriction on a product. If $u = XT$ is a nonzero classical product, then $XT' = X''T$. Choose a time with $T \neq 0$ and points where $X \neq 0$. The ratios satisfy $T'/T = X''/X = -\lambda$, independent of both variables. The differential identity $X'' = -\lambda X$ extends across spatial zeros by continuity; the temporal ODE $T' = -\lambda T$ extends likewise. Zero boundary values give the Dirichlet eigenproblem. The trivial zero product is separate. Division at zeros is unnecessary and cannot create extra eigenfunctions.

Step 3: Use a factorization invariant. Every product has determinant

$$u(x_1,t_1)u(x_2,t_2) - u(x_1,t_2)u(x_2,t_1) = 0.$$

For the stated points, the four entries are

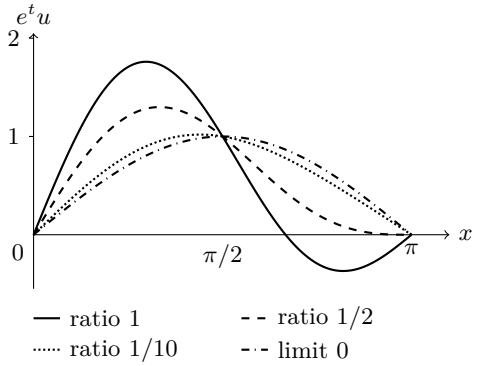
$$u(x_1,0) = \sqrt{3}, \quad u(x_2,0) = 0, \quad u(x_1,\log 2) = \frac{9\sqrt{3}}{32}, \quad u(x_2,\log 2) = \frac{7\sqrt{3}}{32}.$$

Their determinant is $\boxed{21/32 \neq 0}$, proving that the exact field is not one product.

Step 4: Quantify approximate separation. The normalized field is $e^t u = \sin x + e^{-3t}\sin(2x)$, so

$$\boxed{\|e^t u - \sin x\|_\infty = e^{-3t}, \qquad t \geq \tfrac{1}{3}\log(1/\varepsilon) \implies \|e^t u - \sin x\|_\infty \leq \varepsilon.}$$

Equality in the norm follows at a point where $|\sin(2x)| = 1$. The plotted ratios correspond to $t = 0, (\log 2)/3, (\log 10)/3$ and the limiting profile. Approximation by a surviving mode does not retroactively justify a single-product initial ansatz.



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Question 10

Let $f(x) = \sum_{n\geq 1} a_n \sin(nx)$ on $0 \leq x \leq \pi$, where $|a_n| \leq n^{-2}$. Use it as initial displacement for zero-endpoint heat and wave problems $H_t = H_{xx}$, $W_{tt} = W_{xx}$, with zero initial wave velocity. Also use it as the top trace of a harmonic rectangle of height $b > 0$, with zero side and bottom values. In each field retain modes $n \leq N$, $N \geq 1$.

Tasks

1. Write the three complete expansions and justify the original data and the relevant convergence claims.
2. Derive uniform truncation bounds for heat at $t \geq \tau > 0$, for the wave at all times, and for the harmonic field at $0 \leq y \leq b - d$, $0 < d < b$.
3. For $\tau = 0.1$, $b = 1$, $d = 0.25$ and error tolerance 10^{-2}, find the smallest integer certified by each of your displayed bounds. Plot the three bound functions and mark these sufficient cutoffs.
4. Explain why the different cutoffs reflect different mechanisms, and why waiting alone does not create an all-time smoothing estimate for the wave.

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Question 10 – Solution

Strategy. Apply the same coefficient envelope to three different modal factors, keeping the location where each error bound is valid.

Step 1: Construct the fields and their traces. The three factors give

$$H = \sum a_n e^{-n^2 t} \sin(nx), \quad W = \sum a_n \cos(nt) \sin(nx), \quad P = \sum a_n \frac{\sinh(ny)}{\sinh(nb)} \sin(nx).$$

The summable coefficient envelope ensures uniform displacement and boundary-series convergence. Heat derivatives converge away from $t = 0$, and harmonic derivatives away from the top edge. The wave has finite energy since $\sum n^2 a_n^2 < \infty$; its data and PDE have the corresponding energy/weak meaning without an automatic higher-smoothness claim.

Step 2: Bound the three tails. Using $\sum_{n>N} n^{-2} \le 1/N$ gives

$$\boxed{B_H(N) = \frac{e^{-(N+1)^2\tau}}{N}, \qquad B_W(N) = \frac{1}{N}.}$$

For $y \le b - d$, write the hyperbolic ratio as $e^{-n(b-y)}(1 - e^{-2ny})/(1 - e^{-2nb})$. It is at most $e^{-nd}/(1 - e^{-2b})$, so

$$\boxed{B_P(N) = \frac{e^{-(N+1)d}}{N(1 - e^{-2b})}.}$$

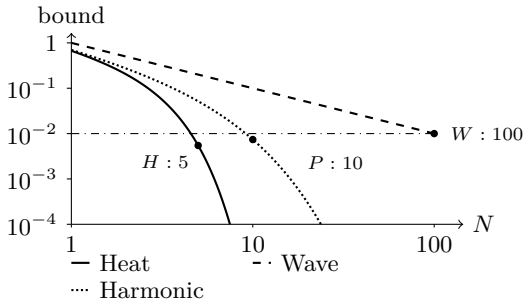
These respectively bound the heat error for $t \ge \tau$, the wave displacement error at all times, and the harmonic error in the stated lower strip.

Step 3: Choose certified integer cutoffs. Each bound decreases with N. With the specified parameters,

	preceding bound	accepted bound	N
H	$B_H(4) \approx .02052$	$B_H(5) \approx .005465$	5
W	$B_W(99) \approx .01010$	$B_W(100) = .01000$	100
P	$B_P(9) \approx .01055$	$B_P(10) \approx .007394$	10

These are the smallest integers certified by these bounds, not necessarily the fewest modes required by particular data. The graph uses continuous bound envelopes on logarithmic axes.

Step 4: Interpret the filtering mechanisms. Heat damps high modes by $e^{-n^2 t}$; distance from the driven harmonic edge gives spatial attenuation of order e^{-nd}. Wave modes retain their amplitudes and only oscillate: at every $t = 2m\pi$, the entire displacement returns to f. Consequently waiting cannot supply a uniform future decay factor for every wave tail. The heat and harmonic improvements require positive waiting time or positive distance; they cannot be silently extended to $t = 0$ or to the top trace.



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