

Question 1

A stationary rod has cross-sectional area $A(x) > 0$, conductivity $k(x) > 0$ and volumetric heat capacity $C(x) > 0$. Temperature $u(x, t)$ is uniform across each section; the lateral surface is insulated. A volumetric source $s(x, t)$ has units W/m^3 . Define signed heat power toward increasing x by $Q = -kAu_x$, in watts. All coefficients are independent of time.

Tasks

1. Apply energy conservation to an arbitrary interval $[a, b]$ and derive the local heat equation. State the units of every term in the balance.
2. Expand the spatial derivative and identify the diffusivity. Explain when the equation reduces to $u_t = \alpha u_{xx}$.
3. After nondimensionalization, take $0 < x < 1$, $A = e^x$, $k = C = 1$, $s = 0$ and an instantaneous profile $u = x$. Find Q , the storage rate Au_t and u_t . Sketch the first two quantities.
4. Explain why this linear temperature profile need not be steady, even though its curvature is zero. Distinguish heat flux per area from total heat power.

Question 1 – Solution

Strategy. Balance total heat power before dividing by the local cross-sectional area.

Step 1: Derive the conservation law. The stored thermal energy relative to a fixed reference has time derivative $\int_a^b CAu_t dx$. Inflow minus outflow plus generation gives

$$\int_a^b CAu_t dx = Q(a, t) - Q(b, t) + \int_a^b As dx.$$

Since this holds on every interval, with Fourier's law it yields

$$CAu_t = -Q_x + As = (kAu_x)_x + As.$$

Each local term has units W/m: C is J/(m³K), A is m² and u_t is K/s. Each integrated term, including $Q(a) - Q(b)$, is in watts.

Step 2: Identify the geometric contribution. Dividing by CA gives

$$u_t = \frac{k}{C}u_{xx} + \frac{(kA)'}{CA}u_x + \frac{s}{C}.$$

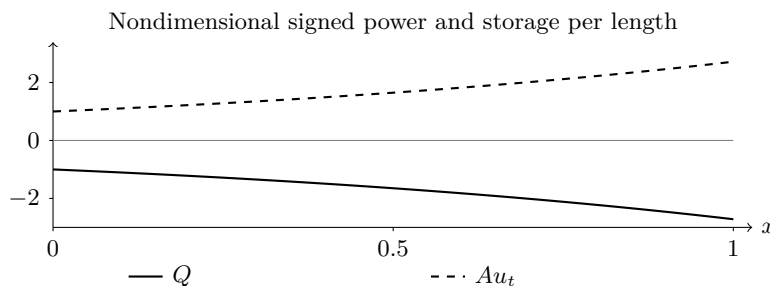
The diffusivity k/C has units m²/s. For a homogeneous constant-area rod without a source, this is $u_t = \alpha u_{xx}$ with constant $\alpha = k/C$. More generally, the first-derivative term vanishes when kA is constant; constant diffusivity additionally requires k/C constant.

Step 3: Evaluate the nondimensional example. Here $u_x = 1$ and $u_{xx} = 0$, but

$$Q(x) = -e^x, \quad Au_t = -Q_x = e^x, \quad u_t = 1.$$

The signed power is negative because heat moves toward decreasing temperature, which here is the negative x direction. Its magnitude increases toward the larger cross section.

Step 4: Interpret the nonzero storage. Flux per unit area is $q = -ku_x = -1$, constant in this example. Total power is $Q = Aq = -e^x$, which is not constant. More heat enters a small interval through its larger right face than leaves through its smaller left face, so energy accumulates. Zero curvature alone is a steady-state test only under the coefficient and geometry assumptions that remove the extra term.



Question 2

A homogeneous rod of length L and constant area A obeys $Cu_t = ku_{xx}$, with $C, k > 0$ and insulated sides. Both ends exchange heat with an ambient temperature T_a . The outward heat flux per area equals $h_j(u - T_a)$ at end j , where $h_0, h_L \geq 0$. Set $v = u - T_a$ and use the signed flux $q = -ku_x$ toward increasing x .

Tasks

1. Derive the two boundary conditions, explaining the different signs at the left and right ends.
2. Derive the rate of change of the total excess heat $E = CA \int_0^L v \, dx$. Must this signed quantity always decrease?
3. Derive a dissipation identity for $V = (CA/2) \int_0^L v^2 \, dx$.
4. Use the identity to classify every steady state, including $h_0 = h_L = 0$, and prove uniqueness for a prescribed initial temperature profile.

Question 2 – Solution

Strategy. Outward flux has opposite coordinate signs at the two ends; quadratic energy provides a sign-definite quantity.

Step 1: Orient both boundary fluxes. At the left end, outward flux is $-q(0) = ku_x(0)$; at the right it is $q(L) = -ku_x(L)$. Thus

$$kv_x(0, t) = h_0v(0, t), \quad -kv_x(L, t) = h_Lv(L, t).$$

Using the same coordinate sign at both ends would reverse one heat transfer.

Step 2: Balance the signed excess heat. Integrating the equation gives

$$E' = kA[v_x(L) - v_x(0)] = -A(h_Lv(L) + h_0v(0)).$$

It is negative when both ends are hotter than ambient and positive when both are colder. It need not always decrease, since E is a signed excess rather than a nonnegative measure of departure from ambient.

Step 3: Derive the dissipative quantity. Multiplying by Av and integrating by parts yields

$$V' = -kA \int_0^L v_x^2 dx - Ah_0v(0)^2 - Ah_Lv(L)^2 \leq 0.$$

The endpoint terms have the correct negative signs precisely because the boundary conditions were oriented outward.

Step 4: Classify steady states and prove uniqueness. For a steady state $V' = 0$, every nonnegative term on the right must vanish. Thus $v_x = 0$. If at least one $h_j > 0$, its boundary term forces this constant to be zero, so the sole equilibrium is $u = T_a$. If both coefficients vanish, every spatially constant temperature is steady. For uniqueness, the difference of two sufficiently smooth solutions with the same initial data obeys the same homogeneous equation and boundary conditions. Its quadratic energy starts at zero and cannot increase, so the difference vanishes everywhere by continuity. This also covers the insulated case: its possible equilibrium constants do not cause nonuniqueness for fixed initial data.

Question 3

A rod has constant area, insulated sides, no internal source and conductivity $k(x) = k_0(1 + x/L)$, with $k_0, L > 0$. Its end temperatures are $u(0) = T_h$, $u(L) = T_c$, with $T_h > T_c$. The steady equation is $(k(x)u_x)_x = 0$, and flux per area toward the right is $q = -k(x)u_x$.

Tasks

1. Derive the exact temperature and heat flux.
2. Verify the equation and both boundary values, and locate the largest magnitude of the temperature gradient.
3. Test the straight-line interpolant between the same endpoints in the correct variable-conductivity equation. Explain its physical failure.
4. Define k_{eff} by $q = k_{\text{eff}}(T_h - T_c)/L$. Compute it and compare it with the arithmetic spatial average of k . Sketch the normalized exact and linear temperature profiles.

Question 3 – Solution

Strategy. Constant steady heat flux requires the gradient to vary inversely with conductivity.

Step 1: Integrate the flux law. Since $(ku_x)_x = 0$, the flux q is constant. Integrating $u_x = -q/k(x)$ gives

$$T_c - T_h = -\frac{qL}{k_0} \log 2.$$

Consequently

$$q = \frac{k_0(T_h - T_c)}{L \log 2}, \quad u(x) = T_h - (T_h - T_c) \frac{\log(1 + x/L)}{\log 2}.$$

The flux is positive: heat travels from the hotter left end toward the right.

Step 2: Check the profile and gradient. The logarithm is zero at 0 and $\log 2$ at L , giving the two endpoint values. Also

$$u_x = -\frac{T_h - T_c}{L \log 2 (1 + x/L)}, \quad k(x)u_x = -\frac{k_0(T_h - T_c)}{L \log 2}.$$

Thus the product is constant as required. The gradient magnitude is largest at $x = 0$ and decreases by a factor of two from left to right.

Step 3: Diagnose the linear candidate. For $\ell = T_h - (T_h - T_c)x/L$, one has $\ell_{xx} = 0$, but

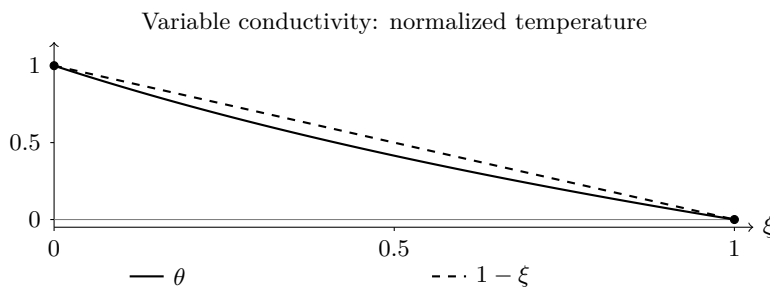
$$(k\ell_x)_x = -\frac{k_0(T_h - T_c)}{L^2} \neq 0.$$

Its rightward flux grows along the rod, so more heat leaves each small interval than enters. Without a source this profile would cool locally, not remain steady. Using $ku_{xx} = 0$ discards the essential term $k'u_x$.

Step 4: Interpret the effective conductivity. The definition gives $k_{\text{eff}} = k_0/\log 2$, while the arithmetic average is $3k_0/2$. In fact $\log 2 > 2/3$: the function $1/t$ is strictly convex and its average on $[1, 2]$ exceeds its midpoint value $2/3$. Hence

$$k_{\text{eff}} = \frac{k_0}{\log 2} < \frac{3k_0}{2}.$$

Steady conduction uses the reciprocal average $k_{\text{eff}} = L/\int_0^L k^{-1} dx$. The graph uses $\xi = x/L$ and $\theta = (u - T_c)/(T_h - T_c)$.



Question 4

Two homogeneous layers in series have thicknesses L_1, L_2 , conductivities $k_1, k_2 > 0$ and the same cross-sectional area. The outer temperatures are $T_h > T_c$. There are no sources. A contact resistance per area $R \geq 0$ satisfies $u_- - u_+ = Rq$, where u_-, u_+ are the left and right interface traces and q is the steady rightward heat flux. The interface stores no energy.

Tasks

1. Explain why flux is continuous at the interface while temperature may jump. State all equations determining the steady profiles.
2. Derive q, u_-, u_+ and the two linear temperature profiles.
3. In consistent nondimensional units, take $L_1 = L_2 = 1$, $k_1 = 1$, $k_2 = 2$, $R = 1/2$, $T_h = 100$ and $T_c = 0$. Compute and sketch the profiles, showing both interface traces.
4. Analyze the limits $R \rightarrow 0$ and $R \rightarrow \infty$. Explain why imposing temperature continuity when $R > 0$ would generally contradict the model.

Question 4 – Solution

Strategy. Add the two layer resistances and the contact resistance, retaining the interface temperature drop.

Step 1: Apply the interface balance. No interfacial storage or source means incoming flux equals outgoing flux. Thus the same q satisfies $u_x = -q/k_1$ in the first layer and $u_x = -q/k_2$ in the second. The boundary and contact relations are

$$u(0) = T_h, \quad u(L_1 + L_2) = T_c, \quad u_- - u_+ = Rq.$$

Temperature continuity is a special case of zero resistance, not the general interface law.

Step 2: Add all temperature drops. Writing $\mathcal{R} = L_1/k_1 + R + L_2/k_2$ gives

$$q = \frac{T_h - T_c}{\mathcal{R}}, \quad u_- = T_h - \frac{qL_1}{k_1}, \quad u_+ = T_c + \frac{qL_2}{k_2}.$$

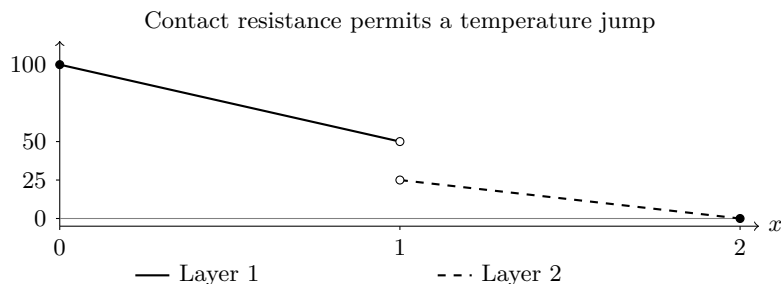
The profiles are $T_h - qx/k_1$ on the first layer and $T_c + q(L_1 + L_2 - x)/k_2$ on the second. Their traces differ by Rq .

Step 3: Verify the numerical example. Here $\mathcal{R} = 1 + 1/2 + 1/2 = 2$, so

$$q = 50, \quad u_- = 50, \quad u_+ = 25, \quad u(x) = \begin{cases} 100 - 50x, & 0 \leq x < 1, \\ 50 - 25x, & 1 < x \leq 2. \end{cases}$$

The flux is 50 on both sides, although the slopes differ. The temperature jump is $25 = Rq$. Open circles show separate traces, not a single assigned temperature at an idealized zero-thickness contact.

Step 4: Take the resistance limits. As $R \rightarrow 0$, the flux tends to $(T_h - T_c)/(L_1/k_1 + L_2/k_2)$ and the traces meet. As $R \rightarrow \infty$, $q \rightarrow 0$, each layer becomes nearly constant at its own outer temperature, and $Rq \rightarrow T_h - T_c$. A large resistance can therefore sustain a finite temperature jump while the flux vanishes. For finite $R > 0$, forcing $u_- = u_+$ would give $q = 0$, contradicting $T_h > T_c$ and the finite total resistance.



Question 5

A homogeneous rod with insulated ends is observed to have temperature

$$u(x, t) = T_a + Be^{-\lambda t} \cos(\pi x/L), \quad 0 \leq x \leq L, \quad B, \lambda, L > 0.$$

The unknown constant conductivity and volumetric heat capacity satisfy $Cu_t = ku_{xx}$ and $k, C > 0$. Assume B, L, λ are known.

Tasks

1. Verify the boundary conditions and infer the diffusivity $\alpha = k/C$ from the observed field.
2. Prove that temperature observations of this field alone do not identify k and C separately. Describe the full family of positive parameter pairs.
3. An independent measurement gives rightward heat flux per area $q_m > 0$ at $x = L/2, t = 0$. Recover k and C uniquely and check their units.
4. Explain why the midpoint is useful for this flux measurement but useless for measuring the temperature decay. Identify which temperature sensor locations can recover λ from two distinct measurement times.

Question 5 – Solution

Strategy. The temperature equation determines a ratio of material parameters; a flux measurement provides an independent scale.

Step 1: Verify the field and recover diffusivity. Differentiation gives

$$u_t = -\lambda B e^{-\lambda t} \cos(\pi x/L), \quad u_{xx} = -\frac{\pi^2}{L^2} B e^{-\lambda t} \cos(\pi x/L).$$

Also $u_x = -(\pi B/L)e^{-\lambda t} \sin(\pi x/L)$, so $u_x = 0$ at both ends. Matching the equation at points where the cosine is nonzero gives

$$\alpha = \frac{k}{C} = \frac{\lambda L^2}{\pi^2}.$$

The same equality then verifies the equation everywhere, including the node.

Step 2: Identify the nonuniqueness. For every choice $C_* > 0$, the pair

$$C = C_*, \quad k = \frac{\lambda L^2}{\pi^2} C_*$$

produces exactly the observed field. Multiplying both parameters by the same positive factor leaves the equation and insulated boundary conditions unchanged. More exact observations of this same temperature field cannot resolve that scaling ambiguity.

Step 3: Use the independent flux data. At the midpoint initially, $q = -ku_x = kB\pi/L$. Therefore

$$k = \frac{q_m L}{B\pi}, \quad C = \frac{k}{\alpha} = \frac{q_m \pi}{B\lambda L}.$$

With q_m in W/m², B in kelvins and λ in s⁻¹, these have units W/(mK) and J/(m³K), respectively. Positivity of the measured quantities ensures physically positive parameters.

Step 4: Choose informative sensor locations. At $x = L/2$, $u = T_a$ for all time: both u_t and u_{xx} vanish, so a ratio of these derivatives would be 0/0. But $|u_x|$ is largest there, giving a strong flux signal. At any fixed $x \neq L/2$ in $[0, L]$, the excess temperature is nonzero and

$$\lambda = \frac{1}{t_2 - t_1} \log \left| \frac{u(x, t_1) - T_a}{u(x, t_2) - T_a} \right|, \quad t_2 > t_1.$$

The absolute values allow sensors on either side of the node; the ratio eliminates the unknown local amplitude.

Question 6

On an infinite homogeneous rod, consider

$$u(x, t) = T_a + \frac{B}{\sqrt{t + \tau}} \exp\left(-\frac{x^2}{4\alpha(t + \tau)}\right), \quad B, \tau, \alpha > 0, \quad t \geq 0.$$

Use $C > 0$ for volumetric heat capacity and interpret integrated heat per unit cross-sectional area. You may use $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} z^2 e^{-z^2} dz = \sqrt{\pi}/2$.

Tasks

1. Verify $u_t = \alpha u_{xx}$ directly, retaining the derivative of the time-dependent prefactor.
2. Compute the total excess heat per area and show that it is constant.
3. Compute the variance of the excess-temperature profile about zero and describe its peak and width as time increases.
4. Determine where the temperature is increasing at a given time, and the time of maximum temperature at a fixed point x . Sketch the profiles for $\alpha = B = \tau = 1$ at $t = 0, 3$, marking the inflection points.

Question 6 – Solution

Strategy. Differentiate the amplitude and width together; local warming can coexist with a falling global peak and conserved heat.

Step 1: Verify the differential equation. Put $v = u - T_a$ and $\sigma = t + \tau > 0$. Logarithmic differentiation gives

$$v_t = v \left(-\frac{1}{2\sigma} + \frac{x^2}{4\alpha\sigma^2} \right), \quad v_{xx} = v \left(-\frac{1}{2\alpha\sigma} + \frac{x^2}{4\alpha^2\sigma^2} \right).$$

Thus $v_t = \alpha v_{xx}$. Omitting the derivative of $\sigma^{-1/2}$ would remove the first term and invalidate the equation.

Step 2: Integrate the excess heat. With $z = x/\sqrt{4\alpha\sigma}$,

$$\mathcal{H} = C \int_{\mathbb{R}} v \, dx = 2CB\sqrt{\pi\alpha}.$$

The factor from the increasing width cancels the decreasing prefactor. The Gaussian flux tends to zero at both infinities, consistent with no net heat loss from the infinite rod.

Step 3: Measure spreading. Symmetry gives center zero. The provided Gaussian integrals yield

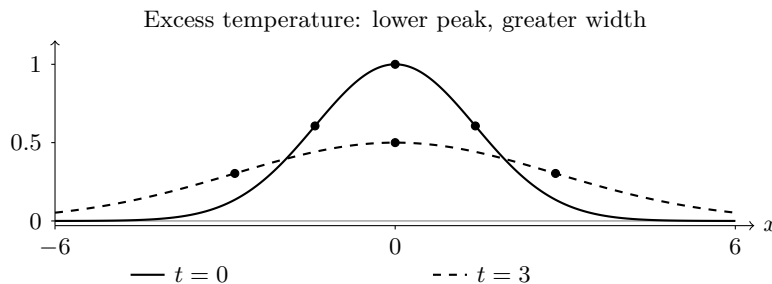
$$\frac{\int_{\mathbb{R}} x^2 v \, dx}{\int_{\mathbb{R}} v \, dx} = 2\alpha\sigma, \quad \max_x v = \frac{B}{\sqrt{\sigma}}.$$

Thus the root-mean-square width grows as $\sqrt{2\alpha(t + \tau)}$, while the peak decreases as $(t + \tau)^{-1/2}$. The inflection points occur at $x = \pm\sqrt{2\alpha\sigma}$.

Step 4: Locate local warming. Since $v > 0$, $u_t > 0$ precisely when $x^2 > 2\alpha(t + \tau)$. A fixed point reaches its maximum on $t \geq 0$ at

$$t_* = \max\left(0, \frac{x^2}{2\alpha} - \tau\right).$$

Outside the initial inflection points it first warms and then cools. Inside them it cools from the start; equality gives an initially zero derivative. Therefore diffusion does not force temperature to decrease at every location. The graph uses excess temperature v , so the ambient offset is removed.



Question 7

A constant-area rod with constant $k, C > 0$ is prescribed the candidate field

$$u(x, t) = T_a + at + bx(L - x), \quad a, b, L > 0, \quad 0 \leq x \leq L.$$

The model permits a volumetric source s and obeys $Cu_t = ku_{xx} + s$. Its lateral surface is insulated; end conditions must be determined consistently.

Tasks

1. Determine the unique source field that makes the candidate satisfy the equation.
2. State the initial and Dirichlet boundary data realized by the candidate. Decide whether it also satisfies insulated-end conditions.
3. Compute both outward end heat fluxes and verify the integrated energy balance, using cross-sectional area A .
4. Suppose the source is removed while the same formula is retained but a is allowed to be real. Find the required a and explain why changing only the boundary values cannot fix a nonzero interior residual.

Question 7 – Solution

Strategy. A manufactured temperature field determines its source and boundary data; those data cannot be prescribed independently afterward.

Step 1: Compute the interior residual. The derivatives are $u_t = a$, $u_x = b(L - 2x)$ and $u_{xx} = -2b$. Substitution requires

$$s = Ca + 2kb.$$

This is a positive spatially and temporally constant source, in W/m^3 . It supplies both the rising stored energy and heat lost at the ends.

Step 2: State the compatible data. At $t = 0$, $u(x, 0) = T_a + bx(L - x)$. Both endpoint temperatures equal $T_a + at$. The endpoint derivatives are bL and $-bL$, neither zero, so this field does not satisfy insulated-end conditions when $b > 0$. Prescribing insulation in addition to the realized end temperatures would contradict the candidate rather than add useful information.

Step 3: Verify total energy accounting. The outward fluxes are $ku_x(0) = kbL$ and $-ku_x(L) = kbL$. The stored heat relative to T_a is

$$E = CA \int_0^L (u - T_a) dx = CA \left(atL + \frac{bL^3}{6} \right), \quad E' = CAaL.$$

Total source power minus total outward power is

$$AL(Ca + 2kb) - 2AkbL = CAaL = E'.$$

This independently checks the source sign and both boundary signs.

Step 4: Remove the source consistently. For $s = 0$, the formula can satisfy the PDE only if

$$a = -\frac{2kb}{C}.$$

Its initially warm curved profile then decreases at a constant rate while both endpoint temperatures follow the same decreasing law. If a retains a different value, the residual $Ca + 2kb$ is nonzero at every interior point. Boundary values alone cannot repair that local equation while keeping the same candidate field. Changing boundary data generally requires changing the field itself.

Question 8

Let u be continuous on $[0, L] \times [0, T]$, sufficiently differentiable in the interior, and satisfy $u_t = \alpha u_{xx}$ with $\alpha > 0$. Assume $0 \leq u(x, 0) \leq M$ and $0 \leq u(0, t), u(L, t) \leq M$, where $M > 0$.

Tasks

1. Prove $u \leq M$ by applying a maximum argument to $w = u - M - \varepsilon t$ and then letting $\varepsilon \downarrow 0$. Address a maximum occurring at the final observation time.
2. Prove $u \geq 0$ and explain why these bounds concern the global range rather than monotonic cooling at every fixed position.
3. Test $v(x, t) = Me^{\alpha(\pi/L)^2 t} \sin(\pi x/L)$ against the source-free heat equation and the range bound.
4. Find the source needed to realize this growing candidate in $Cv_t = kv_{xx} + s$, with $k/C = \alpha$. Give and verify a source-free decaying candidate with the same initial and endpoint data.

Question 8 – Solution

Strategy. A small time-dependent perturbation makes the maximum argument strict, while a growing test field exposes a missing source.

Step 1: Prove the upper bound. For $\varepsilon > 0$, w is nonpositive initially and at the spatial endpoints. If it has a positive maximum on the space-time rectangle, that maximum occurs at an interior spatial point and a time $t_* > 0$. There $w_{xx} \leq 0$ and $w_t \geq 0$: at the latest time, use the derivative from earlier times; at an interior time it is zero. Yet

$$w_t - \alpha w_{xx} = -\varepsilon < 0,$$

a contradiction. One can first apply the argument on any shorter time rectangle and pass to T by continuity if needed. Thus $u \leq M + \varepsilon t$, and letting $\varepsilon \downarrow 0$ gives $u \leq M$.

Step 2: Prove the lower bound. Apply the same argument to $-u$ with upper bound zero to obtain $-u \leq 0$. Hence $0 \leq u \leq M$ throughout the rectangle. The theorem does not say $u_t \leq 0$ at every point. At a local temperature minimum with positive curvature, the equation gives $u_t = \alpha u_{xx} > 0$. Redistribution can warm a cooler location without exceeding the allowed global range.

Step 3: Diagnose the growing sine. Write $\mu = (\pi/L)^2$. Then $v_t = \alpha \mu v$ and $v_{xx} = -\mu v$, so

$$v_t - \alpha v_{xx} = 2\alpha \mu v \neq 0$$

in the spatial interior. It satisfies zero endpoint values and the allowed initial profile, but for $t > 0$ its midpoint exceeds M . This violates the conclusion because it fails the source-free PDE hypothesis.

Step 4: Restore the missing physics or correct the time factor. The required source is $s = C(v_t - \alpha v_{xx}) = 2k\mu v > 0$ in the interior. With that source, the earlier source-free maximum argument no longer applies. Alternatively,

$$\tilde{v}(x, t) = M e^{-\alpha \mu t} \sin(\pi x/L)$$

has $\tilde{v}_t = -\alpha \mu \tilde{v} = \alpha \tilde{v}_{xx}$, the same initial profile and zero endpoints. It remains between zero and M .

Question 9

Two homogeneous rods have lengths L and $2L$, the same $k, C > 0$, insulated sides and no sources. At both ends each rod exchanges heat with the same ambient T_a by an outward convection law. The first rod has coefficient $h > 0$. The two initial profiles have the same shape as functions of relative position. Set $\alpha = k/C$ and choose a nonzero temperature scale ΔT .

Tasks

1. Nondimensionalize the first problem using $\xi = x/L$, $\eta = \alpha t/L^2$ and $\theta = (u - T_a)/\Delta T$. Identify the dimensionless boundary parameter.
2. Verify the PDE and initial shape for the proposed rescaling $u_2(x, t) = u_1(x/2, t/4)$ on the longer rod.
3. Derive exactly which convection coefficient h_2 makes this rescaling satisfy both boundary laws.
4. Assess the claim that doubling length always quadruples every thermal response time when the same h is used. State when the proposed scaling is valid, including the perfectly insulated case.

Question 9 – Solution

Strategy. Match the dimensionless boundary conditions as well as the interior diffusion time scale.

Step 1: Identify the dimensionless problem. The chain rule gives $\theta_\eta = \theta_{\xi\xi}$. The outward convection laws become

$$\theta_\xi(0, \eta) = \text{Bi } \theta(0, \eta), \quad -\theta_\xi(1, \eta) = \text{Bi } \theta(1, \eta), \quad \text{Bi} = \frac{hL}{k}.$$

This dimensionless coefficient is the Biot number for the chosen length scale. The interior time scale is L^2/α , but the boundary parameter also affects the dimensionless response.

Step 2: Verify the rescaled interior field. For $u_2(x, t) = u_1(x/2, t/4)$,

$$(u_2)_t = \frac{1}{4}(u_1)_t, \quad (u_2)_{xx} = \frac{1}{4}(u_1)_{xx}.$$

Thus the same diffusivity equation holds. At $t = 0$, relative positions agree: $x/(2L) = (x/2)/L$, so the stated initial-shape requirement is met.

Step 3: Match both convection conditions. At the left endpoint,

$$k(u_2)_x(0, t) = \frac{1}{2}k(u_1)_x(0, t/4) = \frac{h}{2}(u_2(0, t) - T_a).$$

The right outward derivative scales by the same factor. Therefore the coefficient making this a rescaling of the full problem is

$$h_2 = h/2, \quad h_2(2L)/k = hL/k.$$

Both rods then have the same dimensionless boundary problem.

Step 4: Qualify the length-squared claim. If $h_2 = h$, the longer rod instead has Biot number $2hL/k$. The proposed field generally fails the boundary law; the full response is not obtained merely by multiplying time by four. A special zero-excess equilibrium cannot establish a general scaling rule. The factor-four rescaling is valid when convection is also adjusted to $h/2$ and the initial shapes match. If both rods are perfectly insulated ($h = 0$), the zero-flux conditions are preserved automatically. Interior dimensional analysis alone does not justify ignoring changed boundary parameters.

Question 10

Divide an insulated homogeneous rod into three equal cells of width Δx . Let U_i^n be cell temperatures and $r = \alpha \Delta t / (\Delta x)^2 \geq 0$. Approximate each internal rightward flux by $-k(U_{i+1}^n - U_i^n) / \Delta x$, set outer fluxes to zero, and use a forward Euler time step for the cell energy balances.

Tasks

1. Derive the three-cell update matrix and prove that the sum of temperatures is conserved.
2. Find the exact range of r for which every update preserves the range of every input vector. Prove necessity as well as sufficiency.
3. Using the orthogonal vectors $(1, 1, 1)$, $(1, 0, -1)$ and $(1, -2, 1)$, determine exactly when $\sum_i (U_i^n - \bar{U})^2$ cannot increase in one step.
4. Apply one step with $r = 3/5$ to $(0, 1, 0)$. Check heat conservation and squared norm, and sketch the old and new cell values. Explain why energy stability need not imply preservation of nonnegative temperatures.

Question 10 – Solution

Strategy. Conservation, preservation of the temperature range and decay of a quadratic norm impose different conditions on a numerical scheme.

Step 1: Balance each cell. For the first cell, inflow minus outflow gives $U_1^{n+1} = U_1^n + r(U_2^n - U_1^n)$. Applying the same balance to the others yields

$$U^{n+1} = \begin{pmatrix} 1-r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-r \end{pmatrix} U^n.$$

Every column sums to one, so $\sum_i U_i$ is conserved. Equal cell volumes and constant capacity make this proportional to total heat. Every row also sums to one, so constant profiles are fixed.

Step 2: Characterize range preservation. All entries are nonnegative exactly when $0 \leq r \leq 1/2$. Each new value is then a convex combination of old values, preserving their range. For any $r > 1/2$, input $(0, 1, 0)$ gives a middle value $1 - 2r < 0$, outside $[0, 1]$. Thus the exact range is $\boxed{0 \leq r \leq 1/2}$.

Step 3: Test the quadratic energy on each mode. The stated mutually orthogonal vectors have update factors 1, $1 - r$ and $1 - 3r$, respectively. The mean is fixed; the remaining squared mode components do not grow exactly when $|1 - r| \leq 1$ and $|1 - 3r| \leq 1$. For $r \geq 0$, this is

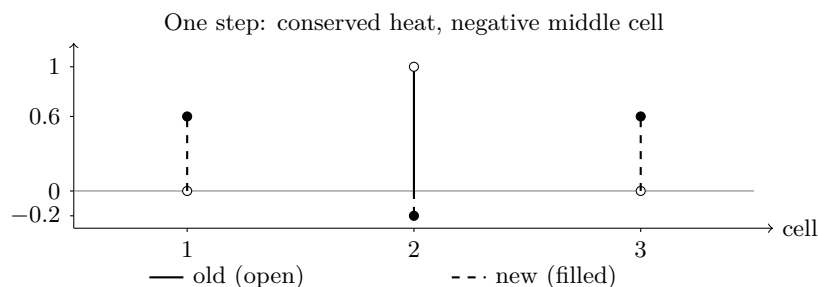
$$\boxed{0 \leq r \leq 2/3}.$$

Necessity follows by choosing the offending eigenvector as input. This threshold belongs to this particular three-cell insulated matrix.

Step 4: Exhibit the distinction numerically. At $r = 3/5$,

$$\boxed{(0, 1, 0) \mapsto (3/5, -1/5, 3/5)}.$$

Both sums equal one. The uncentered squared norm drops from 1 to $9/25 + 1/25 + 9/25 = 19/25$; since the mean is unchanged, the centered squared norm also drops. Nevertheless the middle temperature becomes negative. The stem plot shows individual cell values, without suggesting a smooth interpolated temperature profile between cell centers.



Question 1

An ideal string has constant tension $\mathcal{T} > 0$ and mass per unit length $\mu > 0$. Its small transverse displacement is $u(x, t)$. Assume $|u_x| \ll 1$, negligible longitudinal motion and bending stiffness, and a transverse applied force $p(x, t)$ per unit length. A linear drag force per unit length is $-bu_t$, with $b \geq 0$.

Tasks

1. Apply transverse momentum balance on an arbitrary interval and derive the linear wave model, explaining the small-slope approximation.
2. State the units of $\mu, \mathcal{T}, p, b$ and check each term. Identify the undamped wave speed and the effect of quadrupling μ at fixed tension.
3. Explain why both initial displacement and initial velocity are needed. Give two different undamped, source-free solutions on the whole line with the same zero initial displacement.
4. For $p = b = 0$, test $u = F(x - vt)$ with $F \in C^2$. Determine when this represents a traveling wave of the string and identify the exceptional profiles that do not determine v .

Question 1 – Solution

Strategy. Transverse acceleration is driven by the difference of vertical tension components, not by a first time derivative.

Step 1: Derive the momentum equation. The vertical tension component is $\mathcal{T} \sin \theta \approx \mathcal{T} u_x$ under the small-slope approximation. For an interval $[a, d]$,

$$\int_a^d \mu u_{tt} dx = \mathcal{T} u_x(d, t) - \mathcal{T} u_x(a, t) + \int_a^d (p - bu_t) dx.$$

Localization gives

$$\mu u_{tt} + bu_t = \mathcal{T} u_{xx} + p.$$

The linearization neglects changes of length and tension of higher order in the slope.

Step 2: Check dimensions and speed. The units are $\mu : \text{kg/m}$, $\mathcal{T} : \text{N}$, $p : \text{N/m}$ and $b : \text{Ns/m}^2$. Since u is a length, each term in the local equation has units N/m. Without drag or forcing,

$$u_{tt} = c^2 u_{xx}, \quad c = \sqrt{\mathcal{T}/\mu}.$$

The speed is in m/s. Quadrupling μ at fixed tension halves c .

Step 3: Explain the two initial data. Acceleration determines how velocity changes, so displacement alone cannot specify the motion. For example, on the whole line, both $u = 0$ and $u = Vt$, with constant $V \neq 0$, solve the source-free undamped equation and have $u(x, 0) = 0$. Their initial velocities differ. A prescribed pair $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ removes this particular ambiguity.

Step 4: Verify a moving profile. For $u = F(x - vt)$, $u_{tt} = v^2 F''$ and $u_{xx} = F''$. Hence

$$(v^2 - c^2)F'' = 0.$$

If F'' is not identically zero, the admissible speeds are $v = \pm c$, corresponding to opposite directions. If F is affine, both second derivatives vanish for every v ; a translating straight line cannot by itself identify a propagation speed. A non-affine smooth profile is needed to draw that inference from the PDE.

Question 2

For the whole-line wave equation $u_{tt} = c^2 u_{xx}$, $c > 0$, you may use

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Take $f = 0$ and $g(s) = (1 - s^2)^2$ for $|s| < 1$, $g(s) = 0$ for $|s| \geq 1$, in consistent nondimensional units. Here g is continuously differentiable.

Tasks

1. Differentiate the integral formula to verify the equation and both initial conditions for these data.
2. Find the exact center displacement $u(0, t)$ for all $t \geq 0$.
3. Determine where the solution is zero and when a fixed point reaches the constant plateau behind the departing waves. Explain why compact initial velocity can leave nonzero displacement behind.
4. Sketch the profiles for $c = 1$ at $t = 1/2, 3/2, 3$. Check that the plateau height does not grow without bound and distinguish zero displacement from zero velocity.

Question 2 – Solution

Strategy. Initial velocity contributes an integral over a growing interval; it is not simply two displaced copies of the initial displacement.

Step 1: Verify the integral solution. Leibniz differentiation gives

$$u_t = \frac{g(x+ct) + g(x-ct)}{2}, \quad u_{tt} = \frac{c}{2}(g'(x+ct) - g'(x-ct)), \quad u_{xx} = \frac{1}{2c}(g'(x+ct) - g'(x-ct)).$$

Thus $u_{tt} = c^2 u_{xx}$. At $t = 0$, the integral is zero and $u_t(x, 0) = g(x)$. The C^1 regularity of g makes these second derivatives continuous.

Step 2: Integrate at the center. Symmetry gives $u(0, t) = c^{-1} \int_0^{\min(ct, 1)} (1-s)^2 ds$. Consequently

$$u(0, t) = \begin{cases} t - \frac{2c^2 t^3}{3} + \frac{c^4 t^5}{5}, & 0 \leq ct \leq 1, \\ \frac{8}{15c}, & ct \geq 1. \end{cases}$$

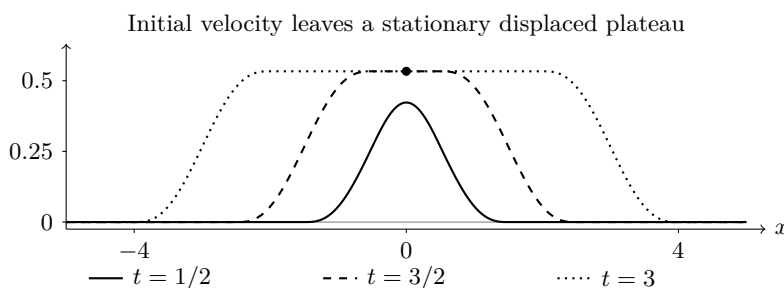
The two expressions agree at $ct = 1$.

Step 3: Locate the moving fronts and plateau. For $t > 0$, the integral is positive exactly when $[x-ct, x+ct]$ overlaps $(-1, 1)$ in positive length. It is therefore zero for $|x| \geq 1+ct$. When $ct \geq |x|+1$, the integration interval contains the entire support of g , so

$$u(x, t) = \frac{1}{2c} \int_{-1}^1 g(s) ds = \frac{8}{15c}.$$

The same value holds at equality because the endpoints of g contribute nothing. Behind the fronts, the string is displaced but locally stationary and flat. Displacement itself need not return to zero after a velocity pulse.

Step 4: Interpret the plotted profiles. Since $g \geq 0$, every interval integral lies between zero and the full integral. Thus $0 \leq u \leq 8/(15c)$ for every x, t . For $c = 1$, the fronts are at $x = \pm(1+t)$; a plateau appears once $t \geq 1$ and occupies $|x| \leq t-1$. Its height stays $8/15$ as its width grows. Zero local velocity on that plateau does not mean zero local displacement.



Question 3

Use the whole-line wave formula

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds, \quad c > 0.$$

At an observation time $t_0 > 0$, compare two solutions with initial data (f, g) and $(\tilde{f}, \tilde{g})$.

Tasks

1. Identify exactly which initial positions can influence an observation (x_0, t_0) in this formula.
2. If $|f - \tilde{f}| \leq \varepsilon$ and $|g - \tilde{g}| \leq \eta$ on that interval, prove $|u(x_0, t_0) - \tilde{u}(x_0, t_0)| \leq \varepsilon + t_0\eta$. Show the two constants are sharp.
3. Take $c = 1$, $g = 0$ and $f(x) = (x - 2)^3(3 - x)^3$ on $2 < x < 3$, zero elsewhere. Find the entire time trace at $x = 0$, including its first and last nonzero times and maximum.
4. Sketch the normalized trace $128u(0, t)$ for $0 \leq t \leq 4$. Explain why changes to initial data outside the backward interval cannot affect a given observation, however large those changes are.

Question 3 – Solution

Strategy. Read causality directly from the interval sampled by the solution formula.

Step 1: Identify the backward interval. Only the initial positions in

$$[x_0 - ct_0, x_0 + ct_0]$$

enter: f is sampled at its two endpoints, while g is integrated across it. This interval is a sufficient domain of dependence for both types of data; interior values of f alone do not enter this one-dimensional formula.

Step 2: Bound and attain the data error. The endpoint contribution is at most $(\varepsilon + \varepsilon)/2$. The integral contribution is at most $(2ct_0)\eta/(2c)$. Hence

$$|u(x_0, t_0) - \tilde{u}(x_0, t_0)| \leq \varepsilon + t_0\eta.$$

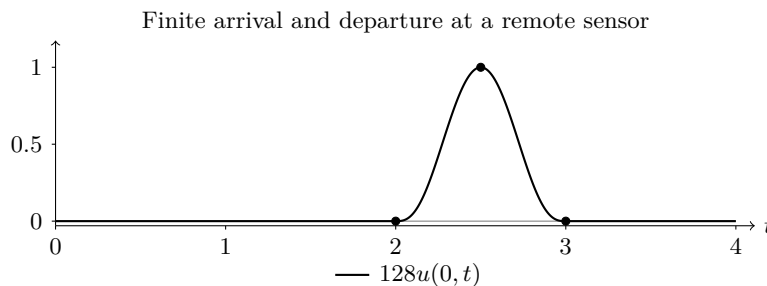
Constant differences $f - \tilde{f} = \varepsilon$ and $g - \tilde{g} = \eta$ attain equality. Taking either difference zero proves the corresponding coefficient cannot be improved.

Step 3: Compute the sensor trace. At $x = 0$, $u(0, t) = [f(-t) + f(t)]/2$. For $t \geq 0$, the first term is always zero. Thus

$$u(0, t) = \begin{cases} \frac{1}{2}(t-2)^3(3-t)^3, & 2 < t < 3, \\ 0, & 0 \leq t \leq 2 \text{ or } t \geq 3. \end{cases}$$

The trace starts to become nonzero immediately after $t = 2$ and returns to zero at $t = 3$. Since $(t-2)(3-t) = 1/4 - (t-5/2)^2$, its unique maximum is $1/128$ at $t = 5/2$. The initial profile is C^2 even at its support edges.

Step 4: Interpret finite propagation. At times before 2, the backward interval does not reach the initial pulse, so the sensor cannot respond. No alteration outside the sampled interval changes either endpoint value or the integral in the formula. This conclusion places no bound on the magnitude of the remote alteration. It is a statement about finite propagation and dependence on initial data, not about small remote effects that have merely been neglected.



Question 4

Two semi-infinite strings meet at $x = 0$, share tension $\mathcal{T} > 0$ and have densities $\mu_1, \mu_2 > 0$. Set $c_j = \sqrt{\mathcal{T}/\mu_j}$ and $Z_j = \mathcal{T}/c_j$. The massless junction has continuous displacement and continuous transverse force $\mathcal{T}u_x$. For a smooth incident pulse with $0 < \int_{\mathbb{R}} (F')^2 < \infty$, write

$$u_1 = F(t - x/c_1) + RF(t + x/c_1) \quad (x < 0), \quad u_2 = DF(t - x/c_2) \quad (x > 0).$$

Tasks

1. Verify each traveling term solves its local wave equation and derive the two junction equations for R, D .
2. Solve for R, D and classify the sign of the reflected displacement when the second string is denser or lighter.
3. Using rightward power $\mathcal{P} = -\mathcal{T}u_t u_x$, derive the reflected and transmitted energy fractions and prove their sum is one.
4. Evaluate the case $\mu_2 = 9\mu_1$. Analyze the limits $Z_2/Z_1 \rightarrow 0$ and $\rightarrow \infty$, and explain why a transmitted displacement amplitude above one need not violate energy conservation.

Question 4 – Solution

Strategy. Displacement amplitudes and energy fractions differ because the strings have different wave impedances.

Step 1: Apply the interface conditions. Every term of the form $F(t \mp x/c_j)$ has $u_{tt} = c_j^2 u_{xx}$. At the junction, displacement continuity gives $1 + R = D$. The slope condition gives

$$\frac{\mathcal{T}}{c_1}(R - 1)F'(t) = -\frac{\mathcal{T}}{c_2}DF'(t), \quad \boxed{Z_1(1 - R) = Z_2D.}$$

Since F is nonconstant, these amplitude equations follow.

Step 2: Solve and interpret the amplitudes. Solving the two equations gives

$$\boxed{R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad D = \frac{2Z_1}{Z_1 + Z_2}.}$$

Here $Z_j = \sqrt{\mathcal{T}\mu_j}$. A denser second string gives $R < 0$, an inverted reflected displacement. A lighter string gives $R > 0$. Equal impedances give no reflection and $D = 1$.

Step 3: Account for energy flux. The incident rightward power is $Z_1(F')^2$, the reflected leftward power has magnitude $Z_1R^2(F')^2$, and transmitted power is $Z_2D^2(F')^2$. The cross terms in $-\mathcal{T}u_tu_x$ on the incident side cancel. For a pulse with finite nonzero $\int(F')^2$, integration in time therefore gives

$$\boxed{\mathcal{R} = R^2, \quad \mathcal{D} = \frac{Z_2}{Z_1}D^2, \quad \mathcal{R} + \mathcal{D} = \frac{(Z_1 - Z_2)^2 + 4Z_1Z_2}{(Z_1 + Z_2)^2} = 1.}$$

Step 4: Test examples and extreme impedances. If $\mu_2 = 9\mu_1$, then $Z_2 = 3Z_1$, so $R = -1/2$, $D = 1/2$, $\mathcal{R} = 1/4$ and $\mathcal{D} = 3/4$. As $Z_2/Z_1 \rightarrow 0$, $R \rightarrow 1$ and $D \rightarrow 2$, but $\mathcal{D} \rightarrow 0$. As $Z_2/Z_1 \rightarrow \infty$, $R \rightarrow -1$, $D \rightarrow 0$ and again $\mathcal{D} \rightarrow 0$. These are the free-end and fixed-end reflection limits, respectively. Energy depends on impedance as well as squared displacement amplitude.

Question 5

On $x \geq 0$, use wave speed one and the even profile $F(z) = e^{-z^2}$. For $a > 0$, compare

$$u_D(x, t) = F(x + t - a) - F(x - t + a), \quad u_N(x, t) = F(x + t - a) + F(x - t + a).$$

These formulas include the reflected field from the start; the Gaussian has noncompact tails.

Tasks

1. Verify the wave equation for both fields and show which one satisfies a fixed end and which one satisfies a free end at $x = 0$.
2. Compute their initial displacements and velocities explicitly. Explain why keeping only the incident Gaussian would violate the boundary condition.
3. At $t = a$, compute each field and its velocity. Decide whether zero displacement everywhere implies the string is at rest.
4. For $a = 3$, sketch u_D at $t = 1, 3, 5$ on $0 \leq x \leq 6$. Interpret the sign of the reflected pulse and compare it with the free-end reflection.

Question 5 – Solution

Strategy. An odd or even spatial reflection enforces the corresponding boundary condition.

Step 1: Verify the two boundary models. Each translated profile solves $u_{tt} = u_{xx}$. Since F is even and F' is odd,

$$u_D(0, t) = F(t - a) - F(a - t) = 0, \quad (u_N)_x(0, t) = F'(t - a) + F'(a - t) = 0.$$

Thus u_D satisfies the fixed-end condition and u_N the free-end condition.

Step 2: Recover the actual initial data. Differentiating the time arguments carefully gives

$$\begin{aligned} u_D(x, 0) &= F(x - a) - F(x + a), & (u_D)_t(x, 0) &= F'(x - a) + F'(x + a), \\ u_N(x, 0) &= F(x - a) + F(x + a), & (u_N)_t(x, 0) &= F'(x - a) - F'(x + a), \end{aligned}$$

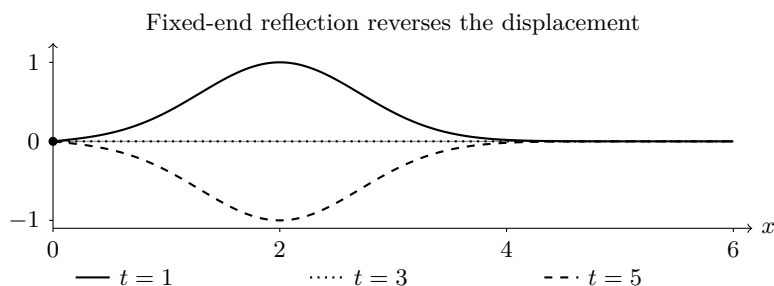
where $F'(z) = -2ze^{-z^2}$. The incident term alone generally has neither zero boundary displacement nor zero boundary slope. Its reflected partner supplies the exact cancellation; small Gaussian tails are not identically zero.

Step 3: Inspect the collision instant. At $t = a$, the two profile arguments both equal x . Therefore

$$u_D(x, a) = 0, \quad (u_D)_t(x, a) = 2F'(x), \quad u_N(x, a) = 2F(x), \quad (u_N)_t(x, a) = 0.$$

The fixed-end string has zero displacement but nonzero velocity for $x > 0$: it is not at rest. The free-end field has a doubled displacement profile at that instant and zero instantaneous velocity. Displacement and velocity must both be known to specify the state.

Step 4: Interpret the reflected sign. For $a = 3$, the incoming peak at $t = 1$ is near $x = 2$. At $t = 3$ the fixed-end displacement is exactly zero everywhere. At $t = 5$ the reflected pulse is near $x = 2$ with opposite sign. Indeed $u_D(x, 5) = -u_D(x, 1)$. A fixed end reverses displacement upon reflection; a free end preserves its sign. The plotted zero snapshot is an exact cancellation, not disappearance of the motion or its energy.



Question 6

A string on $0 \leq x \leq L$ has fixed ends, constant $\mu, \mathcal{T} > 0$ and equation $\mu u_{tt} = \mathcal{T} u_{xx} + p(x, t)$. Assume sufficient smoothness and define

$$E(t) = \frac{1}{2} \int_0^L (\mu u_t^2 + \mathcal{T} u_x^2) dx.$$

Tasks

1. Derive the energy identity, first retaining boundary power and then using the fixed-end conditions.
2. Prove uniqueness for prescribed initial displacement and velocity when the applied force is prescribed.
3. For $p = 0$ and $A \neq 0$, verify $u = A \sin(\pi x/L) \cos(\omega t)$ with $\omega = (\pi/L)\sqrt{\mathcal{T}/\mu}$. Compute its kinetic, elastic and total energies.
4. Sketch the two energy fractions over $0 \leq \omega t \leq 2\pi$. Explain why a zero-displacement snapshot can have positive energy, and why the energy identity does not claim that displacement at each point is monotone.

Question 6 – Solution

Strategy. Integrate the PDE against velocity so that acceleration and curvature become time derivatives of kinetic and elastic energies.

Step 1: Derive the work balance. Multiplying by u_t and integrating by parts gives

$$E'(t) = \mathcal{T}[u_t u_x]_0^L + \int_0^L p u_t dx.$$

At a fixed endpoint, $u = 0$ for all time, hence $u_t = 0$ there. Thus $E' = \int p u_t$. With no applied force, total energy is constant. The sign of the work rate depends on the force and velocity together.

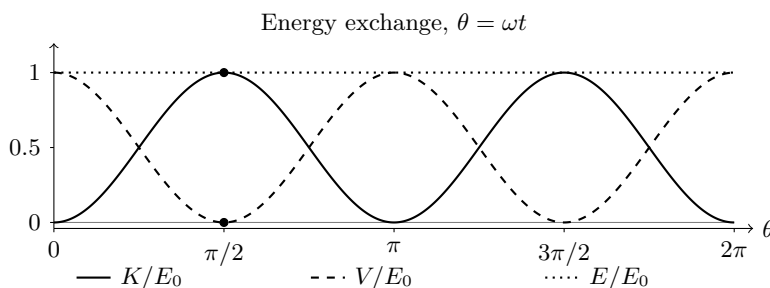
Step 2: Prove uniqueness for the full initial state. The difference w of two solutions with the same force, endpoints and initial data obeys the homogeneous equation and has zero initial energy. Its energy stays zero, so $w_t = w_x = 0$ everywhere by continuity. The fixed endpoints then force the spatial constant to be zero. Initial displacement alone would not make the kinetic part of the initial difference energy vanish; both initial data are essential.

Step 3: Compute the standing-wave energy. Put $k = \pi/L$, so $\omega^2 = (\mathcal{T}/\mu)k^2$. Then $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$, and both endpoints are zero. Using $\int_0^L \sin^2 kx = \int_0^L \cos^2 kx = L/2$ gives

$$K(t) = E_0 \sin^2(\omega t), \quad V(t) = E_0 \cos^2(\omega t), \quad E_0 = \frac{\mathcal{T} A^2 k^2 L}{4}.$$

Their sum is the constant E_0 .

Step 4: Interpret the exchange of energy. For $A \neq 0$, at $\omega t = \pi/2$ the displacement and elastic energy are zero, but the velocity is nonzero and all energy is kinetic. At $\omega t = 0$ the velocity is zero and all energy is elastic. The field exchanges these forms while oscillating; total-energy conservation does not require pointwise monotonic motion. The graph shows K/E_0 , V/E_0 and their constant sum.



Question 7

A fixed-end string is known to have displacement

$$u(x, t) = A \sin(\pi x/L) \cos(c\pi t/L), \quad A, L > 0,$$

with unknown speed $c > 0$. A snapshot at a known time $\Delta > 0$ shows $u(x, \Delta) = 0$ for every x . Initially the shape and zero velocity are known.

Tasks

1. Find every positive speed consistent with the zero snapshot and prove the list is complete.
2. Determine the speed if the independent prior bound $0 < c < L/\Delta$ is available. Explain why the snapshot alone is insufficient without it.
3. Derive the midpoint velocity at time Δ for each admissible speed. Give a consistency test for an exact signed velocity measurement and show that it identifies the speed uniquely when consistent.
4. If the measured velocity is $-5\pi A/(2\Delta)$, find c and predict the midpoint displacement at time $\Delta/2$. Verify the signs rather than using only the velocity magnitude.

Question 7 – Solution

Strategy. A snapshot can hide how many oscillations have occurred; a velocity measurement retains both rate and direction.

Step 1: List all compatible speeds. Since the spatial sine is not identically zero, the snapshot requires $\cos(c\pi\Delta/L) = 0$. For positive c ,

$$c_n = \frac{L}{\Delta} \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

Every listed speed gives the same zero snapshot, as well as the specified initial displacement and zero velocity. The positive zeros of cosine prove completeness.

Step 2: Use an actual prior restriction. The bound $c < L/\Delta$ allows only $n = 0$, so $c = L/(2\Delta)$. Without this independent information, arbitrarily many half-oscillations may have occurred before the snapshot. A zero snapshot does not imply zero speed or determine which crossing of equilibrium was observed.

Step 3: Recover speed from signed velocity. At the midpoint,

$$v_n = u_t(L/2, \Delta) = (-1)^{n+1} \frac{\pi A}{\Delta} \left(n + \frac{1}{2} \right).$$

A measured v is consistent exactly when $n = |v|\Delta/(\pi A) - 1/2$ is a nonnegative integer and its sign is $(-1)^{n+1}$. The strictly increasing magnitudes then select one n uniquely. In particular, a zero velocity measurement is incompatible with the stated nonzero-amplitude zero snapshot.

Step 4: Verify the numerical multiple. For $v = -5\pi A/(2\Delta)$, the magnitude gives $n = 2$ and the predicted sign $(-1)^3$ is negative, as measured. Thus

$$c = \frac{5L}{2\Delta}, \quad u(L/2, \Delta/2) = A \cos(5\pi/4) = -\frac{A}{\sqrt{2}}.$$

The signed result distinguishes this phase from other times at which the displacement vanishes. Inferring the lowest possible speed without a prior bound would produce the wrong prediction for the intermediate snapshot.

Question 8

Compare the ideal wave equation with the given dispersive extension

$$u_{tt} = c^2 u_{xx} - \beta u_{xxxx}, \quad c > 0, \quad \beta \geq 0.$$

For a sinusoidal wave $u = \cos(kx - \omega t)$ with $k > 0, \omega > 0$, define phase speed $v_p = \omega/k$ and group speed $v_g = d\omega/dk$. No general theory of wave packets is required.

Tasks

1. Substitute the sinusoid and derive the frequency relation. Determine the units of β .
2. Compute v_p, v_g and compare them for $\beta > 0$. Find their small- k limits and their values when $\beta = 0$.
3. For $\beta > 0$, consider a sum of two nonzero sinusoidal components with distinct positive wave numbers. Prove that the sum cannot translate rigidly with one fixed speed while both components follow the derived positive-frequency relation.
4. Sketch v_p, v_g for $c = 1, \beta = 1/4, 0 < k \leq 3$, showing the limiting value at zero separately. Explain which feature of the ordinary wave equation is lost.

Question 8 – Solution

Strategy. Test sinusoidal components before claiming that an arbitrary composite profile moves without changing shape.

Step 1: Derive the frequency relation. The derivatives give $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$ and $u_{xxxx} = k^4 u$. Therefore

$$\omega^2 = c^2 k^2 + \beta k^4, \quad \omega = k\sqrt{c^2 + \beta k^2}.$$

Since k has units m^{-1} and ω units s^{-1} , β has units m^4/s^2 .

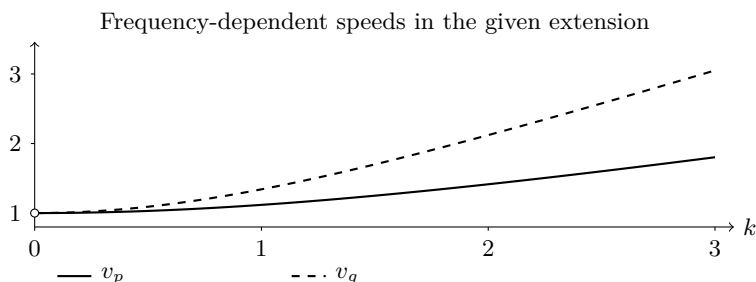
Step 2: Compare the two speeds. Differentiating the positive-frequency branch gives

$$v_p = \sqrt{c^2 + \beta k^2}, \quad v_g = \frac{c^2 + 2\beta k^2}{\sqrt{c^2 + \beta k^2}}.$$

Their difference is $\beta k^2 / \sqrt{c^2 + \beta k^2}$, strictly positive if $\beta > 0$. Both tend to c as $k \downarrow 0$ and both equal c for every k when $\beta = 0$.

Step 3: Test rigid translation of two frequencies. A fixed profile $\cos(k_1 x + \phi_1) + a \cos(k_2 x + \phi_2)$ with $a \neq 0$ translated by speed v gives temporal frequencies vk_1, vk_2 . Independence of the two spatial frequencies requires $v = \omega(k_1)/k_1 = \omega(k_2)/k_2$ for agreement at every x, t . For $\beta > 0$, $v_p(k)$ is strictly increasing for $k > 0$, so distinct wave numbers cannot share this value. Relative phases change in a way that cannot be represented by one common translation.

Step 4: Interpret the comparison. In the ideal equation, all right-moving sinusoidal components travel with the same speed c , so a smooth profile $F(x - ct)$ retains its shape. The given extension makes phase speed depend on frequency; a multi-frequency profile generally changes shape. The graph shows this dependence and the strict separation of the two defined speeds for positive k . The open point at $k = 0$ records their common limit, since the definitions above were made for $k > 0$.



Question 9

Let a sufficiently smooth displacement be periodic in x with period 2π and satisfy $u_{tt} = c^2 u_{xx}$, $c > 0$. Define its mean $m(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} u(x, t) dx$ and momentum $P(t) = \mu \int_{-\pi}^{\pi} u_t dx$, with $\mathcal{T} = \mu c^2$.

Tasks

1. Derive the evolution of the mean and prove momentum conservation.
2. Derive conservation of $E = \frac{1}{2} \int_{-\pi}^{\pi} (\mu u_t^2 + \mathcal{T} u_x^2) dx$. Does bounded energy force bounded displacement for all time?
3. Test the explicit family $u(x, t) = Vt + A \cos x \cos(ct)$. Compute its mean, momentum and energy exactly and use it to answer the preceding question.
4. If the mean initial velocity is zero, prove that a uniform displacement bound does follow from energy and the initial mean. You may use continuity, the intermediate value theorem and Cauchy–Schwarz.

Question 9 – Solution

Strategy. Wave energy controls velocity and slope, but a spatially uniform motion can carry the mean displacement indefinitely.

Step 1: Track the constant spatial mode. Integrating the PDE and using matching periodic slopes gives

$$m''(t) = \frac{c^2}{2\pi} [u_x]_{-\pi}^{\pi} = 0.$$

Hence

$$m(t) = m(0) + m'(0)t, \quad P(t) = 2\pi\mu m'(0).$$

The mean velocity, rather than the mean displacement, is conserved.

Step 2: Derive the periodic energy balance. Multiplication by u_t and integration by parts gives $E' = \mathcal{T}[u_t u_x]_{-\pi}^{\pi} = 0$ by periodicity. The energy contains u_t and u_x , but not u itself. A nonzero constant mean velocity is therefore compatible with a linearly growing mean displacement and a fixed total energy.

Step 3: Verify a drifting example. For $u = Vt + A \cos x \cos(ct)$, the term Vt and the oscillatory term each solve the equation. Orthogonality gives

$$m(t) = Vt, \quad P = 2\pi\mu V, \quad E = \pi\mu V^2 + \frac{\pi\mathcal{T}A^2}{2}.$$

The cosine's kinetic and elastic contributions sum to the second constant. For $V \neq 0$, displacement is unbounded in time even though the energy is finite and constant.

Step 4: Bound displacement when the mean cannot drift. If $m'(0) = 0$, then $m(t) = m(0)$. At each time the continuous function $w = u - m(0)$ has zero spatial mean, so it has a zero at some x_* (or is identically zero). Integrate its derivative from that point over a path of length at most 2π . Cauchy-Schwarz gives

$$|w(x, t)| \leq \sqrt{2\pi} \|u_x(\cdot, t)\|_2 \leq \sqrt{\frac{4\pi E(0)}{\mathcal{T}}}.$$

Thus $|u(x, t)| \leq |m(0)| + \sqrt{4\pi E(0)/\mathcal{T}}$ for all time. The estimate need not be sharp; it identifies the missing mean-velocity condition.

Question 10

A fixed-end string is driven by

$$\mu u_{tt} = \mathcal{T} u_{xx} + F \sin(kx) \cos(\omega t), \quad k = \pi/L, \quad \omega = k\sqrt{\mathcal{T}/\mu}, \quad F > 0.$$

Both initial displacement and velocity are zero. Test solutions of the supplied form $u(x, t) = q(t) \sin(kx)$; no general separation procedure is required.

Tasks

1. Derive the scalar equation for q and verify $q(t) = Ft \sin(\omega t)/(2\mu\omega)$.
2. Check both initial conditions and both fixed-end conditions. Explain why the factor t does not alter the propagation-speed parameter of the PDE.
3. Compute the energy and verify its rate equals the applied work rate. Evaluate the energy at $t_j = j\pi/\omega$ and describe its growth.
4. Replace the driving frequency by $\Omega > 0$, $\Omega \neq \omega$. Verify the zero-initial-data response $q_\Omega = F(\cos \Omega t - \cos \omega t)/(\mu(\omega^2 - \Omega^2))$ and recover the resonant formula as $\Omega \rightarrow \omega$ for fixed t .

Question 10 – Solution

Strategy. A force at the natural frequency produces growing amplitude while the same local wave equation retains its original speed.

Step 1: Verify the resonant amplitude equation. Substitution gives $q'' + \omega^2 q = (F/\mu) \cos \omega t$. If $D = F/(2\mu\omega)$ and $q = Dt \sin \omega t$, then

$$q' = D(\sin \omega t + \omega t \cos \omega t), \quad q'' + \omega^2 q = 2D\omega \cos \omega t.$$

Thus the proposed response satisfies the forced equation.

Step 2: Check the full initial-boundary data. One has $q(0) = q'(0) = 0$. The spatial sine vanishes at $0, L$, giving both fixed ends for all time. The growing factor is a resonant response to sustained external forcing. It changes the amplitude history, not the coefficient $c^2 = T/\mu$ that sets the source-free propagation speed.

Step 3: Measure energy and work. Spatial integration gives

$$E(t) = \frac{\mu L}{4}(q'^2 + \omega^2 q^2) = \frac{F^2 L}{16\mu\omega^2} [(\sin \omega t + \omega t \cos \omega t)^2 + (\omega t \sin \omega t)^2].$$

Differentiating the first expression and using the amplitude equation yields $E' = (FL/2)q' \cos \omega t$, exactly $\int_0^L F \sin(kx) \cos(\omega t) u_t dx$. At $t_j = j\pi/\omega$,

$$E(t_j) = \frac{F^2 L}{16\mu} t_j^2.$$

Energy is unbounded along these times. Its instantaneous rate can have either sign; sustained resonant input produces the growing envelope.

Step 4: Take the frequency limit carefully. The supplied q_Ω has zero value and derivative at zero. Direct differentiation gives $q''_\Omega + \omega^2 q_\Omega = (F/\mu) \cos \Omega t$. For fixed t , differentiating numerator and denominator with respect to Ω at ω gives

$$\lim_{\Omega \rightarrow \omega} q_\Omega(t) = \frac{F}{\mu} \frac{-t \sin \omega t}{-2\omega} = \frac{Ft \sin \omega t}{2\mu\omega}.$$

Each fixed nonresonant response is bounded in time, but those bounds are not uniform as its frequency approaches resonance.

Question 1

Let $u = u(x, t)$ be an unknown scalar field on $\mathbb{R}^2$, and let a, b, c be real constants. Consider the family

$$au_{tt} + u_t - (1 + x^2)u_{xx} + buu_x + c(u_x)^2 = \sin t.$$

The order of a PDE is the highest derivative order actually present. A linear PDE has coefficients depending only on the independent variables, and the unknown and all its derivatives occur linearly. For a linear PDE, homogeneous means that its prescribed right-hand side is zero.

Tasks

1. Identify the independent and dependent variables and determine the order for every a , including $a = 0$. Explain why second order need not mean second order in time.
2. Classify exactly which pairs (b, c) make the equation linear. In that case, decide whether it has constant coefficients and whether it is homogeneous.
3. The nonlinear term uu_x vanishes on every field independent of x . Explain why testing only those fields cannot establish linearity. Give a scaling test that detects each nonzero nonlinear coefficient.
4. Replace the right-hand side by zero. Explain what changes in the linear case, and why a zero right-hand side alone cannot make a nonlinear equation linear.

Question 1 – Solution

Strategy. Classify the differential expression on arbitrary admissible fields, rather than on a special solution.

Step 1: Count the derivatives that remain. The independent variables are x, t and the dependent variable is u . Since $1 + x^2 > 0$, the term u_{xx} never disappears. The order is therefore $\boxed{2}$ for every a . When $a = 0$, the equation is first order in time but still second order as a PDE. When $a \neq 0$, it is also second order in time.

Step 2: Separate coefficients from unknowns. The expression is linear exactly when $\boxed{b = c = 0}$. Then $1 + x^2$ is an allowed variable coefficient, so the PDE is linear with variable coefficients. It is nonhomogeneous because $\sin t$ is not identically zero on the domain. A forcing term may vanish at some times without making the equation homogeneous.

Step 3: Test scaling on informative fields. Write the nonlinear part as $N[u] = buu_x + c(u_x)^2$. It satisfies $N[2u] - 2N[u] = 2N[u]$, whereas a linear operator would give zero. For the smooth test field $u = x$, this difference is

$$2(bx + c).$$

It vanishes for every x only when $b = c = 0$. In particular, if $b \neq 0$ it is not the zero function; if $b = 0$ and $c \neq 0$ it is the nonzero constant $2c$. The restricted tests $u = h(t)$ have $u_x = 0$ and conceal both nonlinear terms. Linearity must hold on all fields in the operator's domain.

Step 4: Classify the equation after changing the forcing. With zero right-hand side and $b = c = 0$, the equation is linear homogeneous. With $(b, c) \neq (0, 0)$, the same scaling obstruction remains, regardless of the right-hand side. Under the stated convention, “homogeneous linear PDE” is not a label for an arbitrary nonlinear equation written with zero on one side.

Question 2

On $\mathbb{R} \times [0, \infty)$ define $L[u] = u_t - u_{xx}$ and consider $L[u] = 2$. Set

$$u_1(x, t) = 2t, \quad u_2(x, t) = 2t + e^{-t} \cos x.$$

No initial or boundary conditions are imposed unless explicitly stated.

Tasks

1. Verify that u_1, u_2 solve the PDE and find the exact condition on real α, β for $\alpha u_1 + \beta u_2$ to solve the same PDE.
2. Determine the equations satisfied by $u_2 - u_1$ and $u_1 + u_2$. Explain why a linear equation need not have a vector space of solutions.
3. Prove that the complete solution set is $u_1 + \ker L$, where $\ker L = \{w : L[w] = 0\}$ in the chosen smooth function class. Explain the word affine using this representation.
4. Now impose $u(x, 0) = \cos x$. Decide which displayed solution satisfies the full problem. Explain why this check alone is neither a construction of all solutions nor a uniqueness proof.

Question 2 – Solution

Strategy. Apply the operator to a combination before applying a superposition rule.

Step 1: Verify the source and its coefficient. For u_1 , $u_{1,t} = 2$ and $u_{1,xx} = 0$. For u_2 , $u_{2,t} = 2 - e^{-t} \cos x$ and $u_{2,xx} = -e^{-t} \cos x$. Thus $L[u_j] = 2$. Linearity gives

$$L[\alpha u_1 + \beta u_2] = 2(\alpha + \beta).$$

Consequently the combination solves the same equation exactly when $\boxed{\alpha + \beta = 1}$, not for all coefficients.

Step 2: Distinguish differences from sums. The difference has source $2 - 2 = 0$, so $u_2 - u_1 \in \ker L$. The sum has source 4 and is not a solution of $L[u] = 2$. The zero field is not a solution either. Thus this solution set is not a vector space, although its defining operator is linear. For $L[w] = 0$, arbitrary linear combinations do remain solutions.

Step 3: Describe every solution by a translation. If $L[u] = 2$, then $L[u - u_1] = 0$, so $u = u_1 + w$ with $w \in \ker L$. Conversely $L[u_1 + w] = 2 + 0 = 2$ for every such w . Hence

$$\boxed{\{u : L[u] = 2\} = u_1 + \ker L.}$$

This is an affine space: a translate of a vector space. Combinations whose coefficients sum to one stay in it. The particular solution provides an origin for the translation, but is not the zero vector of the ambient function space.

Step 4: Check the additional data separately. At $t = 0$, $u_1 = 0$ and $u_2 = \cos x$, so only u_2 among these two fields satisfies the specified initial condition. This proves existence in the smooth class because u_2 is an explicit solution. It does not prove uniqueness among all fields; that requires a theorem or argument in a stated function class, including any necessary behavior at spatial infinity. Comparing two candidates cannot eliminate all other candidates.

Question 3

Let $u(x, t)$ solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $0 < t < T$, where $\kappa, L, T > 0$. For this problem, initial data are prescribed at $t = 0$; spatial boundary data are prescribed at $x = 0, L$. The outward normal derivative is $\partial_n u = n_x u_x$, with $n_x = -1$ on the left and $n_x = 1$ on the right. Consider

$$u(x, 0) = f(x), \quad u(0, t) = A(t), \quad \kappa \partial_n u(L, t) + hu(L, t) = r(t), \quad h > 0.$$

Dirichlet data prescribe u , Neumann data prescribe $\partial_n u$, and Robin data prescribe a linear combination with nonzero coefficients of both.

Tasks

1. Classify each condition, and identify the portions of the space-time rectangle on which they are prescribed. Is $t = T$ a spatial boundary?
2. Rewrite the right condition with u_x . Write the corresponding coordinate form of $\kappa \partial_n u + hu = r$ if it were imposed at the left endpoint instead.
3. Classify the spatial boundary operator when $h = 0$, and decide exactly when the displayed spatial data are homogeneous. Distinguish this from homogeneity of the PDE and of the initial data.
4. Verify the manufactured field $u_*(x, t) = x^2 + 2\kappa t$. Find f, A, r that make it solve the full displayed problem, and illustrate where each datum acts.

Question 3 – Solution

Strategy. Identify both the operator being prescribed and the geometric location of the data.

Step 1: Locate and name the data. The bottom edge $t = 0$ carries initial displacement/temperature data $f(x)$. The left edge carries Dirichlet data $A(t)$; the right carries Robin data $r(t)$. The top $t = T$ is a final-time slice, not a spatial boundary. No data are prescribed there in this initial-boundary-value problem.

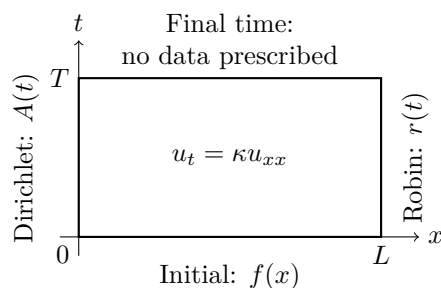
Step 2: Keep the outward signs. At the right, the condition is $\kappa u_x(L, t) + hu(L, t) = r(t)$. At the left, the analogous outward-normal condition would be $-\kappa u_x(0, t) + hu(0, t) = r(t)$. Changing the endpoint changes the sign of the normal derivative, not the sign of the prescribed value by convention.

Step 3: Classify homogeneity at each level. When $h = 0$, the right condition is Neumann, since $\kappa > 0$. The spatial data are homogeneous exactly when $A \equiv 0$ and $r \equiv 0$. The PDE is already linear homogeneous; initial data are homogeneous exactly when $f \equiv 0$. These are three separate assertions. A nonzero Robin coefficient h does not itself make the condition nonhomogeneous.

Step 4: Verify all parts of a full problem. Here $u_{*,t} = 2\kappa$ and $u_{*,xx} = 2$, so the PDE holds. Evaluation gives

$$f(x) = x^2, \quad A(t) = 2\kappa t, \quad r(t) = 2\kappa L + h(L^2 + 2\kappa t).$$

These data also agree at their shared corners. The diagram locates the data; it is a space–time domain, not a graph of the solution value.



Question 4

Consider $u_t = u_{xx}$ for $0 < x < 1$, $0 < t \leq T$, with

$$u(x, 0) = f(x) = x(1 - x), \quad u(0, t) = A(t), \quad u(1, t) = B(t).$$

For this question a solution smooth up to the corners has continuous u, u_t, u_x, u_{xx} on the closed rectangle, satisfies the PDE there by continuity, and has $u_t(x, 0)$ and $u_{xx}(x, 0)$ obtained by differentiating its traces. Assume A, B are continuously differentiable.

Tasks

1. Derive the necessary value compatibility conditions at the two initial corners.
2. Derive the next necessary compatibility conditions involving $A'(0), B'(0)$ and f'' .
3. Set $A = B = 0$. Decide whether value compatibility holds and whether a solution in the stated corner-smooth class can exist. Explain why this does not rule out solutions smooth only for positive time.
4. Keep f fixed and construct linear-in-time boundary data for which a polynomial solution exists. Verify the PDE and every datum explicitly; distinguish this sufficiency proof from checking necessary conditions alone.

Question 4 – Solution

Strategy. At a corner, two different traces must describe the same value and the same time derivative.

Step 1: Match the values. Continuity at $(0, 0)$ and $(1, 0)$ requires

$$A(0) = f(0) = 0, \quad B(0) = f(1) = 0.$$

If these equalities fail, even a continuous solution on the entire closed rectangle is impossible. Interior smoothness alone does not imply these equalities.

Step 2: Match the PDE to the time traces. The left boundary trace gives $u_t(0, 0) = A'(0)$ and the initial trace gives $u_{xx}(0, 0) = f''(0)$. The PDE therefore requires $A'(0) = f''(0)$. The right corner gives the analogous identity. Since $f'' = -2$,

$$A'(0) = B'(0) = -2.$$

These conditions use the stated corner regularity; they cannot be imposed silently on a class that has no continuous u_{xx} at the corners.

Step 3: Diagnose homogeneous endpoint data. For $A = B = 0$, both value conditions hold, but $A'(0) = B'(0) = 0 \neq -2$. Thus no solution exists in the corner-smooth class specified here. This is a regularity obstruction at $t = 0$, not a proof of nonexistence in a weaker class. A solution may be continuous up to the initial edge and smooth for $t > 0$ without its second spatial derivative extending continuously through the initial corners. No such weaker solution is needed for the contradiction.

Step 4: Construct a compatible full example. Take $A(t) = B(t) = -2t$ and

$$u(x, t) = x(1 - x) - 2t.$$

Then $u_t = u_{xx} = -2$, $u(x, 0) = f(x)$ and $u(0, t) = u(1, t) = -2t$. This polynomial has all required corner derivatives and proves existence for the modified problem. Matching a finite list of necessary compatibility conditions would not alone prove existence for arbitrary data; the explicit field supplies the sufficiency argument in this case.

Question 5

For the first-order PDE $u_t + u_x = 0$ on $\mathbb{R}^2$, consider C^1 solutions. A characteristic curve is a curve $(x(t), t)$ along which the PDE forces u to be constant. Suppose data are prescribed on the line $x = at$:

$$u(at, t) = h(t), \quad t \in \mathbb{R},$$

where $a \in \mathbb{R}$ and $h \in C^1(\mathbb{R})$.

Tasks

1. Use the chain rule to identify the characteristic lines, and prove that every C^1 solution has the form $u(x, t) = F(x - t)$.
2. For $a \neq 1$, determine the unique F and verify the resulting full solution for arbitrary h .
3. For $a = 1$, classify exactly which h permit a solution and whether that solution is unique. Give two distinct solutions when $h \equiv H$ is constant.
4. Compare $a = 0, h(t) = \sin t$ with $a = 1, h(t) = \sin t$. Explain geometrically why data on an entire line can determine a unique field, no field, or many fields.

Question 5 – Solution

Strategy. Check whether the data line intersects all characteristic lines or follows just one of them.

Step 1: Recover the characteristic form. Along $x = t + s$, the chain rule gives $\frac{d}{dt}u(t+s, t) = u_x + u_t = 0$. Thus $u(t+s, t) = u(s, 0)$. Defining $F(s) = u(s, 0)$ proves $\boxed{u(x, t) = F(x-t)}$ for every solution. Conversely this form has $u_t = -F'$ and $u_x = F'$, so it solves the PDE.

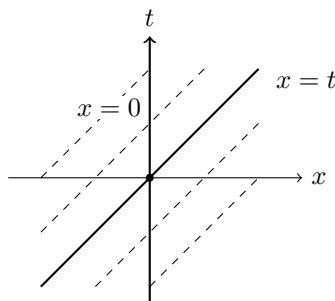
Step 2: Use a noncharacteristic data line. The condition becomes $F((a-1)t) = h(t)$. If $a \neq 1$, the argument ranges over all real numbers exactly once. Consequently

$$\boxed{F(s) = h\left(\frac{s}{a-1}\right), \quad u(x, t) = h\left(\frac{x-t}{a-1}\right)}.$$

This is C^1 , solves the PDE and takes the prescribed trace. The characteristic representation proves uniqueness in the whole stated C^1 class.

Step 3: Diagnose characteristic data. If $a = 1$, the condition reduces to $F(0) = h(t)$ for every t . A solution exists exactly when h is constant, say H . Then any C^1 function with $F(0) = H$ works. For example, $u = H$ and $u = H + x - t$ are distinct solutions. Thus compatible characteristic data leave infinitely many fields undetermined.

Step 4: Interpret the two sine traces. For $a = 0$, $u(x, t) = \sin(t-x)$ is the unique solution. For $a = 1$, the varying trace $\sin t$ contradicts constancy along that characteristic, so no solution exists. The diagram shows the unique intersection with $x = 0$ and the characteristic line $x = t$; the latter supplies information on just one member of the family.



Characteristic family: $x - t = s$

Question 6

For $u_t + u_x = 0$ on $\mathbb{R}^2$, a classical solution here means a C^1 field satisfying the PDE at every point. Compare

$$u(x, t) = |x - t|, \quad u_\varepsilon(x, t) = \sqrt{(x - t)^2 + \varepsilon^2}, \quad \varepsilon > 0.$$

For this question only, an integral solution is a locally integrable field satisfying

$$\int_{\mathbb{R}^2} u(x, t) (\phi_t + \phi_x)(x, t) \, dx \, dt = 0$$

for every smooth compactly supported test function ϕ .

Tasks

1. Check the PDE for u off $x = t$, and decide whether u is a classical solution on the whole plane.
2. Verify that each u_ε is classical and prove a uniform error bound tending to zero as $\varepsilon \downarrow 0$.
3. Explain precisely why uniform convergence of these classical solutions does not establish that u is classical. Inspect $u_{\varepsilon, x}$ near the line $x = t$.
4. Verify that u is an integral solution using $s = x - t$, $r = t$. State which conclusions about u have and have not been proved.

Question 6 – Solution

Strategy. A solution label includes a regularity requirement; checking a formula away from a singular line is not enough.

Step 1: Locate the failure of classical regularity. For $x > t$, $u_x = 1, u_t = -1$; for $x < t$, $u_x = -1, u_t = 1$. Thus the PDE holds on both open regions. At $x = t$, the two one-sided spatial derivatives disagree. The field is not differentiable there, hence is not a classical solution on $\mathbb{R}^2$.

Step 2: Verify the smooth approximations. Writing $s = x - t$, one has

$$u_{\varepsilon,x} = \frac{s}{\sqrt{s^2 + \varepsilon^2}}, \quad u_{\varepsilon,t} = -\frac{s}{\sqrt{s^2 + \varepsilon^2}}.$$

These are smooth and sum to zero. Also

$$0 \leq \sqrt{s^2 + \varepsilon^2} - |s| = \frac{\varepsilon^2}{\sqrt{s^2 + \varepsilon^2} + |s|} \leq \varepsilon.$$

Equality occurs at $s = 0$, so the uniform error on the whole plane is exactly ε .

Step 3: Separate convergence of values from derivatives. For $s \neq 0$, $u_{\varepsilon,x} \rightarrow \text{sgn}(s)$, while its value at $s = 0$ is always zero. This pointwise limit is discontinuous at zero. It cannot be a uniform limit on any closed interval about zero of these continuous derivatives. Uniform convergence of function values does not preserve differentiability, so it does not prove classical solvability.

Step 4: Check the stated integral formulation. Set $\psi(s, r) = \phi(s + r, r)$. The change of variables has Jacobian one, and $\psi_r = \phi_x + \phi_t$. Compact support and local integrability allow integration in either order. Therefore

$$\int_{\mathbb{R}^2} |x - t|(\phi_t + \phi_x) dx dt = \int_{\mathbb{R}} |s| \left(\int_{\mathbb{R}} \psi_r(s, r) dr \right) ds = 0.$$

The inner integral vanishes by compact support. Thus u is an integral solution under the supplied definition, as well as a classical solution off the line. It remains nonclassical on the whole plane; no uniqueness assertion follows from this verification alone.

Question 7

For a real second-order linear PDE in two spatial variables, use the convention

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower-order terms} = 0, \quad D = B^2 - AC.$$

At points where the second-order part is nonzero, call the equation hyperbolic if $D > 0$, elliptic if $D < 0$, and parabolic if $D = 0$. Consider

$$u_{xx} + 2yu_{xy} + xu_{yy} + 3u_x - u = 0.$$

Tasks

1. Identify A, B, C and classify the equation at every point of the plane, including the separating curve. Explain why the coefficient of u_{xy} is not B itself.
2. Classify the points $(0, 1), (2, 1), (1, 1)$, and sketch the type regions with correct axes and boundary curve.
3. At a fixed point, rewrite the principal quadratic form $Q(\xi, \eta) = \xi^2 + 2y\xi\eta + x\eta^2$ as a completed square. Explain its signs and rank in all three regions.
4. Replace $3u_x - u$ by $e^{x+y}u_x + 7u$ and then multiply the entire PDE by a smooth nowhere-zero function $m(x, y)$. Decide which changes can alter the type and justify your answer.

Question 7 – Solution

Strategy. Classify the principal second-order part point by point, keeping the mixed-derivative convention fixed.

Step 1: Read the discriminant correctly. Here $A = 1$, $B = y$ and $C = x$, since the mixed coefficient is $2B = 2y$. Thus $D = y^2 - x$, giving

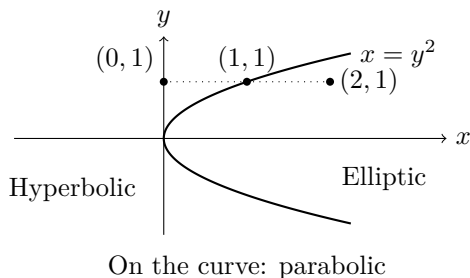
$x < y^2$: hyperbolic, $x = y^2$: parabolic, $x > y^2$: elliptic.
--

Since $A = 1$, the principal part never vanishes completely. The curve is therefore parabolic under the stated convention, not an order-zero exception.

Step 2: Check the points and the geometry. The three discriminants are respectively 1, -1 , 0, so $(0, 1)$ is hyperbolic, $(2, 1)$ elliptic and $(1, 1)$ parabolic. With x horizontal and y vertical, $x = y^2$ is a right-opening parabola; ellipticity lies to its right.

Step 3: Interpret the principal form. Completing the square gives $Q = (\xi + y\eta)^2 + (x - y^2)\eta^2$. For $x > y^2$, it is positive definite, with rank two. For $x < y^2$, the choices $(\xi, \eta) = (1, 0)$ and $(-y, 1)$ give opposite signs; it is indefinite with rank two. On $x = y^2$, it is a nonnegative square of rank one, vanishing on the nonzero direction $(-y, 1)$. This explains the type boundary algebraically.

Step 4: Identify changes that preserve type. Changing lower-order terms leaves A, B, C unchanged. Multiplying by m gives $\bar{D} = (mB)^2 - (mA)(mC) = m^2D$, with the same sign because $m \neq 0$. Neither operation changes type. Allowing $m = 0$ could erase the principal part at those points, which is why the nowhere-zero restriction matters.



Question 8

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and

$$u(0, t) = A(t), \quad u(L, t) = B(t), \quad u(x, 0) = f(x).$$

Assume all data are smooth and $f(0) = A(0)$, $f(L) = B(0)$. Define a boundary lifting

$$\ell(x, t) = \left(1 - \frac{x}{L}\right) A(t) + \frac{x}{L} B(t), \quad v = u - \ell.$$

Tasks

1. Determine the PDE, endpoint data and initial data for v . Classify the homogeneity of its PDE and its spatial boundary conditions separately.
2. Find the exact condition on A, B for this affine lifting to leave the transformed PDE homogeneous for every x, t .
3. For $A(t) = t$, $B(t) = 2t$, $f = 0$, compute the transformed source and initial data. Does making the endpoint conditions homogeneous remove the time-dependent forcing from the full problem?
4. Show that a lifting is not unique. Replace ℓ by $\tilde{\ell} = \ell + q$, where $q(0, t) = q(L, t) = 0$, and derive the new source and initial data. Explain why this is a change of unknown, not a different physical field u .

Question 8 – Solution

Strategy. Subtract the lifting from every part of the problem, including the differential equation and the initial condition.

Step 1: Compute the transformed full problem. Since $\ell_{xx} = 0$,

$$v_t - \kappa v_{xx} = -\ell_t = -\left(1 - \frac{x}{L}\right) A'(t) - \frac{x}{L} B'(t).$$

The endpoints are $v(0, t) = v(L, t) = 0$, and $v(x, 0) = f(x) - \ell(x, 0)$. The endpoint conditions are homogeneous Dirichlet. The transformed PDE is linear, but generally nonhomogeneous. Its initial condition is homogeneous only if $f = \ell(\cdot, 0)$.

Step 2: Determine when the source vanishes. For each t , the displayed affine function of x vanishes throughout $(0, L)$ if and only if both its endpoint limits vanish. Thus $A'(t) = B'(t) = 0$ for all t , or equivalently $\boxed{A, B \text{ are constant in time}}$. Nonzero constant boundary values can therefore produce a homogeneous transformed PDE, although the original boundary data were nonhomogeneous.

Step 3: Inspect time-dependent endpoint values. For $A = t, B = 2t, \ell = t(1 + x/L)$, so

$$v_t - \kappa v_{xx} = -(1 + x/L), \quad v(0, t) = v(L, t) = 0, \quad v(x, 0) = 0.$$

The boundary forcing has moved into the interior equation; it has not been removed from the full problem. In particular, $v = 0$ would fail this PDE. The stated value compatibility holds, but no corner-smooth existence is claimed for these data; higher compatibility would require additional checks.

Step 4: Track the freedom in the lifting. Let $\tilde{v} = u - \tilde{\ell} = v - q$. Direct differentiation gives

$$\tilde{v}_t - \kappa \tilde{v}_{xx} = -\ell_t - q_t + \kappa q_{xx}, \quad \tilde{v}(x, 0) = f(x) - \ell(x, 0) - q(x, 0).$$

The endpoints still vanish. For example, $q = x(L - x)e^{-t}$ is a nonzero smooth choice with zero endpoints. Although the source and transformed initial data change, $u = v + \ell = \tilde{v} + \tilde{\ell}$ is identical. Each lifting gives an equivalent formulation when every datum is transformed consistently.

Question 9

A problem is well posed in specified function spaces if solutions exist, are unique, and depend continuously on the prescribed data. To examine the third requirement, consider the Cauchy data problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y \leq 1; \quad u(x, 0) = f(x), \quad u_y(x, 0) = g(x),$$

with $u(0, y) = u(\pi, y) = 0$. Measure the bottom data by $\|f\|_\infty + \|g\|_\infty$ on $[0, \pi]$ and the observed solution by $\|u(\cdot, 1)\|_\infty$. For integers $n \geq 1$, set

$$u_n(x, y) = \frac{e^{-n}}{n} \sin(nx) \sinh(ny).$$

Tasks

1. Verify the PDE and side conditions and find the exact bottom data f_n, g_n .
2. Compute the bottom data norm and top solution norm. Determine their limits as $n \rightarrow \infty$.
3. Prove that the amplification ratio is unbounded and explain why the calculation rules out a continuous solution map at zero in the stated norms if the problem is uniquely solvable on a linear class containing these fields.
4. Explain why convergence of $u_n(\cdot, 1)$ to zero does not rescue continuity. Rescale the fields to obtain data converging to zero while the top norm stays exactly one, and sketch the vertical amplitude profiles for three n .

Question 9 – Solution

Strategy. Small absolute outputs are not enough for stability; compare their size with the much smaller input, then normalize.

Step 1: Verify the supplied solutions. Two x derivatives multiply u_n by $-n^2$; two y derivatives multiply it by n^2 . Their sum is zero, and $\sin(nx)$ vanishes on both sides. At the bottom, $f_n = 0, \quad g_n = e^{-n} \sin(nx)$. These are smooth fields with compatible side and bottom values.

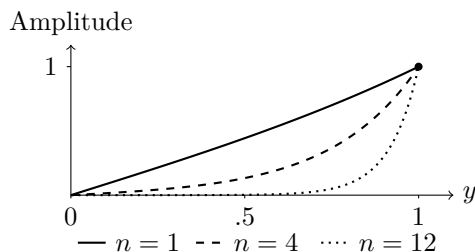
Step 2: Measure the input and output. For each integer n , $\sup_{[0,\pi]} |\sin(nx)| = 1$. Hence the bottom norm is e^{-n} and the top norm is

$$M_n = \frac{e^{-n}}{n} \sinh n = \frac{1 - e^{-2n}}{2n}.$$

Both tend to zero. Thus this sequence by itself does not contradict continuity; the relative amplification is the information needed for the next step.

Step 3: Expose unbounded amplification. The ratio is $M_n/e^{-n} = \sinh n/n \rightarrow \infty$. If uniqueness holds in a linear solution class containing these fields, solving the homogeneous linear PDE with data is a linear map. A continuous linear map between the stated normed spaces must obey a uniform bound $\|u(\cdot, 1)\|_\infty \leq C(\|f\|_\infty + \|g\|_\infty)$ on its domain. The unbounded ratios rule out that bound. No existence or uniqueness theorem for arbitrary Cauchy data has been assumed.

Step 4: Give a direct discontinuity sequence. Set $w_n = u_n/M_n = \sin(nx) \sinh(ny)/\sinh n$. Its top norm is one, whereas its bottom norm is $n/\sinh n \rightarrow 0$. The zero data have the zero solution; uniqueness would force the solution map to select these w_n , contradicting continuity at zero. The vertical envelopes are $\sinh(ny)/\sinh n$, not solution traces at one fixed x for every n .



Question 10

On the unit square $\Omega = (0, 1)^2$, consider a C^2 field smooth up to each edge, satisfying

$$\Delta u = s, \quad \partial_n u = g \text{ on } \partial\Omega,$$

where s is a constant and g is constant on each open edge. Denote these four constants by g_L, g_R, g_B, g_T . At vertices, interpret boundary data by one-sided edge traces. You may use the divergence theorem and Green's identity.

Tasks

1. Identify the boundary type and derive a necessary relation between s and the four edge values, retaining the outward-normal signs.
2. For arbitrary edge constants satisfying that relation, construct a polynomial solution of the form $u = ax^2 + by^2 + cx + dy + C$. Verify that the compatibility condition is also sufficient for this data family.
3. Prove that any two solutions for the same compatible data differ by a constant. Explain why this is uniqueness up to a constant rather than uniqueness of the original problem.
4. Take $(g_L, g_R, g_B, g_T) = (0, 2, 0, 4)$. Find s , impose the normalization $\int_{\Omega} u \, dA = 0$, and give the resulting unique solution. Would the same edge data be compatible with $s = 0$?

Question 10 – Solution

Strategy. Separate existence compatibility from the constant ambiguity left by Neumann data.

Step 1: Integrate the equation before solving it. These are Neumann conditions. Since the square has area one and each edge has length one, the divergence theorem gives

$$s = \int_{\Omega} \Delta u \, dA = \int_{\partial\Omega} \partial_n u \, ds = g_L + g_R + g_B + g_T.$$

In coordinates the derivatives are $-u_x$ on the left, u_x on the right, $-u_y$ on the bottom and u_y on the top. Replacing all by positive coordinate derivatives would give the wrong compatibility relation.

Step 2: Construct every compatible polynomial. For the proposed polynomial the four traces give

$$c = -g_L, \quad 2a + c = g_R, \quad d = -g_B, \quad 2b + d = g_T.$$

Thus

$$u = \frac{g_R + g_L}{2}x^2 + \frac{g_T + g_B}{2}y^2 - g_Lx - g_By + C.$$

Its Laplacian is the sum of the four edge values, exactly s when compatible. Every boundary derivative has the prescribed value. This explicit construction proves sufficiency for constant source and constant edge data on this square; it is not a general sufficiency theorem for arbitrary domains and functions.

Step 3: Prove the exact remaining ambiguity. The difference w of two solutions obeys $\Delta w = 0$, $\partial_n w = 0$. Green's identity yields

$$\int_{\Omega} |\nabla w|^2 \, dA = \int_{\partial\Omega} w \partial_n w \, ds - \int_{\Omega} w \Delta w \, dA = 0.$$

Continuity gives $\nabla w = 0$ everywhere, and connectedness of the square makes w constant. Conversely any added constant leaves all equations and data unchanged. A value or mean normalization is needed for actual uniqueness.

Step 4: Normalize the concrete field. Here $s = 6$ and $u = x^2 + 2y^2 + C$. Its integral is $1/3 + 2/3 + C = 1 + C$. The zero-mean condition fixes $C = -1$, so $u = x^2 + 2y^2 - 1$ is unique among solutions with that normalization. For $s = 0$, the integrated compatibility condition would require $0 = 6$, so no solution exists with these edge data.

Question 1

Let I, J be open intervals, $\kappa > 0$, $r \in \mathbb{R}$, and consider

$$u_t = \kappa u_{xx} + ru \quad ((x, t) \in I \times J).$$

A separated field is $u(x, t) = X(x)T(t)$, with $X \in C^2(I)$ and $T \in C^1(J)$. Assume neither factor is identically zero. Do not assume either factor is nonzero at every point.

Tasks

1. Substitute the product into the PDE. Choose one time where T is nonzero and use it to derive a spatial ODE on all of I , without dividing by X .
2. Show that a single real constant λ gives $X'' + \lambda X = 0$ and $T' = (r - \kappa\lambda)T$ everywhere on their respective intervals. Prove the converse as well.
3. Determine whether a nontrivial T can vanish at a time in J . Explain why zeros of X do not invalidate the separated field and what happens if a factor is identically zero.
4. Verify $X(x) = \sin(2x)$ when $I = \mathbb{R}$. Find all corresponding time factors, determine the growth threshold in r , and explain the harmless freedom to rescale the two factors reciprocally.

Question 1 – Solution

Strategy. Evaluate at a nonzero anchor value of one factor before deriving equations valid even at zeros of the other.

Step 1: Derive a global spatial identity. Substitution yields $XT' = \kappa X''T + rXT$. Choose $t_0 \in J$ with $T(t_0) \neq 0$. Evaluating there gives, for every $x \in I$,

$$\kappa X''(x) = \left(\frac{T'(t_0)}{T(t_0)} - r \right) X(x).$$

Set $\lambda = (r - T'(t_0)/T(t_0))/\kappa$. Then $\boxed{X'' + \lambda X = 0}$ on all of I , including any zeros of X .

Step 2: Recover the time equation and the converse. Substituting $X'' = -\lambda X$ into the product identity gives $X[T' - (r - \kappa\lambda)T] = 0$. Choose x_0 with $X(x_0) \neq 0$ to obtain

$$\boxed{T' = (r - \kappa\lambda)T \quad \text{on } J.}$$

The constant is independent of both variables; a different anchor must give the same value because X is not identically zero. Conversely, these two ODEs imply the original product identity directly, without any divisions. Thus every such pair of factors produces a PDE solution.

Step 3: Treat zeros and the excluded trivial field. The time equation gives $T(t) = Ce^{(r-\kappa\lambda)(t-t_0)}$. For a nontrivial factor $C \neq 0$, it never vanishes. Spatial zeros are allowed; the identities above remain valid there. A factor identically zero produces $u \equiv 0$, which solves the PDE but determines no separation constant from its chosen factors. The trivial field must be retained separately, not mistakenly assigned restrictions obtained by division.

Step 4: Verify a concrete mode and its normalization. For $X = \sin(2x)$, $X'' = -4X$, so $\lambda = 4$ and

$$u(x, t) = C \sin(2x) e^{(r-4\kappa)t}.$$

Its nonzero amplitude grows if $r > 4\kappa$, is constant if $r = 4\kappa$, and decays if $r < 4\kappa$. Direct differentiation verifies the PDE, including at $x = n\pi/2$. Replacing (X, T) by $(aX, T/a)$ for $a \neq 0$ leaves u unchanged; the two factors are not individually unique until a normalization is chosen.

Question 2

Seek nonzero separated solutions of $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and mixed endpoint conditions

$$u(0, t) = 0, \quad u_x(L, t) = 0.$$

Use the convention $X'' + \lambda X = 0$, $T' = -\kappa \lambda T$.

Tasks

1. Derive the endpoint conditions on X . Use integration by parts to prove that every nonzero spatial factor has $\lambda > 0$; address equality explicitly.
2. Solve the spatial problem and list all eigenvalues and eigenfunctions, using an index $n = 0, 1, 2, \dots$. Explain why checking only a sine condition at both ends would give the wrong spectrum.
3. Find the time factor and compare the amplitude half-lives of the first two modes. Distinguish amplitude half-life from the half-life of squared amplitude.
4. Sketch the first three spatial factors normalized to have sine coefficient one, using $\xi = x/L$. Locate their interior zeros and verify their right-end slopes.

Question 2 – Solution

Strategy. Let the actual boundary operators select the admissible separation constants.

Step 1: Prove positivity before solving the ODE. At a time where $T \neq 0$, the endpoint conditions give $X(0) = 0$, $X'(L) = 0$. Multiplication by X and integration yield

$$\lambda \int_0^L X^2 dx = \int_0^L (X')^2 dx - [XX']_0^L = \int_0^L (X')^2 dx.$$

If $\lambda = 0$, then X' vanishes identically; $X(0) = 0$ forces $X = 0$. Thus a nonzero factor has $\boxed{\lambda > 0}$; negative values are impossible.

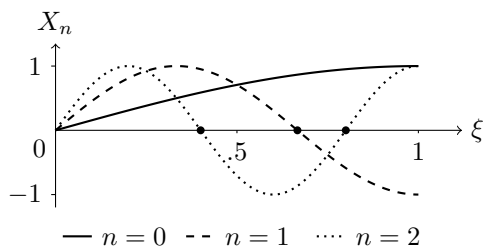
Step 2: Apply the two different endpoint conditions. Put $k = \sqrt{\lambda}$. The general solution is $A \cos(kx) + B \sin(kx)$. The left condition gives $A = 0$, and the right gives $Bk \cos(kL) = 0$. For $B \neq 0$,

$$\boxed{k_n = \frac{(n + 1/2)\pi}{L}, \quad \lambda_n = k_n^2, \quad X_n(x) = \sin(k_n x), \quad n = 0, 1, \dots}$$

The right condition is on the derivative, so it selects zeros of cosine. Using $\sin(kL) = 0$ would impose an unrequested Dirichlet condition there.

Step 3: Compare decay measures. The modes are $CX_n(x)e^{-\kappa\lambda_n t}$. Their amplitude half-lives are $\tau_n = \log 2/(\kappa\lambda_n)$, so $\tau_1 = \tau_0/9$. Squared amplitude decays as $e^{-2\kappa\lambda_n t}$ and has half-life $\tau_n/2$. Confusing those two quantities introduces a factor of two.

Step 4: Check the plotted nodes and slopes. For $n = 0, 1, 2$, the normalized factors are $\sin((n + 1/2)\pi\xi)$. There are respectively zero, one and two interior zeros: none; $2/3$; and $2/5, 4/5$. Every right slope is zero because $\cos((n + 1/2)\pi) = 0$. The right values alternate $1, -1, 1$ and need not vanish.



Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, take $\kappa, L > 0$ and insulated endpoints $u_x(0, t) = u_x(L, t) = 0$. Use $X'' + \lambda X = 0$ and $T' = -\kappa \lambda T$. All fields are smooth enough for the stated derivatives and integrals.

Tasks

1. Solve the spatial eigenvalue problem, including $\lambda = 0$, and exclude $\lambda < 0$ by an integral identity.
2. Show directly from the PDE that the spatial mean is constant in time. Prove that each positive-eigenvalue spatial factor has zero mean, without evaluating its trigonometric integral.
3. Construct a finite separated-mode sum satisfying $u(x, 0) = 2 + 3 \cos(\pi x/L)$. Verify the PDE, both endpoints and the initial data, and identify the limiting field as $t \rightarrow \infty$.
4. Explain precisely what is lost if one divides the eigenvalue calculation by λ and discards the zero case. Can any finite sum of positive-eigenvalue modes reproduce these initial data?

Question 3 – Solution

Strategy. The zero eigenvalue carries a conserved quantity and must be analyzed before any division by the separation constant.

Step 1: Retain the constant spatial mode. Integration by parts gives $\lambda \int X^2 = \int (X')^2 \geq 0$ because both endpoint derivatives vanish. Thus negative eigenvalues are excluded. For $\lambda = 0$, $X = ax + b$ and the endpoints force $a = 0$; nonzero constants are allowed. For $\lambda = k^2 > 0$, the left derivative condition removes the sine term and the right requires $\sin(kL) = 0$. Consequently

$$\lambda_0 = 0, X_0 = 1; \quad \lambda_n = (n\pi/L)^2, X_n = \cos(n\pi x/L), n \geq 1.$$

Step 2: Track the conserved mean. For $m(t) = L^{-1} \int_0^L u(x, t) dx$, $m'(t) = \kappa[u_x]_0^L/L = 0$. Integrating $X'' + \lambda X = 0$ gives $\lambda \int_0^L X dx = -[X']_0^L = 0$. Thus every $\lambda > 0$ factor has zero mean. The zero mode has a constant time factor and can carry a nonzero mean.

Step 3: Verify the two-mode construction. Let $\omega = \kappa(\pi/L)^2$. Then

$$u(x, t) = 2 + 3e^{-\omega t} \cos(\pi x/L).$$

Its time derivative is $-3\omega e^{-\omega t} \cos(\pi x/L)$, equal to κu_{xx} . Its endpoint derivatives vanish, and its initial trace is the prescribed function. Its mean is two for every time. Since the decaying term has absolute value at most $3e^{-\omega t}$, it tends uniformly to $u = 2$. No infinite-series argument is required for this finite construction.

Step 4: Diagnose the missing mode. Dividing by λ loses the legitimate constant eigenfunction and the corresponding stationary PDE solutions. Every finite linear combination of positive-eigenvalue modes has zero spatial mean at each time, while the specified initial data have mean two. Such a combination cannot reproduce these data. This obstruction follows from the integral, not from an inability to find enough trigonometric coefficients.

Question 4

A product field $u(x, t) = X(x)T(t)$ has proportional spatial snapshots. Consider the two-mode heat solution on $0 \leq x \leq \pi$, $t \geq 0$,

$$u(x, t) = e^{-t} \sin x + e^{-4t} \sin(2x).$$

For any two positions and times define

$$D = u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1).$$

Tasks

1. Prove that a product field has $D = 0$. Conversely, if every such determinant vanishes and $u(x_0, t_0) \neq 0$ somewhere, construct factors X, T representing u everywhere.
2. Verify the PDE $u_t = u_{xx}$ and homogeneous Dirichlet endpoints for the displayed sum. Compute D at $x_1 = \pi/2, x_2 = \pi/6, t_1 = 0, t_2 = \log 2$ and decide whether the sum is a single product.
3. Find every interior spatial zero of $u(\cdot, t)$, the time at which it leaves the interior, and the sign of the field after that time.
4. Sketch the normalized snapshots $v(x, t) = e^t u(x, t)$ at $t = 0$, $(\log 2)/6$ and $(\log 2)/3$. Explain why division by a nonzero time factor cannot make a nonseparable field separable.

Question 4 – Solution

Strategy. Test separation with an invariant determinant, then interpret the changing spatial shape.

Step 1: Establish an exact product test. For a product, both terms of D equal $X(x_1)X(x_2)T(t_1)T(t_2)$. Conversely, use the nonzero anchor to write $u(x, t)u(x_0, t_0) = u(x, t_0)u(x_0, t)$. Hence

$$\boxed{X(x) = u(x, t_0), \quad T(t) = u(x_0, t)/u(x_0, t_0)}$$

gives a product everywhere, including zeros. The nonzero-anchor restriction excludes only the identically zero field, which is already a trivial product.

Step 2: Verify the solution and reject one-product form. Each term has time derivative equal to its second spatial derivative; the sum solves the PDE and vanishes at $x = 0, \pi$. At the stated points,

$$D = \frac{\sqrt{3}}{2} \left(\frac{1}{16} - \frac{1}{2} \right) = \boxed{-\frac{7\sqrt{3}}{32} \neq 0}.$$

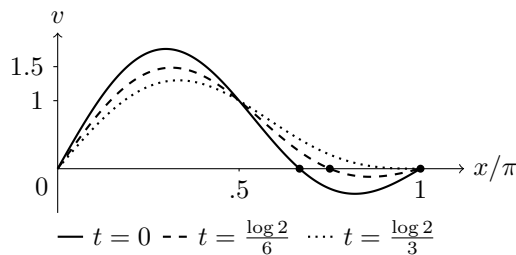
Thus superposition preserves the linear PDE but does not preserve the class of single products. The two spatial modes decay at different rates.

Step 3: Track the moving interior node. Factoring gives $u = e^{-t} \sin x [1 + 2e^{-3t} \cos x]$. On $0 < x < \pi$, $\sin x > 0$, so the only possible interior zero is

$$x_*(t) = \arccos(-e^{3t}/2), \quad 0 \leq t < t_* := \frac{\log 2}{3}.$$

It moves from $2\pi/3$ toward π . At $t = t_*$ it is at the endpoint, not an interior zero. For $t \geq t_*$ the field is strictly positive in the interior. Before then it is positive to the left of the node and negative to its right.

Step 4: Interpret the normalized plots. The three normalized fields are $\sin x + \sin 2x$, $\sin x + 2^{-1/2} \sin 2x$, and $\sin x + \frac{1}{2} \sin 2x$. Their changing zeros reveal changing shape. If $e^t u$ were a product, multiplying its time factor by e^{-t} would make u a product too, contradicting the determinant. Normalization changes amplitudes but cannot repair this failure of separation.



Question 5

For $u_{tt} = c^2 u_{xx}$ on $0 < x < \pi$, $t \in \mathbb{R}$, with $c > 0$ and $u(0, t) = u(\pi, t) = 0$, seek a nonzero product $u = X(x)T(t)$ with both factors C^2 . Use the convention $X'' + \lambda X = 0$.

Tasks

1. Derive the two factor equations using nonzero anchor values instead of assuming T has no zeros. Determine the allowed spatial factors and time factors.
2. Prove the precise restriction a single product places on its two initial functions $f(x) = u(x, 0)$ and $g(x) = u_t(x, 0)$. Include the cases where one of these functions vanishes identically.
3. Determine whether $f = \sin(2x), g = \sin(3x)$ can come from one separated mode. Construct a sum of two modes that satisfies both initial functions and verify it directly.
4. Use $u(x, t) = \sin(2x) \sin(2ct)$ to explain why a zero-displacement snapshot need not mean zero solution or zero velocity. Explain the error in dividing by T at that time.

Question 5 – Solution

Strategy. The time equation is second order and may cross zero; derive it globally and match both initial traces.

Step 1: Separate without excluding oscillation zeros. The product identity is $XT'' = c^2X''T$. A time t_0 with $T(t_0) \neq 0$ gives $X'' = -\lambda X$ everywhere, with $\lambda = -T''(t_0)/(c^2T(t_0))$. A position with $X \neq 0$ then gives $T'' + c^2\lambda T = 0$ for all times. Dirichlet endpoints and $\lambda \int X^2 = \int (X')^2$ exclude $\lambda \leq 0$. Solving the endpoint ODE gives

$$X = \sin(nx), \quad \lambda = n^2, \quad T = A \cos(cnt) + B \sin(cnt), \quad n = 1, 2, \dots,$$

with overall spatial scale absorbed into A, B .

Step 2: Match both initial functions to one spatial shape. A single mode necessarily has $f = A \sin(nx)$ and $g = cnB \sin(nx)$ for the same integer n . Conversely, any such pair gives the displayed mode. Either coefficient may vanish: zero initial displacement permits a pure sine time factor, and zero initial velocity permits a pure cosine time factor. If both vanish, uniqueness for the scalar time ODE gives only the zero field. Thus proportionality alone is insufficient unless the nonzero shape is also an admissible spatial eigenfunction.

Step 3: Construct a field beyond one product. The functions $\sin 2x$ and $\sin 3x$ are not proportional, so one mode cannot provide the prescribed pair. Instead use

$$u = \sin(2x) \cos(2ct) + \frac{1}{3c} \sin(3x) \sin(3ct).$$

Each term satisfies the wave equation and both endpoint values. At $t = 0$, the displacement is $\sin 2x$; differentiating first gives initial velocity $\sin 3x$. These checks establish this finite construction directly.

Step 4: Preserve the solution at a zero time factor. For $u = \sin 2x \sin 2ct$, the initial displacement vanishes everywhere, while $u_t(x, 0) = 2c \sin 2x$ is not identically zero. The time factor crosses zero at $t = 0$ and other times, but its ODE and the PDE remain valid. Dividing by $T(0)$ is undefined; inferring that this mode is forbidden would discard a legitimate oscillatory solution. The anchor argument requires a nonzero value somewhere, not at every time.

Question 6

Consider a single product $u(x, y) = X(x)Y(y)$ satisfying $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < \pi$, $0 < y < H$, where $H > 0$, with

$$u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = \sin(2x).$$

Restrict attention to nonzero smooth separated fields; a general Fourier solution for arbitrary top data is not requested.

Tasks

1. Derive $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$ and determine the allowable λ from the two vertical sides.
2. Use the top data to identify the spatial mode and find Y . Verify all four boundary conditions and the PDE directly.
3. Explain why replacing the hyperbolic sine in Y by an ordinary sine generally fails. Compute the PDE residual for the proposed replacement $\sin(2x) \sin(2y)$.
4. Prove that the solution's vertical amplitude increases from zero to one, yet its absolute value never exceeds one in the rectangle. Compare the signs of the remaining factor equations for heat, wave and Laplace separation with the same positive λ .

Question 6 – Solution

Strategy. The boundary conditions select the spatial eigenvalue, while the PDE determines the sign in the other factor equation.

Step 1: Separate with the correct opposite signs. Substitution yields $X''Y + XY'' = 0$. Nonzero anchors, as in the previous separation arguments, give $X'' = -\lambda X$, $Y'' = \lambda Y$ globally. The vertical sides impose $X(0) = X(\pi) = 0$. Integration by parts excludes $\lambda \leq 0$ for nonzero X ; solving the positive case gives $\lambda = n^2$ and $X = \sin(nx)$, with the spatial constant absorbed into Y .

Step 2: Match the one-mode top trace. The top data are nonzero, so $Y(H) \neq 0$ and X must be proportional to $\sin 2x$. Thus $n = 2$. The remaining conditions are $Y(0) = 0$, $Y(H) = 1$, which determine

$$u(x, y) = \sin(2x) \frac{\sinh(2y)}{\sinh(2H)}.$$

Its second x derivative is $-4u$ and its second y derivative is $4u$. The sum is zero. The sine factor vanishes at the sides, $\sinh 0 = 0$ gives the bottom, and the ratio equals one at the top. The denominator is nonzero since $H > 0$. This uniquely determines the field within the stipulated product class.

Step 3: Reject a plausible sign error. For $\tilde{u} = \sin 2x \sin 2y$, both second derivatives equal $-4\tilde{u}$, so $\Delta \tilde{u} = -8\tilde{u}$, not identically zero. Rescaling a nonzero field to repair top values would not remove this interior residual. The required equation $Y'' - 4Y = 0$ has hyperbolic, rather than sinusoidal, solutions because the Laplace equation sums the second derivatives.

Step 4: Interpret growth in a bounded spatial direction. For $a(y) = \sinh(2y)/\sinh(2H)$, one has $a(0) = 0$, $a(H) = 1$ and $a'(y) = 2 \cosh(2y)/\sinh(2H) > 0$. Hence $0 \leq a \leq 1$ and $|u| \leq 1$. For the same $X'' = -\lambda X$ with $\lambda > 0$, the remaining equations are

$$\begin{aligned} \text{heat:} \quad & T' + \kappa \lambda T = 0, \\ \text{wave:} \quad & T'' + c^2 \lambda T = 0, \\ \text{Laplace:} \quad & Y'' - \lambda Y = 0. \end{aligned}$$

Spatial hyperbolic growth in the last equation is compatible with bounded boundary data on a finite rectangle; it is not a statement about time instability.

Question 7

On $1 < x < e$, $t \geq 0$, consider the variable-coefficient equation

$$\frac{1}{x}u_t = (xu_x)_x, \quad u(1, t) = u(e, t) = 0.$$

For a nonzero product $u = X(x)T(t)$, use the separation convention $(xX')' + \lambda X/x = 0$, $T' = -\lambda T$. Introduce $s = \log x$ and $Z(s) = X(e^s)$.

Tasks

1. Derive the factor equations from the PDE and verify, by the chain rule, the transformed spatial equation and its endpoint conditions in s .
2. Determine every eigenvalue and spatial factor and verify the resulting separated fields in the original x equation.
3. Derive an integral identity that proves eigenvalue positivity. Identify the weight in the squared spatial norm; explain why an unweighted denominator would not follow from this equation.
4. Prove weighted orthogonality of distinct spatial modes by subtracting their ODE identities. Evaluate their squared weighted norms and locate the interior zeros of the third mode in the original coordinate.

Question 7 – Solution

Strategy. A change of spatial coordinate can simplify separation, but the original equation determines the correct integration weight.

Step 1: Transform the spatial equation carefully. Substitution gives $XT'/x = (xX')'T$. Nonzero anchors yield $(xX')' = -\lambda X/x$ and $T' = -\lambda T$. Since $X' = Z_s/x$, one has $xX' = Z_s$ and $(xX')' = Z_{ss}/x$. Multiplying the spatial equation by x gives

$$Z_{ss} + \lambda Z = 0, \quad Z(0) = Z(1) = 0.$$

The physical endpoints $1, e$ become $0, 1$, not an interval of length $e - 1$.

Step 2: Recover the modes in the original variable. The transformed Dirichlet problem has $\lambda_n = n^2\pi^2$ and $Z_n(s) = \sin(n\pi s)$ for $n \geq 1$. Thus

$$u_n(x, t) = C \sin(n\pi \log x) e^{-n^2\pi^2 t}.$$

Indeed $xX'_n = n\pi \cos(n\pi \log x)$ and $(xX'_n)' = -n^2\pi^2 X_n/x$. Multiplication by the time factor gives exactly u_t/x . At $x = 1, e$ the sine values vanish. Zero and negative eigenvalues produce only the trivial Dirichlet factor.

Step 3: Derive the weighted energy quotient. Multiplication of $-(xX')' = \lambda X/x$ by X and integration give

$$\lambda \int_1^e \frac{X^2}{x} dx = \int_1^e x(X')^2 dx.$$

The endpoint term vanishes. For a nonzero factor the denominator is positive; zero numerator would force a constant and then the zero field. Hence $\lambda > 0$. The weight $1/x$ comes from the coefficient of u_t ; dropping it would change the identity and the associated eigenvalue quotient.

Step 4: Verify orthogonality, norms and nodes. For distinct modes m, n , subtracting the two integrated spatial identities cancels the derivative terms and yields $(\lambda_n - \lambda_m) \int_1^e X_n X_m dx/x = 0$. Thus their weighted inner product vanishes. With $s = \log x$,

$$\int_1^e \frac{X_n^2}{x} dx = \int_0^1 \sin^2(n\pi s) ds = \frac{1}{2}.$$

For $n = 3$, the interior zeros occur at $s = 1/3, 2/3$, hence at $x = e^{1/3}, e^{2/3}$. Equal spacing in s is not equal spacing in x .

Question 8

For $u_t = u_{xx}$ on $0 < x < L$, $t \geq 0$, with $L > 0$, impose

$$u(0, t) = 0, \quad u_x(L, t) + hu(L, t) = 0, \quad h \in \mathbb{R}.$$

A separated mode obeys $X'' + \lambda X = 0$, $T' = -\lambda T$. Define the dimensionless parameter $q = hL$. No sign restriction is imposed on h ; a negative value need not model passive heat loss.

Tasks

1. Derive the identity $\lambda \int_0^L X^2 = \int_0^L (X')^2 + hX(L)^2$. Show that $h \geq 0$ guarantees positive eigenvalues, and determine exactly when $\lambda = 0$ has a nonzero factor.
2. For $\lambda > 0$, derive an eigenvalue equation using $z = L\sqrt{\lambda}$. For $\lambda < 0$, use $\rho = L\sqrt{-\lambda}$ to derive $\rho \coth \rho = -q$.
3. Prove that the negative-eigenvalue equation has exactly one root if $q < -1$ and none if $q \geq -1$. Determine the time behavior of this mode and of the zero mode.
4. For $q = -2$, bracket the positive root ρ to within 0.001, state the corresponding eigenvalue in terms of L , and sketch the intersection defining it. Mark the limiting value at $\rho = 0$ as a limit, not a negative eigenvalue.

Question 8 – Solution

Strategy. A Robin boundary term can change the sign of the eigenvalue; analyze zero before treating the oscillatory and hyperbolic cases.

Step 1: Retain the boundary contribution. Integration by parts gives $\lambda \int X^2 = \int (X')^2 - [XX']_0^L = \int (X')^2 + hX(L)^2$. For $h \geq 0$, equality at zero would force $X' = 0$ and then $X = 0$; nonzero factors therefore have $\lambda > 0$. At $\lambda = 0$, $X = Ax$ and the right condition is $A(1 + hL) = 0$. Thus a nonzero zero mode exists exactly when $\boxed{q = -1}$.

Step 2: Separate the positive and negative cases. For $\lambda > 0$, $X = A \sin(\sqrt{\lambda}x)$ and

$$\boxed{z \cos z + q \sin z = 0, \quad z > 0.}$$

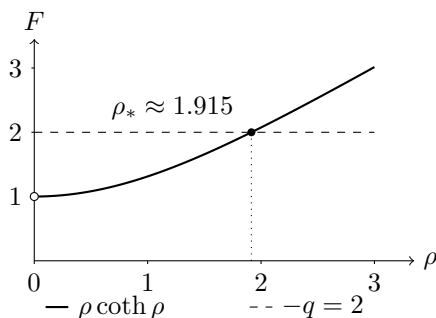
For $\lambda = -\mu^2 < 0$, $X = A \sinh(\mu x)$ and the right condition becomes $\mu \cosh(\mu L) + h \sinh(\mu L) = 0$. Since $\rho = \mu L > 0$,

$$\boxed{\rho \coth \rho = -q, \quad \lambda = -\rho^2/L^2.}$$

Dividing by $\sinh \rho$ here is valid only in this strictly negative case.

Step 3: Prove the threshold and uniqueness. Let $F(\rho) = \rho \coth \rho$. It has limit 1 as $\rho \downarrow 0$ and $F(\rho) \rightarrow \infty$. Its derivative is $F' = (\sinh \rho \cosh \rho - \rho) / \sinh^2 \rho > 0$: the numerator is zero at zero and has derivative $2 \sinh^2 \rho > 0$ for $\rho > 0$. Therefore its range is $(1, \infty)$, giving exactly one negative eigenvalue iff $q < -1$. That mode grows as $e^{\rho^2 t/L^2}$. At $q = -1$, the zero mode $X = x$ is stationary; it is not a negative eigenmode.

Step 4: Quantify the concrete crossing. For $q = -2$, numerical substitution gives $F(1.915) < 2 < F(1.916)$. Strict monotonicity certifies one root in this interval, approximately $\rho = 1.915008$. Thus $\lambda \approx -3.667256/L^2$. The open point at $(0, 1)$ records the excluded endpoint limit of F .



Question 9

Consider smooth product candidates $u(x, t) = X(x)T(t)$, neither factor identically zero, on an open rectangle $I \times J$, where each interval has positive length. Compare

$$(A) \quad u_t = u_{xx} + xtu, \quad (B) \quad u_t = u_{xx} + [a(x) + b(t)]u.$$

In (B), the given coefficient functions are smooth. No boundary data are imposed.

Tasks

1. Show that a nonzero product solving (A) would imply $T'/T - X''/X = xt$ on some smaller open rectangle where both factors are nonzero.
2. Prove that this identity is impossible by comparing two positions and two times. Explain why zeros of the factors cannot avoid the contradiction.
3. Derive separated ODEs for (B), using $X'' + a(x)X = -\lambda X$. Explain how the additive coefficient structure permits separation even though the PDE has variable coefficients.
4. Take $a(x) = -x^2$, $b(t) = t$. Construct a nonzero product with $X = e^{-x^2/2}$, determine λ and T , and verify the resulting field directly in (B).

Question 9 – Solution

Strategy. Test the coefficient's dependence on both variables before assuming that the product ansatz can succeed.

Step 1: Divide only on a legitimate local rectangle. A nonzero value of each continuous factor has a neighborhood where that factor is nonzero. On the product of those neighborhoods, substitution into (A) gives

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} = xt.$$

The left side is a function of time minus a function of position. This local identity is already enough to test whether any nonzero product can exist.

Step 2: Use a mixed difference, not extra derivatives. Choose distinct x_1, x_2 and distinct t_1, t_2 in that rectangle. Subtract the four identities in the pattern $(x_1, t_1) - (x_1, t_2) - (x_2, t_1) + (x_2, t_2)$. All terms on the left cancel, whereas the right gives

$$0 = (x_1 - x_2)(t_1 - t_2),$$

which is false. Thus (A) has no nonzero product solution on an open rectangle. Factor zeros elsewhere cannot undo this contradiction in a nonzero neighborhood. The zero solution still exists; the conclusion does not assert that all solutions of (A) vanish.

Step 3: Separate an additive coefficient instead. For (B), substitution rearranges to $X[T' - b(t)T] = [X'' + a(x)X]T$. Nonzero anchors give, with one constant λ ,

$$X'' + [a(x) + \lambda]X = 0, \quad T' = [b(t) - \lambda]T.$$

The remaining coefficients each depend on only the appropriate variable. Conversely these ODEs imply the product PDE everywhere. Variable coefficients do not alone prevent separation; their dependence on the variables matters.

Step 4: Verify a nonconstant-coefficient example. For $X = e^{-x^2/2}$, $X'' = (x^2 - 1)X$, so $X'' - x^2X = -X$ and $\lambda = 1$. The time equation $T' = (t - 1)T$ gives $T = Ce^{t^2/2 - t}$. Hence

$$u = Ce^{-x^2/2 + t^2/2 - t}, \quad C \neq 0.$$

Directly $u_t = (t - 1)u$ and $u_{xx} = (x^2 - 1)u$; adding $(-x^2 + t)u$ to the second expression gives the first. This is an explicit product solution of (B), not a solution of (A).

Question 10

For $u_t = u_{xx} + ru$ on $0 < x < \pi$, $t \geq 0$, impose homogeneous Dirichlet endpoints. The separated spatial modes are $\sin(nx)$ with eigenvalues n^2 , $n = 1, 2, \dots$. Consider $1 < r < 4$ and the finite-mode initial field

$$u(x, 0) = \sin(2x) + \varepsilon \sin x, \quad 0 < \varepsilon < 1.$$

Only this finite-mode construction is requested; no arbitrary-data expansion or infinite-series convergence theorem is needed.

Tasks

1. Derive the time equation for each mode and construct a solution with the given initial field. Verify its PDE and endpoints.
2. Compare the behavior when $\varepsilon = 0$ with every fixed $\varepsilon > 0$. Prove that in the latter case $e^{-(r-1)t}u$ converges uniformly to $\varepsilon \sin x$.
3. Using $\|v\|_2^2 = \int_0^\pi v^2 dx$, compute the squared norm of each modal component and determine the time when the growing component overtakes the decaying component in this norm.
4. For $r = 2$, $\varepsilon = 1/8$, sketch the two positive modal amplitudes and mark their crossing. Explain why finding a decaying separated solution alone does not show that all solutions decay, and state the first-mode threshold in r .

Question 10 – Solution

Strategy. Separation reveals individual growth rates; superposition shows how a small unstable component can dominate a decaying one.

Step 1: Construct and verify the finite sum. For $X_n = \sin nx$, $X_n'' = -n^2 X_n$, so the factor equation is $T_n' = (r - n^2)T_n$. Consequently

$$u = e^{(r-4)t} \sin 2x + \varepsilon e^{(r-1)t} \sin x.$$

Each time derivative has coefficient $r - n^2$, matching $u_{xx} + ru$. Both sine factors vanish at $0, \pi$, and their initial coefficients are $1, \varepsilon$. Linearity justifies adding these two verified modes.

Step 2: Compare exact cancellation with small contamination. For $\varepsilon = 0$, the solution decays uniformly because $r - 4 < 0$. For fixed $\varepsilon > 0$, the first mode grows because $r - 1 > 0$. Moreover

$$e^{-(r-1)t} u = \varepsilon \sin x + e^{-3t} \sin 2x, \quad \|e^{-(r-1)t} u - \varepsilon \sin x\|_\infty = e^{-3t}.$$

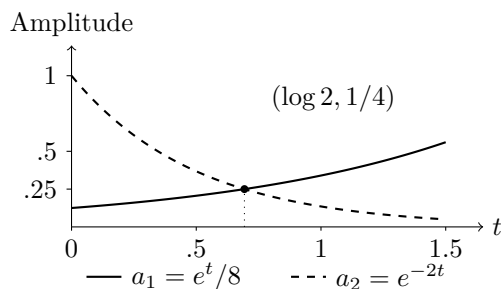
This proves uniform convergence of the normalized shape. The unnormalized solution grows without bound in sup norm, as is also seen at $x = \pi/2$.

Step 3: Compare modal norms exactly. Since $\int_0^\pi \sin^2(nx) dx = \pi/2$, the squared component norms are

$$N_2^2 = \frac{\pi}{2} e^{2(r-4)t}, \quad N_1^2 = \frac{\pi}{2} \varepsilon^2 e^{2(r-1)t}.$$

Their equality is $\varepsilon^2 e^{6t} = 1$, giving $t_c = \frac{1}{3} \log(1/\varepsilon)$. The growing component is larger for $t > t_c$. The crossing time is independent of r in the stated range because the difference of modal rates is always three. Orthogonality also gives $\|u\|_2^2 = N_1^2 + N_2^2$.

Step 4: Interpret the crossing and the threshold. For $r = 2$, $\varepsilon = 1/8$, the amplitudes are $a_1 = e^t/8$ and $a_2 = e^{-2t}$. They cross at $t_c = \log 2$, with common value $1/4$. A decaying mode only describes its own initial shape. A nonzero first-mode component decays for $r < 1$, is stationary for $r = 1$, and grows for $r > 1$. Thus one decaying example cannot establish decay for every admissible field.



Question 1

Solve $u_t = u_{xx}$ for $0 < x < \pi$, $t > 0$, with

$$u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = \sin x + \frac{1}{2} \sin(2x).$$

The temperature is measured relative to the zero endpoint temperature.

Tasks

1. Construct the solution from the initial modes and verify the PDE, both endpoint conditions and the initial trace.
2. Prove directly from the formula that the initial temperature and every positive-time temperature are nonnegative. Does this force the temperature at every fixed interior point to decrease?
3. At $x = 5\pi/6$, determine the initial rate of change and the exact time of the unique temperature maximum. Justify the sign change of the time derivative.
4. Compute the total heat $H(t) = \int_0^\pi u(x, t) dx$ and its boundary-flux balance. Sketch profiles at $t = 0, 1/5, 1$ and explain how local warming is consistent with net heat loss.

Question 1 – Solution

Strategy. Different spatial modes decay at different rates; local temperature and total heat need not have the same monotonicity.

Step 1: Evolve each initial mode. Since $(\sin nx)'' = -n^2 \sin nx$, the solution is

$$u = e^{-t} \sin x + \frac{1}{2} e^{-4t} \sin 2x.$$

Termwise differentiation of this finite sum gives $u_t = u_{xx}$. Both sines vanish at the endpoints, and setting $t = 0$ gives the prescribed field. The formula is smooth on the closed space interval for all $t \geq 0$.

Step 2: Prove nonnegativity without a plot. Factor the solution as $u = e^{-t} \sin x [1 + e^{-3t} \cos x]$. For $0 \leq x \leq \pi$ and $t \geq 0$, $\sin x \geq 0$ and the bracket is nonnegative. At positive time the bracket is strictly positive; hence the interior temperature is positive. This sign statement concerns values, not their time derivatives. Diffusion can transport heat into a cooler interior region.

Step 3: Find the local warming interval. At $x = 5\pi/6$,

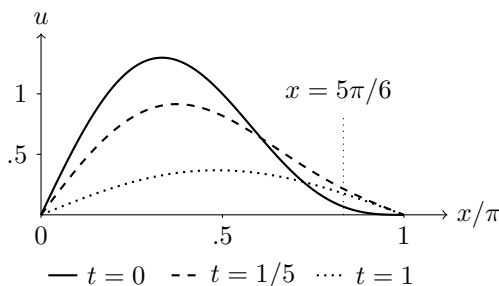
$$u = \frac{1}{2} e^{-t} - \frac{\sqrt{3}}{4} e^{-4t}, \quad u_t = -\frac{1}{2} e^{-t} + \sqrt{3} e^{-4t}.$$

Thus $u_t(x, 0) = \sqrt{3} - 1/2 > 0$. Factoring out e^{-4t} shows that the derivative changes sign exactly when $e^{3t} = 2\sqrt{3}$. The unique maximum occurs at

$$t_* = \frac{1}{3} \log(2\sqrt{3}).$$

Before t_* the derivative is positive; afterward it is negative. This gives a concrete counterexample to pointwise monotone cooling.

Step 4: Compare with the global heat balance. Integration gives $H(t) = 2e^{-t}$, since the second sine has zero integral. Also $u_x = e^{-t} \cos x + e^{-4t} \cos 2x$, so $H'(t) = [u_x]_0^\pi = -2e^{-t} < 0$. The right endpoint derivative minus the left is the signed flux balance obtained by integrating the PDE. Total heat falls even while a neighborhood of $5\pi/6$ initially warms.



Question 2

Let $\kappa, L > 0$ and solve $u_t = \kappa u_{xx}$ on $0 < x < L, t > 0$, with zero endpoint values and initial temperature $f(x) = x(L - x)$. You may use Fourier sine convergence for piecewise smooth odd extensions.

Tasks

1. Compute every coefficient $b_n = (2/L) \int_0^L f(x) \sin(n\pi x/L) dx$ and construct the solution series.
2. Justify uniform convergence to the initial trace as $t \downarrow 0$ and termwise verification of the PDE for $t > 0$. Do not infer corner smoothness from uniform convergence of the values.
3. Explain the absence of the even modes from reflection symmetry. Determine the total heat $H(t)$ as a convergent series and evaluate $H(0)$ directly.
4. Differentiate H for positive time and verify $H' = \kappa[u_x]_0^L$. Explain why H strictly decreases and why the solution is not a single separated product.

Question 2 – Solution

Strategy. Exact coefficients control the initial convergence, while Gaussian mode decay justifies later differentiation.

Step 1: Compute the sine coefficients. Put $k = n\pi/L$. Twice integrating by parts, using $f(0) = f(L) = 0$ and $f'' = -2$, gives $\int_0^L f \sin(kx) dx = 2(1 - (-1)^n)/k^3$. Thus $b_n = 4L^2(1 - (-1)^n)/(n^3\pi^3)$, and

$$u(x, t) = \frac{8L^2}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x/L)}{n^3} e^{-\kappa(n\pi/L)^2 t}.$$

Step 2: Verify the full initial-value solution. The coefficients are absolutely summable, so the series at $t = 0$ converges uniformly. Fourier convergence identifies its sum as f on $[0, L]$, including the zero endpoints. Dominated convergence of the coefficient sum then gives $\|u(\cdot, t) - f\|_\infty \rightarrow 0$. For $t \geq \tau > 0$, any fixed number of spatial or time derivatives only adds powers of n ; the factor $e^{-\kappa(n\pi/L)^2 \tau}$ makes these series uniformly summable. Hence $u_t = \kappa u_{xx}$ termwise for positive time. At an initial corner, $f'' = -2$ conflicts with the zero boundary time derivative; continuity of all second spatial derivatives through those corners is not claimed.

Step 3: Use symmetry and integrate the solution. Since $f(L - x) = f(x)$ and $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$, even coefficients vanish. Integrating the uniformly convergent series gives

$$H(t) = \frac{16L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^4}, \quad H(0) = \int_0^L x(L - x) dx = \frac{L^3}{6}.$$

Step 4: Check heat loss and the mode content. For $t > 0$, differentiation gives

$$H'(t) = -\frac{16\kappa L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^2} = \kappa[u_x]_0^L.$$

The equality on the right follows by differentiating the spatial sine series and using the odd-mode cosine endpoint values. Every summand is positive, so $H' < 0$. Infinitely many nonzero coefficients decay at distinct rates; in particular the ratio of the third coefficient to the first changes with time. A single product would have fixed ratios wherever its time factor is nonzero.

Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$ with $\kappa, L > 0$, zero endpoint values, and $0 < a < L$, take the continuous tent-shaped initial temperature

$$f(x) = \begin{cases} x/a, & 0 \leq x \leq a, \\ (L-x)/(L-a), & a \leq x \leq L. \end{cases}$$

Its height is one. You may use Fourier convergence for continuous piecewise smooth functions with zero endpoint values.

Tasks

1. Derive all sine coefficients by splitting the integral at a , retaining the change in slope, and write the heat solution.
2. Justify the initial trace and positive-time smoothing even though f' jumps at a .
3. Show how the location a can be recovered from the ratio b_2/b_1 . State its allowed range and explain why the recovery is unique when $0 < a < L$.
4. For $L = \kappa = 1$, $a = 1/3$, simplify the coefficients, identify the missing modes, and sketch the initial tent and the profiles at $t = 0.01, 0.05, 0.15$.

Question 3 – Solution

Strategy. The corner in the initial profile appears as a slope jump in the coefficient calculation, not as a failure of the positive-time heat solution.

Step 1: Compute the contribution of the slope jump. With $k = n\pi/L$, integration by parts once gives $\int f \sin(kx) = k^{-1} \int f' \cos(kx)$ since f vanishes at both endpoints. Using $f' = 1/a$ to the left and $-1/(L-a)$ to the right gives

$$\int_0^L f \sin(kx) dx = \frac{L \sin(ka)}{a(L-a)k^2}.$$

Therefore

$$b_n = \frac{2L^2 \sin(n\pi a/L)}{\pi^2 n^2 a(L-a)}, \quad u = \sum_{n=1}^{\infty} b_n e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L).$$

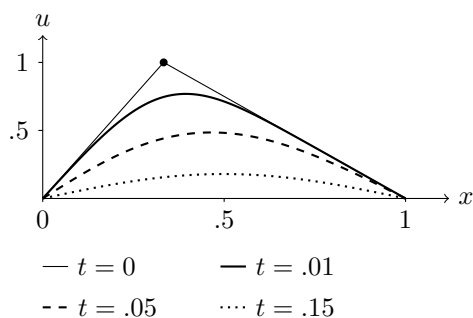
Step 2: Separate initial regularity from later regularity. Since $|b_n| \leq C/n^2$, the sine series converges absolutely and uniformly, and Fourier convergence identifies its initial sum as the continuous tent. Summable domination gives uniform convergence to f as $t \downarrow 0$. For every $\tau > 0$, polynomial factors from any finite number of derivatives are dominated by Gaussian decay for $t \geq \tau$. Thus the series is smooth for positive time, satisfies the PDE termwise and retains zero endpoints. Its initial derivative jump does not persist as a corner in a positive-time profile.

Step 3: Recover the peak location from two modes. Put $\theta = \pi a/L \in (0, \pi)$. Then $b_1 > 0$ and

$$\frac{b_2}{b_1} = \frac{\sin 2\theta}{4 \sin \theta} = \frac{1}{2} \cos \theta, \quad a = \frac{L}{\pi} \arccos(2b_2/b_1).$$

The ratio must lie strictly between $-1/2$ and $1/2$. Cosine is strictly decreasing on $(0, \pi)$, so this reconstruction is unique in the stated family.

Step 4: Specialize and interpret the profiles. For $L = \kappa = 1$, $a = 1/3$, $b_n = 9 \sin(n\pi/3)/(\pi^2 n^2)$. Exactly the multiples of three vanish, and $b_2/b_1 = 1/4$ recovers $a = 1/3$. The figure uses the exact piecewise initial trace and rapidly convergent positive-time series. The peak rounds as the profile diffuses toward zero.



Question 4

Solve $u_t = u_{xx}$ on $0 < x < \pi$, $t > 0$, with $u(0, t) = u(\pi, t) = 0$ and initial data in the L^2 sense

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 1/2, & x = \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

The L^2 norm is $\|v\|_2^2 = \int_0^\pi |v|^2 dx$. You may use sine-series convergence at continuity points and to the average at a jump, and Parseval's identity.

Tasks

1. Compute the sine coefficients and construct the heat solution.
2. Prove that the solution is smooth and solves the PDE for positive time, and that it approaches f in L^2 as $t \downarrow 0$.
3. State the pointwise limit at the interior jump and at each boundary when boundary values are retained as zero. Explain why convergence cannot be uniform on $[0, \pi]$, even if the endpoints of f are defined to be zero.
4. Prove uniqueness among solutions smooth up to the spatial boundaries for $t > 0$ and continuous into L^2 at $t = 0$, with these zero endpoint values. Use an energy estimate starting at $t = \delta > 0$ and then let $\delta \downarrow 0$.

Question 4 – Solution

Strategy. Interpret the initial condition in its stated norm; incompatible corners and jumps do not invalidate a positive-time series solution.

Step 1: Compute all modal amplitudes. Integration over the nonzero half of the interval gives

$$b_n = \frac{2}{\pi n} (1 - \cos(n\pi/2)), \quad u(x, t) = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx).$$

The point value assigned at $\pi/2$ does not affect the integral. Modes divisible by four vanish; the other coefficients must not all be replaced by the odd-only coefficients for a constant temperature on the entire interval.

Step 2: Verify positive-time smoothness and the norm trace. For $t \geq \tau > 0$, $|b_n| \leq 4/(\pi n)$ and the exponential makes all finitely differentiated series uniformly convergent. Thus the PDE and endpoints hold. Parseval and the sine expansion of f give

$$\|u(\cdot, t) - f\|_2^2 = \frac{\pi}{2} \sum_{n \geq 1} b_n^2 (1 - e^{-n^2 t})^2 \rightarrow 0,$$

by summable domination, since $\sum b_n^2 < \infty$. This is precisely the specified initial condition.

Step 3: State the pointwise limits correctly. The limits are 1 on $(0, \pi/2)$, 0 on $(\pi/2, \pi)$ and $1/2$ at the jump. To justify Gaussian damping of a pointwise convergent Fourier series, apply summation by parts: the decreasing weights $e^{-n^2 t}$ average its partial sums, have total weight tending to one, and move past each fixed index as $t \downarrow 0$. Thus they preserve the Fourier limit. At both endpoints u is identically zero for $t > 0$, so the limit is zero. A uniform limit of continuous functions on $[0, \pi]$ would be continuous; the jump and the left corner mismatch preclude uniform convergence to these data.

Step 4: Prove uniqueness in the stated class. Let w be the difference of two such solutions. For $t \geq \delta > 0$,

$$\frac{d}{dt} \|w\|_2^2 = 2 \int_0^\pi w w_{xx} dx = -2 \int_0^\pi w_x^2 dx \leq 0.$$

The boundary term vanishes. Hence $\|w(\cdot, t)\|_2 \leq \|w(\cdot, \delta)\|_2$. Both solutions have the same L^2 initial trace, so the right side tends to zero as $\delta \downarrow 0$. Positive-time continuity then makes $w = 0$ pointwise. No smoothness at the discontinuous initial trace was assumed.

Question 5

Let $\kappa, L > 0$. Solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with insulated ends $u_x(0, t) = u_x(L, t) = 0$ and initial temperature $f(x) = x$. Use the convention $f = a_0 + \sum_{n \geq 1} a_n \cos(n\pi x/L)$, where $a_0 = L^{-1} \int_0^L f$ and $a_n = (2/L) \int_0^L f \cos(n\pi x/L)$. You may use cosine-series convergence and Parseval's identity.

Tasks

1. Compute the coefficients and construct the solution, retaining the zero mode.
2. Prove conservation of the mean and the reflection identity $u(L - x, t) = L - u(x, t)$. Determine the limiting equilibrium.
3. Justify uniform recovery of the initial values, and explain why the spatial derivative cannot extend continuously through the initial boundary corners.
4. Compute $V(t) = \int_0^L (u - L/2)^2 dx$ as a series, find $V(0)$ and establish $V(t) \leq e^{-2\kappa(\pi/L)^2 t} V(0)$. Sketch profiles for $L = \kappa = 1$ at $t = 0, 0.03, 0.15$.

Question 5 – Solution

Strategy. Insulation preserves the constant component while all nonconstant cosine modes decay.

Step 1: Compute and evolve the cosine coefficients. Integration gives $a_0 = L/2$ and $a_n = 2L((-1)^n - 1)/(\pi^2 n^2)$. Therefore

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \cos(n\pi x/L)}{n^2}.$$

Gaussian decay justifies the PDE and derivative boundary conditions termwise for $t > 0$. The constant mode remains stationary.

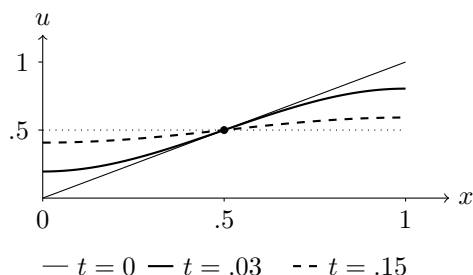
Step 2: Identify the conserved and symmetric parts. Integrating the PDE yields $(\int u)' = \kappa[u_x]_0^L = 0$, so the mean is $L/2$. For odd n , $\cos(n\pi(L-x)/L) = -\cos(n\pi x/L)$. The series therefore gives $u(L-x, t) = L - u(x, t)$, in particular $u(L/2, t) = L/2$ at all times. The absolutely summable coefficients and positive modal decay imply uniform convergence to the equilibrium $L/2$ as $t \rightarrow \infty$.

Step 3: Check values and derivatives separately. The n^{-2} coefficients give absolute uniform convergence at $t = 0$; the cosine convergence theorem identifies the sum as x , including both endpoints. Summable domination proves uniform recovery of those values as $t \downarrow 0$. But the initial trace has derivative one, whereas $u_x = 0$ at the endpoints for every positive time. A continuously extending spatial derivative would have to take both values at each initial corner, which is impossible. Uniform convergence of temperatures does not imply convergence of derivatives.

Step 4: Quantify equilibration. Orthogonality gives

$$V(t) = \frac{8L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-2\kappa(n\pi/L)^2 t}}{n^4}, \quad V(0) = \int_0^L (x - L/2)^2 dx = \frac{L^3}{12}.$$

Each exponential is at most $e^{-2\kappa(\pi/L)^2 t}$, proving the required bound. The profiles flatten toward their conserved mean, rather than toward zero. Only the initial profile has the incompatible endpoint slopes.



Question 6

Solve $u_t = \kappa u_{xx}$ for $0 < x < L$, $t > 0$, with $\kappa, L > 0$,

$$u(0, t) = 0, \quad u_x(L, t) = 0, \quad u(x, 0) = x(2L - x).$$

You may use the complete orthogonal family $\sin(k_n x)$, $k_n = (n + 1/2)\pi/L$, $n = 0, 1, \dots$, with squared norm $L/2$, and its Fourier convergence for the odd-at-zero, even-at- L extension of these data.

Tasks

1. Compute the expansion coefficients using the actual mixed conditions and construct the solution.
2. Verify the initial trace, PDE and two different endpoint conditions, justifying the series operations.
3. Compute total heat and its rate of loss. Explain why only the left endpoint contributes to the heat flux.
4. Determine the limiting shape of $e^{\kappa k_0^2 t} u$ uniformly on $[0, L]$. Compare its slowest exponential decay rate with that of a rod of the same length and diffusivity with both endpoints fixed at zero.

Question 6 – Solution

Strategy. Use half-integer modes for the mixed boundary problem; integer sine modes would solve a different problem.

Step 1: Evaluate the coefficient integral. For $f = x(2L - x)$, $f(0) = 0$, $f'(L) = 0$ and $f'' = -2$. Two integrations by parts, using $\cos(k_n L) = 0$, give

$$\int_0^L f \sin(k_n x) dx = \frac{2}{k_n^3}, \quad u(x, t) = \frac{4}{L} \sum_{n=0}^{\infty} \frac{e^{-\kappa k_n^2 t} \sin(k_n x)}{k_n^3}.$$

The vanishing boundary terms use the derivative condition at L , not a zero value of $f(L)$, which is actually L^2 .

Step 2: Verify the complete solution. The coefficients are summable, so the initial series is uniformly convergent; the supplied Fourier theorem identifies it as f . Summable domination gives uniform convergence to f as $t \downarrow 0$. For $t \geq \tau > 0$, differentiated series converge uniformly by exponential decay and verify the PDE. At zero, all sine factors vanish; at L , all derivative cosine factors vanish. The initial function also satisfies these value/first-derivative data, although higher corner compatibility is a separate issue.

Step 3: Compute heat loss through the open thermal boundary. Since $\int_0^L \sin(k_n x) dx = 1/k_n$,

$$H(t) = \frac{4}{L} \sum_{n \geq 0} \frac{e^{-\kappa k_n^2 t}}{k_n^4}, \quad H(0) = \frac{2L^3}{3}.$$

For $t > 0$, $H' = -(4\kappa/L) \sum e^{-\kappa k_n^2 t}/k_n^2 = -\kappa u_x(0, t)$. This is $\kappa[u_x]_0^L$ because the right derivative is zero. Every term in the loss sum is positive; insulation blocks flux at the right endpoint only.

Step 4: Extract the slowest mode rigorously. Multiplying the series by $e^{\kappa k_0^2 t}$ leaves the first term and a remainder bounded by $e^{-\kappa(k_1^2 - k_0^2)t} \sum_{n \geq 1} 4/(Lk_n^3)$, which tends to zero. Hence uniformly,

$$e^{\kappa k_0^2 t} u \longrightarrow \frac{32L^2}{\pi^3} \sin(\pi x/(2L)).$$

The slowest rate is $\kappa\pi^2/(4L^2)$, one quarter of the Dirichlet–Dirichlet rate $\kappa\pi^2/L^2$. Equal material properties and length do not imply equal decay rates when the boundary conditions differ.

Question 7

For the zero-endpoint heat problem $u_t = u_{xx}$ on $(0, \pi)$, let $f \in L^2(0, \pi)$ with $\|f\|_2 \leq M$, where $M > 0$. Its solution for $t > 0$ is

$$u = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx), \quad u_N = \sum_{n=1}^N b_n e^{-n^2 t} \sin(nx), \quad N \geq 1.$$

You may use Parseval: $\sum b_n^2 = 2\|f\|_2^2/\pi$, and Cauchy-Schwarz.

Tasks

1. Prove $\|u - u_N\|_2 \leq M e^{-(N+1)^2 t}$. Show that the bound is sharp over the stated data class for each N, t .
2. Derive a uniform-in- x error bound using Cauchy-Schwarz and the tail sum $\sum_{n > N} e^{-2n^2 t}$.
3. Bound that sum by a geometric series using $n = N + 1 + j$, $j \geq 0$, and obtain an explicit bound requiring no infinite summation.
4. For $M = 1, t \geq 0.1$, certify that six modes give a uniform error below 0.01. Explain why the available bound does not certify five modes, and why this does not prove that five modes fail for every particular initial field.

Question 7 – Solution

Strategy. A truncation certificate needs a bound on the omitted coefficients, not just a graph of the retained modes.

Step 1: Prove the sharp norm estimate. Orthogonality gives

$$\|u - u_N\|_2^2 = \frac{\pi}{2} \sum_{n>N} b_n^2 e^{-2n^2 t} \leq e^{-2(N+1)^2 t} \|f\|_2^2.$$

Taking square roots proves the bound. Equality is attained by $f = M\sqrt{2/\pi} \sin((N+1)x)$: all its energy lies in the first omitted mode. Thus the rate and constant are sharp for the whole L^2 ball.

Step 2: Control pointwise errors uniformly. Using $|\sin nx| \leq 1$ and Cauchy–Schwarz,

$$|u - u_N| \leq \left(\sum_{n>N} b_n^2 \right)^{1/2} \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2} \leq \sqrt{\frac{2}{\pi}} M \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2}.$$

This bound is independent of x and finite for $t > 0$. It also justifies uniform convergence of the positive-time solution for arbitrary L^2 data.

Step 3: Replace the tail by a geometric bound. For the integer $j \geq 0$, $(N+1+j)^2 \geq (N+1)^2 + (2N+3)j$, since $j^2 \geq j$. Therefore

$$\sum_{n>N} e^{-2n^2 t} \leq \frac{e^{-2(N+1)^2 t}}{1 - e^{-2(2N+3)t}}, \quad \|u - u_N\|_\infty \leq \sqrt{\frac{2}{\pi}} \frac{M e^{-(N+1)^2 t}}{\sqrt{1 - e^{-2(2N+3)t}}}.$$

The denominator is positive because $t > 0$. Each factor makes this certificate decrease as time increases.

Step 4: Make a numerical guarantee with the correct scope. At $t = 0.1$, the bound is less than 0.00611 for $N = 6$ and greater than 0.0226 for $N = 5$. Hence six modes certify the requested 0.01 uniform error for every $t \geq 0.1$ and every datum with $\|f\|_2 \leq 1$. The five-mode certificate is too large; it does not prove failure for each field. For example, a datum supported entirely in the first five modes has zero truncation error with five modes. The bound is a sufficient test, not an exact error formula for unknown data.

Question 8

For $\alpha > 0$, solve the forced heat problem

$$u_t = u_{xx} + e^{-\alpha t} \sin x, \quad 0 < x < \pi, \quad t > 0, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = 0.$$

Use a field of the form $u(x, t) = q_\alpha(t) \sin x$. This is interior forcing; the boundary temperatures remain zero.

Tasks

1. Derive and solve the scalar initial-value problem for q_α , treating $\alpha = 1$ separately.
2. Verify the PDE and all data, and prove that the constructed amplitude is positive for $t > 0$. Explain why uniqueness holds in the class of smooth solutions continuous into L^2 initially.
3. Determine the unique time of maximum amplitude for $\alpha \neq 1$ and its limit as $\alpha \rightarrow 1$. Check the maximum directly for $\alpha = 1$.
4. Derive the total-heat balance and compare matching temporal decay rates here with unbounded oscillatory resonance. Does the factor t in the exceptional solution cause unbounded temperature?

Question 8 – Solution

Strategy. The source has one spatial eigenfunction, so the full problem reduces to a forced scalar decay equation.

Step 1: Solve the amplitude equation including the exceptional rate. Substitution gives $q'_\alpha + q_\alpha = e^{-\alpha t}$, $q_\alpha(0) = 0$. Multiplying by e^t and integrating yields

$$q_\alpha(t) = \begin{cases} (e^{-\alpha t} - e^{-t})/(1 - \alpha), & \alpha \neq 1, \\ te^{-t}, & \alpha = 1. \end{cases}$$

The exceptional formula follows from the same integrating-factor integral; it is not obtained by substituting into a zero denominator.

Step 2: Verify existence, sign and uniqueness. The scalar ODE verifies the PDE, while the sine factor and $q_\alpha(0) = 0$ verify endpoints and initial data. The integral representation

$$q_\alpha(t) = \int_0^t e^{-(t-s)} e^{-\alpha s} ds$$

is strictly positive for $t > 0$. The difference of two solutions with the same source solves the homogeneous zero-data problem. Its squared L^2 norm is nonincreasing by integration by parts; starting at a positive time and using the zero initial norm proves uniqueness in the stated class.

Step 3: Locate the unique maximum. For $\alpha \neq 1$, $q'_\alpha = (e^{-t} - \alpha e^{-\alpha t})/(1 - \alpha)$. The equation $q'_\alpha = 0$ is $e^{(\alpha-1)t} = \alpha$, giving

$$t_\alpha = \frac{\log \alpha}{\alpha - 1} > 0.$$

There is exactly one root; $q'_\alpha(0) = 1 > 0$ and the derivative is negative for sufficiently large time, proving it is the unique maximum. As $\alpha \rightarrow 1$, $t_\alpha \rightarrow 1$. Directly $(te^{-t})' = e^{-t}(1 - t)$, with maximum $1/e$ at $t = 1$.

Step 4: Interpret the forced heat balance. The total heat is $H = 2q_\alpha$. Since $[u_x]_0^\pi = -2q_\alpha$ and the source integral is $2e^{-\alpha t}$,

$$H' = -2q_\alpha + 2e^{-\alpha t},$$

which agrees with the scalar ODE. At $\alpha = 1$, matching the source decay rate to the homogeneous decay produces te^{-t} . This field is bounded and tends to zero; its time factor does not imply unbounded oscillatory resonance. For every fixed $\alpha > 0$, the source and the solution ultimately decay.

Question 9

For $u_t = u_{xx}$ on $(0, \pi)$ with zero endpoint values, let $\phi_n(x) = \sqrt{2/\pi} \sin(nx)$ be the orthonormal spatial modes. Write $u(x, 0) = \sum c_n \phi_n(x)$ and $u(x, T) = \sum d_n \phi_n(x)$, where $T > 0$. All norms in this question are $L^2(0, \pi)$ norms.

Tasks

1. Relate c_n and d_n . Prove that there is at most one L^2 initial field for given exact final data, and state the precise coefficient condition for such an initial field to exist.
2. Use initial fields ϕ_n to show that uniqueness of backward reconstruction does not imply continuous dependence on final data in these norms.
3. Suppose measurement noise in the final field has norm at most δ . Reconstruct only modes $1, \dots, N$ and prove that the reconstructed noise has norm at most $e^{N^2 T} \delta$. Show that this bound is sharp.
4. For $T = 0.1$, $\delta = 10^{-6}$, choose the largest integer N certified to keep reconstructed noise below 0.01. Plot $\log_{10}$ of modal amplification against integer mode number, and explain why controlling noise alone does not bound the omitted true initial modes.

Question 9 – Solution

Strategy. Each forward-decaying mode becomes an exponentially amplified mode when time is reversed.

Step 1: Characterize exact inversion. Forward evolution gives $d_n = c_n e^{-n^2 T}$, so the only possible coefficients are $c_n = e^{n^2 T} d_n$. Completeness implies at most one initial field. It exists in L^2 exactly when

$$\sum_{n \geq 1} e^{2n^2 T} |d_n|^2 < \infty.$$

Indeed this condition constructs an L^2 sine expansion whose forward solution has the given coefficients. Arbitrary L^2 final data need not satisfy this stronger condition.

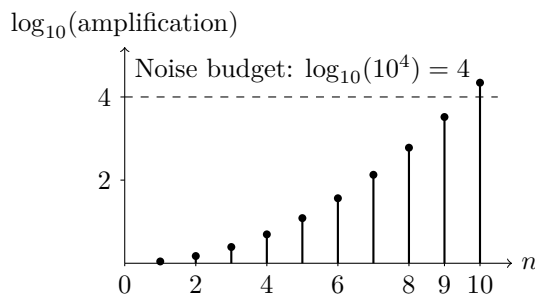
Step 2: Disprove continuity despite uniqueness. Take $f_n = \phi_n$, whose initial norm is one. Its final field has norm $e^{-n^2 T} \rightarrow 0$. Thus final data tend to zero while their uniquely determined initial fields do not. The backward map on its attainable-data domain is not continuous at zero in the stated norms. No contradiction arises with the stable forward problem, which damps these same modes.

Step 3: Bound noise after truncating the inverse. Let η_n be the noise coefficients, so $\sum |\eta_n|^2 \leq \delta^2$. The retained reconstructed noise satisfies

$$\left\| \sum_{n=1}^N e^{n^2 T} \eta_n \phi_n \right\|_2^2 = \sum_{n=1}^N e^{2n^2 T} |\eta_n|^2 \leq e^{2N^2 T} \delta^2.$$

Taking square roots gives the bound, attained by noise $\delta \phi_N$.

Step 4: Choose a noise cutoff and state its limitation. The requirement is $e^{0.1 N^2} 10^{-6} < 10^{-2}$, or $N^2 < 40 \log 10$. Thus $\lfloor N = 9 \rfloor$ is the largest certified integer: the bounds for 9, 10 are approximately 0.003295 and 0.022027. The plot uses discrete stems, since n is an integer; their heights are $0.1 n^2 / \log 10$. The omitted initial tail has norm $(\sum_{n > N} |c_n|^2)^{1/2}$, for which no numerical bound was supplied. A noise cutoff controls amplification but requires additional prior information to control that approximation error.



Question 10

Let $u_t = \kappa u_{xx}$ on $(0, L)$ with $\kappa, L > 0$, zero endpoint values, and initial field $f \in L^2(0, L)$. Let b_n be its sine coefficients and $E(t) = \frac{1}{2} \int_0^L u(x, t)^2 dx$ its quadratic energy. You may use Parseval's identity, positive-time termwise differentiation and sine completeness.

Tasks

1. Derive the energy identity and prove $E(t) \leq e^{-2\kappa(\pi/L)^2 t} E(0)$. Characterize every nonzero datum attaining equality at a fixed positive time.
2. Suppose $f(L - x) = -f(x)$ almost everywhere. Prove that only even modes remain and obtain the improved bound with rate $8\kappa\pi^2/L^2$. Identify when that bound is attained.
3. Show that a nonnegative, nonzero datum cannot have $b_1 = 0$. Explain why the enhanced asymptotic decay caused by removing the first mode requires sign changes in nonzero real initial data.
4. Prove the long-time L^2 asymptotic formula $e^{\kappa(\pi/L)^2 t} u \rightarrow b_1 \sin(\pi x/L)$, with a quantitative remainder bound. State what the formula says when $b_1 = 0$.

Question 10 – Solution

Strategy. Orthogonality turns energy decay and symmetry restrictions into precise statements about which coefficients are present.

Step 1: Prove the sharp basic energy decay. For $t > 0$, integration by parts gives $E' = \kappa \int u u_{xx} = -\kappa \int u_x^2$ because $u = 0$ at both ends. With $\lambda_1 = (\pi/L)^2$, Parseval yields

$$E(t) = \frac{L}{4} \sum_{n \geq 1} b_n^2 e^{-2\kappa n^2 \lambda_1 t} \leq e^{-2\kappa \lambda_1 t} E(0).$$

For fixed $t > 0$, every term with $n \geq 2$ has strictly smaller decay factor. Equality for nonzero data holds exactly when $f = b_1 \sin(\pi x/L)$ with $b_1 \neq 0$. This also proves sharpness of the uniform energy rate.

Step 2: Use antisymmetry to improve the rate. Changing variable $x \mapsto L - x$ in the coefficient integral and using $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$ gives $b_n = (-1)^n b_n$. Thus odd coefficients vanish. The smallest remaining index is two, giving

$$E(t) \leq e^{-8\kappa \pi^2 t/L^2} E(0).$$

Nonzero equality data are precisely scalar multiples of $\sin(2\pi x/L)$. The symmetry is preserved by the evolved even-mode series.

Step 3: Identify the sign constraint. If $f \geq 0$ almost everywhere and is nonzero in L^2 , then it is positive on a set of positive measure. The first sine is strictly positive on $(0, L)$; therefore $b_1 = (2/L) \int f \sin(\pi x/L) > 0$. Similarly a nonpositive nonzero datum has $b_1 < 0$. Thus any nonzero real datum with $b_1 = 0$ must have both positive and negative parts of positive measure. Removing the first mode is possible, but not for a nontrivial nonnegative initial temperature relative to the zero boundaries.

Step 4: Prove the normalized asymptotic statement. After removing the first term, Parseval gives

$$\|e^{\kappa \lambda_1 t} u - b_1 \sin(\pi x/L)\|_2^2 = \frac{L}{2} \sum_{n \geq 2} b_n^2 e^{-2\kappa(n^2-1)\lambda_1 t} \leq e^{-6\kappa \lambda_1 t} \|f\|_2^2.$$

The remainder norm is at most $e^{-3\kappa \lambda_1 t} \|f\|_2$ and tends to zero. If $b_1 = 0$, the normalized limit is zero and the actual field decays at least at the second-mode norm rate $e^{-4\kappa \lambda_1 t}$. The next nonzero coefficient, if present, can give an even faster decay rate.

Question 1

Let $\kappa, L > 0$ and $A, B, C \in \mathbb{R}$. Solve

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0, t) = A, \quad u(L, t) = B,$$

with $u(x, 0) = A + (B - A)x/L + C \sin(\pi x/L)$. Define the rightward diffusive flux as $j = -\kappa u_x$; the outward flux at the right endpoint is $j(L, t)$.

Tasks

1. Determine the unique stationary temperature and explain why it need not have zero flux.
2. Subtract that stationary field, solve the transformed initial-boundary problem, and verify the reconstructed solution against every original datum.
3. Take $L = \kappa = 1$, $A = 1$, $B = 3$, $C = 2$. Determine when the right reservoir receives heat from the rod and when it supplies heat to the rod. Find the exact switching time.
4. Compute total heat and its flux balance in this example. Sketch profiles at $t = 0, 0.08, 0.3$, together with the stationary field, and explain why the hotter endpoint can initially receive heat.

Question 1 – Solution

Strategy. Separate the maintained stationary gradient from the transient temperature excess.

Step 1: Find the stationary conduction state. Stationarity requires $S'' = 0$, and the two boundary values give

$$S(x) = A + (B - A)x/L, \quad j_S = -\kappa(B - A)/L.$$

This state is unique because a linear function is determined by two endpoint values. Unless $A = B$, it carries a constant nonzero flux. Stationary temperature means zero local storage rate, not an absence of heat transport.

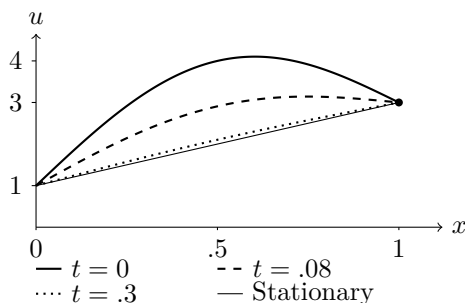
Step 2: Solve the homogeneous-boundary transient. With $v = u - S$, $v_t = \kappa v_{xx}$, $v(0, t) = v(L, t) = 0$ and $v(x, 0) = C \sin(\pi x/L)$. Thus, writing $\lambda = \kappa\pi^2/L^2$,

$$u(x, t) = S(x) + Ce^{-\lambda t} \sin(\pi x/L).$$

Since $S'' = 0$, differentiation verifies the PDE. The sine vanishes at both ends, and at $t = 0$ the original initial field is recovered exactly.

Step 3: Determine the direction of endpoint heat flow. For the given numbers, $u_x(1, t) = 2 - 2\pi e^{-\pi^2 t}$, so $j(1, t) = -2 + 2\pi e^{-\pi^2 t}$. The right reservoir receives heat when $j > 0$, that is $0 \leq t < t_* := \log \pi / \pi^2$. At t_* the right flux vanishes; for $t > t_*$ it supplies heat to the rod. The eventual rightward flux is -2 .

Step 4: Verify the global balance and the profiles. Here $H(t) = \int_0^1 u dx = 2 + (4/\pi)e^{-\pi^2 t}$, so $H' = -4\pi e^{-\pi^2 t}$. Directly $j(0, t) = -2 - 2\pi e^{-\pi^2 t}$ and $j(1, t) = -2 + 2\pi e^{-\pi^2 t}$, giving $H' = j(0, t) - j(1, t)$. Initially the interior exceeds the right endpoint temperature, so heat can leave through that endpoint despite its being hotter than the left one.



Question 2

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ and is initially at zero temperature in $L^2(0, 1)$. At $t = 0$ the left endpoint is raised to one and held there, while the right remains zero: $u(0, t) = 1$, $u(1, t) = 0$ for $t > 0$. You may use Fourier sine convergence, Parseval, and $\int_0^\infty e^{-\pi^2 s^2} ds = 1/(2\sqrt{\pi})$.

Tasks

1. Subtract the stationary field, compute the sine coefficients of the transformed initial data and construct the full solution.
2. Justify the PDE for positive time and the L^2 initial trace. State the initial limit at interior points and at the left endpoint; explain why uniform convergence to the zero initial field is impossible.
3. Derive a convergent total-heat formula and prove that total heat increases for every positive time, although the source-free PDE has no interior forcing.
4. Find the rightward flux entering at $x = 0$ and its leading behavior as $t \downarrow 0$. Use a Riemann-sum limit to justify the singular rate rather than evaluating a divergent series at $t = 0$.

Question 2 – Solution

Strategy. The sudden boundary change creates an initial corner mismatch; subtracting the stationary profile isolates its decaying sine expansion.

Step 1: Construct the boundary step response. The stationary field is $S = 1 - x$. For $v = u - S$, the endpoints are zero and $v(x, 0) = x - 1$. Integration gives $2 \int_0^1 (x - 1) \sin(n\pi x) dx = -2/(n\pi)$. Hence

$$u(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n\pi x)}{n}.$$

Step 2: Verify the appropriate initial trace. For $t \geq \tau > 0$, Gaussian decay makes every finitely differentiated transient series uniformly convergent. It satisfies the heat equation and zero endpoint values; adding S gives the required boundary data. The square-summable coefficients and Parseval prove $v \rightarrow x - 1$ in L^2 , hence $u \rightarrow 0$ in L^2 . At each interior point, the Fourier limit is zero after reconstruction; Gaussian damping preserves that limit by summation by parts. At $x = 0$, the positive-time value is always one, so its limit is one. Uniform convergence to zero on the closed interval is therefore impossible. No corner-continuous solution is claimed.

Step 3: Integrate the boundary-driven heating. For positive time,

$$H(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-n^2 \pi^2 t}}{n^2}, \quad H'(t) = 4 \sum_{n \text{ odd}} e^{-n^2 \pi^2 t} > 0.$$

The initial norm trace gives $H(0+) = 0$ and the final heat is $1/2$. Differentiating the spatial series gives $H' = [u_x]_0^1 = j(0, t) - j(1, t)$, where $j = -u_x$. The endpoint supplies heat; no interior source is needed.

Step 4: Quantify the singular initial input. At the left endpoint,

$$j(0, t) = 1 + 2 \sum_{n \geq 1} e^{-n^2 \pi^2 t}.$$

With $h = \sqrt{t}$, the decreasing, integrable Gaussian gives $h \sum_{n \geq 1} e^{-\pi^2 (nh)^2} \rightarrow \int_0^\infty e^{-\pi^2 s^2} ds$ by upper/lower integral bounds. Thus $\sqrt{t} j(0, t) \rightarrow 1/\sqrt{\pi}$, or $j(0, t) \sim 1/\sqrt{\pi t}$. The singular rate is integrable in time and is consistent with vanishing initial heat. Substituting $t = 0$ into the differentiated series would be invalid.

Question 3

For $\kappa, L, Q > 0$, consider uniform internal heating with unequal fixed endpoints:

$$u_t = \kappa u_{xx} + Q, \quad u(0, t) = A, \quad u(L, t) = B, \quad u(x, 0) = \ell(x) := A + (B - A)x/L.$$

You may use the sine expansion of a smooth zero-endpoint function.

Tasks

1. Determine the stationary field and find exactly when its maximum lies strictly inside the rod. Locate that maximum.
2. Subtract the stationary field, compute the transient coefficients and construct the solution satisfying the original linear initial profile.
3. Verify the reconstructed PDE and data, explain its initial and positive-time regularity, and check the stationary heat-flux balance against the source integral.
4. For $L = \kappa = 1$, $A = 1$, $B = 2$, $Q = 4$, determine the stationary maximum and sketch the initial field, the profiles at $t = 0.02, 0.1$, and the stationary field. Explain why the maximum may exceed both boundary values.

Question 3 – Solution

Strategy. Subtract a stationary solution of the forced equation, not just the line joining its boundary values.

Step 1: Solve the stationary forced problem. The equation $\kappa S'' + Q = 0$ and the endpoint data give

$$S = \ell + \frac{Q}{2\kappa}x(L-x).$$

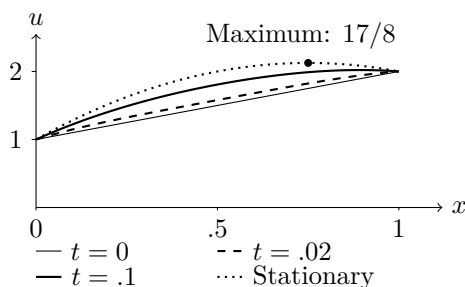
It is strictly concave, with critical point $x_* = L/2 + \kappa(B-A)/(QL)$. This is strictly interior exactly when $|B-A| < QL^2/(2\kappa)$. If equality holds, the maximum is at an endpoint; otherwise the maximum is at the endpoint in the direction of the larger value.

Step 2: Construct the transient from the actual initial difference. For $v = u - S$, the PDE and boundaries are homogeneous and $v(x, 0) = -Qx(L-x)/(2\kappa)$. Two integrations by parts give the nonzero coefficients $-4QL^2/(\kappa\pi^3n^3)$ for odd n . Therefore

$$u = S - \frac{4QL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.$$

Step 3: Check the PDE, trace and maintained fluxes. The transient solves the homogeneous heat equation; $\kappa S'' + Q = 0$ then verifies the forced equation for u . The transient vanishes at the ends. Summable coefficients and Fourier convergence recover $u(\cdot, 0) = \ell$ uniformly. Gaussian decay permits differentiation for $t > 0$. The initial linear profile has $\kappa\ell'' + Q = Q$, inconsistent with a zero endpoint time derivative at an initial corner, so full corner smoothness is not asserted. For $j_S = -\kappa S'$, one has $j_S(L) - j_S(0) = QL$. Hence the steady balance $0 = j_S(0) - j_S(L) + QL$ accounts for all generated heat.

Step 4: Interpret the internally heated profile. In the example, $S = 1 + x + 2x(1-x)$, $x_* = 3/4$ and $S(x_*) = 17/8$. The final maximum exceeds both boundary values because heat is generated inside the rod. The source-free maximum principle cannot be applied unchanged to this forced equation. The transient decays uniformly toward the plotted stationary field.



Question 4

On $0 < x < 1$, let the storage and conductivity vary so that

$$\frac{u_t}{1+x} = ((1+x)u_x)_x, \quad u(0,t) = 1, \quad u(1,t) = 3.$$

Define $j = -(1+x)u_x$, $\ell = \log 2$, and let the initial field be

$$f(x) = 1 + \frac{2 \log(1+x)}{\ell} + \sin\left(\frac{\pi \log(1+x)}{\ell}\right).$$

Tasks

1. Derive the stationary temperature and stationary flux. Explain why the straight line $1 + 2x$ is not stationary for this equation.
2. Use $s = \log(1+x)$ to transform the entire PDE and its endpoints. Construct and verify the full solution for the given initial field.
3. Compute the stored quantity $H(t) = \int_0^1 u(x,t)/(1+x) dx$ and verify $H' = j(0,t) - j(1,t)$.
4. For the error $v = u - S$, derive the weighted energy identity. Evaluate its weighted squared norm for this solution and explain why the weight cannot simply be omitted from the identity.

Question 4 – Solution

Strategy. The stationary gradient and the natural storage norm both follow from the variable coefficients.

Step 1: Integrate constant stationary flux. The stationary equation is $((1+x)S')' = 0$, so $(1+x)S' = C$. The endpoint difference gives $C = 2/\ell$, hence

$$S = 1 + 2 \log(1+x)/\ell, \quad j_S = -2/\ell.$$

For $1+2x$, the quantity $((1+x)u_x)' = 2$ is nonzero, so it is not stationary. Equal endpoint values do not make different interior profiles solve the same PDE.

Step 2: Transform and solve the full problem. Write $U(s, t) = u(e^s - 1, t)$ on $0 < s < \ell$. Then $u_x = U_s/(1+x)$ and $((1+x)u_x)_x = U_{ss}/(1+x)$. The transformed equation is $U_t = U_{ss}$, with endpoints one and three. Subtracting $1 + 2s/\ell$ gives one sine mode. With $\lambda = \pi^2/\ell^2$,

$$u = S + e^{-\lambda t} \sin(\pi \log(1+x)/\ell).$$

The chain-rule identities verify the original PDE, and the sine gives zero endpoint correction and exactly the prescribed initial field.

Step 3: Check storage against boundary flux. Since $dx/(1+x) = ds$,

$$H(t) = 2\ell + \frac{2\ell}{\pi} e^{-\lambda t}, \quad H' = -\frac{2\pi}{\ell} e^{-\lambda t}.$$

Also $j = -U_s = -2/\ell - (\pi/\ell)e^{-\lambda t} \cos(\pi s/\ell)$. Its left value minus its right value equals the displayed derivative. The unweighted integral of u is not the stored quantity in this model.

Step 4: Derive the weighted error dissipation. Multiplying $v_t/(1+x) = ((1+x)v_x)_x$ by v and integrating, with zero error at both ends, gives

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{1+x} dx = - \int_0^1 (1+x) v_x^2 dx.$$

For this mode the squared norm is $(\ell/2)e^{-2\lambda t}$ and the right-hand integral is $(\pi^2/(2\ell))e^{-2\lambda t}$, agreeing with the identity. The weight is the storage coefficient; dropping it changes the energy calculation.

Question 5

Let both endpoint temperatures rise at a constant rate $r > 0$:

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0, t) = A + rt, \quad u(L, t) = B + rt,$$

where $\kappa, L > 0$ and $u(x, 0) = \ell(x) := A + (B - A)x/L$. You may use the sine expansion of $x(L - x)$.

Tasks

1. Subtract the instantaneous boundary line $\ell(x) + rt$ and derive the PDE and initial data for the remaining field. Explain why that line alone is not a solution.
2. Find a time-independent zero-endpoint correction $P(x)$ so that $\ell + rt + P$ solves the PDE. Add the transient needed to recover the original initial field.
3. Verify the reconstructed solution, justify the initial trace and positive-time differentiations, and determine its limiting offset relative to the moving boundary line.
4. Compute the limiting midpoint lag, the limiting mean lag and the eventual total-heat growth rate. Explain why there is no stationary equilibrium even though the offset approaches a fixed shape.

Question 5 – Solution

Strategy. A boundary lifting with nonzero time derivative creates a source in the transformed equation; a persistent lag balances that source.

Step 1: Keep the source created by the lifting. For $w = u - \ell - rt$,

$$w_t = \kappa w_{xx} - r, \quad w(0, t) = w(L, t) = 0, \quad w(x, 0) = 0.$$

The line $\ell + rt$ has time derivative r and zero second spatial derivative, so it fails the original source-free PDE when $r > 0$.

Step 2: Find the lag profile and the initial correction. A fixed P must satisfy $\kappa P'' = r$, $P(0) = P(L) = 0$, giving $P = -rx(L-x)/(2\kappa)$. The remaining transient has initial data $-P$. Its odd sine coefficients are $4rL^2/(\kappa\pi^3n^3)$. Thus

$$u = \ell + rt + P + \frac{4rL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.$$

Step 3: Verify all data and the moving-frame limit. The field $\ell + rt + P$ satisfies $u_t = \kappa u_{xx} = r$; its added transient satisfies the homogeneous heat equation. All correction terms vanish at the endpoints. At $t = 0$, the summable sine expansion equals $-P$ uniformly and recovers ℓ . For $t \geq \tau > 0$, the exponential transient permits all termwise derivatives. Initial corner smoothness is not implied: the boundary time derivative is r , whereas $\kappa \ell'' = 0$. As $t \rightarrow \infty$, the transient tends uniformly to zero, so $u - \ell - rt \rightarrow P$ uniformly.

Step 4: Quantify the persistent lag and heat input. The midpoint offset is $P(L/2) = -rL^2/(8\kappa)$, and the mean offset is $L^{-1} \int P = -rL^2/(12\kappa)$. Their negatives are the positive lags behind the boundary line. Since $P'(L) - P'(0) = rL/\kappa$, the eventual flux balance is $H' = \kappa[u_x]_0^L \rightarrow rL$, also obtained by differentiating the integrated solution. Boundary values grow indefinitely; it is the offset, not the absolute temperature, that approaches a fixed state.

Question 6

A semi-infinite rod has $u_t = \kappa u_{xx}$ on $x > 0$, with $\kappa, \omega > 0$ and a maintained oscillating boundary $u(0, t) = A + B \cos(\omega t)$, where $A > |B| > 0$. Seek its periodic response of the form

$$u(x, t) = A + Be^{-\alpha x} \cos(\omega t - \alpha x), \quad \alpha > 0,$$

with $u(x, t) \rightarrow A$ as $x \rightarrow \infty$. The initial field is to be the trace of this periodic response, so no additional startup transient is requested. Use the rightward flux convention $j = -\kappa u_x$.

Tasks

1. Determine α by direct substitution, and state the initial field that makes the response a solution of the full initial-boundary problem.
2. Derive the amplitude attenuation and phase delay at distance x . Find the depth where the oscillation amplitude falls to $1/e$ of its boundary value, and express it in terms of the period $P = 2\pi/\omega$.
3. Compute the boundary flux and its phase relative to the imposed temperature. Explain why constant far-field temperature does not imply zero instantaneous boundary flux.
4. Prove positivity of the temperature and approach to the far-field value as $x \rightarrow \infty$, uniformly in time. For $A = 2, B = 1, \kappa = 1, \omega = 2$, sketch profiles at phases $\omega t = 0, \pi/2, \pi$.

Question 6 – Solution

Strategy. A periodic boundary signal diffuses inward with both amplitude attenuation and a spatial phase shift.

Step 1: Match the PDE and the initial trace. Put $\theta = \omega t - \alpha x$. Then $u_t = -\omega B e^{-\alpha x} \sin \theta$ and $u_{xx} = -2\alpha^2 B e^{-\alpha x} \sin \theta$. Thus $\alpha = \sqrt{\omega/(2\kappa)}$. The boundary trace is the imposed cosine, and the compatible initial field is $f(x) = A + B e^{-\alpha x} \cos(\alpha x)$. It is generally not the constant A ; a constant initial field would require an additional transient.

Step 2: Quantify attenuation and delay. The local oscillation amplitude is $|B|e^{-\alpha x}$ and its phase lag is αx . Relative to the boundary waveform, the time delay is $\alpha x/\omega$, understood modulo the period. The penetration depth is

$$d = 1/\alpha = \sqrt{2\kappa/\omega} = \sqrt{\kappa P/\pi}.$$

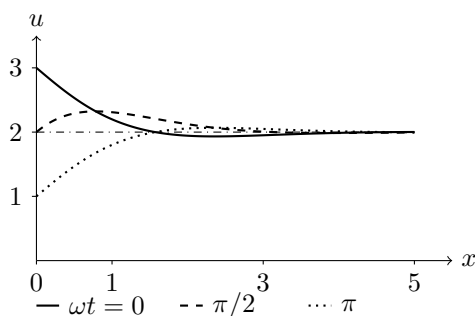
Longer periods penetrate farther, with depth proportional to the square root of the period rather than the period itself.

Step 3: Determine the phase of the boundary flux. Differentiation gives $u_x = \alpha B e^{-\alpha x} (-\cos \theta + \sin \theta)$. Hence

$$j(0, t) = B\sqrt{\kappa\omega} \cos(\omega t + \pi/4).$$

The flux leads the boundary temperature oscillation by $\pi/4$ in phase and has zero average over a full period. Instantaneous storage and release near the driven boundary are compatible with a constant far-field limit.

Step 4: Verify bounds and interpret the profiles. For all $x \geq 0, t$, $|u - A| \leq |B|e^{-\alpha x} \leq |B|$, so $u \geq A - |B| > 0$. Also $\sup_t |u(x, t) - A| \leq |B|e^{-\alpha x} \rightarrow 0$ as $x \rightarrow \infty$. This is a spatial far-field limit uniform in time, not decay of the maintained oscillation at a fixed point as time increases. For the plotted parameters, $\alpha = 1$ and the penetration depth is one.



Question 7

On $0 < x < 1$, let $u_t = u_{xx}$ and $u(x, 0) = A$. The right endpoint stays at A . The left is held at $A + B$ for $0 < t < \tau$ and returns to A for $t \geq \tau$, where $B, \tau > 0$. You may use the unit boundary-step response

$$S(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n \pi x)}{n}, \quad t > 0,$$

which has zero initial interior trace, left boundary one and right boundary zero. Set $S(x, 0) = 0$ for $0 < x < 1$ when discussing the switching time.

Tasks

1. Construct the solution before and after the switch by superposing boundary step responses. Verify the boundary traces on both open time intervals.
2. Derive a homogeneous-boundary sine series for $t > \tau$, with coefficients at time τ . Explain continuity at the switch for interior points and why continuity at the switching boundary corner is impossible.
3. Compute the total excess heat after the switch and prove that it strictly decreases. Determine the long-time equilibrium.
4. For $A = B = 1$, $\tau = 0.1$, sketch profiles at $t = 0.03, 0.08, 0.13, 0.3$. Explain why the rod does not return instantly to its initial field when the boundary pulse ends.

Question 7 – Solution

Strategy. Switching off a boundary step is a delayed negative step; its instantaneous boundary effect does not erase the stored interior heat.

Step 1: Superpose the two boundary events. Linearity gives

$$u = A + BS(x, t) \quad (0 < t < \tau), \quad u = A + B[S(x, t) - S(x, t - \tau)] \quad (t > \tau).$$

Each term solves the heat equation in its time interval. Before the switch, the left trace is $A + B$; afterward the two unit left traces cancel and leave A . Both right traces are always A . The initial interior trace is A .

Step 2: Identify the post-switch initial data. The stationary lines cancel after the switch, leaving

$$u - A = \frac{2B}{\pi} \sum_{n \geq 1} \frac{1 - e^{-n^2 \pi^2 \tau}}{n} e^{-n^2 \pi^2 (t - \tau)} \sin(n \pi x).$$

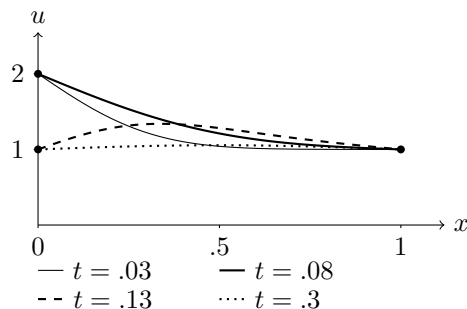
These are the sine coefficients of $BS(x, \tau)$ on the open interval. For every interior x , $S(x, t - \tau) \rightarrow 0$ as $t \downarrow \tau$, so the interior trace agrees with the pre-switch value. The left boundary itself jumps from $A + B$ to A ; no single continuous trace at that corner can match both. For times strictly after the switch, all differentiated series converge uniformly by Gaussian decay. The switching trace is also recovered in L^2 .

Step 3: Track the retained heat. For $t > \tau$, integration yields

$$H_{\text{ex}}(t) = \frac{4B}{\pi^2} \sum_{n \text{ odd}} \frac{1 - e^{-n^2 \pi^2 \tau}}{n^2} e^{-n^2 \pi^2 (t - \tau)}.$$

Its derivative is $-4B \sum_{n \text{ odd}} (1 - e^{-n^2 \pi^2 \tau}) e^{-n^2 \pi^2 (t - \tau)} < 0$. This equals $[u_x]_0^1$ after the switch. At any later fixed starting time the exponentially damped coefficients are summable, which proves uniform convergence to the equilibrium $u = A$ as $t \rightarrow \infty$.

Step 4: Interpret the finite pulse. The profiles before the switch have left value two; those after it have left value one. The right value is one throughout. At the switch the excess heat is $B \int_0^1 S(x, \tau) dx > 0$, as follows from the positive derivative of step-response heat and its zero initial value. It must subsequently diffuse out. A changed boundary value is not an instantaneous reset of the interior.



Question 8

Consider $u_t = u_{xx}$ on $0 < x < 1$, with

$$u(0, t) = e^t, \quad u(1, t) = e^t \cosh 1, \quad u(x, 0) = \cosh x.$$

Compare the affine boundary lifting $\ell(x, t) = e^t[1 + (\cosh 1 - 1)x]$ with the nonaffine lifting $P(x, t) = e^t \cosh x$. You may use the sine expansion of a smooth function vanishing at both endpoints.

Tasks

1. Verify a closed-form solution of the original problem using P , including the PDE and every datum.
2. For $v = u - \ell$, derive the homogeneous-boundary forced PDE and its initial data. Explain why setting $v = 0$ would give the wrong physical field.
3. Set $k_n = n\pi$ and $D_n = 1 - (-1)^n \cosh 1$. Compute the sine coefficients of the initial difference and the transformed source. Solve the resulting modal initial-value equations.
4. Reconstruct u from the modal solution and prove that it agrees with the closed form. Explain why different liftings can produce different source terms without producing different solutions of the original problem.

Question 8 – Solution

Strategy. A convenient lifting can solve the PDE as well as its boundary traces; a less convenient lifting is still valid if every transformed term is retained.

Step 1: Verify the exact nonaffine solution. The field $\boxed{u = e^t \cosh x}$ has $u_t = u_{xx} = e^t \cosh x$. Its two endpoint values and its initial trace are exactly the prescribed ones. With lifting P , the remaining field is zero, with zero source and initial data.

Step 2: Transform all parts under the affine lifting. Write $p(x) = 1 + (\cosh 1 - 1)x$, so $\ell = e^t p$. Since $\ell_{xx} = 0$,

$$v_t - v_{xx} = -e^t p(x), \quad v(0, t) = v(1, t) = 0, \quad v(x, 0) = \cosh x - p(x).$$

The zero field fails both the interior source equation and the initial trace. The affine lifting satisfies the boundary values but is not itself a heat solution.

Step 3: Solve the transformed modal equations. Direct integration gives

$$2 \int_0^1 p(x) \sin(k_n x) dx = \frac{2D_n}{k_n}, \quad 2 \int_0^1 \cosh x \sin(k_n x) dx = \frac{2k_n D_n}{1 + k_n^2}.$$

Thus the initial coefficient is $c_n = -2D_n/[k_n(1 + k_n^2)]$. The modal equation is $v'_n + k_n^2 v_n = -(2D_n/k_n)e^t$, $v_n(0) = c_n$. Since $(1 + k_n^2)c_n = -2D_n/k_n$, its solution is $\boxed{v_n(t) = c_n e^t}$. No additional decaying term is needed for these data.

Step 4: Reconstruct the same physical temperature. The coefficients $c_n = O(n^{-3})$ are summable. Their sine series equals the smooth zero-endpoint function $\cosh x - p(x)$ uniformly on $[0, 1]$. Consequently

$$u = \ell + e^t \sum_{n \geq 1} c_n \sin(k_n x) = e^t p + e^t (\cosh x - p) = e^t \cosh x.$$

This identity verifies the reconstructed PDE directly without claiming absolute uniform convergence of every differentiated sine series at the boundary. Different liftings change the auxiliary source and initial data together; retaining both transformations preserves the original solution.

Question 9

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ with unknown constant boundary temperatures A, B . It is known that its full field has the form

$$u(x, t) = A + (B - A)x + Ce^{-\pi^2 t} \sin(\pi x).$$

Sensors at $x = 1/3, 2/3$ record $y_L(t), y_R(t)$. Set $m(t) = (y_L(t) + y_R(t))/2$ and let $\Delta > 0$ be known.

Tasks

1. Derive which combination of the readings identifies the stationary gradient immediately, even before the transient has decayed.
2. Use readings at times $0, \Delta$ to recover A, B, C uniquely. State the consistency condition on the differences of the two pairs of readings.
3. For $\Delta = (\log 2)/\pi^2$, use readings $(8/3, 10/3)$ at zero and $(13/6, 17/6)$ at Δ to find the complete field and verify the readings and boundary values.
4. Explain why one pair of simultaneous readings cannot identify all three parameters. For small Δ , quantify amplification of errors in the two averaged readings when estimating the mean of the boundary temperatures.

Question 9 – Solution

Strategy. Equal sine values at the two sensors let their difference cancel the transient; a second observation time separates the remaining steady and decaying parts.

Step 1: Cancel the equal transient values. At both sensor positions, $\sin(\pi x) = \sqrt{3}/2$. Thus

$$d := B - A = 3(y_R - y_L), \quad m(t) = S + Ke^{-\pi^2 t}, \quad S = (A + B)/2, \quad K = \sqrt{3}C/2.$$

The difference determines the stationary gradient at any measurement time. It must be unchanged between the two observations if the assumed model holds.

Step 2: Invert the two-time averages. Put $r = e^{-\pi^2 \Delta} \in (0, 1)$. From $m_0 = S + K$, $m_\Delta = S + rK$,

$$K = \frac{m_0 - m_\Delta}{1 - r}, \quad S = \frac{m_\Delta - rm_0}{1 - r}, \quad A = S - d/2, \quad B = S + d/2, \quad C = 2K/\sqrt{3}.$$

The denominator is nonzero because $\Delta > 0$. Equality of the observed differences is the only additional consistency condition; with it these parameters reconstruct all four readings under the stated model.

Step 3: Reconstruct the numerical example. Here $r = 1/2$, $d = 2$, $m_0 = 3$ and $m_\Delta = 5/2$. Thus $S = 2$, $K = 1$ and

$$A = 1, \quad B = 3, \quad C = 2/\sqrt{3}, \quad u = 1 + 2x + \frac{2}{\sqrt{3}}e^{-\pi^2 t} \sin(\pi x).$$

The endpoint values are one and three. The sensor transients are one at zero and one half at Δ , giving exactly the four supplied readings. The linear stationary part and the decaying sine verify $u_t = u_{xx}$.

Step 4: State the identifiability and conditioning limits. A single pair gives d and $S + K$, leaving infinitely many choices of S, K . For two times, errors e_0, e_Δ in the averages cause $\delta S = (e_\Delta - re_0)/(1 - r)$. If $|e_0|, |e_\Delta| \leq \eta$,

$$|\delta S| \leq \frac{1 + r}{1 - r} \eta \sim \frac{2\eta}{\pi^2 \Delta} \quad (\Delta \downarrow 0).$$

Opposite-signed errors attain the bound. Very close observations are therefore poorly conditioned even though the noiseless parameters are unique.

Question 10

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, with fixed endpoint temperatures A, B , $\kappa, L > 0$, and stationary field $S = A + (B - A)x/L$. For smooth compatible data define the error energy $E_v = \frac{1}{2} \int_0^L (u - S)^2 dx$ and the raw quadratic quantity $E_u = \frac{1}{2} \int_0^L u^2 dx$. You may use the zero-endpoint Poincare inequality $\int v_x^2 \geq (\pi/L)^2 \int v^2$ and the parabolic maximum principle.

Tasks

1. Derive the energy identities for E_v and E_u , retaining all boundary terms. Prove the sharp bound $E_v(t) \leq e^{-2\lambda t} E_v(0)$, where $\lambda = \kappa \pi^2 / L^2$.
2. For $A = B = 1$ and $u(x, 0) = 1 - \varepsilon \sin(\pi x / L)$, $0 < \varepsilon \leq 1$, construct the solution. Prove that its raw quadratic quantity increases while its error energy decreases.
3. If $|u(x, 0) - S(x)| \leq C \sin(\pi x / L)$, $C > 0$, use comparison to prove a pointwise error bound for all later times. Give a sufficient time for uniform temperature error at most δ , where $0 < \delta < C$.
4. Give a sufficient time for reducing error energy to a fraction $\eta \in (0, 1)$ of its initial value, and explain why neither total heat nor raw quadratic energy alone measures distance from a maintained nonzero equilibrium.

Question 10 – Solution

Strategy. Dissipation applies naturally to deviations from the maintained equilibrium; the boundary reservoirs can increase the unshifted quantities.

Step 1: Identify which energy has no boundary work. For $v = u - S$, the endpoints vanish and $v_t = \kappa v_{xx}$. Therefore

$$E'_v = -\kappa \int_0^L v_x^2 dx \leq -2\lambda E_v, \quad E_v(t) \leq e^{-2\lambda t} E_v(0).$$

A first-sine error attains equality, so the rate is sharp. For the raw quantity,

$$E'_u = \kappa [uu_x]_0^L - \kappa \int_0^L u_x^2 dx.$$

The first term need not vanish with nonzero maintained boundary values.

Step 2: Exhibit simultaneous raw growth and error decay. The stated initial data give $u = 1 - \varepsilon e^{-\lambda t} \sin(\pi x/L)$. Then

$$E_u = \frac{L}{2} - \frac{2\varepsilon L}{\pi} e^{-\lambda t} + \frac{\varepsilon^2 L}{4} e^{-2\lambda t}, \quad E_v = \frac{\varepsilon^2 L}{4} e^{-2\lambda t}.$$

Writing $z = \varepsilon e^{-\lambda t} \in (0, 1]$ gives $E'_u = \lambda L z(2/\pi - z/2) > 0$, since $2/\pi > 1/2$. Meanwhile $E'_v = -2\lambda E_v < 0$. Heating by the reservoirs moves this field closer to equilibrium while increasing its raw quadratic quantity.

Step 3: Establish a pointwise stopping criterion. The barriers $\pm C e^{-\lambda t} \sin(\pi x/L)$ solve the homogeneous zero-endpoint heat equation and bound the initial error. Applying comparison to their differences with v on any finite time interval gives

$$|u - S| \leq C e^{-\lambda t} \sin(\pi x/L) \leq C e^{-\lambda t}.$$

Thus $t \geq \lambda^{-1} \log(C/\delta)$ suffices for uniform error at most δ . This conclusion uses the supplied pointwise initial bound, not merely an initial L^2 norm.

Step 4: Distinguish the two stopping criteria. The energy inequality gives $E_v(t)/E_v(0) \leq \eta$ once $t \geq (2\lambda)^{-1} \log(1/\eta)$ for nonzero initial error; zero error is already at equilibrium. The factor of two comes from squaring a decaying amplitude. Total heat can conceal spatial differences, and raw energy includes the maintained background and boundary work. Neither is a substitute for an explicit norm of $u - S$ when measuring equilibration.

Question 1

A square plate occupies $0 < x < 1$, $0 < y < 1$. Its steady temperature satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = u(x, 0) = 0, \quad u(x, 1) = \sin(\pi x) + \frac{1}{2} \sin(3\pi x).$$

The conductivity is one, so the outward heat flux is $-\partial_n u$.

Tasks

1. Construct the harmonic temperature by separation of variables and verify all four boundary values.
2. At height $y = 1/2$, determine the attenuation of each boundary mode and their amplitude ratio. Explain which spatial feature penetrates farther into the plate.
3. Compute the total outward heat flux on each edge and verify global balance. Use outward normals consistently.
4. Sketch temperature profiles at $y = 1/4, 1/2, 1$. Prove that the temperature is strictly positive in the interior despite the sign changes of the third sine mode.

Question 1 – Solution

Strategy. Separate the two imposed spatial frequencies and track their normal derivatives as well as their temperatures.

Step 1: Extend each boundary mode. The zero side values give $\sin(n\pi x)$; the corresponding vertical equation is $Y'' - (n\pi)^2 Y = 0$. The bottom value selects $\sinh(n\pi y)$ and the top value normalizes it:

$$u = \frac{\sinh(\pi y)}{\sinh \pi} \sin(\pi x) + \frac{1}{2} \frac{\sinh(3\pi y)}{\sinh(3\pi)} \sin(3\pi x).$$

Each term has zero Laplacian. Substitution gives all four traces, including compatible corner values. The maximum principle gives uniqueness among continuous harmonic extensions.

Step 2: Compare penetration of the modes. Since $\sinh(a/2)/\sinh a = 1/(2 \cosh(a/2))$, the first and third modes are attenuated by $1/(2 \cosh(\pi/2))$ and $1/(2 \cosh(3\pi/2))$. Their third-to-first amplitude ratio is

$$\frac{\cosh(\pi/2)}{2 \cosh(3\pi/2)}.$$

It is smaller than the boundary ratio $1/2$: the higher spatial frequency loses amplitude faster away from the driven edge.

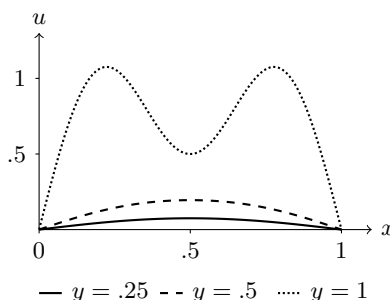
Step 3: Balance the four edge fluxes. Integrating $-\partial_n u$ along each unit edge gives

$$F_{\text{top}} = -2 \coth \pi - \coth(3\pi), \quad F_{\text{bottom}} = 2 \operatorname{csch} \pi + \operatorname{csch}(3\pi),$$

$$F_{\text{left}} = F_{\text{right}} = \tanh(\pi/2) + \frac{1}{2} \tanh(3\pi/2).$$

For example, at the bottom the outward normal is $(0, -1)$, so the outward heat flux is u_y , not $-u_y$. The identity $\coth a - \operatorname{csch} a = \tanh(a/2)$ proves that the four totals sum to zero.

Step 4: Interpret the profiles and their sign. Writing $s = \sin(\pi x) \in [0, 1]$, the top value is $s + \frac{1}{2}(3s - 4s^3) = s(5/2 - 2s^2) \geq 0$, and it is positive for $0 < x < 1$. The weak maximum principle gives $u \geq 0$; the strong maximum principle gives $u > 0$ inside because the boundary data are not constant. Individual modes need not be positive.



Question 2

Let u be harmonic on the unit square, with zero values on the left, right and bottom edges and

$$u(x, 1) = x(1 - x).$$

You may use Fourier sine convergence for a smooth function vanishing at both endpoints and Green's first identity.

Tasks

1. Derive the sine coefficients of the top boundary function and construct the complete harmonic extension.
2. Justify continuity up to every edge and harmonicity in the interior. Explain why boundary convergence and interior differentiation require different estimates.
3. Prove the pointwise bounds $0 \leq u(x, y) \leq y/4$. Determine whether equality can occur at an interior point.
4. Compute the Dirichlet energy $\mathcal{E} = \int_0^1 \int_0^1 |\nabla u|^2 dx dy$ as an explicitly convergent positive series. Justify the limiting operation.

Question 2 – Solution

Strategy. Use the boundary Fourier coefficients for construction, a harmonic barrier for bounds, and orthogonality for energy.

Step 1: Compute the boundary expansion. Two integrations by parts give

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4(1 - (-1)^n)}{(n\pi)^3}.$$

Thus $b_n = 8/(n\pi)^3$ for odd n , and $b_n = 0$ for even n . The solution is

$$u(x, y) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{8}{(n\pi)^3} \frac{\sinh(n\pi y)}{\sinh(n\pi)} \sin(n\pi x).$$

Step 2: Establish the appropriate convergence. The hyperbolic ratio is between zero and one for $0 \leq y \leq 1$. Since $\sum |b_n| < \infty$, the series converges uniformly on the closed square. Its top trace is the sine series of $x(1-x)$, and its other traces vanish. On any strip $y \leq 1 - \epsilon$, derivatives of every fixed order are bounded by a polynomial in n times $e^{-n\pi\epsilon}$. They converge uniformly there, so the Laplacian vanishes term by term. Uniform convergence of the original series alone would not justify taking two derivatives.

Step 3: Compare with a harmonic barrier. The boundary values are nonnegative. The function $h(x, y) = y/4$ is harmonic, is nonnegative on the sides, vanishes on the bottom, and dominates $x(1-x) \leq 1/4$ on the top. Applying the maximum principle to u and $u - h$ gives

$$0 \leq u(x, y) \leq y/4.$$

Both inequalities are strict in the interior: either equality there would make the corresponding harmonic difference constant, contrary to its boundary values.

Step 4: Evaluate energy without losing boundary terms. For a finite partial sum, Green's identity leaves only the top contribution $\int_0^1 u(x, 1) u_y(x, 1) dx$. Sine orthogonality gives

$$\mathcal{E}_N = \frac{1}{2} \sum_{\substack{n \leq N \\ n \text{ odd}}} b_n^2 n\pi \coth(n\pi).$$

The same identity applied to a difference of partial sums proves their gradients are Cauchy in L^2 , since the summands are $O(n^{-5})$. The uniform limit has that weak gradient and is the harmonic field already constructed. Therefore

$$\mathcal{E} = \frac{32}{\pi^5} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\coth(n\pi)}{n^5}.$$

Every term is positive; truncating this energy series gives a lower bound.

Question 3

Use centered coordinates $X = x - \frac{1}{2}$, $Y = y - \frac{1}{2}$ on the unit square. A harmonic temperature has the following boundary data:

$$\begin{aligned} u(x, 0) &= X^2 - X - \frac{1}{4}, & u(x, 1) &= X^2 + X - \frac{1}{4}, \\ u(0, y) &= \frac{1}{4} - Y - Y^2, & u(1, y) &= \frac{1}{4} + Y - Y^2. \end{aligned}$$

Seek a harmonic polynomial of degree at most two.

Tasks

1. Recover the polynomial from the four traces. Check their corner compatibility and prove uniqueness among all continuous harmonic solutions.
2. Find and classify every interior critical point. Explain why its existence does not contradict the maximum principle.
3. Find the exact global maximum and minimum on the closed square and all points where they occur.
4. Draw the zero-temperature lines and label the signs of the regions they separate. Verify that the total outward conductive flux, with conductivity one, is zero.

Question 3 – Solution

Strategy. Boundary polynomials can identify an exact harmonic field, but its interior stationary point must be classified rather than assumed to be an extremum.

Step 1: Reconstruct and verify the polynomial. A general quadratic $aX^2 + bXY + cY^2 + dX + eY + f$ is harmonic precisely when $c = -a$. The top and bottom traces force $a = 1$, $b = 2$, $d = e = 0$, and $f = 0$. The side traces then agree:

$$u = X^2 + 2XY - Y^2.$$

At the corners its values are $1/2, -1/2, -1/2, 1/2$, in the order $(0,0), (1,0), (0,1), (1,1)$, whether computed from horizontal or vertical edges. The difference of two continuous harmonic solutions has zero boundary data; its maximum and minimum are zero.

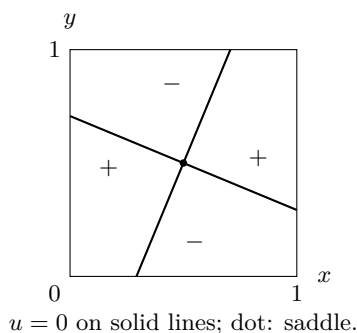
Step 2: Classify the critical point. The gradient is $(2X+2Y, 2X-2Y)$, so its sole zero is $(X, Y) = (0, 0)$. The Hessian has eigenvalues $2\sqrt{2}$ and $-2\sqrt{2}$, hence the center is a saddle. In particular $u(X, 0) = X^2$ while $u(0, Y) = -Y^2$. The maximum principle prohibits a nonconstant harmonic field's interior maximum or minimum, not a saddle.

Step 3: Locate the actual extrema. On the bottom, $X^2 - X - 1/4$ decreases on $[-1/2, 1/2]$, with range $[-1/2, 1/2]$; the top has the same range. On the left, $1/4 - Y - Y^2$ decreases over the allowed interval, also with that range, and the right behaves symmetrically. Therefore

$$\max u = \frac{1}{2} \text{ at } (0,0), (1,1), \quad \min u = -\frac{1}{2} \text{ at } (1,0), (0,1).$$

No interior point attains either value by the strong maximum principle.

Step 4: Draw the zero set and check flux. Solving $X^2 + 2XY - Y^2 = 0$ gives the two lines $Y = (1 \pm \sqrt{2})X$, clipped to the square. The horizontal regions are positive and the vertical ones negative. Direct integration of $-\partial_n u$ gives flux -1 on each vertical edge and $+1$ on each horizontal edge; the sum is zero, also following from $\Delta u = 0$.



Question 4

Inside the unit disk, seek the bounded harmonic extension of the boundary function

$$g(\theta) = \begin{cases} 1, & -\pi/2 < \theta < \pi/2, \\ 0, & \pi/2 < \theta < 3\pi/2. \end{cases}$$

Angles are interpreted modulo 2π . Values assigned at the two jump points do not affect the Poisson integral

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} g(\phi) d\phi, \quad 0 \leq r < 1.$$

You may use that this kernel is positive, integrates to 2π , and has Fourier expansion $1 + 2 \sum_{n \geq 1} r^n \cos(n\alpha)$.

Tasks

1. Compute the boundary Fourier coefficients and the resulting harmonic series.
2. Prove $0 < u < 1$ in the disk and find the center value. Explain the uniqueness class for these discontinuous data.
3. Determine the radial boundary limits on both open semicircles and at the two jump points. Explain why no continuous extension to the entire closed disk exists.
4. Find a closed expression for $u(x, 0)$ along the horizontal diameter and sketch it, including its limiting endpoint values.

Question 4 – Solution

Strategy. The Poisson kernel supplies a bounded harmonic extension without imposing impossible continuity at jumps.

Step 1: Compute and damp the Fourier modes. The average is $1/2$. Symmetry makes every sine coefficient zero, while

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(n\phi) d\phi = \frac{2 \sin(n\pi/2)}{n\pi}.$$

The kernel expansion therefore gives

$$u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{k \geq 0} \frac{(-1)^k r^{2k+1} \cos((2k+1)\theta)}{2k+1}.$$

On every smaller closed disk all derivatives converge uniformly by geometric decay. Each polar harmonic is the real part of $(x + iy)^{2k+1}$.

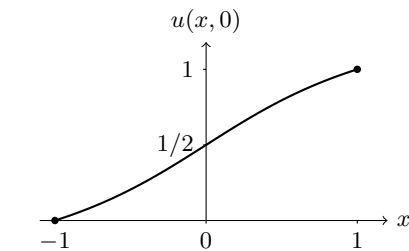
Step 2: Use positivity and specify uniqueness. Both semicircles have positive length, and the kernel is strictly positive inside. Thus their harmonic weights lie strictly between zero and one, giving $0 < u < 1$ and $u(0) = 1/2$. Uniqueness holds among bounded harmonic functions with these radial traces almost everywhere. Indeed, for the difference w , the Poisson formula on a circle of radius $R < 1$ expresses a fixed interior value through $w(R, \phi)$. As $R \uparrow 1$, boundedness and the a.e. zero trace let dominated convergence give $w = 0$. No common continuous trace at the jumps is required.

Step 3: Distinguish the boundary limits. At a point strictly within either semicircle the kernel is an approximate identity: away from any fixed angular neighborhood it tends uniformly to zero, while its total mass is fixed. The radial limit is therefore the local constant, one or zero. At $\theta = \pm\pi/2$ every displayed cosine term vanishes, so $u(r, \pm\pi/2) = 1/2$ for all $r < 1$. Neighboring boundary limits disagree; assigning $1/2$ at the jumps cannot make the closed-disk extension continuous.

Step 4: Sum the diameter profile. For $-1 < x < 1$, integrate the geometric series for $1/(1+x^2)$ from zero to x to obtain $\arctan x = \sum_{k \geq 0} (-1)^k x^{2k+1}/(2k+1)$. Hence

$$u(x, 0) = \frac{1}{2} + \frac{2}{\pi} \arctan x.$$

Its derivative is $2/[\pi(1+x^2)] > 0$; its limits at $x = -1, 1$ are zero and one. These endpoints lie on the open constant-data arcs, not at the jumps.



Endpoint dots show boundary limits.

Question 5

A disk of radius one has prescribed boundary temperature

$$g(\theta) = 2 + \cos \theta + \frac{1}{2} \cos(2\theta).$$

Let u be its continuous harmonic extension. For $r > 0$, the polar Laplacian is $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$.

Tasks

1. Construct u in polar and Cartesian coordinates. Explain why excluding singular radial terms is essential at the center.
2. Determine the exact boundary range and the locations of its extrema. Deduce sharp strict bounds for interior temperature.
3. Calculate the center temperature and the angular mean on every concentric circle. Explain their relation to the boundary average.
4. Find $\partial_n u$ on the boundary, verify zero total outward heat flux for conductivity one, and compute $\int_{\text{disk}} |\nabla u|^2 dA$.

Question 5 – Solution

Strategy. Regular disk modes preserve the mean, damp angular oscillations and diagonalize the boundary energy calculation.

Step 1: Retain only regular radial factors. For mode $n \geq 1$, separation gives r^n and r^{-n} ; the mean mode gives a constant and $\log r$. The singular factors cannot occur in a harmonic function on the full disk. Thus

$$u = 2 + r \cos \theta + \frac{1}{2} r^2 \cos(2\theta) = 2 + x + \frac{1}{2}(x^2 - y^2).$$

The Cartesian polynomial verifies $\Delta u = 1 - 1 = 0$ even at the center, and setting $r = 1$ gives the prescribed data.

Step 2: Find the sharp range. Put $c = \cos \theta$. Then $g = 3/2 + c + c^2 = (c + 1/2)^2 + 5/4$, with $-1 \leq c \leq 1$. The minimum $5/4$ occurs at $\theta = 2\pi/3, 4\pi/3$; the maximum $7/2$ occurs at $\theta = 0$ modulo 2π . Consequently

$$\frac{5}{4} < u(x, y) < \frac{7}{2} \quad \text{inside the disk.}$$

The inequalities are strict by the strong maximum principle. The constants cannot be improved over the whole interior, since approaching the stated boundary points approaches those values.

Step 3: Identify the preserved mean. Both cosine terms have zero angular integral, so

$$\frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta = 2 = u(0) \quad (0 < r \leq 1).$$

This explicitly realizes the mean-value property. The center equals the boundary average, although most boundary points and most interior points have different temperatures.

Step 4: Compute flux and energy. At $r = 1$, the outward derivative is $u_r = \cos \theta + \cos(2\theta)$. Its integral is zero, hence so is the total conductive flux $-\int u_r d\theta$. Green's identity and trigonometric orthogonality give

$$\int_{\text{disk}} |\nabla u|^2 dA = \int_0^{2\pi} g(\theta) u_r(1, \theta) d\theta = \pi + \frac{1}{2}\pi = \boxed{\frac{3}{2}\pi}.$$

Zero net heat flux does not imply zero gradient or zero energy: heat enters through some boundary portions and leaves through others.

Question 6

A conducting annulus $a < r < R$ has constant inner and outer temperatures T_i, T_o , where $0 < a < R$ and $T_i \neq T_o$. Conductivity is one. Seek a radial harmonic field. Total outward heat flux means the integral of $-\partial_n u$ over the indicated boundary circle.

Tasks

1. Derive the temperature and prove it is the unique continuous harmonic solution of these boundary data, including among nonradial competitors.
2. Compute both total boundary fluxes with the annulus's outward normals. Identify its conductance, defined as the magnitude of transmitted total heat flux divided by $|T_i - T_o|$.
3. Compute the Dirichlet energy and describe its relation to the temperature difference and conductance.
4. Hold R, T_i, T_o fixed and let $a \downarrow 0$. Determine the local limit of temperature away from the origin, the energy limit and the largest gradient. Explain the failure of uniform convergence to T_o . Sketch the profiles for $R = 1, T_i = 1, T_o = 0$ and $a = 0.1, 0.3, 0.5$.

Question 6 – Solution

Strategy. The logarithmic radial mode is admissible on an annulus, but its small-hole limit is not uniform near the shrinking inner boundary.

Step 1: Solve the radial equation. The equation $(ru_r)' = 0$ gives $u = A \log r + B$. Writing $D = \log(R/a) > 0$ and $\Delta T = T_i - T_o$, the boundary data give

$$u(r) = T_o + \Delta T \frac{\log(R/r)}{D}.$$

Its radial Laplacian is zero and its two traces are correct. Any other continuous harmonic solution differs by a harmonic function with zero boundary data; the maximum principle on the annulus proves uniqueness.

Step 2: Respect the inner normal. Here $u_r = -\Delta T/(rD)$. At $r = R$ the outward normal points toward increasing r , whereas at $r = a$ it points toward decreasing r . Thus

$$F_{\text{outer}} = \frac{2\pi\Delta T}{D}, \quad F_{\text{inner}} = -\frac{2\pi\Delta T}{D}, \quad G = \frac{2\pi}{D}.$$

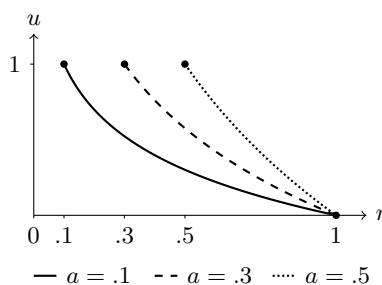
The sum is zero. When $\Delta T > 0$, heat enters the annulus at the inner boundary and leaves at the outer boundary.

Step 3: Integrate the energy. Direct radial integration yields

$$\mathcal{E} = 2\pi \int_a^R \frac{(\Delta T)^2}{r^2 D^2} r dr = \frac{2\pi(\Delta T)^2}{D} = G(\Delta T)^2.$$

Equivalently, Green's identity gives $\mathcal{E} = -T_i F_{\text{inner}} - T_o F_{\text{outer}}$, with the same sign and value.

Step 4: Separate local and uniform limits. As $a \downarrow 0$, $D \rightarrow \infty$. On each fixed $r_0 \leq r \leq R$ the field tends uniformly to T_o , and $\mathcal{E} \rightarrow 0$. However $\sup_{a \leq r \leq R} |u - T_o| = |\Delta T|$ for every a . Also $\max |\nabla u| = |\Delta T|/[a \log(R/a)] \rightarrow \infty$ at the inner circle. The concentration near a shrinking boundary is compatible with vanishing integrated energy.



Question 7

On the unit disk consider the Neumann problem

$$\Delta u = 0, \quad \partial_n u = 1 + 3 \cos \theta + 4 \sin(2\theta) \quad \text{on } r = 1.$$

Here $\partial_n u$ is the outward normal derivative, not the outward conductive heat flux.

Tasks

1. Decide whether a classical solution exists. Derive the necessary compatibility condition and evaluate it for these data.
2. Subtract a constant c from the prescribed derivative to make the data compatible. Find c and construct a regular harmonic solution whose area average on the disk is zero.
3. Prove uniqueness under that normalization, and state precisely the nonuniqueness without it.
4. For the corrected solution, compute its boundary trace, Dirichlet energy and net conductive heat flux. Check the energy independently by integrating the Cartesian gradient.

Question 7 – Solution

Strategy. Test flux compatibility before separation; then divide each nonconstant disk mode by its normal-derivative multiplier.

Step 1: Test the solvability condition. The divergence theorem gives

$$0 = \int_{\text{disk}} \Delta u \, dA = \int_{\partial \text{disk}} \partial_n u \, ds.$$

The prescribed derivative integrates to 2π , since both oscillatory terms integrate to zero. It violates a necessary condition, so the original problem has no classical solution.

Step 2: Correct the mean and construct the field. The corrected integral is $2\pi(1 - c)$, so the only constant correction is $c = 1$. Since $\partial_r(r^n) = n$ at $r = 1$, the regular solution is

$$u = 3r \cos \theta + 2r^2 \sin(2\theta) + C = 3x + 4xy + C.$$

The nonconstant terms have zero area average by symmetry. Thus normalization gives $C = 0$, and

$$\boxed{u(x, y) = 3x + 4xy.}$$

Its Cartesian Laplacian vanishes, and its outward derivative is exactly $3 \cos \theta + 4 \sin(2\theta)$.

Step 3: Establish uniqueness up to a constant. For two solutions, their difference w has $\Delta w = 0$ and $\partial_n w = 0$. Green's identity gives $\int |\nabla w|^2 = \int_{\partial \text{disk}} w \partial_n w = 0$. Hence $\nabla w = 0$, so w is constant on the connected disk. The zero-area-mean condition forces that constant to vanish. Without a normalization, every constant translate is another solution and these are all possibilities.

Step 4: Check energy in two coordinate systems. The boundary trace is $3 \cos \theta + 2 \sin(2\theta)$. Orthogonality gives

$$\mathcal{E} = \int_0^{2\pi} [3 \cos \theta + 2 \sin(2\theta)][3 \cos \theta + 4 \sin(2\theta)] \, d\theta = \boxed{17\pi}.$$

Independently, $|\nabla u|^2 = (3 + 4y)^2 + 16x^2 = 9 + 24y + 16r^2$. Its disk integral is $9\pi + 0 + 16(\pi/2) = 17\pi$. The corrected normal derivative integrates to zero, so its negative, the conductive flux, has zero total as required.

Question 8

A unit square has insulated vertical sides and prescribed horizontal temperatures:

$$\Delta u = 0, \quad u_x(0, y) = u_x(1, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = 1 + \cos(2\pi x).$$

Seek a solution smooth up to the boundary. Conductivity is one.

Tasks

1. Construct the solution by separation, treating the zero spatial eigenvalue explicitly.
2. Verify every datum and prove uniqueness for this mixed boundary problem using Green's identity.
3. Compute the horizontal average at each height and the integrated outward heat flux on each edge. Explain why zero side flux does not force constant temperature.
4. Calculate the Dirichlet energy. Prove $0 < u < 2$ at every interior point, justifying how the insulated sides are handled.

Question 8 – Solution

Strategy. Neumann side conditions select cosine modes, including a zero mode whose vertical factor is linear rather than hyperbolic.

Step 1: Keep the zero mode. The side eigenfunctions are $1, \cos(n\pi x)$ with eigenvalues $0, (n\pi)^2$. The mean mode satisfies $Y'' = 0$ with bottom value zero and top value one, so it is y . The second cosine mode gives

$$u(x, y) = y + \frac{\sinh(2\pi y)}{\sinh(2\pi)} \cos(2\pi x).$$

Dropping the zero mode would lose the entire prescribed mean temperature.

Step 2: Verify and prove mixed uniqueness. The displayed terms are harmonic. The sine factor in u_x vanishes at both vertical sides; the bottom and top traces are exactly zero and $1 + \cos(2\pi x)$. For the difference w of two solutions, Green's identity yields $\int |\nabla w|^2 = \int_{\partial\Omega} w \partial_n w = 0$: $w = 0$ on the horizontal edges and $\partial_n w = 0$ on the vertical ones. Thus w is constant, and its bottom value forces $w = 0$.

Step 3: Track the mean and edge fluxes. The cosine has zero horizontal integral, so $\bar{u}(y) = \int_0^1 u(x, y) dx = y$. The integrated outward conductive fluxes are

$$F_{\text{top}} = -1, \quad F_{\text{bottom}} = 1, \quad F_{\text{left}} = F_{\text{right}} = 0.$$

They balance. Insulation forbids heat crossing the vertical sides; it does not forbid a vertical gradient or heat transfer between the horizontal edges.

Step 4: Compute energy and bound the field. Only the top contributes to $\int_{\partial\Omega} u \partial_n u$. At the top, $u_y = 1 + 2\pi \coth(2\pi) \cos(2\pi x)$, so orthogonality gives

$$\mathcal{E} = 1 + \pi \coth(2\pi).$$

For $0 < y < 1$, strict convexity of $\sinh(2\pi y)$ gives $0 < \sinh(2\pi y)/\sinh(2\pi) < y$. Therefore $0 < y - \sinh(2\pi y)/\sinh(2\pi) \leq u \leq y + \sinh(2\pi y)/\sinh(2\pi) < 2y < 2$. This direct estimate avoids incorrectly treating the unspecified side temperatures as Dirichlet data.

Question 9

Let u be regular and harmonic on the unit disk. Its boundary temperature is known to have the form

$$g(\theta) = A + B \cos(N\theta), \quad N \geq 1,$$

where N is known and A, B are unknown. The temperature is measured on the circle $r = \rho$, where $0 < \rho < 1$.

Tasks

1. Derive exact recovery formulas for A, B from the measured angular mean and cosine coefficient. Specify the consistency conditions on the other measured Fourier coefficients.
2. For $\rho = 1/2$, $N = 6$, and measured trace $2 + 0.03 \cos(6\theta)$, recover the boundary data and the full field.
3. If the measured N th cosine coefficient has error at most η , find the sharp error bound for the recovered B . Derive the largest allowable mode number when the desired per-coefficient error is at most $\varepsilon > \eta$. Evaluate it for $\rho = 1/2, \eta = 10^{-4}, \varepsilon = 10^{-2}$.
4. Contrast forward boundary stability with inverse instability when arbitrarily high modes are allowed. Give an explicit sequence showing why small interior-circle errors need not imply small boundary errors.

Question 9 – Solution

Strategy. Harmonic continuation damps a mode by ρ^N ; inversion must divide by exactly that small factor.

Step 1: Recover the two parameters. The regular harmonic extension is $u = A + Br^N \cos(N\theta)$. For the measured trace $h(\theta) = u(\rho, \theta)$, define

$$m = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta, \quad c_N = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(N\theta) d\theta.$$

Orthogonality gives

$$A = m, \quad B = \rho^{-N} c_N.$$

Consistency with the stated model requires every sine coefficient and every nonconstant cosine coefficient except c_N to vanish. Since $\rho > 0$, the two parameters are unique.

Step 2: Reconstruct the given field. Here $\rho^N = 1/64$, $m = 2$, and $c_6 = 0.03 = 3/100$, so $B = 48/25 = 1.92$. Thus

$$g(\theta) = 2 + \frac{48}{25} \cos(6\theta), \quad u(r, \theta) = 2 + \frac{48}{25} r^6 \cos(6\theta).$$

Substitution at $r = 1/2$ reproduces the measured coefficient $3/100$. The polar mode is a Cartesian harmonic polynomial and is regular at the center.

Step 3: Quantify inverse amplification. A coefficient error e , $|e| \leq \eta$, produces error $\rho^{-N}e$ in B . Consequently $|\delta B| \leq \eta \rho^{-N}$, with equality when $|e| = \eta$. The desired bound is met exactly when

$$N \leq \frac{\log(\varepsilon/\eta)}{\log(1/\rho)}, \quad N_{\max} = \left\lfloor \frac{\log(\varepsilon/\eta)}{\log(1/\rho)} \right\rfloor.$$

If this integer is zero, no positive mode qualifies. The given numbers give $N_{\max} = 6$: mode six has error bound 0.0064, whereas mode seven has 0.0128. This is a per-coefficient criterion, not a bound on a sum of many mode errors.

Step 4: Separate forward and inverse stability. For continuous boundary perturbations, the maximum principle gives $\|u_1 - u_2\|_{\infty, \text{disk}} \leq \|g_1 - g_2\|_{\infty, \partial \text{disk}}$. In contrast, take $g_n(\theta) = \cos(n\theta)$ and $u_n = r^n \cos(n\theta)$. Then $\|g_n\|_{\infty} = 1$ but $\|u_n(\rho, \cdot)\|_{\infty} = \rho^n \rightarrow 0$. Thus no fixed bound can control boundary sup-norm errors by measured-circle sup-norm errors across all modes, although each exact finite-mode inversion is unique.

Question 10

Let u be harmonic on the unit square Ω , continuous on its closure, with prescribed boundary values g . A proposed approximation v is twice continuously differentiable on the closed square and satisfies

$$|v - g| \leq \varepsilon \quad \text{on } \partial\Omega, \quad |\Delta v| \leq \delta \quad \text{in } \Omega, \quad \varepsilon, \delta \geq 0.$$

You may use the weak maximum principle for subharmonic functions: if $\Delta w \geq 0$, then $\max_{\overline{\Omega}} w \leq \max_{\partial\Omega} w$.

Tasks

1. Construct a quadratic nonnegative barrier ψ with $-\Delta\psi = 1$ and maximum $1/8$ on the square.
2. Prove $|u - v| \leq \varepsilon + \delta\psi \leq \varepsilon + \delta/8$. Check the signs of both comparison functions.
3. Test the estimate for $g = x + y$ and $v = x + y + ax(1-x)y(1-y)$, where $a \neq 0$. Determine the exact boundary error, residual bound and actual maximum error, and compare with the certificate.
4. Explain why checking only the PDE residual cannot certify the answer. Show that the coefficient of ε in the estimate is sharp, and state whether this example proves sharpness of $1/8$.

Question 10 – Solution

Strategy. Convert a bounded residual into an explicit comparison barrier, while retaining a separate bound for incorrect boundary values.

Step 1: Construct the barrier. Set

$$\psi(x, y) = \frac{x(1-x) + y(1-y)}{4}.$$

It is nonnegative on the closed square and satisfies $\Delta\psi = -1$. Each quadratic is at most $1/4$, so $\max\psi = 1/8$ at $(1/2, 1/2)$. The barrier need not vanish on every edge; its nonnegative boundary values are sufficient for the comparison.

Step 2: Compare the error in both directions. Put $e = u - v$, so $\Delta e = -\Delta v$. For $w_+ = e - \varepsilon - \delta\psi$ and $w_- = -e - \varepsilon - \delta\psi$,

$$\Delta w_+ = -\Delta v + \delta \geq 0, \quad \Delta w_- = \Delta v + \delta \geq 0.$$

On the boundary, $|e| = |g - v| \leq \varepsilon$ and $\psi \geq 0$, so both $w_{\pm} \leq 0$. The stated maximum principle yields

$$|u - v| \leq \varepsilon + \delta\psi(x, y) \leq \varepsilon + \delta/8.$$

This is an analytic bound over the entire square, not an inference from sampled residuals.

Step 3: Compare the certificate with the exact error. The exact harmonic solution for $g = x + y$ is $u = x + y$. The added factor vanishes on every edge, so $\varepsilon = 0$. Direct differentiation gives

$$\Delta v = -2a[x(1-x) + y(1-y)], \quad \|\Delta v\|_{\infty} = |a|.$$

Thus the certificate gives $\|u - v\|_{\infty} \leq |a|/8$. The exact error is $|a|x(1-x)y(1-y)|$, with maximum $|a|/16$ at the center. The certificate is valid and overestimates this example's maximum by a factor of two.

Step 4: Identify what must be checked and what is sharp. The field $v = u + M$ has zero residual for every constant M , yet its error is $|M|$ everywhere. Without a boundary-error check, an arbitrarily inaccurate temperature can pass the PDE test. Taking $|M| = \varepsilon$ and $\delta = 0$ attains the bound $\|u - v\|_{\infty} = \varepsilon$, proving that coefficient one is sharp. The polynomial example does not attain the residual coefficient $1/8$ and does not prove it sharp. The barrier supplies a rigorous convenient certificate without a claim of optimality.

Question 1

A string on $0 < x < 1$ has unit tension and unit mass per length, fixed endpoints, and initial data

$$u_{tt} = u_{xx}, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = \sin(\pi x) + \frac{1}{2} \sin(2\pi x), \quad u_t(x, 0) = 2\pi \sin(3\pi x).$$

Its energy is $E(t) = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx$.

Tasks

1. Construct the complete modal solution, including the correct coefficient of the velocity-generated mode.
2. Verify the PDE, both endpoint conditions and both initial data. Determine whether the entire string can ever have zero displacement.
3. Compute the energy carried by each active mode and the total energy. Explain why cross terms do not contribute.
4. Find the smallest positive period of the full state (u, u_t) . Prove minimality rather than merely giving a return time.

Question 1 – Solution

Strategy. Initial displacement and initial velocity set different temporal coefficients; spatial orthogonality separates both energy and state recurrence.

Step 1: Normalize the three temporal modes. Fixed ends select $\sin(n\pi x)$ with frequencies $\omega_n = n\pi$. A modal initial velocity g_n gives coefficient g_n/ω_n multiplying $\sin(\omega_n t)$. Therefore

$$u = \sin(\pi x) \cos(\pi t) + \frac{1}{2} \sin(2\pi x) \cos(2\pi t) + \frac{2}{3} \sin(3\pi x) \sin(3\pi t).$$

The last coefficient is $2/3$, not 2π .

Step 2: Verify the state and test for a flat snapshot. Each term has equal second derivatives in x and t ; each spatial sine vanishes at both endpoints. At $t = 0$ the first two terms give the stated displacement. Differentiation gives $2\pi \sin(3\pi x)$ as the initial velocity. If $u(\cdot, t)$ were identically zero, orthogonality would force both $\cos(\pi t) = 0$ and $\cos(2\pi t) = 0$. But the first condition gives $t = k + 1/2$, at which $\cos(2\pi t) = -1$. Thus the string is never entirely flat.

Step 3: Add the orthogonal modal energies. For $q_n(t) \sin(n\pi x)$, the energy is $E_n = \frac{1}{4}[q_n'^2 + (n\pi)^2 q_n^2]$. Distinct sines are orthogonal in the kinetic term, and distinct cosines in the elastic term, so no cross terms remain. Evaluating at $t = 0$ gives

$$E_1 = \frac{\pi^2}{4}, \quad E_2 = \frac{\pi^2}{4}, \quad E_3 = \pi^2, \quad E = \frac{3\pi^2}{2}.$$

Each modal energy is constant because $q_n'' + (n\pi)^2 q_n = 0$.

Step 4: Find the fundamental state period. All three temporal modes return with their derivatives after time 2. Conversely, a return of the full state forces its nonzero first mode to satisfy $\cos(\pi P) = 1$ and $\sin(\pi P) = 0$. Thus P must be a positive even integer. The smallest is

$$P = 2.$$

A state recurrence requires displacement and velocity together; isolated zeros of one modal displacement do not establish a period.

Question 2

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ has fixed endpoints and wave speed $c = \sqrt{T/\mu}$. It is plucked to height $h > 0$ at $x = a$, where $0 < a < L$, then released from rest:

$$f(x) = \begin{cases} hx/a, & 0 \leq x \leq a, \\ h(L-x)/(L-a), & a \leq x \leq L, \end{cases} \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Use the finite-energy solution of $u_{tt} = c^2 u_{xx}$.

Tasks

1. Derive the sine coefficients and the full solution. State in what sense the initial data and PDE are satisfied near the moving corners.
2. Set $a = L/3$. Identify precisely which modes are absent and compute the initial, hence conserved, energy.
3. Prove $u(x, L/c) = -f(L-x)$ and $u(x, 2L/c) = f(x)$. Include the velocity state at these times.
4. Let F be the odd, $2L$ -periodic extension of f . Use it to obtain exact piecewise-linear profiles at $t = 0, L/(2c), L/c$ for $L = c = h = 1$, $a = 1/3$, and sketch them without truncating a Fourier series.

Question 2 – Solution

Strategy. Fourier coefficients reveal the missing harmonics, while reflected traveling profiles preserve the corners exactly.

Step 1: Compute the pluck coefficients. Integrating separately on $[0, a]$ and $[a, L]$ gives

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2hL^2}{a(L-a)n^2\pi^2} \sin \frac{n\pi a}{L}.$$

Thus

$$u(x, t) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

The $O(n^{-2})$ coefficients give uniform convergence of displacement. The initial velocity is zero in L^2 , and the solution has finite conserved energy. It solves the PDE weakly and classically away from the propagating slope corners; no globally C^2 solution is claimed.

Step 2: Identify the spectrum and energy. For $a = L/3$, $b_n = 9h \sin(n\pi/3)/(n^2\pi^2)$, so exactly the positive multiples of three are absent. Initially all energy is elastic:

$$E = \frac{T}{2} \int_0^L f'^2 dx = \frac{Th^2L}{2a(L-a)} = \boxed{\frac{9Th^2}{4L}} \quad (a = L/3).$$

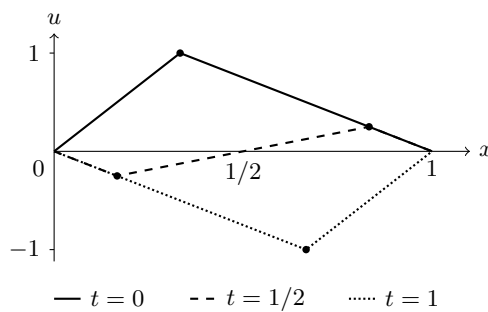
The finite-energy wave evolution conserves this value.

Step 3: Verify the reflected and full returns. At $t = L/c$, each cosine is $(-1)^n$, and $\sin(n\pi(L-x)/L) = (-1)^{n+1} \sin(n\pi x/L)$. Hence $u(x, L/c) = -f(L-x)$. At $2L/c$ every cosine equals one, so $u = f$. The modal velocities vanish at both times, giving zero velocity in L^2 .

Step 4: Construct exact corner-preserving snapshots. The reflected d'Alembert form is $u(x, t) = \frac{1}{2}[F(x-ct) + F(x+ct)]$. Oddness and periodicity enforce both endpoints and recover both initial data. For the stated numbers, the half-time profile is

$$u(x, \tfrac{1}{2}) = \begin{cases} -3x/2, & 0 \leq x \leq 1/6, \\ 3x/4 - 3/8, & 1/6 \leq x \leq 5/6, \\ 3(1-x)/2, & 5/6 \leq x \leq 1. \end{cases}$$

The other two profiles are $f(x)$ and $-f(1-x)$; their corners are plotted at their exact positions.



Question 3

A unit string has fixed endpoints and unit tension and density. It starts straight and is struck so that its initial velocity is

$$u(x, 0) = 0, \quad g(x) = u_t(x, 0) = \begin{cases} V, & 1/3 < x < 2/3, \\ 0, & 0 < x < 1/3 \text{ or } 2/3 < x < 1, \end{cases} \quad V > 0.$$

Values assigned at the two jump points do not change the finite-energy solution.

Tasks

1. Derive the velocity sine coefficients and construct the complete fixed-end solution of $u_{tt} = u_{xx}$.
2. Explain the symmetry and identify exactly which modes vanish. State the appropriate initial trace and regularity claims.
3. Find an exact formula for the midpoint displacement for $0 \leq t \leq 1/2$. Identify when its velocity first changes and explain why the displacement then stays positive.
4. Compute the conserved energy directly from the initial data. Derive the corresponding modal Parseval identity, including every normalization factor.

Question 3 – Solution

Strategy. A velocity profile must be integrated in time; its jumps do not permit a globally classical interpretation of the initial state.

Step 1: Convert the strike to temporal sine amplitudes. The velocity coefficients are

$$g_n = 2V \int_{1/3}^{2/3} \sin(n\pi x) dx = \frac{2V}{n\pi} \left(\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3} \right).$$

Because the initial displacement is zero,

$$u(x, t) = \sum_{n \geq 1} \frac{g_n}{n\pi} \sin(n\pi t) \sin(n\pi x).$$

The additional division by $n\pi$ is essential to recover $u_t(\cdot, 0) = g$.

Step 2: Interpret the symmetry and traces. The strike satisfies $g(1-x) = g(x)$, so only the symmetric, odd-indexed sine modes can occur. The displayed coefficient vanishes for every even n and is nonzero for every odd n : the cosine difference for odd n is either 1 or -2 . Displacement converges uniformly since its coefficients are $O(n^{-2})$. Velocity recovers g in L^2 ; its initial sine series takes the average $V/2$ at either jump. The PDE holds weakly everywhere and classically away from the reflected characteristic fronts.

Step 3: Compute the midpoint before reflected arrivals. For $0 \leq t \leq 1/2$, the interval $[1/2-t, 1/2+t]$ remains within $[0, 1]$. The velocity-only d'Alembert formula therefore gives

$$u(\tfrac{1}{2}, t) = \frac{1}{2} \int_{1/2-t}^{1/2+t} g(s) ds = \begin{cases} Vt, & 0 \leq t \leq 1/6, \\ V/6, & 1/6 \leq t \leq 1/2. \end{cases}$$

At $t = 1/6$, both edges of the moving integration interval pass the strike edges. The one-sided midpoint velocities change from V to zero. A zero subsequent velocity does not undo the displacement already accumulated.

Step 4: Match physical and modal energies. Initially $u_x = 0$, so

$$E = \frac{1}{2} \int_0^1 g^2 dx = \frac{V^2}{6}.$$

Parseval with $\int_0^1 \sin^2(n\pi x) dx = 1/2$ gives $\int g^2 = \frac{1}{2} \sum g_n^2$, hence $E = \frac{1}{4} \sum g_n^2$. Equivalently,

$$\sum_{n \geq 1} \frac{[\cos(n\pi/3) - \cos(2n\pi/3)]^2}{n^2} = \frac{\pi^2}{6}.$$

Each mode trades kinetic and elastic energy while preserving its contribution $g_n^2/4$.

Question 4

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. Its left end is fixed and its right end is free for $t > 0$:

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u_x(L, t) = 0.$$

At release its displacement is $f(x) = hx/L$, $h > 0$, and its velocity is zero. Interpret this as a finite-energy problem, since the initial slope need not satisfy the newly released end condition.

Tasks

1. Derive the mixed-end eigenfunctions, frequencies and coefficients of f . Construct the solution.
2. Explain the initial corner incompatibility. Prove that the full state reverses sign after $2L/c$ and has smallest positive period $4L/c$.
3. Show that at $t = L/c$ the string is flat but is not at rest. Find its velocity in the open interval and verify the conserved energy.
4. For $L = c = h = 1$, sketch exact profiles at $t = 0, 1/2, 1$. Use the odd reflection at the fixed end and even reflection at the free end, and mark the moving corner.

Question 4 – Solution

Strategy. A free end selects half-integer modes; the release launches a corner rather than making the initial slope compatible retroactively.

Step 1: Construct the mixed-mode solution. The conditions $X(0) = 0$, $X'(L) = 0$ give $k_n = (2n+1)\pi/(2L)$, $n \geq 0$, and $\omega_n = ck_n$. Since $\int_0^L \sin^2(k_n x) dx = L/2$,

$$b_n = \frac{2}{L} \int_0^L \frac{hx}{L} \sin(k_n x) dx = \frac{8h(-1)^n}{(2n+1)^2\pi^2}, \quad \boxed{u = \sum_{n \geq 0} b_n \sin(k_n x) \cos(\omega_n t)}.$$

Step 2: State the trace and recurrence precisely. Initially $f'(L) = h/L \neq 0$, so a solution with continuous u_x at the release corner cannot satisfy the new free-end condition there. The displayed finite-energy solution recovers f uniformly and zero initial velocity in L^2 ; its boundary condition has the usual weak/natural meaning at wavefront events. Every active frequency is an odd multiple of $\pi c/(2L)$. Thus both u and u_t change sign after $2L/c$ and return after $4L/c$. The nonzero lowest mode requires that full period, proving minimality.

Step 3: Resolve the flat but moving state. At $t = L/c$ every temporal cosine vanishes. For $0 < x < L$,

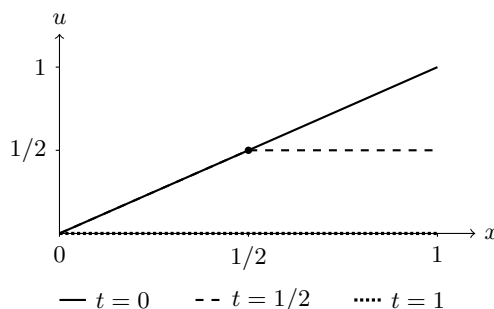
$$u_t(x, L/c) = -\frac{4hc}{L\pi} \sum_{n \geq 0} \frac{\sin((2n+1)\pi x/(2L))}{2n+1} = \boxed{-hc/L}.$$

Here $\sum_{n \geq 0} \sin((2n+1)z)/(2n+1) = \pi/4$ for $0 < z < \pi$, the sine series of a constant on that interval. The fixed-end velocity remains zero; the interior velocity trace is an L^2 statement, not corner continuity. Initially $E = Th^2/(2L)$. At this flat snapshot, the kinetic energy is $\mu L(hc/L)^2/2 = Th^2/(2L)$, so the energy is unchanged.

Step 4: Follow the released-end corner. Extend f oddly across zero and evenly across L , with period $4L$, and take the average of its two translates. For the stated units and $0 \leq t \leq 1$ this gives

$$\boxed{u(x, t) = \min(x, 1-t), \quad 0 \leq x \leq 1.}$$

For $0 < t < 1$, the corner is at $x = 1-t$: the left part retains slope one and the right part is flat and moving downward. The snapshots below use exact line segments.



Question 5

A unit string with fixed ends, unit density and unit tension experiences uniform viscous drag:

$$u_{tt} + 2\beta u_t = u_{xx}, \quad \beta = 2\pi, \quad u(x, 0) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x), \quad u_t(x, 0) = 0.$$

Tasks

1. Derive the modal equations and classify each of the three active modes as over-, critically or under-damped.
2. Construct the full solution, satisfying both initial conditions for every mode.
3. Derive the energy dissipation law for $E = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx$. Explain why energy is nonincreasing even when a modal displacement oscillates.
4. Find the leading long-time displacement profile and its exact exponential decay rate. More generally, state when all positive string modes are underdamped.

Question 5 – Solution

Strategy. The same drag rate competes with different natural frequencies, so one string can exhibit all three damping regimes at once.

Step 1: Classify the modal roots. With $u = \sum q_n(t) \sin(n\pi x)$, each coefficient satisfies $q_n'' + 2\beta q_n' + (n\pi)^2 q_n = 0$. For $\beta = 2\pi$, the discriminant sign is the sign of $4 - n^2$. Thus mode one is overdamped, mode two is critical, and mode three is underdamped.

Step 2: Impose both initial conditions. The required coefficient functions, all with $q_n(0) = 1$ and $q_n'(0) = 0$, are

$$\begin{aligned} q_1 &= e^{-2\pi t} \left[\cosh(\sqrt{3}\pi t) + \frac{2}{\sqrt{3}} \sinh(\sqrt{3}\pi t) \right], \\ q_2 &= e^{-2\pi t} (1 + 2\pi t), \\ q_3 &= e^{-2\pi t} \left[\cos(\sqrt{5}\pi t) + \frac{2}{\sqrt{5}} \sin(\sqrt{5}\pi t) \right]. \end{aligned}$$

Then $u = \sum_{n=1}^3 q_n(t) \sin(n\pi x)$ satisfies the damped PDE and both endpoints. The added sinh, linear and sine terms are necessary to make the initial velocities zero.

Step 3: Account for dissipated energy. Differentiate E and integrate $u_x u_{xt}$ by parts:

$$E' = [u_x u_t]_0^1 + \int_0^1 u_t (u_{tt} - u_{xx}) dx = \left[-2\beta \int_0^1 u_t^2 dx \leq 0 \right].$$

The boundary term is zero because fixed ends have zero velocity. Oscillation changes how energy is partitioned; drag still removes energy whenever the velocity is not identically zero.

Step 4: Identify the slow surviving mode. Expanding q_1 into two exponentials, its slower coefficient is $C = (1 + 2/\sqrt{3})/2$ and its slower decay rate is $\gamma = (2 - \sqrt{3})\pi$. The other two modes decay as $e^{-2\pi t}$ times bounded or linear factors. Therefore

$$e^{\gamma t} u(x, t) \longrightarrow C \sin(\pi x) \quad \text{uniformly for } 0 \leq x \leq 1.$$

For a nonnegative general drag parameter, mode n is underdamped when $\beta < n\pi$; all positive modes are underdamped exactly when $0 \leq \beta < \pi$. The threshold compares drag with the lowest frequency, not with the highest one.

Question 6

A unit string with unit tension and density has fixed ends and is initially at rest. A spatially sinusoidal force is held constant for a chosen duration $\tau > 0$:

$$u_{tt} - u_{xx} = F_0 \sin(\pi x) \chi_{(0,\tau)}(t), \quad u(x, 0) = u_t(x, 0) = 0, \quad F_0 > 0.$$

The indicator is one during the pulse and zero otherwise; isolated switch-time values do not affect the motion.

Tasks

1. Derive the modal displacement during and after the pulse, imposing continuity of displacement and velocity at switch-off.
2. Determine every positive duration that leaves the string exactly at rest after switch-off. Prove that both state variables vanish for those durations.
3. Compute the residual energy as a function of τ and verify it equals the net work of the applied force.
4. Find the durations that maximize residual energy. Plot its normalized dependence on pulse duration and explain how a force of one sign can do zero net work over a nontrivial pulse.

Question 6 – Solution

Strategy. Switch-off is a delayed negative step in forcing; controlling pulse duration can cancel the entire excited state.

Step 1: Superpose the two forcing steps. Only the first spatial mode is driven. Writing $u = q(t) \sin(\pi x)$ gives $q'' + \pi^2 q = F_0 \chi_{(0,\tau)}$ with zero initial data. Hence

$$q(t) = \frac{F_0}{\pi^2} \begin{cases} 1 - \cos(\pi t), & 0 \leq t \leq \tau, \\ \cos(\pi(t - \tau)) - \cos(\pi t), & t \geq \tau. \end{cases}$$

The two expressions and their first derivatives agree at $t = \tau$; only acceleration can jump. They verify the forcing on each open time interval.

Step 2: Find exact cancellation durations. For $t \geq \tau$, the cosine difference becomes

$$q(t) = \frac{2F_0}{\pi^2} \sin \frac{\pi\tau}{2} \sin(\pi(t - \tau/2)).$$

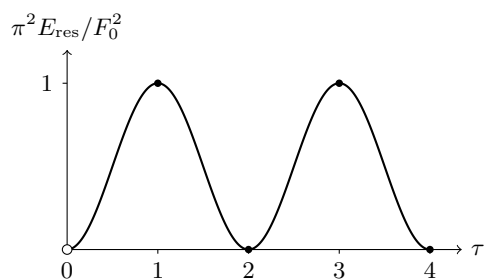
Its displacement and velocity vanish identically precisely when $\sin(\pi\tau/2) = 0$, giving $\tau = 2m, m = 1, 2, \dots$. Equivalently, the switch-off state $q(\tau) = F_0(1 - \cos \pi\tau)/\pi^2$, $q'(\tau) = F_0 \sin \pi\tau/\pi$ is zero exactly for those durations.

Step 3: Evaluate residual energy and work. After switch-off,

$$E_{\text{res}} = \frac{1}{4}(q'^2 + \pi^2 q^2) = \frac{F_0^2}{\pi^2} \sin^2 \frac{\pi\tau}{2}.$$

During forcing, the power is $\int_0^1 F_0 \sin(\pi x) u_t dx = F_0 q'/2$. Its integral from zero to τ is $F_0 q(\tau)/2 = F_0^2(1 - \cos \pi\tau)/(2\pi^2)$, the same energy.

Step 4: Interpret the pulse-duration curve. The maximum is F_0^2/π^2 , attained for $\tau = 2m + 1, m \geq 0$. For an even-integer duration, positive work during motion in the force direction is canceled by negative work while the string moves against the force. A positive force need not supply positive power at every instant.



Open dot: the excluded zero-duration limit.

Question 7

A unit fixed-end string with speed one is known to involve only modes $1, \dots, N$:

$$u(x, t) = \sum_{n=1}^N \left[f_n \cos(n\pi t) + \frac{g_n}{n\pi} \sin(n\pi t) \right] \sin(n\pi x).$$

A displacement sensor at $x = s \in (0, 1)$ records $h_s(t) = u(s, t)$ for $0 \leq t \leq 2$.

Tasks

1. Derive recovery formulas for f_n, g_n from the trace whenever the n th mode is visible.
2. For a rational sensor position $s = p/q$ in lowest terms, identify exactly the invisible modes. Explain why continuous-time sampling cannot recover those modes.
3. Use sensors at $s = 1/2$ and $s = 1/3$. Determine their common invisible modes and the largest N for which the entire stated finite-mode state is uniquely recoverable.
4. Prove that an irrational sensor sees every individual mode but can be arbitrarily poorly conditioned as N grows. You may use the pigeonhole fact that for every integer $Q \geq 1$ there are integers $1 \leq n \leq Q, k$ with $|ns - k| \leq 1/Q$.

Question 7 – Solution

Strategy. Time orthogonality separates frequencies, but it cannot restore a spatial mode whose node lies at the sensor.

Step 1: Project the temporal record. On $[0, 2]$, distinct integer-frequency sines and cosines are orthogonal and each squared norm is one. Consequently

$$f_n = \frac{\int_0^2 h_s(t) \cos(n\pi t) dt}{\sin(n\pi s)}, \quad g_n = \frac{n\pi \int_0^2 h_s(t) \sin(n\pi t) dt}{\sin(n\pi s)}$$

whenever $\sin(n\pi s) \neq 0$. These recover both initial displacement and velocity coefficients.

Step 2: Identify rational-position blindness. For $s = p/q$ in lowest terms, $\sin(n\pi s) = 0$ exactly when q divides n . Every such mode contributes zero to $h_s(t)$ for every time and for every choice of its two coefficients. Thus even a perfect continuous trace cannot distinguish states that differ only in those modes.

Step 3: Combine the two sensors. The midpoint misses the even modes; the one-third sensor misses multiples of three. Their common invisible modes are exactly the multiples of six. For $N \leq 5$, each mode has a nonzero factor at at least one sensor, so the formulas recover the entire state. For $N \geq 6$, arbitrary sixth-mode displacement or velocity can be added without changing either record. The largest fully identifiable cutoff is therefore

$$N = 5.$$

Step 4: Distinguish visibility from stability. If s is irrational, no ns is an integer, so each denominator is nonzero. But the supplied approximation gives $|\sin(n\pi s)| = |\sin(\pi(ns - k))| \leq \pi/Q$. As $Q \rightarrow \infty$, the selected indices cannot stay in a finite set: each such index has a fixed positive distance from the integers. Hence arbitrarily large modes have arbitrarily small sensor factors. An error e_n in the cosine projection becomes $e_n/\sin(n\pi s)$ in f_n ; the sine projection has the additional factor $n\pi$ for g_n . Exact visibility at all modes does not provide a uniform noise bound.

Question 8

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ is fixed at both ends and carries a point mass $M > 0$ at its midpoint. Away from that mass, $u_{tt} = c^2 u_{xx}$ with $c = \sqrt{T/\mu}$. Displacement is continuous at $x = L/2$, and the mass obeys

$$Mu_{tt}(L/2, t) = T[u_x(L/2+, t) - u_x(L/2-, t)].$$

Tasks

1. For a normal mode $u = \phi(x) \cos(\omega t)$, derive the interface condition and classify antisymmetric modes about the midpoint.
2. Derive the frequency equation for symmetric modes using $z = \omega L/(2c)$ and $\alpha = M/(\mu L)$. Prove there is exactly one positive symmetric root in each interval $(j\pi, j\pi + \pi/2)$, $j = 0, 1, \dots$
3. Find the limiting lowest symmetric frequency as $M \downarrow 0$ and its leading behavior as $M \rightarrow \infty$. Interpret the latter as an effective spring-mass system.
4. Write the total conserved energy, including the attached mass, and prove conservation by checking the interface terms.

Question 8 – Solution

Strategy. The midpoint mass changes the dynamic matching condition, but modes with a node there do not move the added mass.

Step 1: Separate and use symmetry. On each half, $\phi'' + k^2\phi = 0$, $k = \omega/c$, and the mass equation becomes

$$T(\phi'_+ - \phi'_-) + M\omega^2\phi(L/2) = 0.$$

Antisymmetry gives $\phi(L/2) = 0$ and matching slopes. The fixed endpoints then yield

$$\phi(x) = A \sin(2m\pi x/L), \quad \omega = 2m\pi c/L, \quad m = 1, 2, \dots$$

These frequencies are independent of M .

Step 2: Derive and locate the symmetric frequencies. A symmetric shape has $\phi = A \sin(kx)$ on the left and $\phi = A \sin(k(L-x))$ on the right. Its slope jump is $-2Ak \cos z$ and midpoint value is $A \sin z$. Substitution gives

$$2Tk \cos z = Mc^2 k^2 \sin z, \quad \cot z = \alpha z, \quad \alpha = \frac{M}{\mu L}.$$

On $(j\pi, j\pi + \pi/2)$, $\cot z - \alpha z$ decreases strictly from positive infinity to a negative value. It has exactly one root. On the intervening intervals the cotangent is negative, so no positive root occurs there. Values $\sin z = 0$ do not solve the original matching equation. There is no nonzero static mode with both endpoints fixed.

Step 3: Interpret the lowest root. For the root $z_0 \in (0, \pi/2)$, $\alpha \downarrow 0$ gives $z_0 \rightarrow \pi/2$, recovering $\omega_0 \rightarrow \pi c/L$. As $\alpha \rightarrow \infty$, $z_0 \rightarrow 0$ and $z \cot z \rightarrow 1$ gives $\alpha z_0^2 \rightarrow 1$. Thus

$$\omega_0 \sim 2\sqrt{\frac{T}{ML}} \quad (M \rightarrow \infty).$$

A small midpoint deflection d gives each half-string slope magnitude $2d/L$; their restoring forces add to $4Td/L$. The effective stiffness is $4T/L$, consistent with the displayed mass-spring frequency.

Step 4: Include the concentrated kinetic energy. For piecewise smooth motion, the energy is

$$E = \frac{1}{2} \int_0^L (\mu u_t^2 + T u_x^2) dx + \frac{M}{2} u_t(L/2, t)^2.$$

Integration by parts on each half leaves $T u_t(L/2)(u_x^- - u_x^+)$; the fixed-end contributions vanish. The point-mass term differentiates to $M u_t(L/2) u_{tt}(L/2) = T u_t(L/2)(u_x^+ - u_x^-)$. They cancel, so $\boxed{E' = 0}$. Omitting the mass term would falsely suggest loss or gain of energy at the interface.

Question 9

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. It is initially straight and at rest in the open interval. For $t > 0$, its left end stays fixed while its right end moves upward at constant speed $V > 0$:

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u(L, t) = Vt.$$

Only times $0 \leq t \leq 2L/c$ are considered. Define $r_+ = \max(r, 0)$.

Tasks

1. Construct the incident wave before it reaches the left end. Locate its front and verify the boundary and initial data in their appropriate senses.
2. Add the fixed-end reflected wave and obtain one formula valid through $t = 2L/c$. Verify both endpoint displacements and explain the time restriction.
3. Compute the total string energy on each side of the first reflection time and verify the boundary power balance.
4. Explain the release-corner and wavefront regularity. For $L = c = V = 1$, sketch exact profiles at $t = 1/2, 1, 3/2, 2$ and describe the interior state at the last time.

Question 9 – Solution

Strategy. A prescribed end motion launches a ramp front; reflection reverses its displacement sign and transfers the injected energy into a growing slope.

Step 1: Launch the incoming ramp. For $0 \leq t \leq L/c$,

$$u_{\text{in}}(x, t) = V(t - (L - x)/c)_+.$$

It vanishes ahead of the front $x = L - ct$ and equals Vt at $x = L$. Every function of $t + x/c$ satisfies the wave equation distributionally; here it is classical away from the front. At $t = 0$, displacement is zero and velocity is zero at every interior point.

Step 2: Reflect at the fixed left end. The reflected ramp must cancel the incoming displacement at $x = 0$:

$$u = V(t - (L - x)/c)_+ - V(t - (L + x)/c)_+.$$

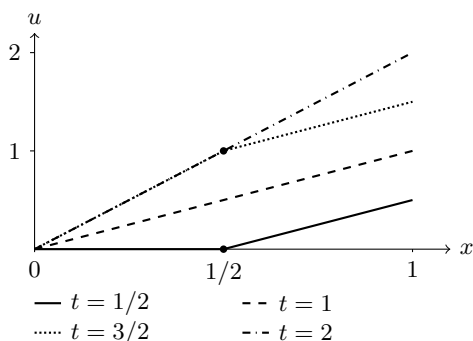
The left trace is identically zero. At $x = L$, the second term is zero through $t = 2L/c$, giving $u(L, t) = Vt$. After that time it would change the right trace, so additional reflections would be needed. The formula is not asserted for later times.

Step 3: Balance energy and injected power. Before the first reflection, the moving portion has length ct , with $u_t = V$ and $u_x = V/c$. Since $T/c^2 = \mu$, its energy is $E = \mu c V^2 t$. For $L/c < t < 2L/c$, the reflected left portion has length $ct - L$, velocity zero and slope $2V/c$; the right portion has length $2L - ct$, velocity V and slope V/c . Hence again

$$E = 2\mu V^2(ct - L) + \mu V^2(2L - ct) = \boxed{\mu c V^2 t}.$$

For $0 < t < 2L/c$, the right-end slope is V/c , so the input power is $Tu_x(L, t)u_t(L, t) = TV^2/c = \mu c V^2 = E'$.

Step 4: Interpret the corner and the final snapshot. The initial interior velocity is zero, but the driven endpoint velocity jumps to V : no continuous velocity at that release corner is possible. Displacement stays continuous across each front, while first derivatives jump; the weak wave equation remains valid. At $t = 2L/c$, $u = 2Vx/c$ and the open-interval velocity is zero; energy is entirely elastic, $2\mu LV^2$. The endpoint still moves, so this is not a smooth state of rest that persists afterward.



Question 10

A unit fixed-end string with unit tension and density is released from rest with $u(x, 0) = x(1 - x)$. Let u_N retain only sine modes with index $n \leq N$, where $N \geq 1$ is an integer, from the exact finite-energy solution. Define the error energy

$$E_{\text{err}}(t) = \frac{1}{2} \int_0^1 [(u_t - u_{N,t})^2 + (u_x - u_{N,x})^2] dx.$$

Tasks

1. Derive the Fourier solution and its total energy. State the regularity caveat at the initial boundary corners.
2. Derive an exact series for E_{err} and prove the all-time estimate $E_{\text{err}} \leq 16/(3\pi^4 N^3)$.
3. Prove the all-time displacement estimate $\|u - u_N\|_{\infty} \leq 4/(\pi^3 N^2)$. Find a sufficient integer N for error at most 10^{-3} everywhere.
4. Find a sufficient integer N for relative energy error at most 10^{-4} . Explain why neither estimate depends on elapsed time and why wave evolution supplies no heat-like smoothing factor.

Question 10 – Solution

Strategy. Orthogonality gives a conserved tail energy, while absolute Fourier tails give a different uniform displacement certificate.

Step 1: Construct the solution and normalize energy. Two integrations by parts give

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4(1-(-1)^n)}{(n\pi)^3}.$$

Thus

$$u(x, t) = \frac{8}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x) \cos(n\pi t)}{n^3}, \quad E = \frac{1}{2} \int_0^1 (1-2x)^2 dx = \frac{1}{6}.$$

Displacement and first-derivative series converge uniformly, and the PDE has a finite-energy interpretation. A globally C^2 corner solution is impossible: fixed endpoints would have $u_{tt} = 0$ there, while the initial second spatial derivative is -2 .

Step 2: Bound the conserved energy tail. Each omitted mode contributes $(n\pi)^2 b_n^2/4$, independent of time. Orthogonality therefore gives

$$E_{\text{err}} = \frac{16}{\pi^4} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^4} \leq \frac{16}{\pi^4} \sum_{n > N} \frac{1}{n^4} \leq \boxed{\frac{16}{3\pi^4 N^3}}.$$

The last inequality is the decreasing-function integral bound $\sum_{n > N} n^{-4} \leq \int_N^\infty s^{-4} ds$.

Step 3: Certify displacement independently. Using $|\sin|, |\cos| \leq 1$ and the same tail comparison,

$$\|u - u_N\|_\infty \leq \frac{8}{\pi^3} \sum_{n > N} \frac{1}{n^3} \leq \frac{4}{\pi^3 N^2}.$$

This holds simultaneously for every x and every t . For tolerance $\delta = 10^{-3}$, it suffices to take $N \geq \sqrt{4/(\pi^3 \delta)} \approx 11.36$, so $\boxed{N = 12}$ suffices. The bound includes even indices for convenience; it is sufficient, not a claim of the smallest possible retained set.

Step 4: Choose an energy-based cutoff. Since $E = 1/6$, $E_{\text{err}}/E \leq 32/(\pi^4 N^3)$. For tolerance $\eta = 10^{-4}$, it suffices that

$$N \geq \left(\frac{32}{\pi^4 \eta} \right)^{1/3} \approx 14.87, \quad \boxed{N = 15}.$$

Each wave mode preserves its energy and merely oscillates. There is no factor decaying with $n^2 t$ as in diffusion, so waiting does not reduce the truncation's energy error. These two cutoffs measure different errors and need not agree.

Question 1

Three problems use the interval $0 < x < \pi$ with zero values at $x = 0, \pi$:

$$H_t = \frac{1}{2}H_{xx}, \quad H(x, 0) = \sin(2x); \quad W_{tt} = 9W_{xx}, \quad W(x, 0) = \sin(2x), \quad W_t(x, 0) = 3\sin(2x).$$

The third is $P_{xx} + P_{yy} = 0$ on $0 < x < \pi$, $0 < y < 1$, with $P(x, 0) = 0$ and $P(x, 1) = \sin(2x)$, as well as the zero side values.

Tasks

1. Derive the common spatial eigenproblem using $-X'' = \lambda X$. Explain why neither $\lambda = 0$ nor $\lambda < 0$ gives a nonzero Dirichlet eigenfunction.
2. Derive the remaining ordinary differential equation for each PDE and solve all three specified problems.
3. Verify every original datum, including the wave velocity. Explain why the three remaining factors require different kinds or numbers of conditions.
4. A proposed rule says a positive spatial eigenvalue always produces an exponential decay factor. Diagnose the rule and state the correct behavior for these three problems.

Question 1 – Solution

Strategy. Keep one spatial sign convention, then let the original PDE determine whether the remaining equation is first-order decay, oscillation or a spatial boundary problem.

Step 1: Fix the spatial spectrum. The conditions are $X(0) = X(\pi) = 0$. At $\lambda = 0$, the affine solution must vanish. For $\lambda = -\sigma^2 < 0$, $X = A \sinh(\sigma x)$ after the left condition, and the right condition forces $A = 0$. For $\lambda > 0$, the nonzero possibilities are

$$\lambda_n = n^2, \quad X_n(x) = \sin(nx), \quad n = 1, 2, \dots$$

All supplied data occupy the $n = 2$ mode.

Step 2: Derive the three remaining equations. Substitution gives $a' = -\lambda a/2$ for heat, $b'' = -9\lambda b$ for the wave, and $d'' = \lambda d$ for Laplace's equation. At $\lambda = 4$, the specified solutions are

$$H = e^{-2t} \sin(2x), \quad W = (\cos(6t) + \tfrac{1}{2} \sin(6t)) \sin(2x), \quad P = \frac{\sinh(2y)}{\sinh 2} \sin(2x).$$

Step 3: Check the original conditions. Each spatial sine vanishes at both ends. At $t = 0$, the two time-dependent fields have displacement $\sin(2x)$, while $W_t(x, 0) = 6(\frac{1}{2}) \sin(2x) = 3 \sin(2x)$. The harmonic field is zero at $y = 0$ and equals $\sin(2x)$ at $y = 1$. Direct second derivatives verify all three PDEs. The heat factor needs one initial value; the wave factor needs position and velocity; the harmonic factor uses two values on different spatial edges.

Step 4: Correct the proposed rule. A positive eigenvalue produces heat decay $e^{-\lambda t/2}$, wave frequency $3\sqrt{\lambda}$, and hyperbolic spatial factors for the harmonic extension. The last factor increases with y because it connects a zero bottom trace to a nonzero top trace; y is not time. The sign of the spatial eigenvalue alone does not identify the remaining factor without the PDE and its data.

Question 2

Impose insulated side conditions: $H_x = W_x = P_x = 0$ at $x = 0, \pi$. Consider

$$H_t = H_{xx}, \quad H(x, 0) = 2 + \cos x; \quad W_{tt} = W_{xx}, \quad W(x, 0) = 2 + \cos x, \quad W_t(x, 0) = 3 + 2 \cos x.$$

On $0 < x < \pi$, $0 < y < 1$, let P be harmonic with $P(x, 0) = 0$ and $P(x, 1) = 2 + \cos x$.

Tasks

1. Derive the Neumann eigenfunctions, including the zero eigenvalue. State the norm of the constant eigenfunction and of each positive cosine mode.
2. Construct all three solutions, treating their zero modes explicitly.
3. Compute the horizontal mean in each problem and explain exactly which data would be lost if the zero mode were omitted.
4. Compute the wave energy $\frac{1}{2} \int_0^\pi (W_t^2 + W_x^2) dx$. Explain why a constant energy does not imply bounded wave displacement in this setting.

Question 2 – Solution

Strategy. The spatial constant is an actual eigenfunction for insulated sides; its remaining equation changes with the type of PDE.

Step 1: Retain the zero eigenvalue. The problem $-X'' = \lambda X$, $X'(0) = X'(\pi) = 0$ has $X_0 = 1$, $\lambda_0 = 0$, and $X_n = \cos(nx)$, $\lambda_n = n^2$ for $n \geq 1$. Their squared $L^2(0, \pi)$ norms are π and $\pi/2$, respectively. The mean coefficient therefore uses $1/\pi$, not the positive-mode factor $2/\pi$.

Step 2: Solve the zero and first-mode equations. For $\lambda = 0$, the remaining equations are $a' = 0$, $b'' = 0$ and $d'' = 0$. Using all the supplied data gives

$$\begin{aligned} H(x, t) &= 2 + e^{-t} \cos x, \\ W(x, t) &= 2 + 3t + (\cos t + 2 \sin t) \cos x, \\ P(x, y) &= 2y + \frac{\sinh y}{\sinh 1} \cos x. \end{aligned}$$

The cosine derivatives vanish at the side boundaries. The scalar factors verify the PDEs, both wave initial data and the two harmonic edge traces.

Step 3: Compare the three means. Since $\int_0^\pi \cos x \, dx = 0$, their horizontal means are

$$\overline{H} = 2, \quad \overline{W} = 2 + 3t, \quad \overline{P} = 2y.$$

Omitting the constant spatial eigenfunction would lose the heat initial mean, both the wave initial mean and mean velocity, and the harmonic top mean. The zero mode preserves a heat mean, permits a freely drifting wave mean, and linearly joins two spatial mean values in the harmonic problem.

Step 4: Separate drift energy from modal energy. Put $q = \cos t + 2 \sin t$. Orthogonality gives

$$E = \frac{1}{2} \left[9\pi + \frac{\pi}{2} (q'^2 + q^2) \right] = \frac{9\pi}{2} + \frac{5\pi}{4} = \boxed{\frac{23\pi}{4}}.$$

The spatially constant displacement $2 + 3t$ contributes no elastic energy, although its velocity contributes a fixed kinetic energy. Thus W becomes unbounded through its mean while its energy remains constant. A displacement bound from this energy alone would require control of that mean.

Question 3

Consider the separated spatial problem

$$-X'' = \lambda X, \quad 0 < x < L, \quad X(0) = 0, \quad X'(L) + hX(L) = 0,$$

where $L > 0$ and $h \in \mathbb{R}$. The same boundary conditions are imposed on the heat equation $u_t = \kappa u_{xx}$ and the wave equation $v_{tt} = c^2 v_{xx}$, with $\kappa, c > 0$.

Tasks

1. Derive the energy identity for λ . Use $X(L)^2 \leq L \int_0^L X'^2 dx$ to identify when every eigenvalue is positive and when a zero eigenvalue occurs.
2. For $h < -1/L$, derive the negative-eigenvalue equation and prove there is exactly one negative eigenvalue.
3. Choose $L = 1$ and $h = -\coth 1$. Construct a heat solution and a wave solution with initial profile $\sinh x$, taking zero initial wave velocity.
4. Explain why these solutions invalidate a rule that separated heat modes always decay and separated wave modes always oscillate. Identify the boundary term responsible.

Question 3 – Solution

Strategy. The admissible spatial spectrum depends on the boundary operator, including the sign of a Robin coefficient.

Step 1: Include the Robin boundary term. Multiplication by X and integration by parts give

$$\lambda \int_0^L X^2 dx = \int_0^L X'^2 dx + hX(L)^2.$$

For $h \geq 0$ the right side is positive for nonzero X with $X(0) = 0$. For $-1/L < h < 0$, the supplied trace inequality bounds it below by $(1 + hL) \int X'^2 > 0$. At $h = -1/L$ it is nonnegative, with equality precisely for an affine function through zero; $X = x$ is the zero mode. Conversely, solving $X'' = 0$ shows a nonzero zero mode occurs only at $h = -1/L$.

Step 2: Find the negative mode. For $\lambda = -\sigma^2$, $\sigma > 0$, the left condition gives $X = A \sinh(\sigma x)$. The right condition is

$$\sigma \coth(\sigma L) = -h.$$

The left side increases strictly from $1/L$ to infinity: with $z = \sigma L$, the derivative of $z \coth z$ is $(\sinh z \cosh z - z)/\sinh^2 z > 0$. Thus there is exactly one positive σ when $-h > 1/L$, and none otherwise. This counts all negative eigenvalues because every such eigenfunction must have the displayed hyperbolic form.

Step 3: Build the two growing examples. For the specified L, h , $\sigma = 1$ and $\lambda = -1$. Hence

$$u(x, t) = e^{\kappa t} \sinh x, \quad v(x, t) = \cosh(ct) \sinh x.$$

Their spatial second derivatives equal the fields; their time derivatives satisfy the respective PDEs. The left traces are zero, and at $x = 1$ the Robin expression is proportional to $\cosh 1 - \coth 1 \sinh 1 = 0$. The initial profiles are $\sinh x$ and the initial wave velocity is zero.

Step 4: Locate the failure of the rule. Heat factors satisfy $a' = -\kappa \lambda a$; wave factors satisfy $b'' + c^2 \lambda b = 0$. A negative eigenvalue therefore gives growth rather than decay or pure oscillation. In the spatial energy identity, $hX(L)^2$ is negative here. The usual nonnegative energy form for dissipative or fixed boundaries cannot be assumed for this Robin condition. The counterexample changes the spectrum, not the separation calculation.

Question 4

On $0 < x < 1$, compare $H_t = H_{xx}$ and $W_{tt} = W_{xx}$ with the moving boundary values

$$H(0, t) = W(0, t) = t, \quad H(1, t) = W(1, t) = 2t.$$

Take $H(x, 0) = W(x, 0) = 0$ and $W_t(x, 0) = 1 + x$.

Tasks

1. For general boundary values $A(t), B(t)$, derive the transformed PDE and initial data after subtracting $\ell(x, t) = (1 - x)A(t) + xB(t)$ in each problem.
2. For the specified data, decide whether the affine lifting itself solves each original PDE.
3. Construct the complete heat solution using a stationary correction to its transformed forcing and a sine expansion. Derive the required coefficients.
4. Verify the reconstructed original data and find the long-time heat offset relative to the moving line. Contrast it with the wave field and state the heat release-corner regularity caveat.

Question 4 – Solution

Strategy. Homogenizing boundary values creates a source involving the first or second time derivative of the lifting, depending on the PDE.

Step 1: Transform the PDE and every initial datum. Since $\ell_{xx} = 0$, the zero-endpoint fields $v = H - \ell$ and $z = W - \ell$ obey

$$v_t - v_{xx} = -\ell_t, \quad z_{tt} - z_{xx} = -\ell_{tt}.$$

Their initial displacements are the respective original profiles minus $\ell(x, 0)$; the wave initial velocity is the original velocity minus $\ell_t(x, 0)$. Transforming only the endpoints would omit essential source and initial terms.

Step 2: Test the actual moving line. Here $\ell = t(1 + x)$, so $\ell_t = 1 + x$ but $\ell_{tt} = \ell_{xx} = 0$. It fails the heat PDE but solves the wave PDE. The wave's transformed displacement and velocity are both zero, giving

$$W(x, t) = t(1 + x).$$

The heat correction has $v_t - v_{xx} = -(1 + x)$ and $v(x, 0) = 0$.

Step 3: Solve the forced heat correction. Seek $P'' = 1 + x$ with $P(0) = P(1) = 0$. Integration gives

$$P(x) = \frac{x^2}{2} + \frac{x^3}{6} - \frac{2x}{3} = \frac{x(x-1)(x+4)}{6}.$$

The sine coefficients of $1 + x$ are $a_n = 2[1 - 2(-1)^n]/(n\pi)$. Twice integrating P'' against a sine gives $p_n = -a_n/(n\pi)^2$. Thus

$$H = t(1 + x) + P(x) + \sum_{n \geq 1} \frac{2[1 - 2(-1)^n]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x).$$

Step 4: Verify and interpret the offset. At both ends, P and every sine vanish. At $t = 0$ the summable sine coefficients reconstruct $-P$ uniformly, giving zero initial temperature. For $t > 0$, Gaussian damping permits differentiated convergence; substitution verifies $H_t = H_{xx}$. The initial flat field has $H_{xx} = 0$ while the boundary time derivatives are one and two, so joint classical corner smoothness is not asserted. Finally $H - \ell \rightarrow P < 0$ on $0 < x < 1$, uniformly on the closed interval, whereas $W - \ell = 0$ at all times. An identical boundary lifting can have different physical consequences in the two PDEs.

Question 5

On the square $0 < x, y < \pi$, let

$$U_t = U_{xx} + U_{yy}, \quad U = 0 \text{ on every edge}, \quad U(x, y, 0) = \sin x \sin(2y) + 2 \sin(2x) \sin y.$$

Tasks

1. Derive the product eigenfunctions and explain why the two active products have the same decay rate. Construct U .
2. Determine whether U is separable as $T(t)F(x, y)$ and whether its nonzero spatial profile can be written as $X(x)Y(y)$. Prove both conclusions.
3. Find the interior zero-temperature curve, its limiting edge endpoints and the signs on either side. Sketch that geometry.
4. Replace the initial coefficients by unknown A, B . Explain why one fixed-point time trace cannot distinguish them, but traces at $(\pi/2, \pi/4)$ and $(\pi/4, \pi/2)$ can. Give exact recovery formulas.

Question 5 – Solution

Strategy. A repeated eigenvalue allows a common time factor without forcing the spatial combination to be a single coordinate product.

Step 1: Identify the repeated eigenvalue. Zero edges give $\sin(mx)\sin(ny)$ with eigenvalues $m^2 + n^2$. The pairs $(1, 2)$ and $(2, 1)$ both give five. Therefore

$$U = e^{-5t} [\sin x \sin(2y) + 2 \sin(2x) \sin y].$$

Every edge and the initial field are correct, and applying the Laplacian multiplies the bracket by -5 .

Step 2: Distinguish two meanings of separation. The displayed formula is exactly one product $T(t)F(x, y)$. It is not one product $T(t)X(x)Y(y)$. Indeed, the spatial slice at $y = \pi/2$ is $2 \sin(2x)$, whereas at $y = \pi/4$ it is $\sin x + \sqrt{2} \sin(2x)$. These two nonzero slices are not proportional by sine orthogonality, as they would have to be for $X(x)Y(y)$.

Step 3: Determine the nodal geometry. Factor the bracket as $2 \sin x \sin y (\cos y + 2 \cos x)$. The first two factors are positive inside the square, so the interior zero set is

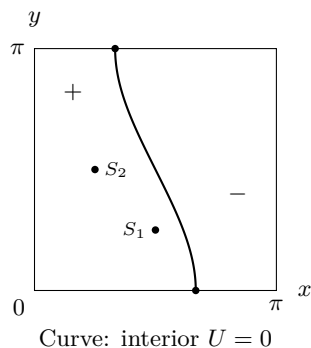
$$\cos y = -2 \cos x, \quad \pi/3 < x < 2\pi/3.$$

Its limiting endpoints are $(\pi/3, \pi)$ and $(2\pi/3, 0)$. At fixed y , the field is positive for $x < \arccos(-\cos y/2)$ and negative to its right. The zero curve is unchanged by the positive decay factor.

Step 4: Resolve the equal-rate inverse problem. At any one sensor, the entire trace has the form $e^{-5t}(A\alpha + B\beta)$ with fixed spatial factors α, β ; it supplies only one scalar equation. At the two specified sensors the unwanted product vanishes and the wanted product equals one. Hence

$$A = e^{5t}U(\pi/2, \pi/4, t), \quad B = e^{5t}U(\pi/4, \pi/2, t).$$

Distinct time samples at one point cannot separate equal decay rates; spatially independent observations can.



Question 6

Consider diffusion with variable spatial coefficient:

$$u_t = (x^2 u_x)_x, \quad 1 < x < e, \quad u(1, t) = u(e, t) = 0,$$

$$u(x, 0) = x^{-1/2} [\sin(\pi \log x) + 2 \sin(2\pi \log x)].$$

Use $s = \log x$ and $U(s, t) = u(e^s, t)$.

Tasks

1. Derive the transformed PDE and find a substitution $U = e^{-s/2}V$ that removes its first spatial derivative.
2. Derive the original spatial eigenfunctions, eigenvalues and their $L^2(1, e; dx)$ norms. Construct the complete solution.
3. Derive the coefficient formula for general initial data in these eigenfunctions. Explain why ordinary unweighted sine projection in s of $u(e^s, 0)$ is incorrect.
4. Verify the original PDE and data and derive the energy identity $\frac{1}{2}(d/dt) \int_1^e u^2 dx = - \int_1^e x^2 u_x^2 dx$. Check it on each separated eigenfunction.

Question 6 – Solution

Strategy. Changing coordinates changes the differential operator and the integration measure; a further amplitude factor restores ordinary sine orthogonality.

Step 1: Transform the operator, not just the variable. The chain rule gives $u_x = U_s/x$ and $(x^2 u_x)_x = U_{ss} + U_s$. With $U = e^{-s/2}V$, direct differentiation yields

$$V_t = V_{ss} - \frac{1}{4}V, \quad 0 < s < 1, \quad V(0, t) = V(1, t) = 0.$$

Ignoring U_s would shift every eigenvalue incorrectly.

Step 2: Recover the spectrum and field. The sine modes give

$$X_n(x) = x^{-1/2} \sin(n\pi \log x), \quad \lambda_n = (n\pi)^2 + \frac{1}{4}.$$

Since $dx = e^s ds$, the two $x^{-1/2}$ factors cancel the measure in a product:

$$\int_1^e X_m X_n dx = \int_0^1 \sin(m\pi s) \sin(n\pi s) ds.$$

Thus the squared norms are $1/2$, with zero cross products. The solution is

$$u = e^{-\lambda_1 t} X_1 + 2e^{-\lambda_2 t} X_2.$$

Step 3: Use the correct projection. For initial data f ,

$$b_n = 2 \int_1^e f(x) x^{-1/2} \sin(n\pi \log x) dx = 2 \int_0^1 e^{s/2} f(e^s) \sin(n\pi s) ds.$$

It is the transformed field $V(s, 0) = e^{s/2} f(e^s)$ that has ordinary sine coefficients. For example, the ungauged sines are not orthogonal in dx :

$$\int_1^e \sin(\pi \log x) \sin(2\pi \log x) dx = -\frac{4\pi^2(e+1)}{(1+\pi^2)(1+9\pi^2)} \neq 0.$$

Step 4: Verify diffusion and dissipation. Each X_n satisfies $-(x^2 X_n')' = \lambda_n X_n$ and vanishes at both endpoints. The temporal factors therefore verify the original PDE and exactly recover the supplied initial coefficients. Integration by parts gives $\frac{1}{2}(d/dt) \int u^2 = [u x^2 u_x]_1^e - \int x^2 u_x^2 = -\int x^2 u_x^2$. On a mode $a e^{-\lambda_n t} X_n$, both sides equal $-\lambda_n a^2 e^{-2\lambda_n t}/2$, since $\int x^2 X_n'^2 = \lambda_n \int X_n^2 = \lambda_n/2$.

Question 7

On $0 < x < \pi$, compare the heat equation $H_t = H_{xx}$ with the wave equation $W_{tt} = W_{xx}$. Both have zero endpoint displacement for positive time and the initial interior profile $f(x) = 1$; the wave has zero initial velocity. Use L^2 convergence for the initial displacement. For the wave, use the reflected bounded function obtained from the odd, 2π -periodic extension F of f , not a presumed finite-energy solution.

Tasks

1. Derive the common sine coefficients and the heat and wave series, stating the appropriate interpretation of each.
2. Use $W(x, t) = \frac{1}{2}[F(x - t) + F(x + t)]$ to determine the wave profile for $0 < t < \pi/2$, including values at its two fronts.
3. Explain the contrast with heat smoothing, and why neither field can converge uniformly to 1 over the whole open interval as $t \downarrow 0$. Sketch both profiles at $t = \pi/4$.
4. Compute the energy of the first N wave modes and show why this problem has no finite-energy wave solution with those Dirichlet initial data. Explain what remains valid in the reflected interpretation.

Question 7 – Solution

Strategy. The same initial Fourier coefficients can lead to a smooth heat field and a discontinuous wave field; convergence alone does not guarantee finite energy.

Step 1: Compute and evolve the coefficients. The coefficients are $b_n = 2(1 - (-1)^n)/(n\pi)$, so

$$H = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{e^{-n^2 t} \sin(nx)}{n}, \quad W = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt) \sin(nx)}{n}.$$

Both recover f in L^2 as $t \downarrow 0$. Heat derivatives converge uniformly for every $t \geq \tau > 0$. The wave series represents the reflected function in L^2 and takes average values at jumps; its zero initial time derivative is understood in distributions.

Step 2: Track the two reflected fronts. The extension is $F(s) = \text{sgn}(\sin s)$ away from multiples of π , with value zero at those multiples. For $0 < t < \pi/2$,

$$W(x, t) = \begin{cases} 0, & 0 < x < t \text{ or } \pi - t < x < \pi, \\ 1, & t < x < \pi - t, \\ 1/2, & x = t \text{ or } x = \pi - t. \end{cases}$$

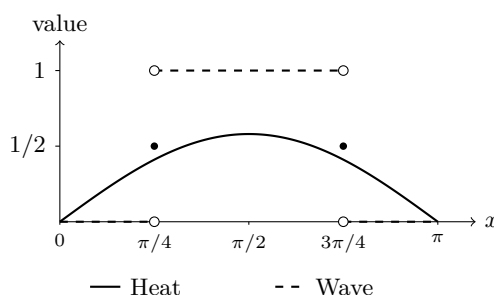
Each endpoint trace is zero. Reflection has produced two moving jumps, not a gradually smoothing profile.

Step 3: Compare smoothing and initial convergence. For $t > 0$, the heat field is smooth and $0 < H < 1$ in the interior, by diffusion and the maximum principle. The wave instead has the step profile above. For any positive time approaching zero, the wave's zero strips give sup-norm error one; the heat's continuous zero endpoint trace makes its supremum error also one over the open interval. Thus L^2 initial convergence does not imply uniform convergence across the incompatible endpoint layer.

Step 4: Test energy rather than assume it. The partial wave energy is

$$E_N = \frac{\pi}{4} \sum_{n \leq N} n^2 b_n^2 = \frac{4}{\pi} \#\{n \leq N : n \text{ odd}\} \rightarrow \infty.$$

No finite-energy Dirichlet wave has this initial profile; it is not in the zero-trace energy space. The reflected bounded field still solves the wave equation distributionally, with the stated initial displacement and zero initial velocity in the corresponding weak sense. At a moving jump, its derivatives need not be ordinary square-integrable functions.



Question 8

On $0 < x < \pi$ with zero endpoint values, consider the stationary problem

$$-u'' - 4u = f(x), \quad f(x) = \sin x + a \sin(2x) + \sin(3x), \quad a \in \mathbb{R},$$

and the associated evolution $v_t = v_{xx} + 4v + f$, initially $v(x, 0) = 0$.

Tasks

1. Derive the necessary stationary solvability condition by testing against the kernel. Show that it is sufficient for these data.
2. Find every stationary solution when one exists, and identify the one orthogonal to the kernel.
3. Solve the evolution from rest. Distinguish growth caused by a kernel forcing from growth caused by a negative eigenvalue of the stationary operator.
4. Replace the zero initial state by $b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x)$. Determine exactly when the evolution stays bounded and give its limiting stationary field.

Question 8 – Solution

Strategy. A separated coefficient equation can be singular or unstable; stationarity, uniqueness and attraction are separate questions.

Step 1: Test the stationary kernel. For $\mathcal{L} = -d^2/dx^2 - 4$, the eigenvalues on $\sin(nx)$ are $n^2 - 4$. The kernel is spanned by $\sin(2x)$. Integration by parts, using both zero endpoints, gives $0 = \int_0^\pi (\mathcal{L}u) \sin(2x) dx = \int_0^\pi f \sin(2x) dx = a\pi/2$. Thus $\boxed{a = 0}$ is necessary.

Step 2: Solve the compatible stationary equations. When $a = 0$, divide the first and third coefficients by $1 - 4 = -3$ and $9 - 4 = 5$:

$$\boxed{u = -\frac{1}{3} \sin x + \frac{1}{5} \sin(3x) + C \sin(2x), \quad C \in \mathbb{R}.}$$

Direct substitution proves sufficiency. Orthogonality to the kernel forces $C = 0$, and without it the stationary field is not unique.

Step 3: Evolve from rest. The modal equations are $q_1' = 3q_1 + 1$, $q_2' = a$, $q_3' = -5q_3 + 1$. Their zero-initial-value solutions give

$$\boxed{v = \frac{e^{3t} - 1}{3} \sin x + at \sin(2x) + \frac{1 - e^{-5t}}{5} \sin(3x).}$$

Nonzero a drives linear growth in the kernel mode. Separately, the first mode grows exponentially because $\mathcal{L}$ has eigenvalue -3 . Therefore even compatible forcing does not make the stationary family an attractor from rest.

Step 4: Identify the bounded initial states. For the prescribed three-mode initial state,

$$q_1 = -\frac{1}{3} + (b_1 + \frac{1}{3})e^{3t}, \quad q_2 = b_2 + at, \quad q_3 = \frac{1}{5} + (b_3 - \frac{1}{5})e^{-5t}.$$

Orthogonality prevents cancellation between spatial modes. Boundedness holds exactly when

$$\boxed{a = 0, \quad b_1 = -1/3.}$$

The coefficients b_2, b_3 are unrestricted. Then the uniform limit is $-\frac{1}{3} \sin x + b_2 \sin(2x) + \frac{1}{5} \sin(3x)$. Eliminating the unstable component and satisfying the kernel compatibility are both necessary.

Question 9

Let $u_t = u_{xx}$ on $0 < x < \pi$, with zero endpoint values and $u(x, 0) = \sin x + \sin(2x)$. A proposed solution strategy assumes that the complete answer is one product $X(x)T(t)$ because the PDE is separable.

Tasks

1. Derive the exact solution and explain why separability of the PDE does not force every solution to be a single product.
2. Prove that a nonzero product solution must use a spatial eigenfunction. Explain how the argument handles zeros of X or T .
3. Test the exact solution with the two-by-two sample determinant at $x_1 = \pi/3$, $x_2 = 2\pi/3$, $t_1 = 0$, $t_2 = \log 2$. Compute it and use it to rule out one-product factorization.
4. Find the exact uniform error when $e^t u(x, t)$ is approximated by $\sin x$, and a sufficient time for that error to be at most $\varepsilon \in (0, 1)$. Sketch normalized profiles for second-mode ratios 1, 1/2, 1/10 and zero.

Question 9 – Solution

Strategy. An eigenfunction expansion can become nearly one mode without ever being exactly one product at a finite time.

Step 1: Superpose the two decays. The eigenvalues are one and four, so

$$u = e^{-t} \sin x + e^{-4t} \sin(2x).$$

The PDE and every datum follow term by term. Separation constructs building blocks; linearity allows their sums, whose distinct temporal factors need not combine into one product.

Step 2: Derive the restriction on a product. If $u = XT$ is a nonzero classical product, then $XT' = X''T$. Choose a time with $T \neq 0$ and points where $X \neq 0$. The ratios satisfy $T'/T = X''/X = -\lambda$, independent of both variables. The differential identity $X'' = -\lambda X$ extends across spatial zeros by continuity; the temporal ODE $T' = -\lambda T$ extends likewise. Zero boundary values give the Dirichlet eigenproblem. The trivial zero product is separate. Division at zeros is unnecessary and cannot create extra eigenfunctions.

Step 3: Use a factorization invariant. Every product has determinant

$$u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1) = 0.$$

For the stated points, the four entries are

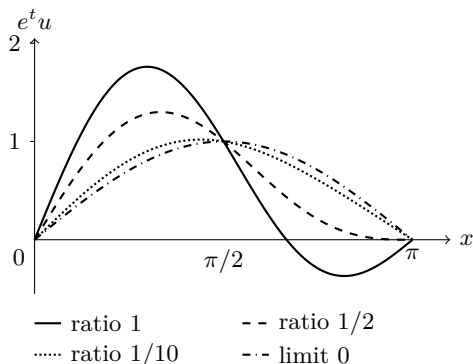
$$u(x_1, 0) = \sqrt{3}, \quad u(x_2, 0) = 0, \quad u(x_1, \log 2) = \frac{9\sqrt{3}}{32}, \quad u(x_2, \log 2) = \frac{7\sqrt{3}}{32}.$$

Their determinant is $\boxed{21/32 \neq 0}$, proving that the exact field is not one product.

Step 4: Quantify approximate separation. The normalized field is $e^t u = \sin x + e^{-3t} \sin(2x)$, so

$$\|e^t u - \sin x\|_\infty = e^{-3t}, \quad t \geq \frac{1}{3} \log(1/\varepsilon) \implies \|e^t u - \sin x\|_\infty \leq \varepsilon.$$

Equality in the norm follows at a point where $|\sin(2x)| = 1$. The plotted ratios correspond to $t = 0, (\log 2)/3, (\log 10)/3$ and the limiting profile. Approximation by a surviving mode does not retroactively justify a single-product initial ansatz.



Question 10

Let $f(x) = \sum_{n \geq 1} a_n \sin(nx)$ on $0 \leq x \leq \pi$, where $|a_n| \leq n^{-2}$. Use it as initial displacement for zero-endpoint heat and wave problems $H_t = H_{xx}$, $W_{tt} = W_{xx}$, with zero initial wave velocity. Also use it as the top trace of a harmonic rectangle of height $b > 0$, with zero side and bottom values. In each field retain modes $n \leq N$, $N \geq 1$.

Tasks

1. Write the three complete expansions and justify the original data and the relevant convergence claims.
2. Derive uniform truncation bounds for heat at $t \geq \tau > 0$, for the wave at all times, and for the harmonic field at $0 \leq y \leq b - d$, $0 < d < b$.
3. For $\tau = 0.1$, $b = 1$, $d = 0.25$ and error tolerance 10^{-2} , find the smallest integer certified by each of your displayed bounds. Plot the three bound functions and mark these sufficient cutoffs.
4. Explain why the different cutoffs reflect different mechanisms, and why waiting alone does not create an all-time smoothing estimate for the wave.

Question 10 – Solution

Strategy. Apply the same coefficient envelope to three different modal factors, keeping the location where each error bound is valid.

Step 1: Construct the fields and their traces. The three factors give

$$H = \sum a_n e^{-n^2 t} \sin(nx), \quad W = \sum a_n \cos(nt) \sin(nx), \quad P = \sum a_n \frac{\sinh(ny)}{\sinh(nb)} \sin(nx).$$

The summable coefficient envelope ensures uniform displacement and boundary-series convergence. Heat derivatives converge away from $t = 0$, and harmonic derivatives away from the top edge. The wave has finite energy since $\sum n^2 a_n^2 < \infty$; its data and PDE have the corresponding energy/weak meaning without an automatic higher-smoothness claim.

Step 2: Bound the three tails. Using $\sum_{n>N} n^{-2} \leq 1/N$ gives

$$B_H(N) = \frac{e^{-(N+1)^2 \tau}}{N}, \quad B_W(N) = \frac{1}{N}.$$

For $y \leq b - d$, write the hyperbolic ratio as $e^{-n(b-y)}(1 - e^{-2ny})/(1 - e^{-2nb})$. It is at most $e^{-nd}/(1 - e^{-2b})$, so

$$B_P(N) = \frac{e^{-(N+1)d}}{N(1 - e^{-2b})}.$$

These respectively bound the heat error for $t \geq \tau$, the wave displacement error at all times, and the harmonic error in the stated lower strip.

Step 3: Choose certified integer cutoffs. Each bound decreases with N . With the specified parameters,

	preceding bound	accepted bound	N
H	$B_H(4) \approx .02052$	$B_H(5) \approx .005465$	5
W	$B_W(99) \approx .01010$	$B_W(100) = .01000$	100
P	$B_P(9) \approx .01055$	$B_P(10) \approx .007394$	10

These are the smallest integers certified by these bounds, not necessarily the fewest modes required by particular data. The graph uses continuous bound envelopes on logarithmic axes.

Step 4: Interpret the filtering mechanisms. Heat damps high modes by $e^{-n^2 t}$; distance from the driven harmonic edge gives spatial attenuation of order e^{-nd} . Wave modes retain their amplitudes and only oscillate: at every $t = 2m\pi$, the entire displacement returns to f . Consequently waiting cannot supply a uniform future decay factor for every wave tail. The heat and harmonic improvements require positive waiting time or positive distance; they cannot be silently extended to $t = 0$ or to the top trace.

