

Question 1

A boundary-value problem prescribes values at two different points:

$$y'' = -6x, \quad y(0) = 1, \quad y(1) = 2.$$

Compare it with the family of initial-value problems having $y(0) = 1$ and $y'(0) = s$, where s is a freely adjustable initial slope.

Tasks

1. Solve the initial-value family and determine the unique slope that meets the right boundary condition. Explain the distinction between shooting for an endpoint and prescribing an initial slope.
2. Verify the resulting differential equation and both endpoint values. Find the maximum and minimum on $[0, 1]$ and determine whether endpoint values bound the entire response.
3. Replace the boundary values by arbitrary α, β . Prove existence and uniqueness for every pair and give the solution explicitly.
4. If the endpoint values change by δ_0, δ_1 while the forcing stays fixed, find the exact change in the solution and a sharp uniform bound. Sketch the original solution and the line joining its endpoints.

Question 1 – Solution

Strategy. Solve the initial-value family first, then use the endpoint equation to select its slope.

Step 1: Select the shooting slope. Integrating twice gives $y_s = 1 + sx - x^3$. The right endpoint is $y_s(1) = s$, so $s = 2$ is the unique successful slope. An initial-value problem fixes s directly; the boundary-value problem determines it from a condition at another point.

Step 2: Verify the solution and its extrema. The solution $y = 1 + 2x - x^3$ has $y'' = -6x$, $y(0) = 1$, $y(1) = 2$. Since $y' = 2 - 3x^2$, its maximum occurs at $r = \sqrt{2/3}$:

$$y_{\max} = 1 + \frac{4}{3}\sqrt{\frac{2}{3}} > 2, \quad y_{\min} = y(0) = 1.$$

Thus the endpoint values do not bound this forced response from above.

Step 3: Allow arbitrary endpoint values. Direct integration gives

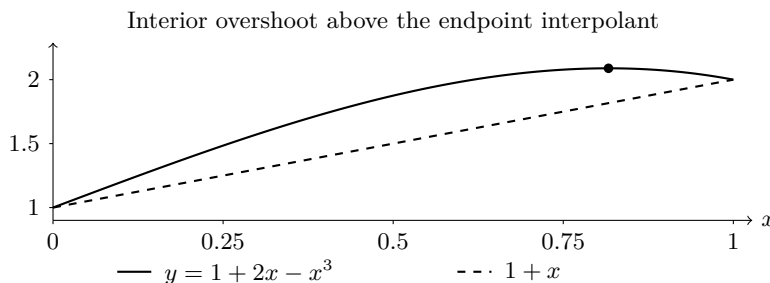
$$y_{\alpha,\beta}(x) = \alpha + (\beta - \alpha + 1)x - x^3.$$

This works for every α, β . The difference of two solutions with the same data satisfies $w'' = 0$, $w(0) = w(1) = 0$, so it is zero, proving uniqueness.

Step 4: Quantify boundary-data sensitivity. The exact response change is $\Delta y = (1 - x)\delta_0 + x\delta_1$. Therefore

$$\max_{0 \leq x \leq 1} |\Delta y| = \max(|\delta_0|, |\delta_1|).$$

The inequality follows from convex weights; equality holds at an endpoint. The original curve exceeds its endpoint line $1 + x$ by $x - x^3 > 0$ inside $(0, 1)$.



Question 2

Consider the fixed-interval problem

$$y'' + y = 1, \quad 0 \leq x \leq \pi, \quad y(0) = \alpha, \quad y(\pi) = \beta.$$

All functions are real. Counting two endpoint conditions does not settle whether a second-order boundary-value problem has exactly one solution.

Tasks

1. Find the complete solution family before applying the boundary conditions. Derive the necessary and sufficient compatibility condition on α, β .
2. Classify the number of solutions for every pair of boundary values. Explain the role of the nonzero homogeneous solution that vanishes at both endpoints.
3. When the problem is compatible, impose $\int_0^\pi y(x) dx = m$. Determine whether this extra condition selects a unique solution for every real m .
4. Replace the right endpoint by $\pi/2$, retaining prescribed values at both ends. Explain why uniqueness returns. For the original interval and $\alpha = \beta = 1$, sketch three distinct solutions with identical endpoint values.

Question 2 – Solution

Strategy. Test whether the endpoint equations independently determine the homogeneous constants.

Step 1: Derive compatibility. The general solution is $y = 1 + A \cos x + B \sin x$. The left endpoint gives $A = \alpha - 1$, while the right endpoint gives $\beta = 1 - A = 2 - \alpha$. Thus

$$\boxed{\text{A solution exists exactly when } \alpha + \beta = 2.}$$

Step 2: Count the compatible solutions. When compatible, the entire family is

$$\boxed{y = 1 + (\alpha - 1) \cos x + B \sin x, \quad B \in \mathbb{R}.}$$

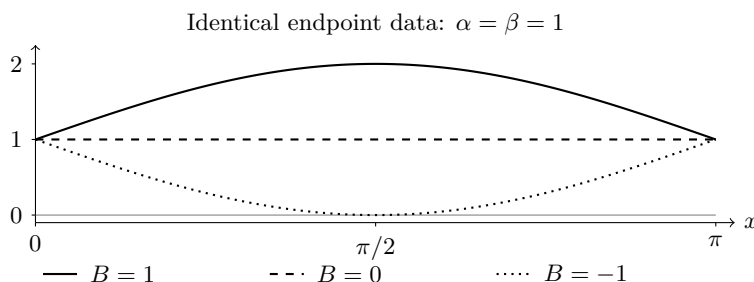
There are infinitely many solutions; otherwise there are none. There is never exactly one with just these two conditions. The nonzero homogeneous solution $\sin x$ vanishes at both endpoints and can be added without changing the data.

Step 3: Use an integral normalization. Since the cosine integrates to zero and the sine integrates to two,

$$\int_0^\pi y \, dx = \pi + 2B = m, \quad \boxed{B = (m - \pi)/2.}$$

For compatible endpoints, every real m selects exactly one solution. If the endpoints are incompatible, an extra integral condition cannot fix that.

Step 4: Change the interval and compare. On $[0, \pi/2]$, the right endpoint gives $B = \beta - 1$ independently of $A = \alpha - 1$, so every pair gives one solution. Independence of the endpoint equations, not their number alone, makes the difference. The graph on the original interval shows $y = 1 + B \sin x$ for $B = -1, 0, 1$.



Question 3

Let the signed flux be $q(x) = -(1+x)y'(x)$ on $[0, 1]$. A variable-conductivity boundary-value problem is

$$-((1+x)y')' = 2x - 1, \quad q(0) = q_0, \quad q(1) = q_1.$$

The sign convention uses the positive x direction at both endpoints.

Tasks

1. Derive the necessary and sufficient compatibility condition on the two prescribed fluxes. Interpret it as a balance over the interval.
2. For $q_0 = q_1 = 0$, find every solution and verify the equation and both flux conditions directly.
3. Add the normalization $\int_0^1 y(x) dx = 0$. Find the resulting constant exactly and prove uniqueness of the normalized solution.
4. Determine whether this normalized solution is increasing, and compute its endpoint values. Explain why flux data alone cannot fix its absolute level and why incompatible fluxes cannot be repaired by adding a constant.

Question 3 – Solution

Strategy. Integrate the flux balance before trying to determine the additive level of the response.

Step 1: Check the interval balance. The equation is $q' = 2x - 1$, so

$$q(x) = q_0 + x^2 - x, \quad q_1 - q_0 = \int_0^1 (2x - 1) dx = 0.$$

Thus $\boxed{q_1 = q_0}$ is necessary. It is sufficient because division by $1 + x > 0$ and one integration produce a solution for every such pair.

Step 2: Solve the zero-flux case. Here $y' = (x - x^2)/(1 + x) = -x + 2 - 2/(1 + x)$, hence

$$\boxed{y = C - x^2/2 + 2x - 2\log(1 + x)}.$$

Multiplication by $-(1 + x)$ gives $q = x^2 - x$, whose derivative is $2x - 1$ and whose values at zero and one are zero. This checks the equation and flux data.

Step 3: Fix the additive constant. Using $\int_0^1 \log(1 + x) dx = 2\log 2 - 1$, we obtain

$$\int_0^1 y dx = C + \frac{17}{6} - 4\log 2.$$

Thus $\boxed{C = 4\log 2 - 17/6}$. The difference of two solutions with the same fluxes has $((1 + x)w')' = 0$ and zero endpoint flux, so $w' = 0$. It is constant; the zero-mean condition forces that constant to vanish.

Step 4: Interpret the resulting profile. Since $y' = x(1 - x)/(1 + x) > 0$ for $0 < x < 1$, the response is strictly increasing. Its endpoint values are

$$\boxed{y(0) = 4\log 2 - 17/6, \quad y(1) = 2\log 2 - 4/3}.$$

Fluxes involve derivatives and therefore leave a constant level undetermined. Adding a constant changes neither flux nor the integrated balance, so it cannot repair a violation of $q_1 = q_0$.

Question 4

A mixed boundary condition couples endpoint value and slope:

$$y'' = 2, \quad y(0) = 0, \quad y'(1) - h y(1) = b,$$

where h, b are real parameters.

Tasks

1. Solve the problem for $h \neq 1$ and verify all data. Classify the number of solutions at $h = 1$ for every b .
2. For fixed $h \neq 1$, determine the exact response to a perturbation $b \mapsto b + \delta$. Find its uniform size on $[0, 1]$.
3. Follow the path $b = 1$ as $h \rightarrow 1$. Explain how a well-defined limit along this path coexists with nonuniqueness at the limiting parameters.
4. For arbitrary real k , use $h_n = 1 - 1/n$ and $b_n = 1 + k/n$ to find the limiting solution. Decide whether the boundary-value solution extends continuously as a single-valued function of (h, b) at $(1, 1)$.

Question 4 – Solution

Strategy. Inspect the denominator in the endpoint equation and distinguish a chosen parameter path from a unique limiting response.

Step 1: Solve and classify the singular case. The left boundary gives $y = x^2 + cx$. The mixed condition becomes $(1 - h)c = b - 2 + h$. For $h \neq 1$,

$$y = x^2 + \left(\frac{b-1}{1-h} - 1 \right) x.$$

Its second derivative is two, its left value is zero, and substitution verifies the mixed condition. At $h = 1$, the condition reduces to $b = 1$ independently of c : infinitely many solutions if $b = 1$, and none otherwise.

Step 2: Quantify sensitivity. At fixed $h \neq 1$, the change in the response is

$$\Delta y(x) = \frac{\delta x}{1-h}, \quad \max_{[0,1]} |\Delta y| = \frac{|\delta|}{|1-h|}.$$

Thus arbitrarily small boundary errors can be strongly amplified near $h = 1$.

Step 3: Examine one compatible path. If $b = 1$ and $h \neq 1$, then $c = -1$, so every solution on this path is $y = x^2 - x$. Its limit is one member of the infinite family at $(h, b) = (1, 1)$. The path selects this member; the limiting boundary conditions alone do not.

Step 4: Compare different approaches. For the proposed sequences, $(b_n - 1)/(1 - h_n) = k$, so

$$y_n = x^2 + (k - 1)x \quad \text{for every } n.$$

All parameter pairs approach $(1, 1)$, but their solution limits depend on k . They range over the entire limiting family. Consequently no choice of a single solution at $(1, 1)$ can make the solution map continuous there along all paths.

Question 5

A dimensionless fourth-order beam model has constant load $y^{(4)} = 24$ on $[0, 1]$. Compare two endpoint designs:

$$\begin{array}{ll} \text{Clamped:} & y(0) = y'(0) = y(1) = y'(1) = 0, \\ \text{Simply supported:} & y(0) = y''(0) = y(1) = y''(1) = 0. \end{array}$$

Tasks

1. Find the solution for each design by integrating the fourth-order equation. Verify all four conditions separately for each solution.
2. Prove uniqueness for both designs by applying integration by parts to the difference of two solutions. Track the boundary terms explicitly.
3. Find each maximum displacement and its location. Explain why symmetry alone would not be a complete proof that the midpoint is a maximum.
4. Compare the two profiles pointwise and compute the ratio of their maximum displacements. Sketch both on the same axes and explain why changing derivative conditions changes the response under the same load.

Question 5 – Solution

Strategy. Determine the integration constants from the correct derivative data, then use an energy identity to check uniqueness.

Step 1: Solve both sets of endpoint equations. Starting from $y = x^4 + ax^3 + bx^2 + cx + d$, the two solutions are

$$y_C = x^2(1-x)^2, \quad y_S = x^4 - 2x^3 + x.$$

Both fourth derivatives equal 24 and both endpoint values are zero. Also $y'_C = 2x(1-x)(1-2x)$ vanishes at both ends, while $y'_S = 12x(x-1)$ vanishes at both ends. These verify the distinct boundary data.

Step 2: Prove uniqueness in each case. A difference w has $w^{(4)} = 0$ and homogeneous boundary conditions. Twice integrating by parts gives

$$0 = \int_0^1 ww^{(4)} dx = [ww''' - w'w'']_0^1 + \int_0^1 (w'')^2 dx.$$

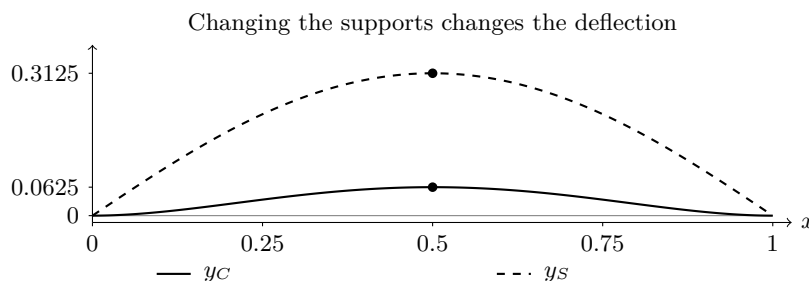
The boundary term is zero for clamped data because $w = w' = 0$, and for simply supported data because $w = w'' = 0$. Thus $w'' = 0$; its zero endpoint values imply $w = 0$. Each design has exactly one solution.

Step 3: Locate and compare the maxima. The derivative of y_C changes from positive to negative at $1/2$. For the other design, $y'_S = (2x-1)(2x^2-2x-1)$, whose second factor is negative on $[0, 1]$. Thus both have their unique maximum at $1/2$, with

$$y_C(1/2) = 1/16, \quad y_S(1/2) = 5/16.$$

The derivative signs establish maxima; symmetry by itself only suggests a stationary midpoint.

Step 4: Compare the complete profiles. The difference is $y_S - y_C = x(1-x) > 0$ inside the interval, and the maximum ratio is five. The same differential load is compatible with different responses because fixing slopes differs from fixing second derivatives at the boundaries.



Question 6

A third-order model is to satisfy four endpoint measurements:

$$y''' = q, \quad y(0) = y'(0) = 0, \quad y(1) = a, \quad y'(1) = b.$$

The constant load q may either be prescribed or be an unknown to infer.

Tasks

1. For a fixed prescribed q , derive the necessary and sufficient relation between a, b for a solution to exist. Determine the number of compatible solutions.
2. Instead regard q as unknown. Recover both the load and the complete profile uniquely for arbitrary real a, b .
3. Apply the inverse problem to $a = 1, b = 0$. Verify the load and all four endpoint conditions, and prove that the resulting displacement is increasing on $(0, 1)$.
4. For a nominal load $q = 6$ and right displacement $a = 1$, determine the required right slope. Explain why the fourth boundary condition can be inconsistent when q is fixed but informative when q is unknown.

Question 6 – Solution

Strategy. Count the integration constants together with any unknown model parameter, and then solve the actual endpoint equations.

Step 1: Find compatibility for a fixed load. The left conditions give $y = qx^3/6 + cx^2/2$. The right conditions require $a = q/6 + c/2$ and $b = q/2 + c$. Subtraction yields

$$b - 2a = q/6.$$

This condition is necessary and sufficient: if it holds, $c = 2a - q/3$ gives the unique solution; otherwise no solution satisfies all the prescribed data.

Step 2: Recover an unknown load. When q is free, the same two equations determine both q and c :

$$q = 6(b - 2a), \quad y = (b - 2a)x^3 + (3a - b)x^2.$$

There is one solution for every pair (a, b) . Direct differentiation verifies the constant load and the two endpoint slopes and values.

Step 3: Check the displacement from rest to rest. For $a = 1, b = 0$,

$$q = -12, \quad y = 3x^2 - 2x^3.$$

Its third derivative is -12 , its endpoint values are zero and one, and $y' = 6x(1 - x)$ vanishes at both ends. Since this derivative is positive inside, the displacement is strictly increasing on $(0, 1)$.

Step 4: Interpret the extra measurement. For $q = 6, a = 1$, compatibility requires $b - 2 = 1$, so $b = 3$ and $y = x^3$. Any other prescribed right slope is inconsistent with those fixed data. The left conditions already remove two of the three integration constants; with fixed q , the remaining constant cannot fit arbitrary two right data. When q is unknown, it supplies one more adjustable quantity, which the fourth measurement identifies. Counting conditions is useful only with this distinction.

Question 7

Consider a singular-endpoint problem on $0 < x \leq 1$:

$$(x^2 y')' = 0, \quad y(1) = 1.$$

An additional condition at zero is to be imposed through a limit or regularity requirement. The differential equation is initially required only for $x > 0$.

Tasks

1. Find every solution on $(0, 1]$ satisfying the right boundary condition. Identify why zero is not a regular endpoint of the normalized equation.
2. Select the solutions that remain bounded as $x \downarrow 0$, and separately those with finite weighted energy $\int_0^1 x^2 (y')^2 dx$. Determine whether these two selections agree.
3. Replace boundedness by $\lim_{x \downarrow 0} xy(x) = k$, with k prescribed. Find the unique solution for every real k and its limiting weighted derivative $\lim_{x \downarrow 0} x^2 y'(x)$.
4. Decide whether a continuous solution can satisfy $y(0) = 0$ and $y(1) = 1$. Explain why two stated endpoint values do not justify invoking the usual regular boundary-value theory at this singular endpoint.

Question 7 – Solution

Strategy. Solve on the open interval first, then apply each proposed endpoint condition to the possible singular term.

Step 1: Find the family on the actual domain. Integration gives $x^2 y' = C$ and $y = A - C/x$. Incorporating $y(1) = 1$, write

$$y = 1 - k + k/x, \quad k \in \mathbb{R}.$$

For $x > 0$ the normalized equation is $y'' + (2/x)y' = 0$. Its coefficient $2/x$ is singular at zero; the original leading coefficient also vanishes there.

Step 2: Test boundedness and energy. Boundedness at zero forces $k = 0$, giving $y \equiv 1$. Since $y' = -k/x^2$,

$$\int_0^1 x^2 (y')^2 dx = \lim_{t \downarrow 0} k^2 \left(\frac{1}{t} - 1 \right).$$

This is infinite for $k \neq 0$ and zero for $k = 0$. Thus boundedness and finite weighted energy select the same single solution in this family.

Step 3: Prescribe a weighted endpoint value. For the displayed family,

$$\lim_{x \downarrow 0} xy(x) = k, \quad \lim_{x \downarrow 0} x^2 y'(x) = -k.$$

Therefore each real prescribed weighted limit selects exactly one solution, including unbounded solutions when $k \neq 0$. A weighted boundary condition is not the same as an ordinary finite value $y(0)$.

Step 4: Test the proposed continuous boundary data. A continuous extension is bounded, so it must be $y \equiv 1$ and must have $y(0) = 1$. Consequently no continuous solution satisfies $y(0) = 0, y(1) = 1$. One cannot assign an unrelated value at zero and retain continuity. Regular endpoint theory requires hypotheses on the normalized coefficients that fail here; the explicit family and its limits decide this problem instead.

Question 8

A linear differential equation is paired with a nonlinear boundary condition:

$$y'' = 0, \quad y(0) = 0, \quad y'(1) = y(1)^2 + \lambda,$$

where $\lambda \in \mathbb{R}$.

Tasks

1. Reduce the boundary-value problem to an algebraic equation and classify the number of real solutions for every λ .
2. At $\lambda = 0$, test whether the average of the two solutions is also a solution. Explain which part of the full problem prevents superposition.
3. Set $\lambda = 1/4 - \varepsilon$ with $\varepsilon > 0$. Measure the uniform distances of the two solutions from the unique critical solution at $\lambda = 1/4$. Decide whether a local Lipschitz estimate in the parameter can hold there.
4. Plot the admissible slopes $c = y'(0)$ against λ , clearly indicating the critical point and where no real solution exists. Explain why uniqueness at a single parameter need not imply stable dependence on nearby parameters.

Question 8 – Solution

Strategy. Solve the differential equation completely, then retain the nonlinear structure of its endpoint constraint.

Step 1: Classify the slope equation. Every solution of $y'' = 0, y(0) = 0$ is $y = cx$. Its remaining boundary condition is $c = c^2 + \lambda$, so

$$c_{\pm} = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}.$$

There are two real solutions for $\lambda < 1/4$, one for $\lambda = 1/4$, and none for $\lambda > 1/4$. The critical solution is $y_* = x/2$.

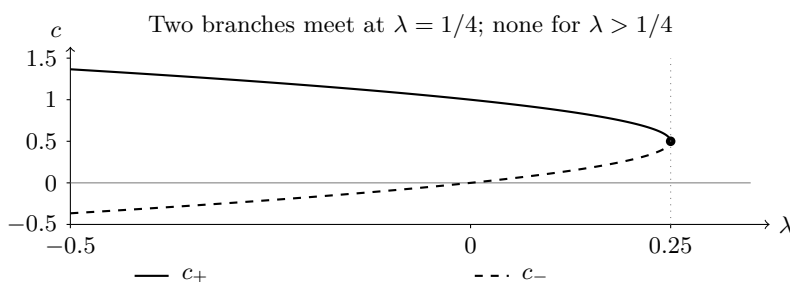
Step 2: Check the superposition claim. At $\lambda = 0$, the solutions are 0 and x . Their average $x/2$ has $y'(1) = 1/2$ but $y(1)^2 = 1/4$, so it fails the boundary condition. The differential equation is linear, but the square in the boundary operator makes the full boundary-value problem nonlinear.

Step 3: Quantify critical sensitivity. At $\lambda = 1/4 - \varepsilon$, the slopes are $1/2 \pm \sqrt{\varepsilon}$. Hence

$$\max_{[0,1]} |y_{\pm} - y_*| = \sqrt{\varepsilon}.$$

A Lipschitz bound with fixed constant K would require $\sqrt{\varepsilon} \leq K\varepsilon$ for all sufficiently small positive ε , which is impossible. Both branches approach y_* continuously from this side, but not at a rate bounded linearly by the parameter error.

Step 4: Read the complete branch picture. The branches meet at $(\lambda, c) = (1/4, 1/2)$ and have no real continuation for larger λ . For $\lambda < 1/4$, $dc_{\pm}/d\lambda = \mp 1/\sqrt{1 - 4\lambda}$, whose magnitude diverges at the join. Critical uniqueness therefore does not guarantee a locally unique family or a Lipschitz-stable response.



Question 9

Let $L[y] = -y'' + q(x)y$ on $[0, 1]$, where $q \geq 0$ is continuous. All solutions below satisfy $y(0) = y(1) = 0$. A useful comparison statement is: if $L[w] \geq 0$ and $w(0), w(1) \geq 0$, then $w \geq 0$ on the interval.

Tasks

1. Prove that $L[y] = f$ has exactly one solution for every continuous f . Use an energy identity for uniqueness and a family of initial-value solutions for existence.
2. Prove the stated comparison result. To handle the case where q can vanish, apply the interior-minimum test first to $w + \eta[1 + x(1 - x)]$ with $\eta > 0$, then let $\eta \downarrow 0$.
3. If an approximation v has zero endpoint values and $|L[v] - f| \leq \epsilon$, prove a pointwise error bound using $\phi = x(1 - x)/2$. Give a sharp constant for the uniform error estimate over all such problems.
4. Apply the result to $q = x^2$, $f = 1$, $v = \phi$. Compute the maximum residual exactly and give rigorous upper and lower bounds for the unknown exact solution, without claiming the approximation solves the equation.

Question 9 – Solution

Strategy. Combine an energy identity, a strict comparison argument and a simple barrier to certify the boundary-value error.

Step 1: Establish existence and uniqueness. A homogeneous difference w with zero endpoint values satisfies $0 = \int_0^1 wL[w]dx = \int_0^1 ((w')^2 + qw^2)dx$, hence $w' = 0$ and $w = 0$. For existence, let p solve $L[p] = f$ with $p(0) = p'(0) = 0$, and let h solve $L[h] = 0$ with $h(0) = 0, h'(0) = 1$. Regular IVP theory supplies both. If $h(1) = 0$, uniqueness just proved would force $h = 0$, a contradiction. Thus $y = p - p(1)h/h(1)$ is the required solution.

Step 2: Make comparison strict before taking a limit. Put $\psi = 1 + x(1 - x) > 0$. Then $L[\psi] = 2 + q\psi > 0$, so $L[w + \eta\psi] > 0$ and its endpoint values are positive. At a nonpositive interior minimum, its second derivative is nonnegative and $q \geq 0$ would give $L[w + \eta\psi] \leq 0$, a contradiction. Thus $w + \eta\psi > 0$. Letting $\eta \downarrow 0$ proves $w \geq 0$.

Step 3: Bound the error with a barrier. Since $L[\phi] = 1 + q\phi \geq 1$, for $e = y - v$ we have $L[\epsilon\phi \pm e] \geq 0$, with zero endpoints. Comparison yields

$$|y(x) - v(x)| \leq \epsilon\phi(x), \quad \max_{[0,1]} |y - v| \leq \epsilon/8.$$

The constant $1/8$ is sharp: take $q = 0, v = 0, f = \epsilon$, for which $y = \epsilon\phi$ attains equality at $x = 1/2$.

Step 4: Certify the proposed approximation. Here $L[\phi] - 1 = x^2\phi = x^3(1 - x)/2$. Its derivative is $x^2(3 - 4x)/2$, so its maximum is $\epsilon = 27/512$ at $x = 3/4$. Also $L[v - y] \geq 0$, hence $y \leq v$. Combining this sign with the error bound gives

$$\left(1 - \frac{27}{512}\right)\phi \leq y \leq \phi, \quad \max_{[0,1]} |y - \phi| \leq \frac{27}{4096}.$$

The residual is not zero; these are certified bounds, not an exact formula for y .

Question 10

A positive point load acts at an unknown location $a \in (0, 1)$:

$$-y'' = F\delta(x - a), \quad y(0) = y(1) = 0, \quad F > 0.$$

Seek a continuous, piecewise linear displacement. The distributional equation requires $y'(a+) - y'(a-) = -F$; no jump in y is allowed.

Tasks

1. Derive the solution on each side of a using the endpoint values, continuity and derivative jump. Verify the distributional equation.
2. Find the peak height and the total area $\int_0^1 y(x) dx$. For fixed F , determine which load location maximizes the height.
3. Suppose only the peak height and total area are measured, while both F and a are unknown. Decide whether these measurements determine the load uniquely, and characterize all compatible loads for a positive measured height H .
4. Instead measure the endpoint slopes $m_L = y'(0)$ and $m_R = y'(1)$. Recover F, a and state their admissibility conditions. Solve the case $m_L = 2, m_R = -1$, and sketch it alongside the reflected load at $1 - a$ with the same F .

Question 10 – Solution

Strategy. Use the boundary conditions to determine two lines, then distinguish independent measurements from redundant ones.

Step 1: Match the two linear pieces. Write $y = mx$ to the left and $y = n(1-x)$ to the right. Continuity requires $ma = n(1-a)$, and the slope jump gives $-n - m = -F$. Thus

$$y(x) = \begin{cases} F(1-a)x, & 0 \leq x \leq a, \\ Fa(1-x), & a \leq x \leq 1. \end{cases}$$

Both endpoint values vanish. The continuous join produces no δ' term; the derivative jump $-F$ gives exactly $-y'' = F\delta(x-a)$, with zero ordinary second derivative elsewhere.

Step 2: Compute the peak and area. The left slope is positive and the right slope negative, so the peak is at a :

$$H = Fa(1-a), \quad \int_0^1 y \, dx = H/2.$$

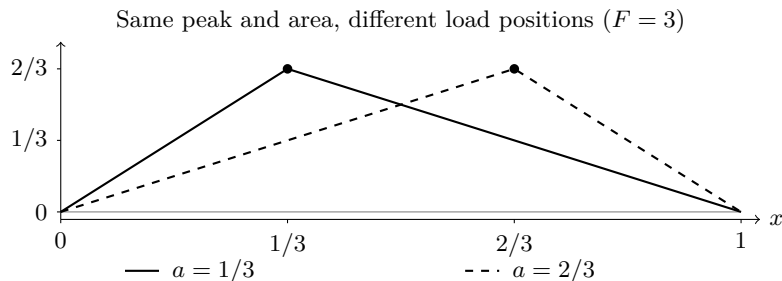
The area is that of a triangle with base one and height H . For fixed F , $a(1-a) = 1/4 - (a - 1/2)^2$ is maximized at $a = 1/2$, giving height $F/4$.

Step 3: Test whether the measurements are independent. If the measured area differs from $H/2$, no load fits the model. If it equals $H/2$, every $a \in (0, 1)$ is compatible with $\boxed{F = H/[a(1-a)]}$. The two measurements supply only one independent number, so there are infinitely many compatible location-strength pairs.

Step 4: Recover the load from endpoint slopes. Since $m_L = F(1-a)$ and $m_R = -Fa$,

$$F = m_L - m_R, \quad a = \frac{-m_R}{m_L - m_R}.$$

The precise admissibility conditions are $m_L > 0$ and $m_R < 0$. For $(m_L, m_R) = (2, -1)$, $F = 3$, $a = 1/3$, with height $2/3$ and area $1/3$. The reflected load has the same height and area but different endpoint slopes.



Question 1

For a fixed $L > 0$, consider

$$-y'' = \lambda y, \quad 0 \leq x \leq L, \quad y(0) = y(L) = 0.$$

An eigenvalue is a real λ for which a nonzero solution exists; any such solution is an eigenfunction. The identically zero solution does not qualify.

Tasks

1. Treat $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, and find every eigenvalue and its full eigenspace.
2. Choose each eigenfunction to have $\int_0^L y^2 dx = 1$ and $y'(0) > 0$. Explain why these two conditions fix its amplitude and sign uniquely.
3. Locate all interior zeros of the n th eigenfunction. For $L = 1$, sketch the first three normalized modes and identify their interior zero counts.
4. If the interval length is doubled, determine the change in each eigenvalue and in the normalization factor. Explain why a function solving the differential equation alone need not be an eigenfunction.

Question 1 – Solution

Strategy. Enforce both homogeneous endpoint conditions in each sign case before normalizing a nonzero solution.

Step 1: Classify all real parameters. For $\lambda = -\mu^2 < 0$, $y(0) = 0$ gives $y = A \sinh(\mu x)$; since $\sinh(\mu L) > 0$, the right condition forces $A = 0$. For $\lambda = 0$, the left condition gives $y = Ax$ and the right gives $A = 0$. For $\lambda = k^2 > 0$, $y = A \sin(kx)$, so a nonzero A requires $kL = n\pi$. Thus

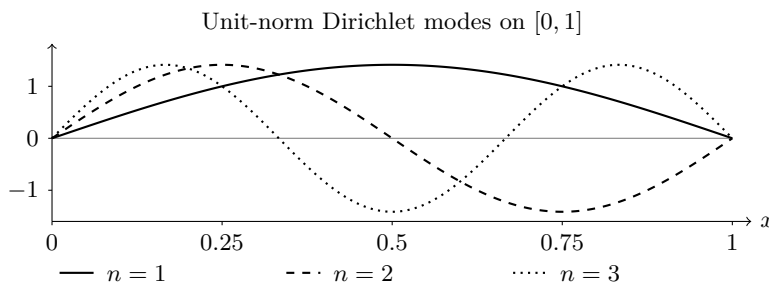
$$\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\sin(n\pi x/L)\}, \quad n = 1, 2, \dots$$

Step 2: Fix amplitude and sign. The identity $\sin^2 t = (1 - \cos 2t)/2$ gives $\int_0^L \sin^2(n\pi x/L) dx = L/2$. The unit norm requires $|A| = \sqrt{2/L}$, and $y'(0) = An\pi/L > 0$ selects the positive sign:

$$\phi_n(x) = \sqrt{2/L} \sin(n\pi x/L).$$

Step 3: Count the interior zeros. The zeros are $x = jL/n$, $j = 0, \dots, n$, of which exactly $n - 1$ lie inside. They are simple because the derivative there is nonzero. The first three modes on the unit interval therefore have zero, one and two interior zeros.

Step 4: Rescale the interval. Replacing L by $2L$ divides each eigenvalue by four and the normalization factor by $\sqrt{2}$. For example, $\cos(kx)$ solves the differential equation with $\lambda = k^2$ but fails the left endpoint condition. The equation, both boundary conditions and nontriviality are all part of the eigenproblem.



Question 2

Study the derivative boundary conditions

$$-y'' = \lambda y, \quad 0 \leq x \leq L, \quad y'(0) = y'(L) = 0, \quad L > 0.$$

A calculation that divides by $\sqrt{\lambda}$ can lose an important mode.

Tasks

1. Find every real eigenvalue and eigenspace, including a separate check at zero and an exclusion of negative eigenvalues.
2. Normalize each mode by $\int_0^L y^2 dx = 1$ and $y(0) > 0$. Explain why the zero mode has a different normalization factor.
3. Integrate the differential equation to prove that every nonzero-eigenvalue mode has zero mean. Decide what happens to the zero eigenspace if zero mean is imposed as an extra constraint.
4. For $L = 2$, test the claims that x is a zero-mode eigenfunction and that $1 + \cos(\pi x/2)$ is an eigenfunction. Distinguish satisfying both endpoint conditions from belonging to a single eigenspace.

Question 2 – Solution

Strategy. Preserve the constant solution by checking the zero parameter before dividing by a frequency.

Step 1: Find the complete spectrum. Integration by parts for any eigenfunction gives $\lambda \int_0^L y^2 dx = \int_0^L (y')^2 dx \geq 0$, so negative eigenvalues are impossible. At zero, $y = A + Bx$ and the derivative conditions force $B = 0$; nonzero constants are eigenfunctions. For $\lambda = k^2 > 0$, $y = A \cos(kx) + B \sin(kx)$; the left derivative condition forces $B = 0$ and the right requires $\sin(kL) = 0$. Thus

$$\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\cos(n\pi x/L)\}, \quad n = 0, 1, 2, \dots$$

Step 2: Normalize the modes. The constant has squared integral L , while every positive-frequency cosine has squared integral $L/2$. The positive value at zero fixes the signs:

$$\phi_0 = 1/\sqrt{L}, \quad \phi_n = \sqrt{2/L} \cos(n\pi x/L) \quad (n \geq 1).$$

Step 3: Interpret the mean constraint. Integrating gives $\lambda \int_0^L y dx = -y'(L) + y'(0) = 0$. Thus all modes with $\lambda \neq 0$ have zero mean. The only constant with zero mean is the zero function, so the additional constraint removes the zero eigenspace from the admissible nonzero functions; it preserves every higher mode.

Step 4: Test the two proposed functions. The function x has derivative one at both ends, so it fails the boundary conditions. The function $g = 1 + \cos(\pi x/2)$ satisfies both derivative conditions, but $-g'' = (\pi^2/4) \cos(\pi x/2)$ is not any constant multiple of g . Indeed, matching the constant term would require $\lambda = 0$, which fails for the cosine term. A sum from different eigenspaces need not be an eigenfunction.

Question 3

An unknown interval length $L > 0$ has mixed boundary conditions:

$$-y'' = \lambda y, \quad y(0) = 0, \quad y'(L) = 0.$$

Two measurements are reported as consecutive eigenvalues, in increasing order: $9\pi^2/16$ and $25\pi^2/16$. The report does not say they are the first two.

Tasks

1. Derive the complete real spectrum and rule out zero and negative values. Index the positive eigenvalues starting at $n = 0$.
2. Recover L and the indices of the two measured modes. Explain why taking the difference of the square roots is useful.
3. Find the true lowest eigenvalue and normalize its eigenfunction by $\int_0^L y^2 dx = 1$ and $y'(0) > 0$.
4. Drop the claim that the measurements are consecutive. Characterize every possible length and pair of indices consistent with the same two numbers. Explain exactly what information the consecutiveness assumption supplied.

Question 3 – Solution

Strategy. Use equally spaced frequencies, not equally spaced eigenvalues, to interpret the measurements.

Step 1: Classify the mixed spectrum. The endpoint term vanishes in integration by parts, so $\lambda \int y^2 = \int (y')^2 > 0$ for a nonzero admissible function; equality would force a constant and then $y = 0$. With $\lambda = k^2 > 0$, the left condition gives $y = A \sin(kx)$ and the right requires $\cos(kL) = 0$. Hence

$$\lambda_n = ((n + 1/2)\pi/L)^2, \quad n = 0, 1, \dots$$

Each eigenspace consists of multiples of the corresponding sine.

Step 2: Recover the consecutive indices. Consecutive positive frequencies differ by π/L . The measured difference is $5\pi/4 - 3\pi/4 = \pi/2$, giving $L = 2$. The first measured frequency satisfies $(n + 1/2)\pi/2 = 3\pi/4$, so its index is $n = 1$ and the next is $n = 2$. They were not the first two eigenvalues.

Step 3: Recover the missing lowest mode. The index-zero eigenvalue is $\lambda_0 = \pi^2/16$. Since $\int_0^2 \sin^2(\pi x/4) dx = 1$ and its initial slope is positive, the requested eigenfunction is $\phi_0 = \sin(\pi x/4)$.

Step 4: Identify the remaining inverse ambiguity. Let the measured indices be $r < s$. Their frequency ratio implies $5(2r + 1) = 3(2s + 1)$. Since three and five are coprime, all solutions are

$$2r + 1 = 3d, \quad 2s + 1 = 5d, \quad L = 2d, \quad r = (3d - 1)/2, \quad s = (5d - 1)/2,$$

where d is any positive odd integer. Conversely, substitution verifies every such choice. The index gap is $s - r = d$; consecutiveness forces $d = 1$ and removes this infinite ambiguity.

Question 4

Let $h \in \mathbb{R}$ control the endpoint condition

$$-y'' = \lambda y, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y'(1) = h y(1).$$

Unlike fixed zero endpoint values, this boundary condition can permit a negative eigenvalue.

Tasks

1. Determine precisely when zero is an eigenvalue and identify its eigenspace.
2. For $\lambda = -\mu^2 < 0$, derive the equation for μ . Prove that there is exactly one negative eigenvalue when $h > 1$ and none when $h \leq 1$.
3. For $\lambda = k^2 > 0$, derive the frequency equation without losing or adding modes by division. Locate all its roots by intervals between multiples of π , including the special first interval.
4. Derive the energy identity and explain why it does not always imply nonnegative eigenvalues. For $h = 2$, bound the negative eigenvalue using $1 < \mu < 2$, and sketch the equation locating μ .

Question 4 – Solution

Strategy. Separate all three signs and prove root counts by monotonicity on the correct domains.

Step 1: Check the zero parameter directly. At $\lambda = 0$, $y = Ax$ and the right condition is $A = hA$. Thus zero is an eigenvalue exactly when $h = 1$, with eigenspace $\text{span}\{x\}$.

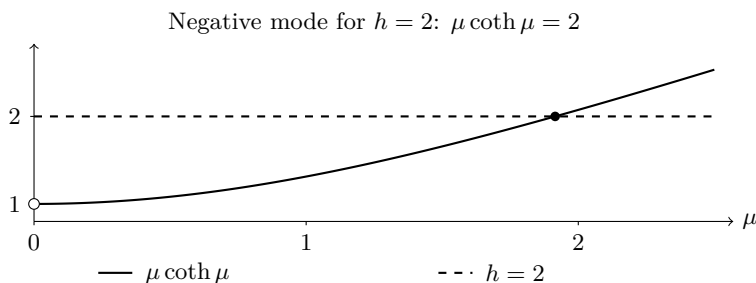
Step 2: Count the negative modes. For $\lambda = -\mu^2$, the nonzero solution is a multiple of $\sinh(\mu x)$, and its endpoint condition is $\mu \coth \mu = h$. The function $g(\mu) = \mu \coth \mu$ tends to one as $\mu \downarrow 0$ and to infinity as $\mu \rightarrow \infty$. Its derivative has numerator $\sinh \mu \cosh \mu - \mu > 0$, since this numerator starts at zero and has derivative $2 \sinh^2 \mu > 0$. Thus precisely $h > 1$ gives one negative mode.

Step 3: Locate every positive mode. Here $y = A \sin(kx)$ and $k \cos k = h \sin k$. A positive multiple of π cannot solve this equation, so division by $\sin k$ is safe at a root:

$$k \cot k = h.$$

Its derivative is $(\sin k \cos k - k)/\sin^2 k < 0$ for $k > 0$ away from poles. On each $(n\pi, (n+1)\pi)$, $n \geq 1$, it decreases from $+\infty$ to $-\infty$, giving exactly one root for every h . On $(0, \pi)$ it decreases from the unattained limit one to $-\infty$: one root if $h < 1$, none if $h \geq 1$.

Step 4: Retain the endpoint energy term. Integration by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 (y')^2 dx - hy(1)^2$, which can be negative. For $h = 2$, $g(1) < 2 < g(2)$: the first inequality follows from $e^2 > 3$ and $g(2) > 2$ follows from $\coth 2 > 1$. Thus $1 < \mu < 2$ and $-4 < \lambda_- < -1$. The graph locates the unique crossing.



Question 5

Consider a ring represented by the interval $[0, 2\pi]$ with matching values and slopes at its ends:

$$-y'' = \lambda y, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

Here the eigenspace dimension need not be one. No Fourier expansion is required.

Tasks

1. Rule out negative eigenvalues using the boundary terms in an energy identity, and determine the zero eigenspace directly.
2. Find all positive eigenvalues and the dimension of each eigenspace. Explain why imposing one endpoint equation alone would be insufficient.
3. At a positive eigenvalue n^2 , impose $y(0) = 0$ and $y'(0) = 1$. Find the resulting eigenfunction and decide whether these normalization data are possible at zero.
4. For the normalized positive modes, test the additional measurements $y(\pi) = 0$ and $y'(\pi) = y'(0)$. Which measurement adds no information, and which selects only part of the spectrum?

Question 5 – Solution

Strategy. Treat value and derivative matching together as a two-dimensional endpoint map.

Step 1: Handle nonpositive parameters. The boundary term $[yy']_0^{2\pi}$ is zero because both values and slopes match. Thus $\lambda \int y^2 = \int (y')^2 \geq 0$. At zero, $y = A + Bx$; value matching forces $B = 0$ and slope matching is then automatic. Hence $E_0 = \text{span}\{1\}$, of dimension one.

Step 2: Find the positive eigenspaces. For $\lambda = k^2 > 0$, write $y = A \cos(kx) + B \sin(kx)$. With $\theta = 2\pi k$, the two conditions are

$$\begin{pmatrix} \cos \theta - 1 & \sin \theta \\ -\sin \theta & \cos \theta - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is $2(1 - \cos \theta)$, so nonzero solutions require $k = n$, a positive integer. Then the entire matrix is zero and both constants are free:

$$\lambda_n = n^2, \quad E_n = \text{span}\{\cos nx, \sin nx\}, \quad n \geq 1.$$

Each positive eigenspace has dimension two. One equation alone would generally leave a nonzero vector even when the other endpoint equation fails.

Step 3: Apply the initial normalization. The data $y(0) = 0, y'(0) = 1$ require $A = 0, nB = 1$, giving $y_n = \sin(nx)/n$. At zero every eigenfunction is constant, so its initial derivative cannot be one. These extra data select a representative from each positive eigenspace, not from the zero eigenspace.

Step 4: Interpret the midpoint measurements. For every positive integer n , $y_n(\pi) = \sin(n\pi)/n = 0$, so that measurement adds no restriction. But $y'_n(\pi) = \cos(n\pi) = (-1)^n$ equals $y'_n(0) = 1$ exactly for even n . The second measurement selects eigenvalues $4, 16, 36, \dots$ from the original positive spectrum.

Question 6

A variable-coefficient eigenproblem uses the positive interval $[1, e^2]$:

$$-(xy')' = \lambda \frac{y}{x}, \quad y(1) = y(e^2) = 0.$$

The natural squared norm for this problem is $\int_1^{e^2} y(x)^2 dx/x$.

Tasks

1. Use $t = \log x$ and $y(x) = u(t)$ to transform the equation and both boundary conditions. Show every derivative factor.
2. Find all real eigenvalues and eigenspaces, checking all parameter signs. Normalize each mode to have weighted norm one and $y'(1) > 0$.
3. Locate the first mode's maximum and all interior zeros of the n th mode. Explain why equal spacing in the transformed coordinate does not imply equal spacing in x .
4. Replace e^2 by a general right endpoint $b > 1$. Give the eigenvalues and weighted normalization factor as functions of b . Sketch the first two normalized modes for the original interval.

Question 6 – Solution

Strategy. Transform the differential equation and its weight together, not just its apparent frequency.

Step 1: Carry out the logarithmic substitution. Since $dt/dx = 1/x$, $y' = u'/x$, $xy' = u'$ and $(xy')' = u''/x$. Multiplying the equation by x gives

$$-u'' = \lambda u, \quad u(0) = u(2) = 0.$$

Also $dx/x = dt$, so the weighted squared norm is exactly $\int_0^2 u^2 dt$.

Step 2: Classify and normalize. At zero the affine function with two zero endpoints is zero. At a negative parameter, the left-zero hyperbolic sine cannot vanish at the right endpoint. For $\lambda = k^2 > 0$, $u = A \sin(kt)$ and $\sin(2k) = 0$. Thus

$$\lambda_n = (n\pi/2)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi \log x}{2}\right), \quad n \geq 1.$$

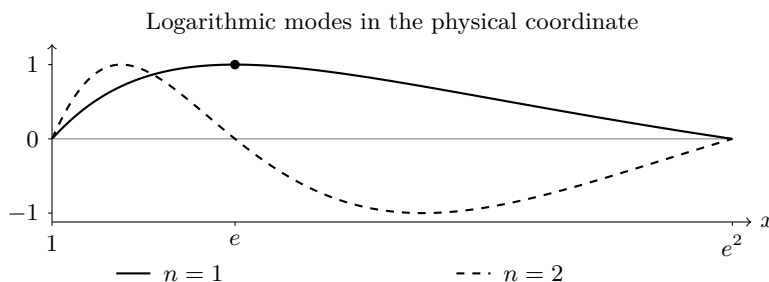
The eigenspace is the span of ϕ_n . Its weighted squared norm is one, and $\phi'_n(1) = n\pi/2 > 0$, fixing both amplitude and sign.

Step 3: Read locations in the original coordinate. For $n = 1$, the sine attains its unique maximum one when $t = 1$, hence $x = e$, the geometric rather than arithmetic midpoint of the endpoints. For mode n , the interior zeros are $x_j = e^{2j/n}$, $j = 1, \dots, n-1$. They form a geometric progression: logarithmic spacing becomes multiplicative spacing in x .

Step 4: Change the endpoint. For $[1, b]$ the transformed length is $\ell = \log b > 0$. Consequently

$$\lambda_n = (n\pi/\log b)^2, \quad \phi_n = \sqrt{\frac{2}{\log b}} \sin\left(\frac{n\pi \log x}{\log b}\right).$$

The graph for $b = e^2$ uses the physical coordinate x , with the landmark $x = e$.



Question 7

For $a \in \mathbb{R}$ and $L > 0$, study

$$-y'' + 2ay' = \lambda y, \quad y(0) = y(L) = 0.$$

The first derivative makes this look different from a sine eigenproblem. Use the weighted norm $\int_0^L e^{-2ax} y^2 dx$ when normalizing.

Tasks

1. Substitute $y = e^{ax}u$ and derive the transformed equation and endpoint conditions. Find every real eigenvalue and eigenspace.
2. Give the weighted unit-norm eigenfunction with positive initial derivative. Explain how the exponential factor affects shape without changing the zeros.
3. For $a = 1, L = \pi$, locate the maximum of the first eigenfunction with amplitude chosen as $y = e^x \sin x$. Sketch it and the underlying sine on the same axes.
4. Suppose the first two eigenvalues are known to be 5 and 8 but both a and L are unknown. Determine everything these data identify, and exhibit the remaining ambiguity.

Question 7 – Solution

Strategy. Remove the first derivative with an invertible exponential change and keep track of the shifted spectral parameter.

Step 1: Transform the operator completely. The derivatives are $y' = e^{ax}(u' + au)$ and $y'' = e^{ax}(u'' + 2au' + a^2u)$. Therefore

$$-y'' + 2ay' = e^{ax}(-u'' + a^2u), \quad \boxed{-u'' = (\lambda - a^2)u, \quad u(0) = u(L) = 0.}$$

The exponential is never zero, so this is a bijection of solution spaces. The shifted parameter must be positive: zero or negative gives only the zero Dirichlet solution. All eigenvalues and eigenfunctions are thus

$$\boxed{\lambda_n = a^2 + (n\pi/L)^2, \quad E_n = \text{span}\{e^{ax} \sin(n\pi x/L)\}, \quad n \geq 1.}$$

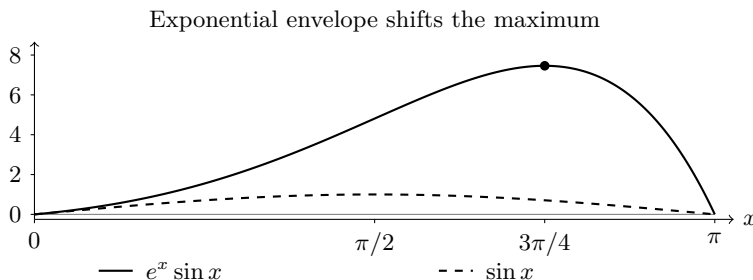
Step 2: Normalize with the correct weight. The factors e^{-2ax} and e^{2ax} cancel in the squared norm. Hence $\boxed{\phi_n = \sqrt{2/L} e^{ax} \sin(n\pi x/L)}$ has unit weighted norm and positive initial derivative. The envelope changes heights and the locations of extrema, but its positivity leaves the zeros jL/n unchanged.

Step 3: Locate the displaced maximum. For $y = e^x \sin x$, $y' = e^x(\sin x + \cos x)$ is positive up to $3\pi/4$ and negative afterwards on $(0, \pi)$. Thus

$$\boxed{x_{\max} = 3\pi/4, \quad y_{\max} = e^{3\pi/4}/\sqrt{2}.}$$

The sine itself peaks at $\pi/2$; multiplying by the increasing envelope shifts the peak without moving either zero.

Step 4: Solve the inverse spectral data. The gap $8 - 5 = 3$ equals $3\pi^2/L^2$, so $L = \pi$. Then $a^2 + 1 = 5$, giving $\boxed{a = 2 \text{ or } a = -2}$. Both choices give the entire same spectrum $4 + n^2$, yet their exponential envelopes point in opposite directions. The eigenvalues alone cannot recover the sign of the drift parameter.



Question 8

Compare an eigenproblem with its forced counterpart on $[0, 1]$:

$$-y'' - \lambda y = \sin(\pi x), \quad y(0) = y(1) = 0.$$

The homogeneous Dirichlet eigenvalues of $-d^2/dx^2$ are $n^2\pi^2$, $n \geq 1$. The identity

$$\int_0^1 \sin(n\pi x) \sin(\pi x) dx = 0 \quad (n \geq 2)$$

may be verified by $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

Tasks

1. Solve the forced problem for every real λ outside the homogeneous spectrum and prove uniqueness.
2. At $\lambda = \pi^2$, prove nonexistence using integration by parts against $\sin(\pi x)$, rather than merely observing a vanishing denominator.
3. At $\lambda = n^2\pi^2$ with $n \geq 2$, find all solutions. Determine which is selected by $\int_0^1 y \sin(n\pi x) dx = 0$.
4. Compare the uniform size of the solution as $\lambda \rightarrow \pi^2$ with its limit as $\lambda \rightarrow 4\pi^2$ through nonspectral values. Explain why a finite limiting particular solution does not imply uniqueness at the limiting parameter.

Question 8 – Solution

Strategy. Separate the existence of a particular solution from the nontrivial homogeneous space at a spectral parameter.

Step 1: Solve away from the spectrum. Since $-(\sin \pi x)'' = \pi^2 \sin \pi x$,

$$y = \frac{\sin \pi x}{\pi^2 - \lambda}$$

solves the equation and both endpoints whenever $\lambda \neq \pi^2$. If λ is outside the entire spectrum, any difference of two solutions is a zero-endpoint homogeneous solution and is therefore zero. This proves uniqueness, including all zero and negative parameter values.

Step 2: Prove failure of compatibility at the first eigenvalue. Let $s = \sin \pi x$. For a proposed solution at $\lambda = \pi^2$, twice integrating by parts gives

$$\int_0^1 s(-y'' - \pi^2 y) dx = [-sy' + s'y]_0^1 + \int_0^1 (-s'' - \pi^2 s)y dx = 0.$$

Both s and y vanish at the endpoints. The forcing instead requires this integral to equal $\int_0^1 s^2 dx = 1/2$, a contradiction. No solution exists.

Step 3: Find every compatible resonant solution. For $n \geq 2$, the particular solution above is valid and the homogeneous space is spanned by $\sin(n\pi x)$. Therefore all solutions are

$$y = \frac{\sin \pi x}{(1 - n^2)\pi^2} + C \sin(n\pi x), \quad C \in \mathbb{R}.$$

The supplied product identity and $\int_0^1 \sin^2(n\pi x) dx = 1/2$ show that the extra integral condition is $C/2 = 0$, selecting exactly $C = 0$.

Step 4: Distinguish two limiting behaviors. Outside the spectrum, $\max |y| = 1/|\pi^2 - \lambda|$, which diverges at the first eigenvalue. As $\lambda \rightarrow 4\pi^2$, the same solution converges uniformly to $-\sin(\pi x)/(3\pi^2)$. This is only the $C = 0$ member of the infinite family at $4\pi^2$. A finite chosen limit does not remove the homogeneous freedom.

Question 9

An inverse design seeks a continuous potential q on $[0, 1]$ for

$$-y'' + q(x)y = \lambda y, \quad y(0) = y(1) = 0.$$

For the constructive part, prescribe $\lambda = \pi^2$ and $y = \sin \pi x + \frac{1}{4} \sin 2\pi x$. Eigenfunctions are required to be C^2 .

Tasks

1. Show that the alternative target $y = x(1 - x)$ cannot be an eigenfunction for any continuous q and any finite real λ . Explain the endpoint obstruction.
2. For the trigonometric target, recover q explicitly and extend it continuously to both endpoints. Compute $q(0)$ and $q(1)$ and verify the differential equation.
3. Prove that the target is positive on $(0, 1)$ and has simple endpoint zeros. For any C^2 test function v with zero endpoints, put $z = v/y$ inside the interval and derive $\int_0^1 [v'^2 + (q - \lambda)v^2]dx = \int_0^1 y^2 z'^2 dx$.
4. Use this identity to prove that no real eigenvalue can lie below π^2 , and that every eigenfunction with eigenvalue π^2 is a multiple of the designed target. Address the boundary term in the identity.

Question 9 – Solution

Strategy. Recover the potential where the target is nonzero, then use endpoint regularity and a factored energy identity to certify the design.

Step 1: Reject an incompatible target. For $y = x(1 - x)$, division inside the interval would require $q = \lambda + y''/y = \lambda - 2/[x(1 - x)]$, which is unbounded at both ends. Equivalently, continuity of the equation at zero would require $-y''(0) + q(0)y(0) = 2 = \lambda y(0) = 0$. No continuous potential can work.

Step 2: Construct a regular potential. Write $c = \cos \pi x$. The target is $y = \sin \pi x(1 + c/2)$ and $y'' = -\pi^2 \sin \pi x(1 + 2c)$. Consequently

$$q(x) = -\frac{3\pi^2 c}{2 + c}, \quad q(0) = -\pi^2, \quad q(1) = 3\pi^2.$$

Since $2 + c \geq 1$, the formula is continuous on the closed interval. Substitution gives $qy = (\lambda y + y'')$, verifying the equation inside; continuity verifies it at the endpoints as well.

Step 3: Factor the energy. The factor $1 + c/2$ lies between $1/2$ and $3/2$, so $y > 0$ inside. The endpoint derivatives are $y'(0) = 3\pi/2$ and $y'(1) = -\pi/2$, both nonzero. With $v = yz$ and $y'' = (q - \lambda)y$, expansion gives

$$v'^2 + (q - \lambda)v^2 = y^2 z'^2 + (yy' z^2)'$$

For zero-endpoint C^2 functions v , the simple zeros of y imply that $z = v/y$ and z' stay bounded near either endpoint, by Taylor expansion. Thus $[yy' z^2]_0^1 = 0$ and integration proves the requested identity.

Step 4: Certify the lowest eigenvalue and its eigenspace. If v is a nonzero eigenfunction with eigenvalue ν , integration by parts and the identity yield

$$(\nu - \pi^2) \int_0^1 v^2 dx = \int_0^1 y^2 z'^2 dx \geq 0.$$

Thus $\nu \geq \pi^2$, and the constructed nonzero y attains this bound. Equality forces $z' = 0$ inside, hence $v = Cy$. Therefore π^2 is the lowest real eigenvalue and its eigenspace is precisely $\text{span}\{y\}$.

Question 10

A two-material interval has stiffness

$$p(x) = \begin{cases} 1, & 0 < x < 1/2, \\ 4, & 1/2 < x < 1, \end{cases} \quad -(py')' = \lambda y, \quad y(0) = y(1) = 0.$$

Require y and the flux py' to be continuous at $x = 1/2$. Solutions are C^2 on each side. Ordinary slopes need not be continuous across the interface.

Tasks

1. Use integration by parts on the two subintervals to show that every real eigenvalue is positive. Explain cancellation of the interface terms.
2. Set $k = \sqrt{\lambda}$ and write the endpoint-compatible sine on each side. Derive the two interface equations for their amplitudes and a determinant condition.
3. With $t = k/4$, factor the determinant and characterize all positive eigenvalues. Treat $\sin t = 0$ separately to avoid losing modes when solving for the amplitude ratio.
4. Identify the smallest eigenvalue and give its eigenfunction with left amplitude one. Verify continuity and the slope ratio at the interface, and sketch the mode with its physical corner.

Question 10 – Solution

Strategy. Match displacement and stiffness-weighted slope, then preserve every branch of the determinant equation.

Step 1: Exclude nonpositive eigenvalues. Adding the two integrations by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 p(y')^2 dx$: endpoint values vanish and the interface terms cancel because both y and py' match. The right side is positive for a nonzero admissible function; equality would make it constant on both sides, hence identically zero. Therefore $\lambda > 0$.

Step 2: Form the interface system. The endpoint-compatible pieces are $A \sin(kx)$ on the left and $B \sin(k(1-x)/2)$ on the right. With $t = k/4$, matching gives

$$A \sin 2t = B \sin t, \quad A \cos 2t = -2B \cos t.$$

The second equation uses flux, not ordinary derivative. The determinant vanishes exactly when $2 \sin 2t \cos t + \cos 2t \sin t = 0$.

Step 3: Preserve both spectral branches. Factoring yields $\sin t(6 \cos^2 t - 1) = 0$. Thus the complete spectrum is

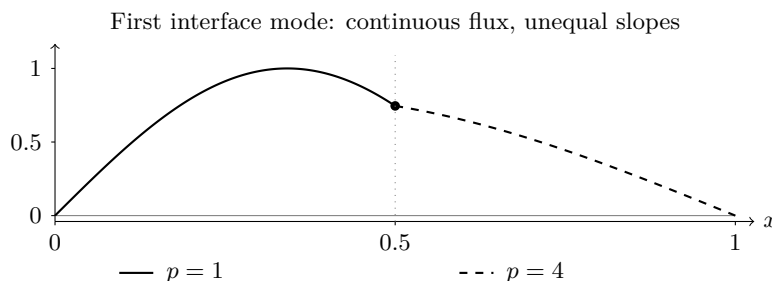
$$\lambda = 16t^2, \quad t > 0, \quad t = m\pi \ (m \geq 1) \text{ or } \cos^2 t = 1/6.$$

At $t = m\pi$, the first matching equation is zero and the second gives $B = (-1)^{m+1}A/2$. On the other branch, $\sin t \neq 0$ and $B = 2A \cos t$. Each branch gives a nonzero solution and a one-dimensional eigenspace. The apparent root $t = 0$ is excluded by Step 1.

Step 4: Construct the first mode and its corner. The smallest positive root is $t_0 = \arccos(1/\sqrt{6}) \in (0, \pi/2)$. Hence $\lambda_1 = 16t_0^2$ and, taking $A = 1$,

$$y = \begin{cases} \sin(4t_0 x), & 0 \leq x \leq 1/2, \\ \sqrt{2/3} \sin(2t_0(1-x)), & 1/2 \leq x \leq 1. \end{cases}$$

At the join both values equal $\sqrt{5}/3$. The one-sided slopes are $-8t_0/3$ and $-2t_0/3$, so $y'(1/2-) = 4y'(1/2+)$ as required by flux continuity. The slope change is physical; smoothing it would violate the model.



Question 1

A positive number T is a period of f if $f(x + T) = f(x)$ for every real x . A fundamental period is the smallest positive period, when one exists. Consider

$$f(x) = \sin 6x + \cos 10x.$$

You may use linear independence of sines and cosines at distinct positive integer frequencies; justify that fact here by integrating products on $[0, 2\pi]$.

Tasks

1. Find a common period of the two summands. Explain why a common period alone does not prove that the sum has that fundamental period.
2. Expand $f(x + T) - f(x)$ and use independence to characterize every positive period of f . Determine the fundamental period exactly.
3. Find the fundamental period of $g(x) = 3f(2x - 1) + 7$. Explain the effects of translation, nonzero amplitude scaling and a constant offset.
4. Decide whether a nonzero constant function and the zero function have fundamental periods. Explain why the definition cannot simply assign either function period zero as a fundamental period.

Question 1 – Solution

Strategy. Prove that a period of the sum must preserve each of its distinct frequency components.

Step 1: Find a candidate period. The summands have periods $\pi/3$ and $\pi/5$, and π is a common period. Thus π is a period of f . In general, cancellation can give a sum a shorter period or even make it constant, so minimality still needs proof.

Step 2: Prove necessity and minimality. Product-to-sum identities show that distinct positive-integer sine/cosine frequencies have zero product integral on $[0, 2\pi]$, while each squared integral equals π . Multiplying a zero linear combination by each component and integrating therefore forces every coefficient to vanish. Expanding the shift of f now gives

$$\cos 6T = 1, \quad \sin 6T = 0, \quad \cos 10T = 1, \quad \sin 10T = 0.$$

Equivalently, $6T = 2\pi m$ and $10T = 2\pi n$ for integers m, n . Then $5m = 3n$, so $m = 3j, n = 5j$ and $T = j\pi$. Conversely every such positive shift works. Hence

Positive periods: $j\pi$ ($j = 1, 2, \dots$), $T_0 = \pi$.

Step 3: Transform the independent variable. Since $2x - 1$ ranges over all real numbers, a shift $T > 0$ preserves g exactly when $2T$ preserves f . Thus $T_g = \pi/2$. Translation, multiplication by a nonzero constant and addition of a constant do not change the set of periods; the argument dilation rescales it.

Step 4: Treat constant functions separately. Every positive number is a period of a constant function, including the zero function. There is no smallest positive number, so neither has a fundamental period. Zero is excluded by the definition and cannot be used to repair this lack of a least positive period.

Question 2

Consider a signal with incommensurate frequencies:

$$f(x) = \cos x + \cos(\sqrt{2}x).$$

A graph over a long interval may suggest repeated patterns without establishing an exact period. Recall that $\sqrt{2}$ is irrational.

Tasks

1. Show that $f(x) \leq 2$ and determine every point where equality holds. Use this to prove that f has no positive period.
2. For a second proof, apply $D^2 + 2$ to f , where $D = d/dx$. Explain why a period of a smooth function is also a period of this derivative combination, and derive a contradiction.
3. Set $T = 140\pi$. Prove a uniform bound on $|f(x+T) - f(x)|$ using $99^2 - 2 \cdot 70^2 = 1$ and $|\cos(u+v) - \cos u| \leq |v|$.
4. Decide whether this small uniform discrepancy makes T an exact period. Contrast the result with $h(x) = \cos x + \cos(3x/2)$, and determine the fundamental period of h .

Question 2 – Solution

Strategy. Distinguish exact simultaneous phase return from a uniformly small phase mismatch.

Step 1: Use the exact maximum. Each cosine is at most one. Equality $f(x) = 2$ requires $x = 2\pi m$ and $\sqrt{2}x = 2\pi n$ for integers m, n . Irrationality forces $m = n = 0$, so the maximum is attained only at $x = 0$. If $T > 0$ were a period, then $f(T) = f(0) = 2$, contradicting this uniqueness.

Step 2: Isolate the frequencies by differentiation. Direct differentiation gives $(D^2 + 2)f = \cos x$. Differentiating an identity $f(x+T) = f(x)$ twice preserves it, so T would also be a period of $\cos x$. Subtracting this cosine from f shows that T preserves $\cos(\sqrt{2}x)$ too. Evaluating both shifted cosines at zero requires $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$, impossible for positive T .

Step 3: Bound an approximate return. At $T = 140\pi$, the first cosine returns exactly. The second phase shift is $2\pi(70\sqrt{2}) = 2\pi \cdot 99 - \delta$, where

$$\delta = 2\pi(99 - 70\sqrt{2}) = \frac{2\pi}{99 + 70\sqrt{2}}.$$

The stated cosine inequality therefore yields

$$\sup_{x \in \mathbb{R}} |f(x + 140\pi) - f(x)| \leq \frac{2\pi}{99 + 70\sqrt{2}} < 0.032.$$

Step 4: Contrast approximate and exact recurrence. The discrepancy bound is positive and does not make T a period; Step 1 rules out every positive exact period. For h , any period must send its value two at zero to another value two. Thus $T = 2\pi m$ and $3T/2 = 2\pi n$, forcing m even. Conversely 4π preserves both summands. Hence $\boxed{T_h = 4\pi}$, an exact fundamental period.

Question 3

Define the two-periodic extension of a ramp by the unique representation

$$x = 2m + r, \quad m \in \mathbb{Z}, \quad -1 \leq r < 1, \quad s(x) = r.$$

The half-open convention specifies the actual function values at the joins. Also let p be the one-periodic extension of $x(1 - x)$ from $[0, 1)$.

Tasks

1. Prove that s has fundamental period two. Find its value and one-sided limits at every odd integer, and sketch it on $[-3, 3]$ with correct open and closed points.
2. Determine whether $s(-x) = -s(x)$ holds everywhere. Change only its join values to produce an odd, two-periodic function, and decide whether that change can make it continuous.
3. Prove that $\int_a^{a+2} s(x) dx = 0$ for every real a . Explain why changing the join values does not affect this statement.
4. Determine the continuity and differentiability of p at integers. Explain why matching function values at the ends of a defining interval is weaker than matching first derivatives.

Question 3 – Solution

Strategy. Separate periodicity, assigned endpoint values, one-sided limits and integral properties.

Step 1: Locate the joins and establish minimality. The representation gives $s(x+2) = s(x)$. Its discontinuities occur precisely at odd integers. Any period must preserve this set, so a positive period is a positive even integer; two is therefore fundamental. At $c = 2m + 1$,

$$s(c) = -1, \quad s(c-) = 1, \quad s(c+) = -1.$$

Each increasing segment has its left endpoint included and its right endpoint excluded. The graph also includes the assigned value $s(3) = -1$.

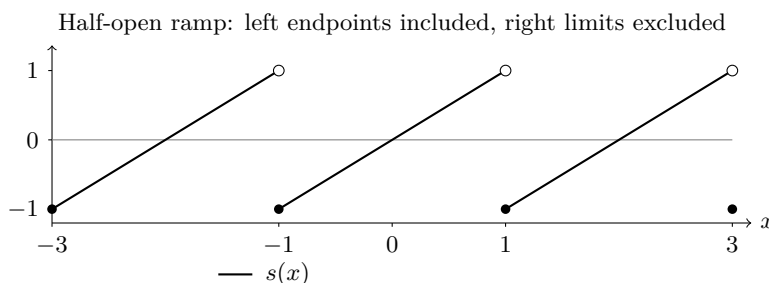
Step 2: Repair oddness, not continuity. Away from odd integers, $s(-x) = -s(x)$. At an odd integer both $s(x)$ and $s(-x)$ equal -1 , so pointwise oddness fails. Assigning zero at every odd integer produces an odd function and preserves two-periodicity. No assigned join value can reconcile the unequal one-sided limits, so the jumps remain.

Step 3: Integrate over any full period. The integral on $[-1, 1]$ is $\int_{-1}^1 x \, dx = 0$. Reduce a modulo two to $b \in [-1, 1)$. Splitting at one and shifting the remaining piece by two gives

$$\int_a^{a+2} s = \int_b^1 s + \int_1^{b+2} s = \int_b^1 s + \int_{-1}^b s = 0.$$

Changing finitely many join values in any bounded interval leaves these Riemann integrals unchanged. Integral cancellation does not require pointwise oddness.

Step 4: Check a continuous periodic join. For p , both endpoint limits of $x(1-x)$ are zero, matching the assigned value at each integer. Thus p is continuous everywhere. The left derivative at an integer is -1 , while the right derivative is 1 , so it is not differentiable there. Value matching alone does not give derivative matching.



Question 4

Orthogonality depends on the interval used in the inner product. For $L > 0$, let

$$\langle u, v \rangle_{a,L} = \int_a^{a+L} u(x)v(x)dx.$$

Examine $u = \sin x$ and $v = \cos x$.

Tasks

1. Compute $\langle u, v \rangle_{a,L}$ exactly. Characterize all lengths L for which u and v are orthogonal for every starting point a .
2. When $L = \pi/2$, find all starting points that give orthogonality. Explain why the same length can work at one location and fail at another.
3. On $[0, \pi/2]$, compute both squared norms and the inner product of the individually normalized functions. Decide whether normalization creates orthogonality.
4. On a window of length $m\pi$, m a positive integer, give an orthonormal pair obtained by rescaling u, v . Sketch their product on $[0, \pi]$ to explain cancellation without treating the graph as proof.

Question 4 – Solution

Strategy. Compute the product integral before deciding whether a pair is orthogonal or merely normalized.

Step 1: Determine the window dependence. Using $\sin x \cos x = \frac{1}{2} \sin 2x$ gives

$$\langle u, v \rangle_{a,L} = \frac{\cos 2a - \cos(2a + 2L)}{4} = \boxed{\frac{1}{2} \sin L \sin(2a + L)}.$$

It vanishes for all a exactly when $\sin L = 0$. The positive lengths are $\boxed{L = m\pi, m = 1, 2, \dots}$. If $\sin L \neq 0$, one can choose a with $\sin(2a + L) = 1$, proving necessity.

Step 2: Locate the shorter orthogonal windows. For $L = \pi/2$, orthogonality requires $\sin(2a + \pi/2) = 0$. Hence

$$\boxed{a = -\pi/4 + j\pi/2, \quad j \in \mathbb{Z}.$$

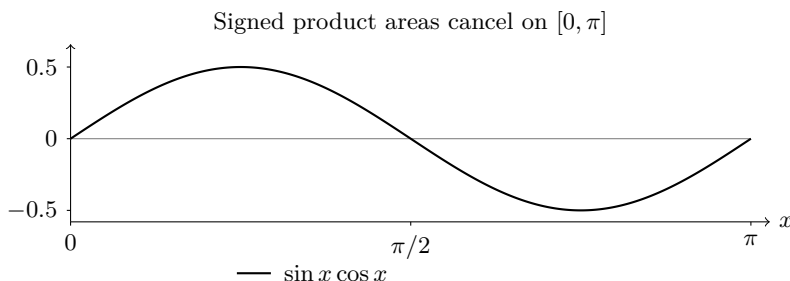
The product changes sign; shifting the window can balance its signed areas. At $a = 0$, however, the product is positive inside the interval.

Step 3: Test individual normalization. On $[0, \pi/2]$, $\|u\|^2 = \|v\|^2 = \pi/4$ and $\langle u, v \rangle = 1/2$. The unit-norm functions are $\hat{u} = 2 \sin x / \sqrt{\pi}$ and $\hat{v} = 2 \cos x / \sqrt{\pi}$, with

$$\boxed{\langle \hat{u}, \hat{v} \rangle = 2/\pi \neq 0}.$$

Nonzero scaling cannot turn a nonzero inner product into zero.

Step 4: Normalize on the correct windows. Over any length $m\pi$, the oscillatory term in either squared integral cancels, so both squared norms equal $m\pi/2$. Thus $\boxed{\sqrt{2/(m\pi)} \sin x, \sqrt{2/(m\pi)} \cos x}$ are orthonormal on every such window. The graph shows equal positive and negative product areas on $[0, \pi]$; the integral calculation establishes their exact cancellation.



Question 5

Use $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. Construct monic orthogonal polynomials from $1, x, x^2, x^3$; monic means the leading coefficient is one.

Tasks

1. Explain which inner products vanish by parity. Starting with $p_0 = 1$ and $p_1 = x$, find monic p_2, p_3 so that all four polynomials are mutually orthogonal.
2. Compute their squared norms exactly and turn them into an orthonormal set. Distinguish monic normalization from unit-norm normalization.
3. Find the unique quadratic-or-lower polynomial minimizing $\int_{-1}^1 [x^3 - q(x)]^2 dx$. Give the exact minimum error and justify uniqueness.
4. A proposed shortcut merely divides each monomial by its norm. Give an explicit inner product showing why this does not produce an orthogonal family.

Question 5 – Solution

Strategy. Use parity to eliminate unnecessary projections, then compute the remaining moments exactly.

Step 1: Construct the monic family. An odd integrand integrates to zero on $[-1, 1]$, so every even polynomial is orthogonal to every odd polynomial. For $p_2 = x^2 - c$, the equation $\langle p_2, 1 \rangle = 2/3 - 2c = 0$ gives $c = 1/3$. For $p_3 = x^3 - dx$, $\langle p_3, x \rangle = 2/5 - 2d/3 = 0$ gives $d = 3/5$. Thus

$$p_0 = 1, \quad p_1 = x, \quad p_2 = x^2 - 1/3, \quad p_3 = x^3 - 3x/5.$$

Parity and these two moment equations verify every distinct pair.

Step 2: Compute and apply the norms. Integrating the squares term by term yields

$$\|p_0\|^2 = 2, \quad \|p_1\|^2 = 2/3, \quad \|p_2\|^2 = 8/45, \quad \|p_3\|^2 = 8/175.$$

An orthonormal set is therefore $\left[p_0/\sqrt{2}, \sqrt{3/2} p_1, \sqrt{45/8} p_2, \sqrt{175/8} p_3 \right]$. Monic normalization fixes the leading coefficient; unit normalization fixes the squared integral and generally changes that coefficient.

Step 3: Find the best quadratic approximation. The decomposition $x^3 = 3x/5 + p_3$ has its residual orthogonal to every quadratic. For any other quadratic q , the cross term vanishes, giving

$$\|x^3 - q\|^2 = \|p_3\|^2 + \|3x/5 - q\|^2.$$

Hence $\left[q_* = 3x/5, \quad \min \|x^3 - q\|^2 = 8/175 \right]$. Equality forces the continuous polynomial $q - 3x/5$ to have zero norm and therefore vanish identically, proving uniqueness.

Step 4: Reject normalization without orthogonalization. The normalized versions of 1 and x^2 are $1/\sqrt{2}$ and $\sqrt{5/2} x^2$. Their inner product is $\sqrt{5}/3 \neq 0$. Normalizing lengths alone does not remove projections onto the preceding functions.

Question 6

On continuous real functions on $[0, 1]$, define

$$\langle f, g \rangle_w = \int_0^1 x f(x) g(x) dx.$$

The weight vanishes at one endpoint. Seek a monic quadratic $q(x) = x^2 + ax + b$ orthogonal to both 1 and x in this inner product.

Tasks

1. Prove positive definiteness despite the zero weight at $x = 0$. Explain the role of continuity of the functions.
2. Determine a, b from the two weighted moment equations. Verify both equations and prove uniqueness of the monic quadratic.
3. Compute $\|q\|_w^2$, give its unit-norm version with positive leading coefficient, and locate its roots. Sketch q on $[0, 1]$.
4. Test whether the same q is orthogonal to 1 for unweighted integration. Also explain why replacing the weight by $x - 1/2$ would fail to define an inner product on this space.

Question 6 – Solution

Strategy. Include the weight in every moment, and distinguish an isolated zero of a nonnegative weight from a sign-changing weight.

Step 1: Establish positive definiteness. Symmetry and bilinearity follow from integration, and $\int_0^1 x f^2 dx \geq 0$. If a continuous f is not identically zero, continuity supplies an interval of positive length inside $(0, 1]$ on which $f^2 > 0$, even if the first known nonzero value is at zero. The weighted integral is then positive. Thus zero weighted norm forces $f = 0$ everywhere.

Step 2: Solve the weighted moment equations. The two conditions are

$$\frac{1}{4} + \frac{a}{3} + \frac{b}{2} = 0, \quad \frac{1}{5} + \frac{a}{4} + \frac{b}{3} = 0.$$

Their coefficient determinant is $1/9 - 1/8 = -1/72 \neq 0$, so the solution is unique:

$$a = -6/5, \quad b = 3/10, \quad q = x^2 - 6x/5 + 3/10.$$

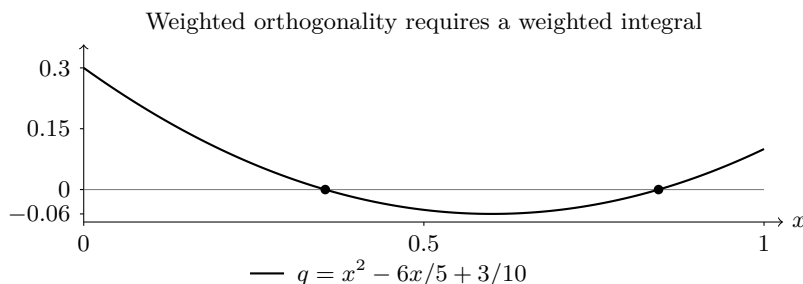
Substitution gives $1/4 - 2/5 + 3/20 = 0$ and $1/5 - 3/10 + 1/10 = 0$.

Step 3: Normalize and locate the sign changes. Expansion and integration give

$$\int_0^1 x q^2 dx = \frac{1}{6} - \frac{12}{25} + \frac{51}{100} - \frac{6}{25} + \frac{9}{200} = \boxed{\frac{1}{600}}.$$

Thus the unit-norm version is $\boxed{\sqrt{600} q}$. The roots are $\boxed{(6 - \sqrt{6})/10, (6 + \sqrt{6})/10}$, both strictly inside $(0, 1)$. The quadratic is positive outside these roots and negative between them.

Step 4: Check two changes of weight. Without the weight, $\int_0^1 q dx = 1/3 - 3/5 + 3/10 = \boxed{1/30 \neq 0}$. Orthogonality therefore changes with the weight. For the proposed signed weight, the nonzero constant function one would have squared “norm” $\int_0^1 (x - 1/2) dx = 0$. This violates positive definiteness, so that bilinear form is not an inner product on the stated space.



Question 7

Let $p \in C^1[a, b]$ be positive, let q, w be continuous with $w > 0$, and let u, v be nonzero real C^2 eigenfunctions:

$$-(pu')' + qu = \lambda wu, \quad -(pv')' + qv = \mu wv.$$

At each endpoint $c = a, b$, both satisfy the same condition $A_c y(c) + B_c p(c) y'(c) = 0$, with real $(A_c, B_c) \neq (0, 0)$.

Tasks

1. Derive an identity relating $(\lambda - \mu) \int_a^b wuv \, dx$ to endpoint terms by multiplying and subtracting the differential equations.
2. Prove the endpoint terms vanish for all the stated boundary conditions, including $B_c = 0$. Deduce weighted orthogonality when $\lambda \neq \mu$.
3. Apply the result to $u = \sin \pi x$ and $v = \sin 2\pi x$ on $[0, 1]$ with zero endpoint values. Give an orthonormal pair.
4. Explain why equal eigenvalues do not force orthogonality. On $[0, 2\pi]$ with periodic value-and-slope matching, use $u = \sin x$ and $v = \sin x + \cos x$ as a counterexample, then orthogonalize and normalize this pair.

Question 7 – Solution

Strategy. Track the boundary expression before dividing by an eigenvalue difference.

Step 1: Derive the boundary identity. Multiply the equation for u by v and that for v by u , then subtract. The q terms cancel, and integration gives

$$(\lambda - \mu) \int_a^b wuv \, dx = [p(uv' - u'v)]_a^b.$$

This identity is valid before any orthogonality conclusion is drawn.

Step 2: Use the actual endpoint conditions. At an endpoint with $B_c = 0$, $A_c \neq 0$ and both function values are zero, so the boundary expression is zero. If $B_c \neq 0$, then $pu' = -A_c u/B_c$ and $p'v = -A_c v/B_c$;

substitution again gives zero. Therefore $\lambda \neq \mu$ implies $\int_a^b wuv \, dx = 0$. Equal eigenvalues give only the identity $0 = 0$.

Step 3: Normalize two distinct Dirichlet modes. Here $p = w = 1, q = 0$, with eigenvalues π^2 and $4\pi^2$. The theorem gives orthogonality, and direct integration of each square gives $1/2$. Thus

$\sqrt{2} \sin \pi x, \sqrt{2} \sin 2\pi x$ are orthonormal. The normalization is separate from the boundary proof of orthogonality.

Step 4: Handle a repeated eigenvalue. Both $u = \sin x$ and $v = \sin x + \cos x$ have eigenvalue one and satisfy periodic matching on $[0, 2\pi]$. Nevertheless, $\langle u, v \rangle = \pi \neq 0$. Since $\|u\|^2 = \pi$, subtracting the projection gives $v - \langle v, u \rangle u / \|u\|^2 = \cos x$. The resulting orthonormal pair is

$$\sin x / \sqrt{\pi}, \quad \cos x / \sqrt{\pi}.$$

Distinct eigenvalues ensure orthogonality under the hypotheses; a repeated eigenspace may instead require an orthogonal choice of basis.

Question 8

Approximate $f(x) = |x|$ on $[-1, 1]$ by a polynomial $g(x) = a + bx^2$. Use the squared error $\|f - g\|_2^2 = \int_{-1}^1 (f - g)^2 dx$. For this inner product, 1 and $p_2 = x^2 - 1/3$ are orthogonal, with squared norms 2 and $8/45$ respectively.

Tasks

1. Compute the projection coefficients onto 1, p_2 and find the unique polynomial g_* minimizing the squared integral error.
2. Verify directly that the residual is orthogonal to both 1 and x^2 . Prove the error decomposition that establishes the minimum and uniqueness.
3. Compute the exact minimum squared error. Sketch f and g_* , and identify where the largest absolute residual occurs.
4. Does the least-squares optimum also minimize the maximum absolute error? Compare it with $h(x) = 1/8 + x^2$ and justify your conclusion by exact uniform error bounds.

Question 8 – Solution

Strategy. Use an orthogonal basis for the integral objective, then test the distinct uniform-error objective separately.

Step 1: Compute the projection. The needed moments are $\langle f, 1 \rangle = 1$ and $\langle f, p_2 \rangle = 1/2 - 1/3 = 1/6$. Thus

$$g_* = \frac{1}{2} + \frac{1/6}{8/45}(x^2 - 1/3) = \boxed{\frac{3}{16} + \frac{15}{16}x^2}.$$

Step 2: Verify residual orthogonality and uniqueness. For $r = f - g_*$,

$$\int_{-1}^1 r \, dx = 1 - \frac{3}{8} - \frac{5}{8} = 0, \quad \int_{-1}^1 rx^2 \, dx = \frac{1}{2} - \frac{1}{8} - \frac{3}{8} = 0.$$

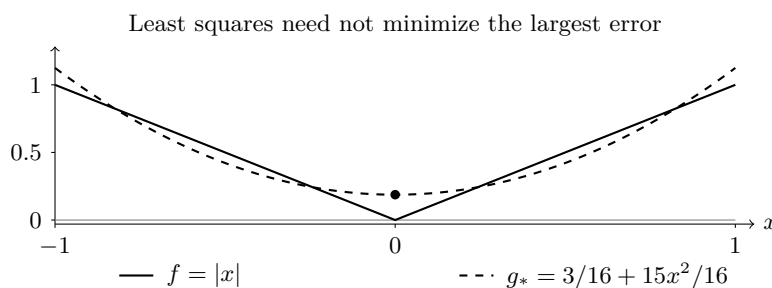
For any other $g = a + bx^2$, the cross term vanishes, so $\|f - g\|_2^2 = \|r\|_2^2 + \|g_* - g\|_2^2$. The second term is zero only when the two continuous polynomials agree, proving both optimality and uniqueness.

Step 3: Evaluate the two errors for the optimum. Subtracting the squared projection norm from $\|f\|_2^2 = 2/3$ gives

$$\|r\|_2^2 = \frac{2}{3} - \frac{1}{2} - \frac{(1/6)^2}{8/45} = \frac{1}{96}.$$

On $[0, 1]$, $r = x - 3/16 - 15x^2/16$ has maximum $19/240$ at $x = 8/15$, and endpoint values $-3/16, -1/8$. By evenness, the largest absolute error is $\|r\|_\infty = 3/16$, attained at zero.

Step 4: Disprove uniform optimality. For $h = 1/8 + x^2$, put $t = |x| \in [0, 1]$. Then $f - h = t(1-t) - 1/8 \in [-1/8, 1/8]$, with both extremes attained. Hence $\|f - h\|_\infty = 1/8 < 3/16$. This competitor proves that the unique least-squares solution is not a uniform best approximation. Different error objectives need not select the same function.



Question 9

A continuous signal f on $[-\pi, \pi]$ is known only through three moments:

$$\int_{-\pi}^{\pi} f \, dx = 2\pi, \quad \int_{-\pi}^{\pi} f \cos x \, dx = \pi, \quad \int_{-\pi}^{\pi} f \sin x \, dx = -2\pi.$$

Use the ordinary integral inner product. The three measurement functions $1, \cos x, \sin x$ are mutually orthogonal.

Tasks

1. Find the unique function in their span that matches all three measurements. Compute its squared norm.
2. Characterize every continuous signal with those measurements as a determined part plus an undetermined residual. Prove a sharp lower bound for $\int_{-\pi}^{\pi} f^2 dx$ and characterize equality.
3. Exhibit infinitely many distinct compatible signals explicitly, and compute their squared norms. Explain why linear independence of the measurement functions does not imply recovery of an arbitrary signal.
4. Suppose a fourth measurement gives $\int_{-\pi}^{\pi} f \cos 2x \, dx = 0$. Does that make the signal unique? Give an explicit counterexample and distinguish a finite orthogonal family from a complete representation.

Question 9 – Solution

Strategy. Orthogonal measurements determine a finite projection, while all perpendicular components remain unseen.

Step 1: Recover the measured component. The squared norms of $1, \cos x, \sin x$ are $2\pi, \pi, \pi$. Dividing each moment by its corresponding squared norm gives

$$p(x) = 1 + \cos x - 2 \sin x, \quad \|p\|^2 = 7\pi.$$

Orthogonality makes these coefficients unique within the three-dimensional span.

Step 2: Derive the sharp energy bound. Every compatible f has the form $f = p + r$, where r is continuous and orthogonal to all three measurement functions. Conversely every such residual preserves the measurements. Since $\langle p, r \rangle = 0$,

$$\|f\|^2 = 7\pi + \|r\|^2 \geq 7\pi.$$

Equality holds exactly when $r = 0$ everywhere: a continuous function with zero squared integral is identically zero. Thus p is the unique minimum-energy signal.

Step 3: Exhibit the invisible freedom. Product-to-sum identities give zero inner products between $\cos 2x$ and each of $1, \cos x, \sin x$. Therefore

$$f_C = p + C \cos 2x, \quad C \in \mathbb{R}, \quad \|f_C\|^2 = 7\pi + C^2\pi.$$

The measurement functions are independent, but their span does not contain all continuous functions. The data determine the projection, not its orthogonal complement.

Step 4: Test the fourth measurement. The added measurement removes the particular $\cos 2x$ freedom above, but $p + D \sin 2x$ satisfies all four conditions for every real D . The required product integrals again vanish by parity or product-to-sum. A finite orthogonal family supplies independent coordinates for its own span; it is not a complete representation of all continuous signals on the interval.

Question 10

Let $u(x) = \sin x$ and $v(x) = \sin 5x$ on $[0, 2\pi]$. Compare the continuous average inner product with its equally spaced sampling version:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} fg \, dx, \quad \langle f, g \rangle_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2\pi j/N)g(2\pi j/N).$$

The sampled form may fail to be positive definite on functions even though it is positive definite on the vectors of sampled values.

Tasks

1. Compute the continuous Gram matrix of u, v ; a Gram matrix has entries consisting of all pairwise inner products.
2. For $N = 4$, list both sample vectors and compute the sampled Gram matrix. Exhibit a nonzero continuous function in their span with zero sampled norm, and sketch the curves with the common sampled points.
3. For integer m , derive $S_m = N^{-1} \sum_{j=0}^{N-1} \cos(2\pi m j/N)$ using a finite geometric sum. Express every entry of the sampled Gram matrix in terms of S_2, S_4, S_6, S_{10} .
4. Find the smallest integer $N \geq 3$ for which the sampled Gram matrix equals the continuous one. Check every smaller candidate and explain why exactness for this pair does not imply exactness for all continuous functions.

Question 10 – Solution

Strategy. Distinguish continuous orthogonality from orthogonality of a finite set of sampled values.

Step 1: Compute the continuous matrix. Product-to-sum gives $\langle u, v \rangle = 0$, while each sine has average square $1/2$. Hence $G = \frac{1}{2}I_2$.

Step 2: Identify the four-point alias. At $0, \pi/2, \pi, 3\pi/2$, both functions give $(0, 1, 0, -1)$. Thus

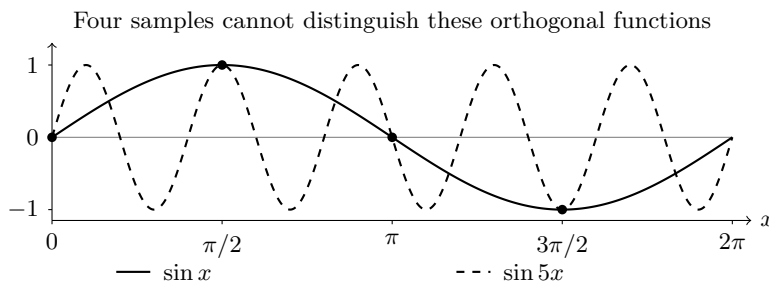
$$G_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The nonzero function $u - v$ vanishes at every sample, so its sampled norm is zero, although its continuous squared norm is one. The samples have lost the distinction.

Step 3: Derive the exact sampling formulas. Put $z = e^{2\pi im/N}$. If N divides m , every summand equals one. Otherwise $z \neq 1$ and $\sum_{j=0}^{N-1} z^j = (1 - z^N)/(1 - z) = 0$. Taking real parts yields $S_m = 1$ when $N \mid m$, and $S_m = 0$ otherwise. Product-to-sum now gives

$$G_N = \frac{1}{2} \begin{pmatrix} 1 - S_2 & S_4 - S_6 \\ S_4 - S_6 & 1 - S_{10} \end{pmatrix}.$$

Step 4: Find the smallest exact sample count. For $N = 3$ the off-diagonal entry is $-1/2$; for $N = 4$ it is $1/2$. For $N = 5$, the second diagonal entry is zero. For $N = 6$ the off-diagonal entry is $-1/2$. None matches G . For $N = 7$, none of $2, 4, 6, 10$ is divisible by seven, so $G_7 = G$ and seven is the smallest allowed count. This is exact only for the tested products: for example, $\sin 7x$ vanishes at all seven sample points but has continuous average square $1/2$.



Question 1

Let $f(x) = x$ on $[0, \pi]$. Define its sine coefficients and partial sums by

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx, \quad S_N = \sum_{n=1}^N b_n \sin(nx).$$

For a piecewise smooth odd periodic extension, the Fourier series converges to the average of its one-sided limits, including at periodic joins.

Tasks

1. Derive b_n by integration by parts. Describe the odd 2π -periodic extension, including its jump at odd multiples of π .
2. Determine the series sum at every $x \in [0, \pi]$. Distinguish the assigned endpoint value $f(\pi)$ from the series limit.
3. Decide whether S_N converges uniformly to f on $[0, \pi]$, or to x on the open interval $(0, \pi)$. Give a quantitative obstruction for both claims.
4. Use the value at $x = \pi/2$ to evaluate $1 - 1/3 + 1/5 - \dots$. Sketch S_{12} with f and explain why the endpoint mismatch does not invalidate the interior representation.

Question 1 – Solution

Strategy. Determine the odd extension before applying the convergence theorem at its endpoints.

Step 1: Compute the coefficients and extension. Integration by parts yields

$$\int_0^{\pi} x \sin(nx) dx = [-x \cos(nx)/n]_0^{\pi} + [\sin(nx)/n^2]_0^{\pi} = -\pi(-1)^n/n.$$

Hence $b_n = 2(-1)^{n+1}/n$. The odd extension equals x on $(-\pi, \pi)$ and repeats every 2π . At an odd multiple of π its left and right limits are π and $-\pi$. The join value may be chosen zero for an odd periodic representative.

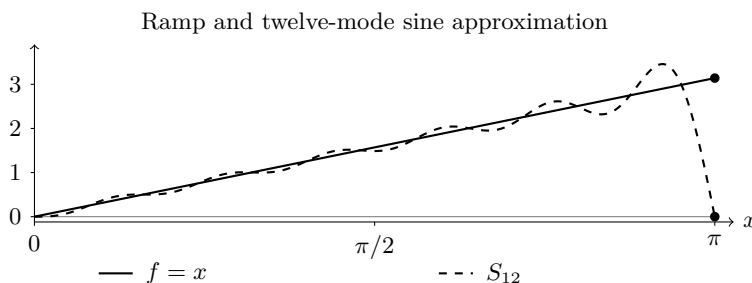
Step 2: Apply the convergence theorem. The series equals x for $0 < x < \pi$. At zero it equals zero by continuity of the extension, and at π it equals the jump average zero:

$$\sum_{n \geq 1} \frac{2(-1)^{n+1}}{n} \sin(nx) = x \quad (0 \leq x < \pi), \quad \text{sum at } \pi = 0.$$

This does not reproduce the separately assigned value $f(\pi) = \pi$.

Step 3: Rule out uniform convergence near the join. On the closed interval, $|f(\pi) - S_N(\pi)| = \pi$ for every N . On the open interval, continuity of each finite sum gives $\lim_{x \uparrow \pi} |x - S_N(x)| = \pi$, so the supremum error is at least π . Neither asserted uniform convergence holds, despite pointwise convergence inside.

Step 4: Evaluate an interior numerical series. At $x = \pi/2$, only odd modes remain and their sine values alternate: $\pi/2 = 2(1 - 1/3 + 1/5 - \dots)$. Therefore the convergent series equals $\pi/4$. The graph displays a finite approximation; the convergence theorem, not endpoint agreement of the original data, justifies the interior sum.



Question 2

Let $f(x) = 1$ for $0 < x < \pi$, with assigned endpoint values zero. The odd 2π -periodic extension is a square wave. Define its first M nonzero modes by

$$P_M(x) = \frac{4}{\pi} \sum_{j=0}^{M-1} \frac{\sin((2j+1)x)}{2j+1}.$$

You may use convergence to the average of the one-sided limits for piecewise smooth periodic functions.

Tasks

1. Derive the sine coefficients and verify the definition of P_M . State the limiting values inside the interval and at its endpoints.
2. Sum the finite cosine progression to prove $P'_M(x) = (2/\pi) \sin(2Mx)/\sin x$ for $0 < x < \pi$. Locate the first local maximum to the right of zero.
3. Express the height of that maximum as an integral. Rescale the integral and prove that the heights tend to $(2/\pi) \int_0^\pi (\sin t)/t \, dt$.
4. Prove that this limiting height is greater than one, using $\sin t/t \geq 1 - t^2/6$ for $0 \leq t \leq \pi/2$ and $1/t \geq 1/\pi$ for $\pi/2 \leq t \leq \pi$. Sketch P_{10} and explain why pointwise convergence does not remove the near-jump overshoot.

Question 2 – Solution

Strategy. Analyze a moving maximum of a finite sum rather than substitute a moving point into a pointwise convergence claim.

Step 1: Find the nonzero modes. Direct integration gives $b_n = 2(1 - (-1)^n)/(\pi n)$: zero for even n and $4/(\pi n)$ for odd n . Thus P_M is the stated truncation. The odd extension has jump averages zero at both endpoints; its sum is one on $(0, \pi)$.

Step 2: Locate the first maximum. A finite geometric sum, or telescoping $2 \sin x \cos((2j+1)x)$, gives $\sum_{j=0}^{M-1} \cos((2j+1)x) = \sin(2Mx)/(2 \sin x)$. Differentiating the finite sum is legitimate, so

$$P'_M(x) = \frac{2}{\pi} \frac{\sin(2Mx)}{\sin x}, \quad x_M = \frac{\pi}{2M}.$$

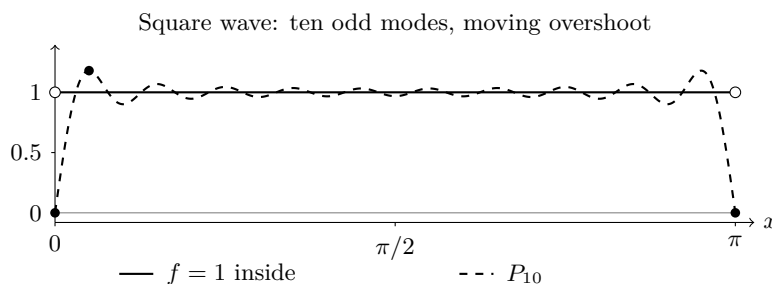
The derivative is positive before x_M and negative just after it, for $M \geq 1$.

Step 3: Pass to the moving-peak limit correctly. Since $P_M(0) = 0$, substitution $t = 2Mx$ gives

$$P_M(x_M) = \frac{2}{\pi} \int_0^\pi \frac{\sin t}{2M \sin(t/(2M))} dt \longrightarrow \boxed{\frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt}.$$

Write the integrand as $(\sin t/t)(z/\sin z)$, where $z = t/(2M)$. Since $z/\sin z$ extends continuously with value one at zero, this factor tends uniformly to one for $0 \leq t \leq \pi$. Also $|\sin t/t| \leq 1$. These facts justify the integral limit; its value is approximately 1.179.

Step 4: Prove a persistent excess. The supplied inequalities imply $\int_0^\pi \sin t/t dt \geq \pi/2 - \pi^3/144 + 1/\pi > \pi/2$, since $\pi^4 < 144$. The limiting peak is strictly above one. The peak location moves toward the jump, so convergence at each fixed interior point is consistent with this persistent overshoot. The square wave jumps by two across zero.



Question 3

A tent on $[0, 1]$ has height $H > 0$ and peak at $a \in (0, 1)$:

$$f(x) = \begin{cases} Hx/a, & 0 \leq x \leq a, \\ H(1-x)/(1-a), & a \leq x \leq 1. \end{cases}$$

Its sine coefficients are $b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$.

Tasks

1. Integrate separately on the two sides of the corner to derive b_n . Explain why the corner contributes even though both endpoint values vanish.
2. For $a = 1/3, H = 1$, identify all missing modes and prove that the sine series converges uniformly to the tent on the entire closed interval.
3. Suppose the only known coefficient data are $b_1 > 0$ and $b_2/b_1 = 1/4$, and the signal is known to be a tent of this form. Recover a, H uniquely in terms of b_1 .
4. Compare a tent with its reflection $f(1-x)$. State how each coefficient changes and explain why the absolute values of all coefficients cannot distinguish the two peak locations. Sketch the $a = 1/3, H = 1$ tent and its first twelve-mode sum.

Question 3 – Solution

Strategy. Retain the change in slope at the interior corner when integrating by parts.

Step 1: Compute the corner contribution. Put $k = n\pi$. One integration by parts removes the zero endpoint terms; integrating each constant slope then gives

$$\int_0^1 f \sin(kx) dx = \frac{1}{k^2} \left(\frac{H}{a} + \frac{H}{1-a} \right) \sin(ka).$$

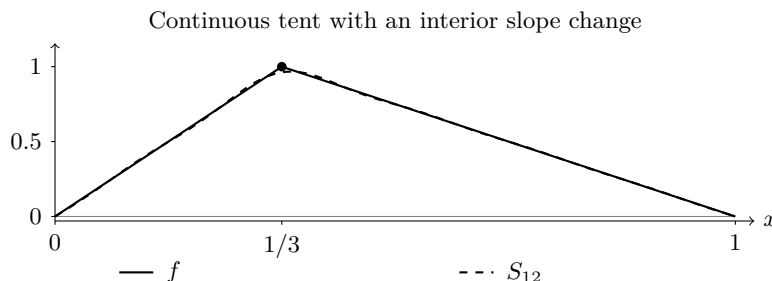
Thus $b_n = 2H \sin(n\pi a) / [\pi^2 a(1-a)n^2]$. The nonzero slope change, not an endpoint displacement, supplies these terms.

Step 2: Identify missing modes and convergence. For $a = 1/3, H = 1$, $b_n = 9 \sin(n\pi/3) / (\pi^2 n^2)$. Exactly the multiples of three vanish. The bound $|b_n| \leq 9/(\pi^2 n^2)$ gives uniform absolute convergence by the Weierstrass test. The continuous, piecewise linear odd extension satisfies the Fourier convergence theorem, so the sum is the tent at every point, including both endpoints and the corner.

Step 3: Recover the unknown tent. Because $\sin(\pi a) > 0$, $b_2/b_1 = \sin(2\pi a)/(4 \sin \pi a) = \frac{1}{2} \cos \pi a$. The measured ratio gives $\cos \pi a = 1/2$, hence $a = 1/3$ uniquely on $(0, 1)$. Substituting into b_1 gives

$$H = \frac{2\pi^2 b_1}{9\sqrt{3}}.$$

Step 4: Interpret reflection and lost sign information. Under $x \mapsto 1-x$, $\sin(n\pi(1-x)) = (-1)^{n+1} \sin(n\pi x)$. A change of variable gives $b_n^{\text{reflected}} = (-1)^{n+1} b_n$. Odd modes are unchanged and even modes reverse sign. All magnitudes therefore coincide, so they cannot distinguish a from $1-a$ when those locations differ.



Question 4

On $[0, \pi]$, let f equal one on $(\pi/3, 2\pi/3)$ and zero elsewhere, including the two jump points. Let S_N denote its first N sine modes.

Tasks

1. Derive every sine coefficient from the defining integral. Use symmetry about $\pi/2$ to explain why all even modes vanish.
2. Determine whether any odd modes vanish. State the sine-series limit at every point of $[0, \pi]$, paying attention to the assigned values at the jumps.
3. Change the assigned values at the two jumps to arbitrary real numbers. Explain what happens to the coefficients, the series and its pointwise limits.
4. Prove that no continuous function can approximate this assigned target with uniform error less than $1/2$. Explain the implication for sine partial sums and sketch S_{15} alongside the pulse.

Question 4 – Solution

Strategy. Use symmetry for coefficient selection, but use one-sided limits to determine convergence at jumps.

Step 1: Compute and simplify the coefficients. Integration over the nonzero portion gives

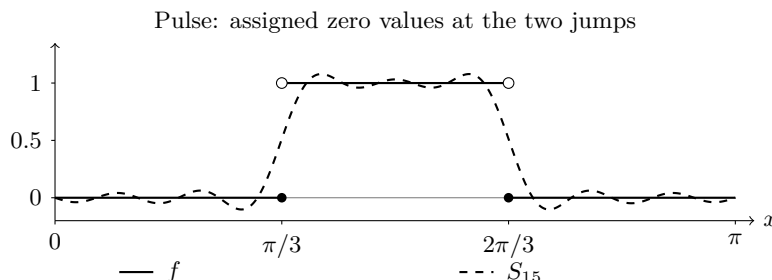
$$b_n = \frac{2}{\pi n} [\cos(n\pi/3) - \cos(2n\pi/3)].$$

The reflection $f(\pi - x) = f(x)$ implies $b_n = (-1)^{n+1}b_n$, so even modes are zero. For odd n , $\cos(2n\pi/3) = -\cos(n\pi/3)$, giving $b_n = 4\cos(n\pi/3)/(\pi n)$.

Step 2: Account for every mode and every limit. For odd integers n , $\cos(n\pi/3)$ is $1/2$ or -1 , never zero. Thus no odd mode vanishes. The odd periodic extension is piecewise constant; the Fourier convergence theorem gives one inside the pulse, zero outside, and $1/2$ at $\pi/3$ and $2\pi/3$. Both endpoints have limit zero. The jump-point sums differ from the assigned target values zero.

Step 3: Change only the assigned values. Changing finitely many points does not change any coefficient integral. The partial sums and their limits are therefore unchanged. The series records the jump averages, not arbitrary isolated assignments at the jumps.

Step 4: Establish the uniform-error obstruction. Let g be continuous and suppose $\sup |f - g| \leq E$. Taking one-sided limits at the first jump gives $|g(\pi/3)| \leq E$ and $|1 - g(\pi/3)| \leq E$. The triangle inequality yields $1 \leq 2E$, so $E \geq 1/2$. Every finite sine sum is continuous and obeys this lower bound. Thus pointwise convergence away from the jumps cannot become uniform convergence to this target on the whole interval. The graph shows the finite-sum transition and oscillations.



Question 5

Let $f(x) = x(\pi - x)$ on $[0, \pi]$ and $S_N = \sum_{n=1}^N b_n \sin(nx)$. You may use Parseval's identity $\int_0^\pi f^2 dx = (\pi/2) \sum_{n \geq 1} b_n^2$, and the standard convergence theorem for piecewise smooth odd periodic extensions.

Tasks

1. Derive all coefficients by integration by parts, and identify the missing modes.
2. Prove a uniform tail bound $\|f - S_N\|_\infty \leq 4/(\pi N^2)$ for $N \geq 1$. State why uniform convergence includes both endpoints here.
3. Find the smallest integer N certified by that bound to give error below 10^{-3} . Distinguish a sufficient certificate from the actual smallest successful truncation.
4. Evaluate $\int_0^\pi f^2 dx$ exactly and use Parseval to derive the sum of the reciprocal sixth powers of the positive odd integers.

Question 5 – Solution

Strategy. Use the zero endpoint values and the constant second derivative to obtain rapidly decaying coefficients.

Step 1: Integrate twice and retain the parity. Because $f(0) = f(\pi) = 0$ and $f'' = -2$, integration by parts gives $\int_0^\pi f \sin(nx) dx = 2(1 - (-1)^n)/n^3$. Therefore

$$b_n = \frac{4(1 - (-1)^n)}{\pi n^3} = \begin{cases} 8/(\pi n^3), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

Step 2: Bound the entire tail uniformly. The coefficient series is absolutely summable, so it converges uniformly. The continuous odd periodic extension and the convergence theorem identify its sum with f everywhere on $[0, \pi]$. For every x ,

$$|f(x) - S_N(x)| \leq \frac{8}{\pi} \sum_{n>N} \frac{1}{n^3} \leq \frac{8}{\pi} \int_N^\infty t^{-3} dt = \boxed{\frac{4}{\pi N^2}}.$$

Both endpoint values of the target and all partial sums are zero; there is no endpoint mismatch or jump obstructing uniform convergence.

Step 3: Choose a certified truncation. The stated bound is below 10^{-3} when $N > \sqrt{4000/\pi}$. Since this threshold lies between 35 and 36, $\boxed{N = 36}$ is the smallest integer certified by this bound. The estimate discarded the missing even modes and possible cancellation; it does not prove that every smaller N fails.

Step 4: Extract an exact numerical sum. Expanding the polynomial square yields $\int_0^\pi x^2(\pi - x)^2 dx = \pi^5/30$. Parseval gives

$$\frac{\pi^5}{30} = \frac{32}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^6}, \quad \boxed{\sum_{j=0}^{\infty} \frac{1}{(2j+1)^6} = \frac{\pi^6}{960}}.$$

This conclusion uses the complete series and Parseval, not a finite numerical fit.

Question 6

For $0 < x < \pi$, the sine series of $f(x) = x$ is

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Its partial sums converge to f in $L^2(0, \pi)$ as well as pointwise inside. Here $\|g\|_2 = (\int_0^\pi |g|^2 dx)^{1/2}$.

Tasks

1. Form the series obtained by formally differentiating once. Test the necessary term-to-zero condition at $x = \pi/2$ and decide whether the formal differentiation is valid there.
2. Integrate each finite partial sum from zero to x . Use Cauchy–Schwarz and the given L^2 convergence to justify passing to the limit uniformly for $0 \leq x \leq \pi$.
3. Derive the resulting cosine identity for $x^2/2$, including both endpoints. Give a uniform tail bound for this integrated series.
4. Evaluate the identity at $x = \pi$ to sum the reciprocal squares of the positive odd integers. Explain why integration is justified here while the attempted differentiation fails.

Question 6 – Solution

Strategy. Test the operation itself: differentiating weakens coefficient decay, while integration improves it.

Step 1: Reject the formal derivative at a specific point. Formal differentiation produces

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

At $x = \pi/2$, the even-indexed terms have magnitude two and do not tend to zero. Thus this series diverges there and cannot be a valid ordinary termwise derivative representation of $f' = 1$.

Step 2: Pass from finite integrals to the limit. For each finite sum S_N , direct integration gives

$$\int_0^x S_N(t) dt = 2 \sum_{n=1}^N \frac{(-1)^{n+1}}{n^2} (1 - \cos nx).$$

Cauchy-Schwarz yields $|\int_0^x (S_N - f) dt| \leq \sqrt{x} \|S_N - f\|_2 \leq \sqrt{\pi} \|S_N - f\|_2 \rightarrow 0$. This bound is independent of x , proving uniform convergence of the integrals.

Step 3: State the integrated identity and its error. Consequently

$$\boxed{\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} (1 - \cos nx), \quad 0 \leq x \leq \pi.}$$

The tail is bounded uniformly by $4 \sum_{n > N} n^{-2} \leq 4/N$. This also proves absolute uniform convergence of the integrated series. At zero both sides vanish; the right endpoint is included by the uniform integral argument, despite the original sine mismatch.

Step 4: Evaluate at the right endpoint. At $x = \pi$ the even terms vanish and each odd term contributes $4/n^2$. Hence $\pi^2/2 = 4 \sum_{j \geq 0} (2j+1)^{-2}$, giving $\boxed{\sum_{j \geq 0} (2j+1)^{-2} = \pi^2/8}$. The integration has a proved limiting argument and summable coefficients; the differentiated series fails even a necessary condition for convergence.

Question 7

On $[0, 1]$, approximate $f(x) = e^x$ using sine functions $\sin(n\pi x)$. To preserve the endpoint data, also consider

$$\ell(x) = 1 + (e - 1)x, \quad r(x) = e^x - \ell(x), \quad A_N(x) = \ell(x) + \sum_{n=1}^N c_n \sin(n\pi x),$$

where c_n are the sine coefficients of r .

Tasks

1. Compute the sine coefficients b_n of e^x exactly. State the ordinary sine-series limits at zero and one.
2. Compute the sine coefficients of ℓ and subtract them to derive c_n . Explain the improvement in decay.
3. Prove that A_N converges uniformly to e^x on $[0, 1]$, with error at most $(1 + e)/(\pi^3 N^2)$. Find an integer N certified to give error below 10^{-3} .
4. Compare endpoint errors of the raw sine partial sum and A_N . Sketch their errors for $N = 12$, and explain why adding the boundary interpolant changes the approximation problem rather than contradicting the endpoint limitation of pure sine sums.

Question 7 – Solution

Strategy. Remove the nonzero boundary values before asking a sine series for uniform approximation.

Step 1: Compute the raw coefficients. For $k = n\pi$, an antiderivative is $e^x(\sin kx - k \cos kx)/(1 + k^2)$. Evaluation gives

$$b_n = \frac{2k(1 - (-1)^n e)}{1 + k^2}.$$

The odd periodic extension has opposite one-sided values at each endpoint, so its jump averages are zero. Every raw sine partial sum is also zero there.

Step 2: Cancel the leading boundary contribution. Integration of the linear interpolant gives $b_n^\ell = 2(1 - (-1)^n e)/k$. Therefore

$$c_n = b_n - b_n^\ell = -\frac{2(1 - (-1)^n e)}{k(1 + k^2)}.$$

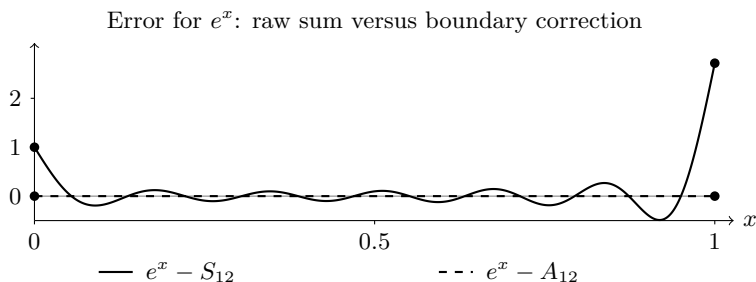
The raw coefficients are of order $1/n$, while c_n are of order $1/n^3$. The residual has zero values at both endpoints; the leading boundary term cancels.

Step 3: Establish a uniform certificate. Since $|c_n| \leq 2(1 + e)/(\pi^3 n^3)$, the residual series converges absolutely and uniformly. Its continuous piecewise smooth odd extension identifies the sum with r throughout the closed interval. Thus

$$\|e^x - A_N\|_\infty \leq \frac{1 + e}{\pi^3 N^2}.$$

For $N = 12$, the bound is below 10^{-3} , so twelve modes suffice.

Step 4: Interpret the endpoint comparison. For the raw sum S_N , $f - S_N$ equals one at zero and e at one. For A_N , both endpoint errors are exactly zero, and the entire error has the small bound above. The graph shows these two errors on the same scale. The corrected approximant includes the non-sine function ℓ ; it is not a pure sine sum and therefore is not constrained to vanish at the endpoints.



Question 8

Solve the boundary-value problem

$$-u'' = x, \quad 0 \leq x \leq \pi, \quad u(0) = u(\pi) = 0$$

using the sine coefficients $b_n = 2(-1)^{n+1}/n$ of the forcing x . You may use L^2 convergence of its sine series and the standard theorem allowing differentiation when the series of derivatives converges uniformly and the original series converges at one point.

Tasks

1. Derive the proposed sine coefficients of u from the action of $-d^2/dx^2$ on a sine mode. Independently solve the differential equation in closed form.
2. Verify that the closed-form solution has the proposed sine coefficients using integration by parts and the boundary values.
3. Justify one termwise differentiation of the solution series. Explain why a second differentiation cannot produce the forcing correctly at the right endpoint as an everywhere pointwise identity.
4. State precisely how the second derivatives of the finite solution sums converge in L^2 . Prove a uniform error bound for the solution itself and explain the smoothing effect of solving the boundary-value problem.

Question 8 – Solution

Strategy. Divide each forcing coefficient by its eigenvalue, then use an independent closed-form solution to verify the spectral construction.

Step 1: Obtain two descriptions of the solution. Since $-(\sin nx)'' = n^2 \sin nx$, the proposed coefficients are $a_n = b_n/n^2 = 2(-1)^{n+1}/n^3$. Direct integration of $u'' = -x$ and the two endpoint conditions gives

$$u(x) = \frac{\pi^2 x - x^3}{6}.$$

Its second derivative is $-x$ and both endpoint values vanish.

Step 2: Verify the actual coefficients. Let $\hat{u}_n = (2/\pi) \int_0^\pi u \sin nx \, dx$. Twice integrating by parts, using zero endpoint values of u and of the sine, gives $(2/\pi) \int_0^\pi (-u'') \sin nx \, dx = n^2 \hat{u}_n$. The left side is b_n , proving $\hat{u}_n = a_n$. The continuous odd extension and the convergence theorem identify the series with this polynomial, including at the endpoints.

Step 3: Justify the first derivative and limit the second. The derivative coefficients have magnitude $2/n^2$, so their cosine series converges uniformly. The original series converges at zero, where it is zero; the stated differentiation theorem therefore applies once. The formal second derivative is $-\sum b_n \sin nx$. At $x = \pi$ every term is zero, whereas $u''(\pi) = -\pi$. Thus the second derivative series cannot equal u'' pointwise everywhere on the closed interval.

Step 4: State the valid convergence and smoothing. For $U_N = \sum_{n=1}^N a_n \sin nx$, $U_N'' = -S_N$ exactly. The supplied forcing convergence gives $\|U_N'' + x\|_2 \rightarrow 0$; isolated endpoint disagreement does not change this norm. Moreover

$$\|u - U_N\|_\infty \leq 2 \sum_{n>N} n^{-3} \leq 1/N^2.$$

Dividing by n^2 suppresses high-frequency forcing coefficients and gives a uniformly convergent displacement with a uniformly convergent first derivative.

Question 9

For $0 < \varepsilon < \pi/2$, define a unit-area narrow pulse centered at $\pi/2$:

$$f_\varepsilon(x) = \begin{cases} (2\varepsilon)^{-1}, & |x - \pi/2| < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \quad 0 \leq x \leq \pi.$$

Use $b_n(\varepsilon) = (2/\pi) \int_0^\pi f_\varepsilon \sin nx \, dx$. You may use Parseval and Bessel's inequality for square-integrable functions.

Tasks

1. Verify the area and derive all sine coefficients. Interpret the value of $\sin z/z$ at $z = 0$ by its limit.
2. Find the limit of each fixed coefficient as $\varepsilon \downarrow 0$. Prove that the limiting coefficient sequence cannot be the sine coefficients of any $L^2(0, \pi)$ function.
3. Compute $\|f_\varepsilon\|_2^2$ and use Parseval to find $\sum_{n \geq 1} b_n(\varepsilon)^2$. Explain why a point-mass limit is not an ordinary square-integrable function.
4. Use $|\sin z/z - 1| \leq z^2/6$ to bound the largest coefficient error among $1 \leq n \leq N$. Derive this inequality from $\sin z/z = \int_0^1 \cos(zt) \, dt$ and explain why the resulting estimate is not uniform over all mode numbers.

Question 9 – Solution

Strategy. Separate a limit at each fixed frequency from a limit of the complete coefficient sequence in squared norm.

Step 1: Integrate the narrow pulse. The height times the width is one. Writing $a = \pi/2$, integration gives

$$b_n(\varepsilon) = \frac{\cos(n(a - \varepsilon)) - \cos(n(a + \varepsilon))}{\pi\varepsilon n} = \boxed{\frac{2}{\pi} \sin(n\pi/2) \frac{\sin(n\varepsilon)}{n\varepsilon}}.$$

The factor $\sin z/z$ has limiting value one at zero. Even modes vanish.

Step 2: Examine the fixed-mode limit. For each fixed n , $b_n(\varepsilon) \rightarrow \beta_n = (2/\pi) \sin(n\pi/2)$. Every odd-indexed β_n has magnitude $2/\pi$, so $\sum \beta_n^2$ diverges. Bessel's inequality would bound this sum for any L^2 function. Hence no square-integrable function has all these limiting coefficients.

Step 3: Track the diverging energy. Direct integration gives $\boxed{\|f_\varepsilon\|_2^2 = 1/(2\varepsilon)}$. Parseval therefore implies

$$\boxed{\sum_{n \geq 1} b_n(\varepsilon)^2 = \frac{1}{\pi\varepsilon}}.$$

The energies grow without bound while area stays one. Indeed, integration against a continuous test function tends to its value at $\pi/2$, by continuity and averaging over the shrinking interval. This is the point-mass limit, not an L^2 function or an ordinary pointwise sine representation of one.

Step 4: Quantify finite-band convergence. From $|1 - \cos s| \leq s^2/2$ and the supplied integral representation, $|\sin z/z - 1| \leq \int_0^1 z^2 t^2/2 dt = z^2/6$. Thus

$$\boxed{\max_{1 \leq n \leq N} |b_n(\varepsilon) - \beta_n| \leq \frac{N^2 \varepsilon^2}{3\pi}}.$$

For each fixed N this tends to zero. It grows with N , so it does not justify uniform convergence across all frequencies or interchange with the infinite energy sum.

Question 10

A signal on $[0, \pi]$ is

$$f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + 2 \sin 10x.$$

An approximation may retain at most three sine modes, with arbitrary real coefficients. Compare this freedom with the usual truncation to the first three frequencies. Use the ordinary squared integral error on $[0, \pi]$.

Tasks

1. Verify all sine coefficients by orthogonality and compute $\|f\|_2^2$.
2. Find the first-three-frequency partial sum and its exact squared error.
3. Find the best approximation using any three sine modes. Prove both the optimal choice of frequencies and their coefficients, and compare its error with the ordinary low-frequency truncation.
4. Suppose only the first three coefficients and the total squared norm are known, without the displayed formula for f . Determine the unseen energy and explain whether these data locate it at frequency ten. Give two distinct compatible signals.

Question 10 – Solution

Strategy. Orthogonality makes the cost of omitting a mode explicit, so optimal selection depends on coefficient size rather than frequency alone.

Step 1: Identify the coefficient energy. The sine functions have squared norm $\pi/2$ and are mutually orthogonal. Thus $b_1 = 1, b_2 = 1/2, b_3 = 1/3, b_{10} = 2$, with all other coefficients zero. Consequently

$$\|f\|_2^2 = \frac{\pi}{2} \left(1 + \frac{1}{4} + \frac{1}{9} + 4 \right) = \frac{193\pi}{72}.$$

Step 2: Evaluate the conventional truncation. The first-three-frequency sum is $S_3 = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$. Its residual is $2 \sin 10x$, so $\|f - S_3\|_2^2 = 2\pi$. Frequency order alone discards the largest-amplitude component here.

Step 3: Optimize the three selected modes. For any selected frequencies, orthogonality expresses the error as $\pi/2$ times the sum of squared coefficient differences on selected modes plus the squared coefficients of omitted modes. Each retained coefficient must therefore equal the true coefficient. To minimize the omitted energy, retain the three largest magnitudes, uniquely 2, 1, 1/2, at frequencies 10, 1, 2. Hence

$$g_* = 2 \sin 10x + \sin x + \frac{1}{2} \sin 2x, \quad \|f - g_*\|_2^2 = \frac{\pi}{18}.$$

The squared error of S_3 is 36 times larger. Adding a zero-coefficient mode cannot improve the selection and using fewer than three omits additional energy.

Step 4: Interpret incomplete spectral information. The measured first-three energy is $(\pi/2)(1 + 1/4 + 1/9) = 49\pi/72$. Subtracting it from the known total leaves 2π of unseen energy. Its location is not determined: $S_3 + 2 \sin 10x$ and $S_3 + 2 \sin 100x$ have the same given coefficients and norm. Total energy constrains the squared size of the tail, not its frequency distribution.

Question 1

For $f(x) = x$ on $[0, \pi]$, use the convention

$$C_N = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n \geq 0).$$

For piecewise smooth periodic extensions, the Fourier series converges to the average of its one-sided limits.

Tasks

1. Compute a_0 and all a_n , and identify the actual constant term. Explain the error caused by confusing a_0 with the mean.
2. Describe the even 2π -periodic extension and determine the sum at the endpoints and inside the interval.
3. Prove uniform convergence on $[0, \pi]$ and give an explicit uniform tail bound. Sketch C_{12} and f .
4. Every C_N has zero endpoint derivatives. Does the limit inherit these slopes? Give a quantitative obstruction to uniform convergence of the derivatives to $f' = 1$.

Question 1 – Solution

Strategy. Separate the mean convention from positive-frequency normalization, and distinguish function convergence from derivative convergence.

Step 1: Compute the mean and coefficients. The zero-frequency integral gives $a_0 = \pi$, so the mean term is $\pi/2$. Integration by parts for $n \geq 1$ gives

$$\int_0^\pi x \cos nx \, dx = \frac{(-1)^n - 1}{n^2}, \quad \boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^2}}.$$

Even modes vanish; odd coefficients are $-4/(\pi n^2)$. Using a_0 in place of $a_0/2$ adds the erroneous constant $\pi/2$.

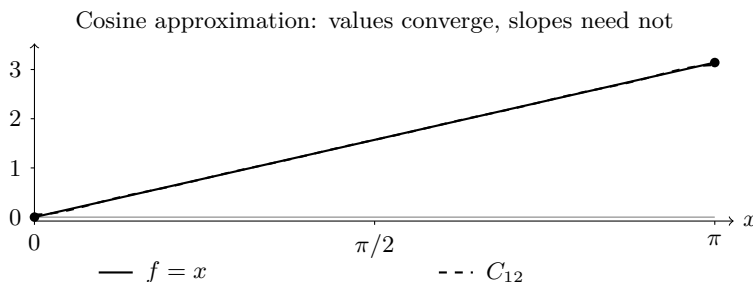
Step 2: Identify the extension and its limits. The extension is $|x|$ on $[-\pi, \pi]$, repeated with period 2π . It is continuous, with corners at multiples of π . At zero its two limits are zero; at π both are π . Therefore

$$\boxed{x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^2}, \quad 0 \leq x \leq \pi.}$$

Step 3: Bound the uniform error. The bound $|a_n| \leq 4/(\pi n^2)$ gives absolute uniform convergence by the Weierstrass test. The Fourier convergence theorem identifies the sum with f , including the endpoints. For $N \geq 1$,

$$\boxed{\|f - C_N\|_\infty \leq \frac{4}{\pi} \sum_{n>N} n^{-2} \leq \frac{4}{\pi N}}.$$

Step 4: Reject an unjustified derivative limit. Every finite derivative is a sine sum, so $C'_N(0) = C'_N(\pi) = 0$. The target has one-sided endpoint derivatives one. Thus $\boxed{\sup_{[0,\pi]} |C'_N - 1| \geq 1}$ for every N . Uniform convergence of functions alone does not permit differentiating the limit or passing endpoint derivative conditions to it.



Question 2

An interior cusp has unknown location $a \in (0, \pi)$:

$$f_a(x) = |x - a|, \quad 0 \leq x \leq \pi.$$

Write $f_a \sim m + \sum_{n \geq 1} a_n \cos nx$, with $m = \pi^{-1} \int_0^\pi f_a$ and $a_n = (2/\pi) \int_0^\pi f_a \cos nx$.

Tasks

1. Compute the mean and all coefficients by splitting at a . Identify the endpoint-slope and interior-corner contributions separately.
2. At $a = \pi/2$, determine exactly which modes survive and find the fundamental period of the even periodic extension. Sketch the target and its first twelve-mode sum.
3. Recover a uniquely from the signed first coefficient a_1 . State the admissible range of that measurement.
4. If only the mean and all coefficient magnitudes are known, is the location unique? Explain the reflection ambiguity and its exceptional symmetric case.

Question 2 – Solution

Strategy. Retain the interior slope jump when integrating a continuous function with a corner.

Step 1: Integrate the linear pieces. The triangular areas give $m = [a^2 + (\pi - a)^2]/(2\pi)$. Twice integrating the pieces separately gives

$$a_n = \frac{2}{\pi n^2} [1 + (-1)^n - 2 \cos(na)], \quad n \geq 1.$$

The endpoint slopes $-1, 1$ contribute $1 + (-1)^n$, and the interior slope jump two contributes $-2 \cos(na)$. None of these terms can be dropped.

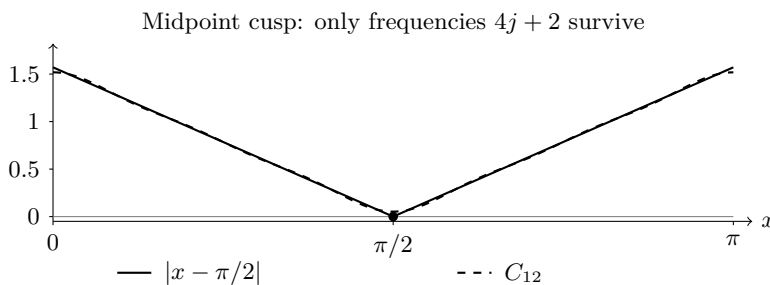
Step 2: Specialize to the midpoint cusp. At $a = \pi/2$ the mean is $\pi/4$. Odd modes vanish, and among the even modes exactly $n = 4j + 2$ survive, with coefficient $8/(\pi n^2)$. The extension has period π ; its maxima occur precisely at multiples of π , so every positive period must be a positive multiple of π . Its fundamental period is therefore $[\pi]$. Absolute coefficient summability gives uniform convergence, and the continuous piecewise linear even extension identifies the sum at all points, including cusps.

Step 3: Invert the first coefficient. The general formula gives $a_1 = -4 \cos a/\pi$. Strict monotonicity of cosine on $(0, \pi)$ yields

$$a = \arccos(-\pi a_1/4), \quad -4/\pi < a_1 < 4/\pi.$$

The open measurement range corresponds exactly to interior cusp locations.

Step 4: Describe what magnitudes lose. Reflection gives $f_{\pi-a}(x) = f_a(\pi-x)$ and $a_n^{\text{reflected}} = (-1)^n a_n$, with unchanged mean. Thus all magnitudes coincide for a and $\pi - a$. In fact $m = \pi/4 + (a - \pi/2)^2/\pi$, so the mean leaves precisely those two possibilities unless $a = \pi/2$, when they coincide. Signed odd coefficients resolve the nonsymmetric ambiguity.



Question 3

Consider the normalized Neumann problem

$$-u'' = x - \pi/2, \quad u'(0) = u'(\pi) = 0, \quad \int_0^\pi u(x)dx = 0.$$

Use the convention $u = a_0/2 + \sum_{n \geq 1} a_n \cos nx$.

Tasks

1. Derive the integral compatibility condition for a general forcing f with these derivative conditions. Explain why replacing the forcing by x makes the problem impossible.
2. Find the cosine coefficients of the stated forcing. Explain how the zero mode differs from every positive-frequency mode when solving for u .
3. Solve in closed form and as a cosine series. Verify the equation, both derivative conditions and the mean.
4. Prove uniqueness with the normalization, and state the full family without it. Explain why adding a constant cannot repair an incompatible forcing.

Question 3 – Solution

Strategy. Check the forcing mean before dividing by eigenvalues; the constant mode has eigenvalue zero.

Step 1: Check compatibility before solving. Integration gives $\int_0^\pi f = -u'(\pi) + u'(0) = 0$. The stated forcing satisfies this condition, whereas $\int_0^\pi x = \pi^2/2 \neq 0$. The replacement is impossible regardless of the additive level of u .

Step 2: Solve the mode equations. The forcing has mean zero and coefficients $F_n = 2((-1)^n - 1)/(\pi n^2)$. Since $-(\cos nx)'' = n^2 \cos nx$,

$$a_n = \frac{2((-1)^n - 1)}{\pi n^4} \quad (n \geq 1).$$

The zero-mode equation is $0 = 0$ and leaves the mean free. The normalization sets $a_0 = 0$; dividing this equation by zero would be invalid.

Step 3: Verify an independent polynomial solution. Direct integration gives

$$u = -\frac{x^3}{6} + \frac{\pi x^2}{4} - \frac{\pi^3}{24} = -\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^4}.$$

Its derivative is $x(\pi - x)/2$, zero at both ends, and $-u'' = x - \pi/2$. The mean of the first two terms is $\pi^3/24$, canceled by the constant. Twice integrating its coefficient integral by parts using $u' = 0$ at the endpoints gives $n^2 a_n = F_n$, verifying the coefficients. The continuous piecewise smooth even extension identifies the series sum.

Step 4: Separate the free constant from compatibility. A difference satisfies $w'' = 0$ and zero endpoint derivatives, hence is constant. Its zero mean forces it to vanish. Without normalization, all solutions are the displayed polynomial plus any real constant. Adding a constant changes neither the equation nor its derivative conditions, so it cannot fix the integrated forcing balance.

Question 4

For $f = e^x$ on $[0, 1]$, improve a cosine approximation by matching slopes:

$$\ell = x + \frac{e-1}{2}x^2, \quad r = e^x - \ell, \quad A_N = \ell + m_r + \sum_{n=1}^N c_n \cos(n\pi x),$$

where $m_r = \int_0^1 r$ and $c_n = 2 \int_0^1 r(x) \cos(n\pi x) dx$.

Tasks

1. Compute the raw mean and cosine coefficients of e^x . Find the endpoint slopes of every raw cosine partial sum.
2. Verify the slopes of ℓ , then compute m_r and all c_n . Explain the improvement from order n^{-2} to order n^{-4} .
3. Prove $\|e^x - A_N\|_\infty \leq 2(1+e)/(3\pi^4 N^3)$ and give an integer N certified to make the bound less than 10^{-4} .
4. Justify one differentiation of the residual series and verify the slopes of every A_N . Sketch the derivative errors of raw and corrected approximants for $N = 8$. State what slope matching does not guarantee.

Question 4 – Solution

Strategy. Remove the endpoint-slope contribution while retaining the residual's generally nonzero mean.

Step 1: Compute the raw data. For $k = n\pi$, direct integration gives

$$\text{mean} = e - 1, \quad a_n = \frac{2(e(-1)^n - 1)}{1 + k^2}.$$

Every raw finite cosine sum has zero endpoint derivatives, disagreeing with $f'(0) = 1$ and $f'(1) = e$.

Step 2: Cancel the slope contribution. The derivative $\ell' = 1 + (e - 1)x$ matches both target endpoint slopes. Two integrations by parts give its positive-frequency coefficient $2(e(-1)^n - 1)/k^2$; the constant second derivative has zero cosine integral. Subtracting yields

$$m_r = \frac{5}{6}(e - 1) - \frac{1}{2}, \quad c_n = -\frac{2(e(-1)^n - 1)}{k^2(1 + k^2)}.$$

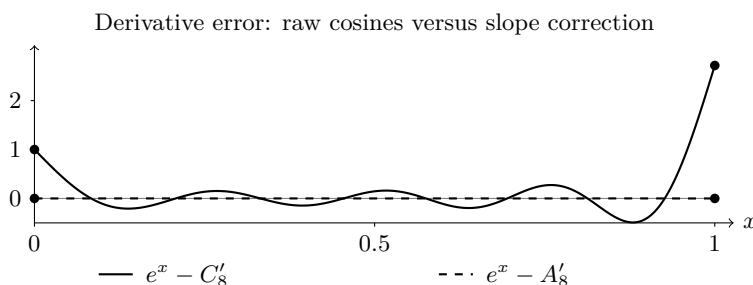
The residual has zero endpoint derivatives, so the leading slope term cancels and fourth-power decay remains. Its mean must still be included.

Step 3: Certify the function error. The bound $|c_n| \leq 2(1 + e)/(\pi^4 n^4)$ gives uniform absolute convergence. The continuous piecewise smooth even extension identifies the sum with r . Consequently

$$\|e^x - A_N\|_\infty \leq \frac{2(1 + e)}{\pi^4} \int_N^\infty t^{-4} dt = \frac{2(1 + e)}{3\pi^4 N^3}.$$

The choice $N = 7$ makes this bound less than 10^{-4} .

Step 4: Verify the derivative behavior. Derivative coefficients are bounded by a constant times n^{-3} , so that series converges uniformly. Together with convergence at one point, this justifies differentiation. Every residual partial sum has zero endpoint slopes; adding ℓ gives $A'_N(0) = 1, A'_N(1) = e$ exactly. Slope matching does not force exact endpoint values at finite N ; those are controlled by the function-error bound and become exact in the limit.



Question 5

On the shifted interval $[1, 4]$, expand $f(x) = (x - 1)(4 - x)$ using a cosine basis adapted to its left endpoint. Put $t = x - 1 \in [0, 3]$.

Tasks

1. State the correct shifted basis and coefficient normalization. Compute the mean and all positive-frequency coefficients.
2. Use reflection about $x = 5/2$ to explain the missing odd modes. Determine the series sum at the endpoints and midpoint.
3. Someone uses $1, \cos(\pi x/3), \cos(2\pi x/3), \dots$ with the usual orthogonal coefficient formula. Compute the inner product of the first two listed functions and diagnose the failure.
4. Derive a uniform truncation bound for the correct series. Explain why interval length alone does not specify the phase origin required by standard cosine orthogonality.

Question 5 – Solution

Strategy. Shift the coordinate in the basis as well as in the target, so the coefficient formulas retain orthogonality.

Step 1: Compute the translated expansion. The basis is $1, \cos(n\pi(x-1)/3)$. The mean is $m = (1/3) \int_1^4 f = 3/2$, and $a_n = (2/3) \int_1^4 f \cos(n\pi(x-1)/3)$. For $f = t(3-t)$, two integrations by parts yield

$$a_n = -\frac{18(1 + (-1)^n)}{\pi^2 n^2}.$$

The endpoint slopes are $3, -3$; the constant second derivative integrates to zero against each positive-frequency cosine.

Step 2: Use symmetry and continuity. Reflection sends t to $3-t$, multiplying mode n by $(-1)^n$ while leaving the target unchanged. Thus odd modes vanish. Coefficient summability and the continuous piecewise smooth even extension give uniform convergence to f . The sums are zero at $x = 1, 4$ and $\boxed{9/4}$ at $x = 5/2$.

Step 3: Diagnose the unshifted family. The constant and first unshifted cosine have inner product

$$\int_1^4 \cos(\pi x/3) dx = \frac{3}{\pi} \left(\sin \frac{4\pi}{3} - \sin \frac{\pi}{3} \right) = -\frac{3\sqrt{3}}{\pi} \neq 0.$$

They are not orthogonal. Dividing independent moments by the usual cosine norms therefore does not compute orthogonal projection coefficients here. The unshifted phase is not an innocuous substitution in that formula.

Step 4: Give a uniform certificate. For the correctly shifted partial sum,

$$\|f - C_N\|_\infty \leq \frac{36}{\pi^2} \sum_{n>N} n^{-2} \leq \frac{36}{\pi^2 N}.$$

Length sets the frequency spacing $\pi/3$; the left endpoint sets the phase. Both are used in translating the standard half-range expansion.

Question 6

Let $f = 1$ on $[0, \pi/2)$, $f(\pi/2) = 1/2$, and $f = 0$ on $(\pi/2, \pi]$. Write $C_N = a_0/2 + \sum_{n=1}^N a_n \cos nx$, including $C_0 = a_0/2$, and define

$$\sigma_N = \frac{1}{N+1} \sum_{j=0}^N C_j.$$

Let F be the even 2π -periodic extension. The ordinary Fourier convergence theorem may be used for this piecewise constant function.

Tasks

1. Compute the mean and coefficients. Show that C_1 becomes negative although the target is nonnegative.
2. Derive the coefficient weights in σ_N . Expand $K_N(t) = (N+1)^{-1} |\sum_{j=0}^N e^{ijt}|^2$ and prove nonnegativity and integral 2π over one period.
3. Verify $\sigma_N(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} F(x-t) K_N(t) dt$, then prove $0 \leq \sigma_N \leq 1$ everywhere.
4. Determine all limits of σ_N on $[0, \pi]$ by averaging convergent sequences of partial sums. Sketch C_9, σ_9 and the target, and explain the tradeoff made by averaging.

Question 6 – Solution

Strategy. Use an average with a nonnegative kernel and unit total mass to preserve the range of the data.

Step 1: Exhibit a negative partial sum. The mean is $1/2$ and $a_n = 2 \sin(n\pi/2)/(\pi n)$. Thus $C_1 = 1/2 + (2/\pi) \cos x$ has minimum $1/2 - 2/\pi < 0$ at π , despite $0 \leq f \leq 1$.

Step 2: Compute the weights and kernel. Mode n appears in $N + 1 - n$ of the partial sums, giving

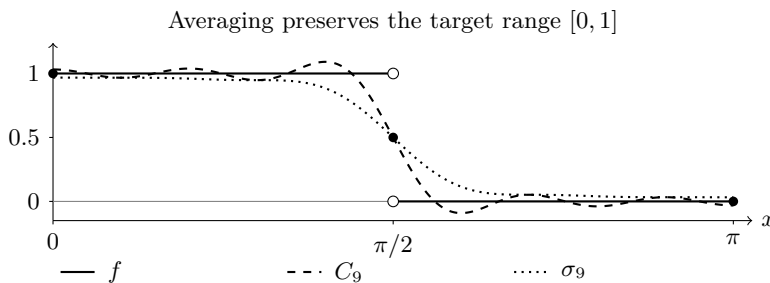
$$\sigma_N = \frac{1}{2} + \sum_{n=1}^N \left(1 - \frac{n}{N+1}\right) a_n \cos nx.$$

Counting pairs with index difference n in the squared modulus gives $K_N = 1 + 2 \sum_{n=1}^N (1 - n/(N+1)) \cos nt$. The squared-modulus definition proves $K_N \geq 0$, and integrating the finite expansion gives $\int_{-\pi}^{\pi} K_N = 2\pi$.

Step 3: Prove preservation of the range. Insert the finite kernel expansion into the convolution. The average integral of $F(x-t) \cos nt$ is $a_n \cos nx/2$; after shifting variables, its sine component vanishes by evenness of F . The constant term gives mean $1/2$. This proves the formula. Because $0 \leq F \leq 1$ and the kernel is nonnegative with mass 2π ,

$$0 \leq \sigma_N(x) \leq 1.$$

Step 4: Identify the limits and tradeoff. Ordinary partial sums tend to one left of the jump, zero right of it, and $1/2$ at the jump. The extension is continuous at zero and π , with values one and zero. Arithmetic means of a convergent sequence have the same limit, so σ_N has exactly these limits. Averaging damps the higher modes and prevents overshoot, but spreads the finite transition instead of fitting the edge as sharply.



Question 7

For $0 < \rho < 1$, define the positive profile

$$P_\rho(x) = \frac{1 - \rho^2}{1 - 2\rho \cos x + \rho^2}, \quad 0 \leq x \leq \pi.$$

Use $P_\rho = a_0/2 + \sum_{n \geq 1} a_n \cos nx$ and, when needed, Parseval: $\int_0^\pi f^2 = \pi(a_0/2)^2 + (\pi/2) \sum_{n \geq 1} a_n^2$.

Tasks

1. Derive the cosine expansion from a geometric series, justifying uniform convergence for fixed ρ . Identify the mean and every coefficient.
2. Compute the integral and squared norm exactly. Locate the maximum and minimum and give their values.
3. Determine the pointwise limit as $\rho \uparrow 1$. Explain why integrating it does not recover the constant mass.
4. For every continuous φ on $[0, \pi]$, prove $\int_0^\pi P_\rho \varphi \rightarrow \pi \varphi(0)$ by splitting at a small fixed $\delta > 0$. Sketch $P_{1/2}$ and $P_{4/5}$ and interpret this limit.

Question 7 – Solution

Strategy. Sum the geometric coefficients exactly, then locate the mass accumulating near the endpoint.

Step 1: Derive the uniformly convergent expansion. The geometric identity gives $1 + 2 \operatorname{Re}[\rho e^{ix}/(1 - \rho e^{ix})] = P_\rho(x)$. Since $\sum \rho^n < \infty$, its cosine series converges absolutely and uniformly:

$$P_\rho = 1 + 2 \sum_{n=1}^{\infty} \rho^n \cos nx, \quad a_0 = 2, \quad a_n = 2\rho^n.$$

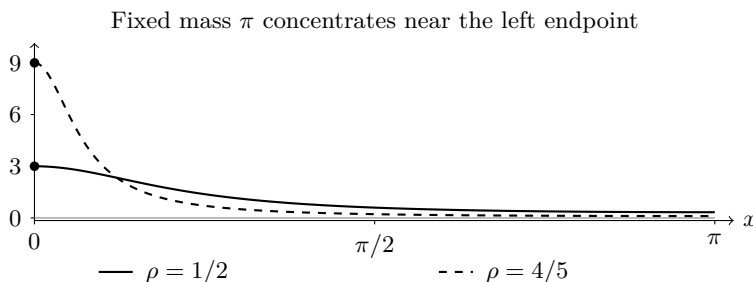
Step 2: Compute mass, energy and extrema. Uniform integration gives $\int_0^\pi P_\rho = \pi$. Parseval and the geometric sum give

$$\int_0^\pi P_\rho^2 = \pi \frac{1 + \rho^2}{1 - \rho^2}.$$

The denominator increases with x . The maximum at zero is $(1+\rho)/(1-\rho)$; the minimum at π is $(1-\rho)/(1+\rho)$.

Step 3: Explain the apparent loss of mass. For fixed $x > 0$, the numerator tends to zero and the denominator to $2(1 - \cos x) > 0$, so $P_\rho(x) \rightarrow 0$. At zero it tends to infinity. The almost-everywhere limit is zero, yet the integral stays π . Uniform convergence and interchange of the integral with this limit are not available; the exact squared norm also diverges.

Step 4: Prove endpoint concentration. Given $\eta > 0$, choose $\delta > 0$ so $|\varphi(x) - \varphi(0)| < \eta$ on $[0, \delta]$. That part of $\int P_\rho(\varphi - \varphi(0))$ is at most $\pi\eta$ in magnitude. On $[\delta, \pi]$, $P_\rho \leq (1 - \rho^2)/[2\rho(1 - \cos \delta)] \rightarrow 0$ uniformly, so the remaining part tends to zero. Letting $\eta \downarrow 0$ proves the claim. The limit is mass π concentrated at zero, not an ordinary integrable profile.



Question 8

Initially f is known to lie in $\text{span}\{1, \cos x, \cos 2x\}$ on $[0, \pi]$. Measurements give

$$\int_0^\pi f = \pi, \quad \int_0^\pi f \cos x = \pi/4, \quad \int_0^\pi f^2 = 5\pi/4.$$

There is also a requirement $f \geq 0$.

Tasks

1. Recover all functions in the stated span satisfying the measurements. Identify where a sign ambiguity enters.
2. Decide whether nonnegativity eliminates either candidate. Prove the answer on the entire interval using $t = \cos x \in [-1, 1]$.
3. A fourth measurement is $f(\pi) = 0$. Find the unique candidate in the stated span and verify all the data.
4. Remove the restriction to that span. Construct another nonnegative continuous function with all four measurements, and explain the limitation of the earlier uniqueness result.

Question 8 – Solution

Strategy. Linear moments preserve coefficient signs, while a squared-energy measurement can lose them.

Step 1: Solve the coefficient equations. Write $f = A + B \cos x + C \cos 2x$. Orthogonality gives $A = 1$, $B = 1/2$ and $\pi[A^2 + (B^2 + C^2)/2] = 5\pi/4$. Thus $C^2 = 1/4$, leaving

$$f_{\pm} = 1 + \frac{1}{2} \cos x \pm \frac{1}{2} \cos 2x.$$

The energy measurement determines the magnitude, but not the sign, of C .

Step 2: Test positivity without sampling. With $t = \cos x$, $f_+ = t^2 + t/2 + 1/2 = (t + 1/4)^2 + 7/16 > 0$. The other candidate is $f_- = 3/2 + t/2 - t^2$. This concave quadratic reaches its minimum on $[-1, 1]$ at an endpoint, with values zero and one. Thus both candidates are nonnegative; positivity does not resolve the ambiguity.

Step 3: Apply the endpoint measurement. At π , $f_+(\pi) = 1$ while $f_-(\pi) = 0$. Hence the unique candidate within the specified span is $f = f_-$. Its mean and first moment are the prescribed ones, and its squared norm is $\pi(1 + 1/8 + 1/8) = 5\pi/4$, verifying all four data.

Step 4: Remove the model restriction. The distinct function

$$g = 1 + \frac{1}{2} \cos x - \frac{1}{2} \cos 4x$$

has the same mean, first cosine moment and squared norm by orthogonality. Also $g(\pi) = 0$ and $g \geq 1 - 1/2 - 1/2 = 0$ everywhere. Thus these measurements and nonnegativity do not give uniqueness among all continuous functions. The finite-span hypothesis was essential.

Question 9

For $\alpha > 0$, consider

$$-u''_{\alpha} + \alpha^2 u_{\alpha} = 1 + \cos x, \quad u'_{\alpha}(0) = u'_{\alpha}(\pi) = 0.$$

The zeroth cosine mode has eigenvalue zero for $-d^2/dx^2$.

Tasks

1. Solve by cosine modes, treating the constant separately. Verify the equation and both boundary conditions.
2. Prove uniqueness for $\alpha > 0$ by an energy identity. Determine the mean and its behavior as $\alpha \downarrow 0$.
3. Subtract the mean to define $v_{\alpha} = u_{\alpha} - \pi^{-1} \int_0^{\pi} u_{\alpha}$. Find its limit and prove uniform convergence of the functions and their first two derivatives.
4. Explain why the unregularized problem with forcing $1 + \cos x$ has no solution, whereas the limit of v_{α} solves a different normalized problem. State that problem precisely.

Question 9 – Solution

Strategy. Track the constant mode separately: it becomes singular as the regularization vanishes.

Step 1: Solve the forced modes. The constant is divided by α^2 , while the first cosine is divided by $1 + \alpha^2$. Therefore

$$u_\alpha = \frac{1}{\alpha^2} + \frac{\cos x}{1 + \alpha^2}.$$

Its derivative is $-\sin x/(1 + \alpha^2)$, zero at both endpoints. Substitution recovers both the constant forcing one and the cosine term.

Step 2: Establish uniqueness and inspect the mean. For a homogeneous difference, integration by parts gives $\int_0^\pi [(w')^2 + \alpha^2 w^2] dx = 0$, so $w = 0$. The mean is $\boxed{1/\alpha^2}$ and diverges as $\alpha \downarrow 0$; the full family cannot have a finite uniform limit.

Step 3: Control the normalized family. Subtracting the mean gives $v_\alpha = \cos x/(1 + \alpha^2)$. The first two derivatives of cosine, as well as cosine itself, have supremum norm one. Thus for $j = 0, 1, 2$,

$$\|v_\alpha^{(j)} - (\cos x)^{(j)}\|_\infty = \frac{\alpha^2}{1 + \alpha^2} \longrightarrow 0.$$

This proves function and derivative convergence on the whole interval.

Step 4: Identify the correct limiting problem. The original forcing has integral $\pi \neq 0$, violating the Neumann condition $\int f = 0$ when $\alpha = 0$. Hence the unregularized original problem is impossible. Subtracting the mean also removes the constant forcing: $-v_\alpha'' + \alpha^2 v_\alpha = \cos x$. The limit solves

$$\boxed{-v'' = \cos x, \quad v'(0) = v'(\pi) = 0, \quad \int_0^\pi v = 0, \quad v = \cos x.}$$

It is the unique zero-mean solution of this compatible problem.

Question 10

Samples at $x_0 = \pi/6$, $x_1 = \pi/2$, $x_2 = 5\pi/6$ give the values $(2, 0, 1)$. Initially assume $f = A + B \cos x + C \cos 2x$ and define $\langle u, v \rangle_d = \frac{1}{3} \sum_{j=0}^2 u(x_j)v(x_j)$.

Tasks

1. List the sample vectors of $1, \cos x, \cos 2x$. Verify discrete orthogonality and compute their squared norms.
2. Recover A, B, C from the discrete moments. Verify every sample and prove uniqueness within the assumed span.
3. Drop the span restriction and exhibit infinitely many smooth functions with the same samples by adding an invisible cosine mode.
4. Do nonnegative samples imply that the reconstructed function is nonnegative throughout $[0, \pi]$? Find its exact minimum using $t = \cos x$ and explain what the samples do not certify.

Question 10 – Solution

Strategy. Distinguish reconstruction in a finite discrete basis from claims about unmeasured modes or intermediate values.

Step 1: Compute the discrete Gram data. The vectors are

$$(1, 1, 1), \quad (\sqrt{3}/2, 0, -\sqrt{3}/2), \quad (1/2, -1, 1/2).$$

Their distinct dot products are zero. Dividing squared lengths by three gives squared norms $\boxed{1, 1/2, 1/2}$.

Step 2: Recover and verify the coefficients. The sample mean gives $A = 1$. Dividing each other discrete moment by $1/2$ gives

$$\boxed{B = 1/\sqrt{3}, \quad C = 1, \quad f = 1 + \frac{1}{\sqrt{3}} \cos x + \cos 2x.}$$

The values at the three nodes are $1 + 1/2 + 1/2 = 2$, $1 - 1 = 0$ and $1 - 1/2 + 1/2 = 1$. Three nonzero orthogonal vectors are independent, so the coefficient triple is unique within the specified span.

Step 3: Exhibit the invisible smooth freedom. The function $\cos 3x$ vanishes at all nodes, whose tripled arguments are $\pi/2, 3\pi/2, 5\pi/2$. Hence $\boxed{f_D = f + D \cos 3x, \quad D \in \mathbb{R}}$ has the same samples. These are distinct smooth functions; reconstruction is not unique among all smooth functions.

Step 4: Check between-node positivity exactly. Since $\cos 2x = 2t^2 - 1$, the reconstructed function is $2t^2 + t/\sqrt{3}$. Completing the square gives

$$\boxed{f = 2 \left(t + \frac{1}{4\sqrt{3}} \right)^2 - \frac{1}{24}.$$

Its minimum is $-1/24$, attained at the interior point $x = \arccos(-1/(4\sqrt{3}))$. Nonnegative samples therefore fail to certify nonnegativity between the nodes, even for this unique finite-span reconstruction.

Question 1

Let $f(x) = x + x^2$ for $-\pi < x < \pi$, extend with period 2π , and assign $f(\pi + 2k\pi) = \pi^2$. Use

$$S_N = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx), \quad (a_n, b_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)(\cos nx, \sin nx) dx.$$

For piecewise smooth periodic functions, the Fourier series converges to the average of the one-sided limits.

Tasks

1. Separate the even and odd contributions and derive all coefficients, including a_0 .
2. Determine the sum at every point, explaining why the two endpoint limits differ despite periodicity. Sketch f and S_8 on one period.
3. Find the best approximation to f in $\text{span}\{1, \cos x, \sin x\}$ for squared integral error. Prove optimality and uniqueness.
4. Compute that minimum error exactly by orthogonality and direct integration. Explain why changing the assigned endpoint value would not change this approximation.

Question 1 – Solution

Strategy. Use parity to separate coefficients, then use orthogonal projection to measure the combined approximation.

Step 1: Separate the two symmetries. The even part is x^2 and the odd part is x . Odd integrands give zero in the cross terms. Integration by parts gives

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = \frac{2(-1)^{n+1}}{n}.$$

For example, $\int_0^\pi x \sin nx = -\pi(-1)^n/n$ and $\int_0^\pi x^2 \cos nx = 2\pi(-1)^n/n^2$.

Step 2: Identify the periodic join. Inside the base interval the sum is $x + x^2$. At the join, the left limit is $\pi^2 + \pi$ and the right limit is $\pi^2 - \pi$, so the sum is π^2 . Periodicity identifies the location of the join, not its two one-sided limits. The assigned value equals this average; the target still has a jump.

Step 3: Prove the projection claim. Orthogonality gives

$$p(x) = \pi^2/3 - 4\cos x + 2\sin x.$$

The residual $f - p$ is orthogonal to the three basis functions. For any other $q = A + B\cos x + C\sin x$,

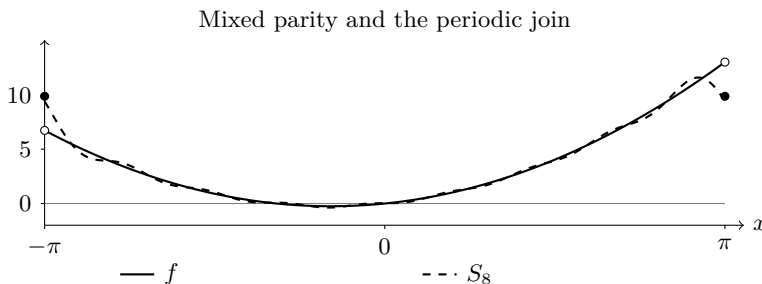
$$\|f - q\|_2^2 = \|f - p\|_2^2 + 2\pi(A - \pi^2/3)^2 + \pi(B + 4)^2 + \pi(C - 2)^2.$$

Thus p is the unique minimizer in this span.

Step 4: Compute the error independently. The odd cross term integrates to zero, giving $\|f\|_2^2 = 2\pi^5/5 + 2\pi^3/3$, while $\|p\|_2^2 = 2\pi^5/9 + 20\pi$. Consequently

$$\min \|f - q\|_2^2 = \frac{8\pi^5}{45} + \frac{2\pi^3}{3} - 20\pi.$$

Changing isolated endpoint values changes neither integrals nor coefficients. It may change whether the assigned values equal the series sums.



Question 2

A 2π -periodic pulse has height one on the circular interval $c - \pi/3 < x < c + \pi/3$ (modulo 2π), height zero elsewhere, and value $1/2$ at its two edges. The unknown center c is defined modulo 2π . Use the real Fourier convention $f = a_0/2 + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$, with coefficients integrated over any full period. The series has the usual one-sided-average value at a jump.

Tasks

1. Derive the mean and all coefficients by centering the integration interval at c .
2. Determine exactly which entire harmonics vanish. Distinguish a missing harmonic from a zero cosine coefficient caused only by phase.
3. Recover c modulo 2π from the signed pair (a_1, b_1) . State an exact consistency condition for a measured pair under this pulse model.
4. Show what is lost if only the harmonic amplitudes $\sqrt{a_n^2 + b_n^2}$ are measured. For $c = \pi/4$, sketch the pulse and its degree-12 partial sum, marking the edge values.

Question 2 – Solution

Strategy. Centering exposes the pulse width; translating back rotates each sine/cosine coefficient pair.

Step 1: Integrate the centered pulse. The area is $2\pi/3$, so the mean is $1/3$ and $a_0 = 2/3$. Put $x = c + t$. The sine terms odd in t integrate to zero, leaving

$$a_n = \frac{2 \sin(n\pi/3)}{\pi n} \cos nc, \quad b_n = \frac{2 \sin(n\pi/3)}{\pi n} \sin nc.$$

These formulas also apply when the pulse crosses the chosen period boundary: integrate instead over a period centered at c .

Step 2: Identify genuine missing modes. The amplitude is

$$R_n = \sqrt{a_n^2 + b_n^2} = \frac{2|\sin(n\pi/3)|}{\pi n}.$$

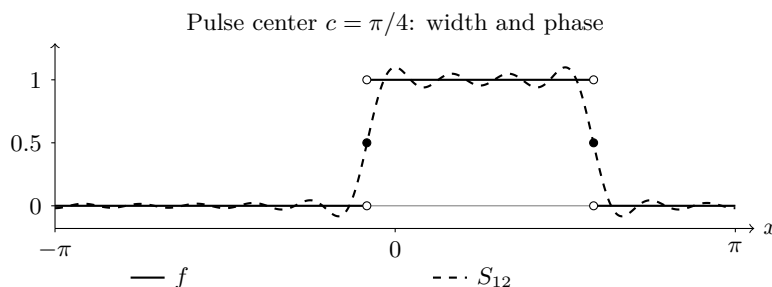
Hence the entire harmonic vanishes exactly when 3 divides n . For other n , a_n alone can vanish because $\cos nc = 0$, while $b_n \neq 0$. Discarding such a sine term would remove a real harmonic.

Step 3: Recover the circular center. For $n = 1$ the prefactor is positive:

$$(a_1, b_1) = \frac{\sqrt{3}}{\pi}(\cos c, \sin c), \quad c = \text{atan2}(b_1, a_1) \pmod{2\pi}.$$

Consistency is exactly $a_1^2 + b_1^2 = 3/\pi^2$ for real measured coefficients. Every pair on that circle corresponds to precisely one center modulo 2π . Using only an ordinary arctangent of b_1/a_1 loses quadrant information.

Step 4: Separate location from amplitude. Every R_n is independent of c , so amplitude measurements leave every center possible, even if all harmonics are known. At $c = \pi/4$, the pulse edges in $[-\pi, \pi]$ are $-\pi/12$ and $7\pi/12$. The sum is one between them, zero outside, and $1/2$ at each edge. The plotted partial sum uses both coefficient families; the finite oscillations do not alter these limiting edge averages.



Question 3

Consider a half-wave rectifier with output $h(x) = \max(\sin x, 0)$, extended with period 2π . Use the mean-term convention $a_0/2$. You may use Parseval:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n \geq 1} (a_n^2 + b_n^2).$$

Tasks

1. Express h using $\sin x$ and $|\sin x|$. Derive its complete real Fourier series.
2. Explain why all sine coefficients except b_1 vanish, and why only even positive cosine frequencies occur.
3. Compute the average squared output and its division into the constant term, the fundamental harmonic, and all higher harmonics.
4. Determine the fundamental periods of h and $|\sin x|$, with a proof that the proposed periods are minimal. Sketch h and its degree-8 Fourier sum.

Question 3 – Solution

Strategy. Rectification combines an odd fundamental with an even function of half the period.

Step 1: Derive the expansion. Since $h = (\sin x + |\sin x|)/2$, its mean is $1/\pi$. Direct integration on $(0, \pi)$ gives $b_1 = 1/2$ and $b_n = 0$ for $n \neq 1$. For $k \geq 1$, product-to-sum integration gives

$$\frac{1}{\pi} \int_0^\pi \sin x \cos(2kx) dx = -\frac{2}{\pi(4k^2 - 1)}.$$

Odd cosine coefficients vanish, including a_1 . Thus

$$h(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 1}.$$

The coefficients are absolutely summable; the standard piecewise smooth Fourier theorem identifies the sum with this continuous target, including its corners.

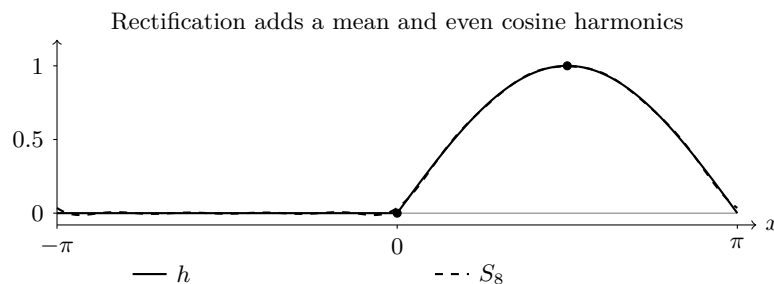
Step 2: Explain the selection rules. The function $|\sin x|$ is even, so contributes only cosines. It is also π -periodic, eliminating odd cosine frequencies in a 2π expansion. The remaining odd part is exactly $\sin x/2$, so it contributes only b_1 . The full rectified waveform is neither even nor odd.

Step 3: Account for the average power. Direct integration gives $(2\pi)^{-1} \int h^2 = 1/4$. Orthogonality separates the nonnegative contributions:

$$P_{\text{constant}} = \frac{1}{\pi^2}, \quad P_{\text{fundamental}} = \frac{1}{8}, \quad P_{\text{higher}} = \frac{1}{8} - \frac{1}{\pi^2}.$$

They sum to $1/4$. The higher-harmonic contribution is positive; the mean squared is only the constant contribution, not the total average squared output.

Step 4: Prove the fundamental periods. The maxima of h occur exactly at $\pi/2 + 2k\pi$. Any positive period must take one maximum to another, hence be a positive multiple of 2π ; 2π works. The maxima of $|\sin x|$ occur at $\pi/2 + k\pi$, so its least positive period is π . Thus the full output has fundamental period 2π despite its even-frequency cosine terms: its nonzero sine fundamental matters.



Question 4

Let $f(x) = e^x$ for $0 < x < 2$ and extend periodically with period 2. At each join assign the average $(1+e^2)/2$. Write

$$f \sim m + \sum_{n \geq 1} (a_n \cos(n\pi x) + b_n \sin(n\pi x)), \quad m = \frac{1}{2} \int_0^2 f, \quad (a_n, b_n) = \int_0^2 f(\cos(n\pi x), \sin(n\pi x)).$$

Set $E = e^2 - 1$.

Tasks

1. Derive m, a_n, b_n . Identify the series sum at the periodic join.
2. Use integration by parts on $f' = f$ inside $(0, 2)$ to obtain two coefficient relations. Explain why omitting the boundary jump would incorrectly force every positive-frequency coefficient to zero.
3. Find a continuous 2-periodic function F with $F' = f - m$ away from the joins and $F(0) = 0$. Prove uniqueness under this normalization.
4. Compute the mean and all Fourier coefficients of F directly. Verify that differentiating its positive-frequency terms formally gives those of $f - m$, and explain why f itself cannot have a continuous periodic primitive.

Question 4 – Solution

Strategy. Keep the endpoint jump in integration by parts, and remove the mean before seeking a periodic primitive.

Step 1: Compute the full coefficients. For $k = n\pi$, exponential-trigonometric antiderivatives at $0, 2$ give

$$m = \frac{E}{2}, \quad a_n = \frac{E}{1+k^2}, \quad b_n = -\frac{kE}{1+k^2}.$$

The Fourier sum at a join is $(1+e^2)/2$, since its two limits are e^2 and 1 . The assigned join value does not enter any integral.

Step 2: Retain the boundary contribution. On the open interval, $f' = f$. Integration by parts gives

$$\int_0^2 f' \cos kx = E + k \int_0^2 f \sin kx, \quad \int_0^2 f' \sin kx = -k \int_0^2 f \cos kx.$$

Thus $a_n = E + kb_n$ and $b_n = -ka_n$, yielding the coefficients above. Dropping E would give $(1+k^2)a_n = 0$ and then $b_n = 0$. The periodic extension has a jump, so it is not a globally smooth periodic solution of $f' = f$.

Step 3: Construct the periodic primitive. Integrating the mean-zero forcing gives, on $[0, 2]$,

$$F(x) = e^x - 1 - \frac{E}{2}x.$$

Both endpoint values are zero, so its periodic extension is continuous. Two such primitives differ by a constant on each open period; continuity connects those constants, and $F(0) = 0$ fixes the remaining freedom.

Step 4: Check the transformed coefficients. Direct integration gives mean $M = -1$. If A_n, B_n denote the coefficients of F , the polynomial terms have zero cosine integral and $\int_0^2 x \sin kx = -2/k$. Consequently

$$A_n = \frac{E}{1+k^2}, \quad B_n = \frac{E}{k(1+k^2)}.$$

Indeed $kB_n = a_n$ and $-kA_n = b_n$. These are coefficient identities; the derivative equation is verified by the closed form on each open period. Finally, a continuous piecewise smooth periodic primitive of f would have net change $\int_0^2 f = E \neq 0$, which is impossible. Subtracting the mean is essential.

Question 5

A change of origin replaces $f(x)$ by $f(t + c)$, where c is defined modulo 2π . Consider

$$f(x) = 2 \cos x + 2 \sin x + \cos 2x.$$

Tasks

1. Derive how a general coefficient pair (a_n, b_n) changes under $x = t + c$. Prove that its amplitude is unchanged.
2. Determine every origin c for which $f(t + c)$ is even, or prove that none exists.
3. Determine every origin c for which $f(t + c)$ is odd. Give the shifted expansion for one such origin.
4. Replace $\cos 2x$ by $\sin 2x$ to obtain g . Find all origins making $g(t + c)$ even and give one resulting cosine-only expansion. Explain why separately aligning each harmonic is not a valid test for a single common symmetry center.

Question 5 – Solution

Strategy. A common origin must satisfy the phase conditions of every nonzero harmonic simultaneously.

Step 1: Derive the coefficient rotation. Angle addition gives the new coefficients

$$a'_n = a_n \cos nc + b_n \sin nc, \quad b'_n = -a_n \sin nc + b_n \cos nc.$$

Squaring and adding cancels the cross terms, so $(a'_n)^2 + (b'_n)^2 = a_n^2 + b_n^2$. For a trigonometric polynomial, evenness is equivalent to every b'_n being zero; oddness requires a zero mean and every a'_n zero. Necessity follows from orthogonality, and sufficiency follows from sine/cosine parity.

Step 2: Test all evenness conditions. At frequency one, $b'_1 = 2(\cos c - \sin c) = 0$ implies $c = \pi/4 + k\pi$. At frequency two, $b'_2 = -\sin 2c$ would then equal -1 . Hence

$$\text{No origin makes } f(t+c) \text{ even.}$$

The first frequency can be aligned, but that shift makes the second purely sine.

Step 3: Find every oddness origin. The first condition $a'_1 = 2(\cos c + \sin c) = 0$ gives $c = 3\pi/4 + k\pi$. At each of these, $a'_2 = \cos 2c = 0$, and the mean is already zero. These are all solutions. At $c = 3\pi/4$,

$$f(t + 3\pi/4) = -2\sqrt{2} \sin t + \sin 2t.$$

Modulo 2π the two origins are $3\pi/4$ and $7\pi/4$.

Step 4: Change the relative harmonic phase. For $g = 2 \cos x + 2 \sin x + \sin 2x$, the first-frequency evenness condition is again $c = \pi/4 + k\pi$. Now the second is $b'_2 = \cos 2c = 0$, which is satisfied. At $c = \pi/4$,

$$g(t + \pi/4) = 2\sqrt{2} \cos t + \cos 2t.$$

All evenness origins are $\pi/4 + k\pi$. Each harmonic admits its own phase alignment, but a symmetry of the full function requires the same c for all frequencies. The two examples have identical harmonic amplitudes and different answers to the evenness question.

Question 6

Find the continuous 2π -periodic steady response of

$$y' + 2y = 1 + q(x), \quad q(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

where q is periodically extended and assigned zero at its jumps. The differential equation is required on the open intervals between jumps. Solutions are continuous and piecewise continuously differentiable.

Tasks

1. Derive the full Fourier coefficients of q , including its mean.
2. Obtain the mean and all cosine and sine coefficients of the periodic response by mode equations.
3. Independently solve on the two intervals and enforce continuity and periodicity. Prove that this response is unique.
4. Verify its mean, extrema and behavior at the switches. Sketch the exact response and its degree-9 Fourier approximation. Explain why a sine-only forcing produces cosine terms in the response.

Question 6 – Solution

Strategy. Solve the harmonic response and the piecewise differential equation independently.

Step 1: Expand the input. The periodic extension of q is odd. Direct integration gives zero mean and zero cosine coefficients, with

$$b_n^{(q)} = \frac{2(1 - (-1)^n)}{\pi n} = \begin{cases} 4/(\pi n), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

The constant in $1 + q$ is one.

Step 2: Solve the coupled mode equations. Write $y = m + \sum (A_n \cos nx + B_n \sin nx)$. Integration by parts is legitimate piecewise: continuity and periodicity cancel the internal and endpoint terms. Thus $2m = 1$, $nB_n + 2A_n = 0$, and $-nA_n + 2B_n = b_n^{(q)}$. Hence

$$m = \frac{1}{2}, \quad A_n = -\frac{4}{\pi(n^2 + 4)}, \quad B_n = \frac{8}{\pi n(n^2 + 4)} \quad (n \text{ odd}),$$

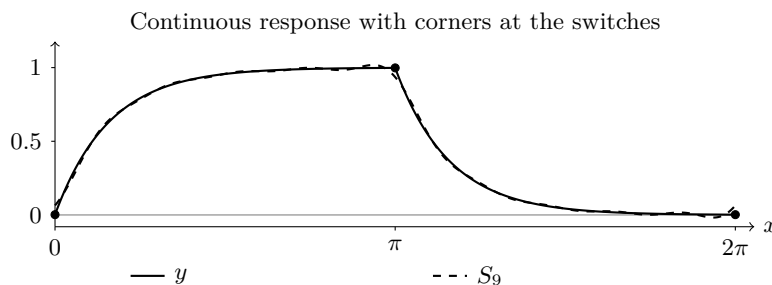
and both coefficients vanish for even n .

Step 3: Verify a closed-form periodic response. Put $r = e^{-2\pi}$. Solving the constant-forcing equations and matching gives

$$y(x) = \begin{cases} 1 - \frac{e^{-2x}}{1+r}, & 0 \leq x \leq \pi, \\ \frac{e^{-2(x-\pi)}}{1+r}, & \pi \leq x \leq 2\pi. \end{cases}$$

Both expressions give $y(\pi) = 1/(1+r)$, and $y(0) = y(2\pi) = r/(1+r)$. A difference of two responses is Ce^{-2x} across the whole period by continuity; periodicity forces $C = 0$. The verified closed form therefore has the coefficients determined above.

Step 4: Interpret the response. The first branch increases and the second decreases, so the extrema are $r/(1+r)$ and $1/(1+r)$. Integrating the equation over a period gives mean $1/2$. The response is continuous, but y' jumps by -2 at π and by 2 at 0 modulo 2π ; the assigned input values do not require a derivative there. Differentiation couples sine and cosine coefficients, creating a phase shift. Absolute coefficient summability and the Fourier theorem identify the plotted series with the continuous response.



Question 7

For real parameters α, β , seek a twice continuously differentiable 2π -periodic solution of

$$y'' + 4y = 3 + \cos x + \alpha \cos 2x + \beta \sin 2x.$$

Tasks

1. Derive necessary conditions on α, β by multiplying by each resonant mode and integrating over one period.
2. Prove those conditions are sufficient and find all periodic solutions when they hold.
3. Under the same conditions, impose $y(0) = y'(0) = 0$. Find the resulting solution and verify all conditions directly.
4. Explain why initial conditions cannot restore periodic solvability when a resonant forcing coefficient is nonzero. Relate this to the failed division by $4 - n^2$ in the Fourier equations.

Question 7 – Solution

Strategy. Test the forcing against the periodic homogeneous modes before dividing by a spectral factor.

Step 1: Derive the compatibility conditions. For a periodic C^2 function, both y and y' match at the period endpoints. Two integrations by parts therefore give

$$\int_0^{2\pi} (y'' + 4y) \cos 2x = 0, \quad \int_0^{2\pi} (y'' + 4y) \sin 2x = 0.$$

Orthogonality makes the corresponding right sides $\pi\alpha$ and $\pi\beta$. Thus

$$\alpha = \beta = 0$$

is necessary. The constant and first harmonic do not contribute to these tests.

Step 2: Construct every compatible solution. With these parameters zero, a particular solution is $3/4 + \cos x/3$. The homogeneous equation has basis $\cos 2x, \sin 2x$, both periodic. Thus

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x + A \cos 2x + B \sin 2x, \quad A, B \in \mathbb{R}.$$

This proves sufficiency as well as completeness: subtract the particular solution from any solution and solve the homogeneous equation.

Step 3: Enforce the additional normalization. The initial values give $3/4 + 1/3 + A = 0$ and $2B = 0$. Hence

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x - \frac{13}{12} \cos 2x.$$

It has $y(0) = 0$, $y'(0) = 0$, and is 2π -periodic together with all derivatives. Substitution gives $y'' + 4y = 3 + \cos x$ because the frequency-two term is annihilated.

Step 4: Identify the resonance obstruction. For frequency n , the mode equation multiplies each coefficient of y by $4 - n^2$. At $n = 2$ it reads $0 = \alpha$ or $0 = \beta$: it is a compatibility condition, not a formula for an unknown coefficient. When it holds, the two coefficients are free until extra conditions are supplied. When it fails, no periodic C^2 solution exists. Although an initial-value problem still has a unique solution, its resonant response cannot be periodic; altering homogeneous constants does not change either integral obstruction.

Question 8

Let $f(x) = 1 + \cos x + 2 \sin x + \cos 2x$. Its periodic autocorrelation is

$$R(\tau) = \frac{1}{2\pi} \int_0^{2\pi} f(x)f(x+\tau) dx.$$

Tasks

1. Derive R using orthogonality. Explain why sine terms disappear from the correlation even though f has a nonzero sine coefficient.
2. Find the exact maximum and minimum of R , including all shifts attaining them modulo 2π .
3. Construct a real trigonometric polynomial g with the same mean and the same R which is not a translate of f . Prove both claims.
4. Derive the identity relating R to the squared distance between f and its translate. Determine every shift leaving f unchanged and connect this with the fundamental period.

Question 8 – Solution

Strategy. Correlation retains harmonic energies but discards their individual phases.

Step 1: Evaluate the correlation. Different frequencies are orthogonal. At frequency n , a pair $a_n \cos nx + b_n \sin nx$ contributes $(a_n^2 + b_n^2) \cos(n\tau)/2$; the two mixed sine terms cancel. The constant contributes the square of the mean. Thus

$$R(\tau) = 1 + \frac{5}{2} \cos \tau + \frac{1}{2} \cos 2\tau.$$

In particular, $R(0) = 4$ is the mean square of f , not the square of its mean. The correlation is even regardless of the parity of f .

Step 2: Optimize over all shifts. Put $t = \cos \tau \in [-1, 1]$. Then $R = t^2 + (5/2)t + 1/2$, whose derivative $2t + 5/2$ is positive throughout this interval. Hence

$$\max R = 4 \quad (\tau = 0 \bmod 2\pi), \quad \min R = -1 \quad (\tau = \pi \bmod 2\pi).$$

A correlation can be negative here; the original signal is not required to be nonnegative.

Step 3: Exhibit phase ambiguity beyond translation. Take $g(x) = 1 + \sqrt{5} \cos x + \cos 2x$. Its mean is one and its harmonic squared amplitudes are 5 and 1, so it gives exactly the same R . If $g(x) = f(x + c)$, equality at frequency two requires $\cos 2c = 1$ and $\sin 2c = 0$, so $c = k\pi$. But then the first sine coefficient of $f(x + c)$ is $2(-1)^k \neq 0$, unlike g . Thus g is not a translate. Correlation does not determine relative harmonic phases.

Step 4: Recover the exact translation invariance. Translation preserves $\int f^2$. Expanding the squared difference gives

$$\frac{1}{2\pi} \int_0^{2\pi} [f(x + \tau) - f(x)]^2 dx = 2(R(0) - R(\tau)).$$

This is zero precisely at $\tau = 0$ modulo 2π . Zero integral of a continuous nonnegative square means equality at every point. Therefore the invariant shifts are exactly $2k\pi$, and the fundamental period is 2π .

Question 9

For real a, b , a modulated waveform is

$$s_{a,b}(x) = (1 + a \cos x) \cos 3x + b \sin x \sin 3x.$$

Use a 2π Fourier expansion, and define average power as $(2\pi)^{-1} \int_0^{2\pi} s_{a,b}^2$.

Tasks

1. Derive the complete Fourier spectrum using product identities. Identify the frequencies surrounding the carrier frequency 3.
2. Recover a, b from measured cosine coefficients C_2, C_4 . Determine all choices canceling the frequency-4 term while leaving a prescribed frequency-2 coefficient d .
3. For $d = 1/2$ with frequency 4 canceled, find the fundamental period. Prove minimality even though the frequency-1 coefficient is zero.
4. Compute the average power for general a, b , then for this special case. Prove a sharp global bound on the special waveform's absolute value and identify points where equality holds.

Question 9 – Solution

Strategy. Resolve the products into actual harmonics before inferring frequency content, period or energy.

Step 1: Resolve the two products. The identities $\cos x \cos 3x = (\cos 2x + \cos 4x)/2$ and $\sin x \sin 3x = (\cos 2x - \cos 4x)/2$ give

$$s_{a,b}(x) = \cos 3x + \frac{a+b}{2} \cos 2x + \frac{a-b}{2} \cos 4x.$$

The mean and every sine coefficient are zero. The neighboring frequencies, often called sidebands, are 2 and 4, not frequency 1 from the modulation alone.

Step 2: Invert the coefficient measurements. The linear equations give

$$a = C_2 + C_4, \quad b = C_2 - C_4.$$

Requiring $C_4 = 0$ and $C_2 = d$ fixes $a = b = d$ uniquely. The two modulation terms reinforce the lower frequency and cancel the upper frequency when their amplitudes agree.

Step 3: Establish the least period. For $d = 1/2$ the signal is $s = \cos 3x + \frac{1}{2} \cos 2x$. If $T > 0$ is a period, translating this finite expansion and using orthogonality forces $3T$ and $2T$ each to be integer multiples of 2π . Subtracting gives T an integer multiple of 2π . Conversely 2π works. Thus the fundamental period is $\boxed{2\pi}$. A nonzero first harmonic is sufficient for this period in a 2π polynomial, but is not necessary: frequencies 2 and 3 already enforce it.

Step 4: Measure power and peak amplitude. Orthogonality gives

$$P = \frac{1}{2} \left(1 + \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4} \right) = \frac{1}{2} + \frac{a^2 + b^2}{4}.$$

At $a = b = 1/2$, $P = 5/8$. Also $|s| \leq |\cos 3x| + \frac{1}{2} |\cos 2x| \leq 3/2$, with equality at $x = 2k\pi$. For negative equality both cosines would have to be -1 ; the equations $3x = (2j+1)\pi$, $2x = (2k+1)\pi$ are incompatible by parity. Positive equality requires both to be 1, giving exactly $x = 2k\pi$.

Question 10

Two periodic signals are $f(x) = \cos x + \cos 2x$ and $g(x) = \sin x + \cos 2x$. To align them by one delay, minimize

$$D(\delta) = \int_0^{2\pi} [f(x - \delta) - g(x)]^2 dx, \quad \delta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Tasks

1. Use orthogonality to derive $D(\delta)$ without expanding a long pointwise square.
2. Determine all globally optimal delays and the exact minimum error. Prove global optimality.
3. Compare delays 0 and $\pi/2$. Explain why exactly aligning the first harmonic need not give the best alignment of the full signal.
4. Decide whether any delay makes the two functions identical. Sketch $D(\delta)/\pi$ over one period, marking every minimizer, and explain the nonuniqueness of the best delay.

Question 10 – Solution

Strategy. One delay rotates each harmonic by a different multiple of the same angle, creating a coupled optimization.

Step 1: Compute distance through correlation. Both signals have squared norm 2π . Angle addition and orthogonality give

$$\int_0^{2\pi} f(x-\delta)g(x) dx = \pi(\sin \delta + \cos 2\delta).$$

Therefore

$$\frac{D(\delta)}{\pi} = 4 - 2(\sin \delta + \cos 2\delta).$$

The sine sign follows from $\cos(x-\delta) = \cos x \cos \delta + \sin x \sin \delta$.

Step 2: Optimize over the whole circle. Put $t = \sin \delta \in [-1, 1]$. Since $\cos 2\delta = 1 - 2t^2$,

$$\frac{D(\delta)}{\pi} = \frac{7}{4} + 4 \left(t - \frac{1}{4} \right)^2.$$

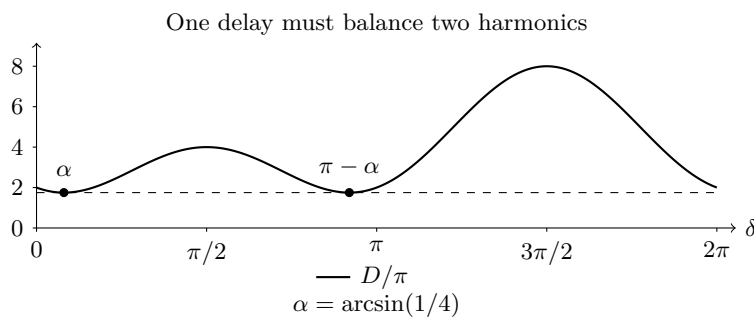
The feasible value $t = 1/4$ is the unique minimizing sine value. Consequently

$$D_{\min} = \frac{7\pi}{4}, \quad \delta = \arcsin(1/4) \text{ or } \pi - \arcsin(1/4) \pmod{2\pi}.$$

Completing the square proves these are all global minimizers.

Step 3: Compare two tempting alignments. At $\delta = 0$, the second harmonics agree and $D = 2\pi$. At $\delta = \pi/2$, the first harmonic matches $\sin x$, but the second becomes $-\cos 2x$, so $D = 4\pi$. Exactly matching one harmonic can worsen the full error. The optimum balances their contributions and improves on both choices.

Step 4: Interpret the two minima. The minimum is strictly positive, so no delay makes the signals identical. The objective depends only on $\sin \delta$; the two distinct delays with sine $1/4$ give the same compromise. They do not produce the same shifted function: their first cosine coefficients are opposite and nonzero. Thus nonuniqueness of the best delay does not imply exact agreement or identical aligned waveforms. The graph shows both minima in $[0, 2\pi)$.



Question 1

Define a 2π -periodic function by $f = -1$ on $(-\pi, 0)$, $f = 2$ on $(0, \pi)$, $f(0) = 7$ and $f(\pi) = -4$. Let S_N be its real Fourier partial sum with mean $a_0/2$. Use the piecewise smooth Fourier theorem (the sum is the average of the one-sided limits) and Parseval. Here $\|u\|_2^2 = \int_{-\pi}^{\pi} |u|^2$.

Tasks

1. Derive the mean and all coefficients. Determine the pointwise series sum everywhere, including both types of assigned exceptional values.
2. Compute $\|f - S_N\|_2^2$ as a coefficient tail and prove it tends to zero.
3. Prove that S_N does not converge uniformly to f . Also prove a lower bound of $3/2$ for the essential supremum error, which ignores sets of measure zero.
4. Decide whether redefining only the jump values can restore uniform convergence. Sketch f and S_9 , displaying the assigned values separately from the open one-sided limits.

Question 1 – Solution

Strategy. Distinguish the assigned function, its Fourier pointwise limit and the equivalence class measured by an integral norm.

Step 1: Compute coefficients and limits. Apart from isolated values, $f = 1/2 + (3/2) \operatorname{sgn}(\sin x)$. Thus $a_0 = 1$, every positive cosine coefficient is zero, and

$$b_n = \frac{3(1 - (-1)^n)}{\pi n} = \begin{cases} 6/(\pi n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The series sum equals -1 or 2 on the open intervals and $1/2$ at every jump. It does not equal either assigned value 7 or -4 there.

Step 2: Verify mean-square convergence. Isolated values do not affect integrals. Parseval and orthogonality give

$$\|f - S_N\|_2^2 = \frac{36}{\pi} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^2} \leq \frac{36}{\pi N} \rightarrow 0.$$

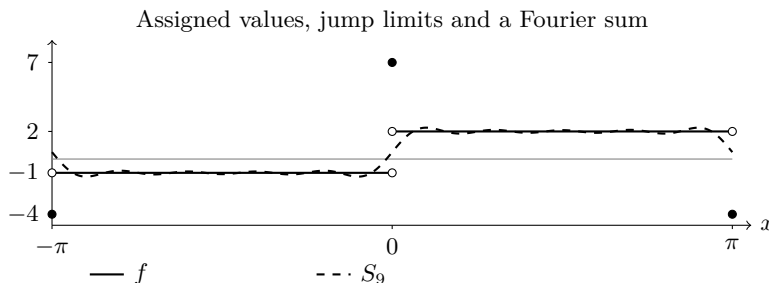
The coefficient tail is square summable despite failure at the assigned jump values.

Step 3: Separate point and essential errors. Every $S_N(0) = 1/2$, so $\|f - S_N\|_\infty \geq 13/2$ for every N . Moreover, by continuity of S_N , as $x \downarrow 0$ its error against the right-hand value 2 tends to $3/2$. For any $\varepsilon > 0$, that error exceeds $3/2 - \varepsilon$ on an interval of positive length. Hence

$$\operatorname{ess\,sup} |f - S_N| \geq 3/2.$$

Removing isolated exceptional values cannot remove this obstruction.

Step 4: Test possible redefinitions. The one-sided limits remain -1 and 2 after any change only at the jumps. The resulting target is still discontinuous. A uniform limit of the continuous trigonometric polynomials S_N would be continuous, so no such redefinition restores uniform convergence. Mean-square convergence and the open-interval pointwise limits are unchanged. Filled points in the graph show the assigned values.



Question 2

For integers $m \geq 1$, let

$$h_m(x) = \max(1 - m|x|, 0), \quad -\pi \leq x \leq \pi,$$

and extend periodically. Write $S_N^{(m)}$ for the degree- N Fourier partial sum of this particular function. The target changes with m ; these are not partial sums of one fixed function.

Tasks

1. Compute $\|h_m\|_\infty$, $\|h_m\|_2^2$ and the pointwise limit as $m \rightarrow \infty$. Distinguish mean-square convergence to zero from pointwise convergence to zero.
2. Derive the mean and all Fourier coefficients of h_m .
3. For each fixed m , prove $S_N^{(m)} \rightarrow h_m$ uniformly as $N \rightarrow \infty$. State a bound whose dependence on m is explicit.
4. Evaluate both iterated limits at $x = 0$, first letting N tend to infinity and then reversing the order. Explain why they differ, and sketch h_1, h_3, h_9 near the origin.

Question 2 – Solution

Strategy. A shrinking support can make integral errors small while retaining a fixed peak.

Step 1: Compare the modes of convergence. The peak is always one and direct integration gives

$$\|h_m\|_\infty = 1, \quad \|h_m\|_2^2 = 2 \int_0^{1/m} (1 - mx)^2 dx = \frac{2}{3m}.$$

The pointwise limit is one at 0 modulo 2π and zero elsewhere. Thus $h_m \rightarrow 0$ in mean square and almost everywhere, but not at every point and not uniformly. The pointwise limit has a single-point spike in each period.

Step 2: Compute the spectrum. The function is even, so $b_n = 0$. Its area is $1/m$, giving mean $1/(2\pi m)$. Integration over its support gives

$$a_n = \frac{2}{\pi} \int_0^{1/m} (1 - mx) \cos nx dx = \frac{2m}{\pi n^2} (1 - \cos(n/m)).$$

For each fixed n , $0 \leq a_n \leq 1/(\pi m)$ by $1 - \cos u \leq u^2/2$. Therefore every fixed coefficient tends to zero as $m \rightarrow \infty$.

Step 3: Keep the target fixed when taking a Fourier limit. For fixed m , $|a_n| \leq 4m/(\pi n^2)$, so the series converges absolutely and uniformly. The piecewise smooth Fourier theorem identifies its sum with the continuous target h_m . In particular,

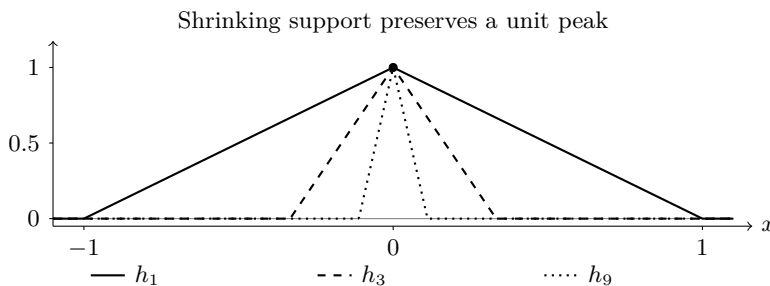
$$\|h_m - S_N^{(m)}\|_\infty \leq \frac{4m}{\pi N}.$$

The factor m prevents using this bound uniformly over all targets.

Step 4: Compare the iterated limits. For fixed m , the inner N -limit at zero is $h_m(0) = 1$. For fixed N , the mean and finitely many coefficients all tend to zero. Thus

$$\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty} S_N^{(m)}(0) = 1, \quad \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} S_N^{(m)}(0) = 0.$$

Fixed-mode convergence does not control the collective contribution of modes whose number grows with m . There is no common uniform estimate permitting this interchange.



Question 3

You may use the Fourier identity

$$F(x) = \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad -\pi \leq x \leq \pi,$$

extended periodically. Let $P_N = \sum_{n=1}^N \cos(nx)/n^2$, $N \geq 1$.

Tasks

1. Prove uniform convergence and find the exact value of $\|F - P_N\|_{\infty}$ as a positive numerical series.
2. Determine all points where this maximum absolute error occurs, modulo 2π .
3. Prove strict two-sided bounds using integrals and determine the smallest integer N for which the uniform error is less than $1/100$.
4. Find the exact squared mean-square error as a series and compare its decay rate with that of the uniform error. Sketch $F - P_5$ and its upper and lower uniform-error levels.

Question 3 – Solution

Strategy. Positive coefficients make the usual triangle bound attain its value at a common phase.

Step 1: Turn the tail estimate into an equality. The Weierstrass test applies because $\sum n^{-2} < \infty$. Put $T_N = \sum_{n>N} n^{-2}$. For every x , $|F(x) - P_N(x)| \leq T_N$, while at $x = 0$ every cosine equals one. Hence

$$\|F - P_N\|_\infty = T_N.$$

This is an exact error, not merely a sufficient upper bound.

Step 2: Identify every equality point. For the tail to equal T_N , every positive summand must have $\cos nx = 1$. Using two consecutive integers $n > N$ forces $x = 0$ modulo 2π . For the tail to equal $-T_N$, every $\cos nx$ would have to be -1 . But then $\cos(2nx) = 1$ for a frequency also in the tail, a contradiction. Thus the maximum absolute error occurs exactly at $x = 0$ modulo 2π .

Step 3: Certify the minimal truncation. Strict monotonicity of t^{-2} gives

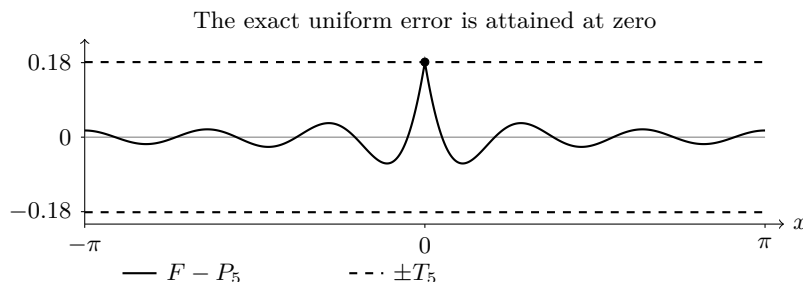
$$\frac{1}{N+1} < T_N < \frac{1}{N}.$$

At $N = 100$, $T_N < 1/100$. At $N = 99$, $T_N > 1/100$; monotonicity excludes every smaller N . Therefore the exact smallest choice is $N = 100$. The two integral bounds straddle the threshold tightly enough to decide it.

Step 4: Compare the two norms. Orthogonality gives

$$\|F - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^4}, \quad \frac{\pi}{3(N+1)^3} < \|F - P_N\|_2^2 < \frac{\pi}{3N^3}.$$

The uniform error is of order N^{-1} , whereas the L^2 error is of order $N^{-3/2}$ (its square is of order N^{-3}). Distinguish the norm from its square.



Question 4

Let $G(x) = (\pi - x)/2$ on $0 < x < 2\pi$, extended periodically with value zero at the joins. Its Fourier coefficients give

$$G(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi).$$

This identity follows from the piecewise smooth Fourier theorem. Set $P_N = \sum_{n=1}^N \sin(nx)/n$ and fix $0 < \delta < \pi$.

Tasks

1. Use a finite geometric sum to bound $\left| \sum_{n=p}^q \sin nx \right|$ uniformly in p, q for $x \in [\delta, 2\pi - \delta]$.
2. Apply summation by parts to prove $\|G - P_N\|_{\infty, [\delta, 2\pi - \delta]} \leq 1/((N+1)\sin(\delta/2))$.
3. Prove that convergence is not uniform even on the full open interval $(0, 2\pi)$. Explain why the preceding bound cannot be used with $\delta = 0$.
4. Compute the squared $L^2(0, 2\pi)$ error and bound it. Sketch the error for $N = 8$, marking the endpoint limits separately from the assigned endpoint errors.

Question 4 – Solution

Strategy. Cancellation controls a tail away from a jump, but the cancellation bound degenerates near the join.

Step 1: Bound finite oscillatory blocks. For $x \notin 2\pi\mathbb{Z}$,

$$\left| \sum_{n=p}^q e^{inx} \right| = \left| \frac{e^{ipx}(1 - e^{i(q-p+1)x})}{1 - e^{ix}} \right| \leq \frac{1}{|\sin(x/2)|}.$$

The same bound holds for the imaginary part. On the specified closed interval, it is at most $B = 1/\sin(\delta/2)$.

Step 2: Sum by parts with a decreasing weight. Put $A_k = \sum_{n=N+1}^k \sin nx$, so $|A_k| \leq B$. Then

$$\sum_{n=N+1}^M \frac{\sin nx}{n} = \frac{A_M}{M} + \sum_{k=N+1}^{M-1} A_k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

The absolute value is bounded by $B/(N+1)$. Passing to the convergent tail gives

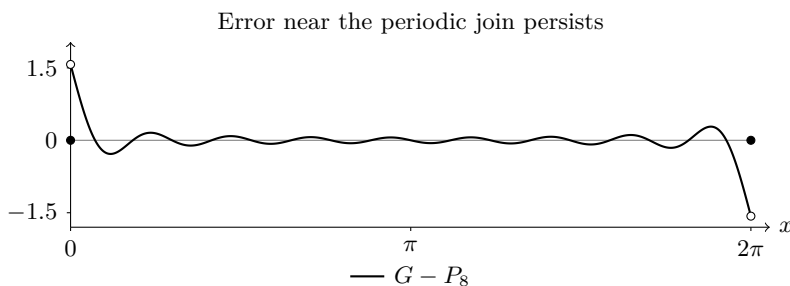
$$\sup_{\delta \leq x \leq 2\pi - \delta} |G - P_N| \leq \frac{1}{(N+1)\sin(\delta/2)} \rightarrow 0.$$

Step 3: Locate the global obstruction. For every fixed N , $P_N(x) \rightarrow 0$ as $x \downarrow 0$, while $G(x) \rightarrow \pi/2$. Therefore $\sup_{0 < x < 2\pi} |G - P_N| \geq \pi/2$ for every N . Excluding the endpoint itself does not exclude points arbitrarily close to it. At $\delta = 0$ the geometric-sum bound has a zero denominator and gives no finite uniform control.

Step 4: Verify convergence in the integral norm. The sine functions have squared norm π over this period. Hence

$$\|G - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^2} < \frac{\pi}{N}.$$

The error tends to zero in mean square despite persistent near-jump suprema. Its assigned endpoint errors are zero, whereas its interior limits are $\pi/2$ at 0 and $-\pi/2$ at 2π .



Question 5

Consider

$$H(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

You may use the theorem: if a series of C^1 functions converges at one point and its derivative series converges uniformly, its sum is C^1 and may be differentiated term by term. Also use

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}, \quad 0 \leq x \leq 2\pi.$$

Tasks

1. Verify every hypothesis needed to differentiate the series once on $[0, 2\pi]$.
2. Integrate the given derivative formula to obtain a closed form for H . Check the endpoint values and slopes of its periodic extension.
3. Determine whether the periodic extension is C^2 . Identify precisely where the formal second derivative series agrees with the classical second derivative.
4. Prove that the second derivative partial sums cannot converge uniformly to the interior second derivative on $(0, 2\pi)$. Explain why the first differentiation remains valid.

Question 5 – Solution

Strategy. Check a separate convergence condition for each differentiation; one valid differentiation does not authorize the next.

Step 1: Justify the first differentiation. Every summand is C^1 . The original series converges at 0 (all terms are zero), and the derivative terms satisfy $|\cos nx/n^2| \leq 1/n^2$. The Weierstrass test gives uniform convergence of the derivative series. The stated theorem therefore gives

$$H'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

This includes one-sided endpoint derivatives on the closed interval.

Step 2: Verify the explicit function. Integrating the provided formula and using $H(0) = 0$ gives

$$H(x) = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}, \quad 0 \leq x \leq 2\pi.$$

Both endpoint values are zero. Both endpoint slopes equal $\pi^2/6$, so the periodic extension is C^1 across the join.

Step 3: Identify the second derivative obstruction. Inside the period, $H''(x) = -\pi/2 + x/2$. Its right limit at zero is $-\pi/2$, whereas its left periodic limit is $\pi/2$. Thus the periodic function is not twice differentiable at the join and is not C^2 . The formal second derivative series is $-\sum \sin(nx)/n$. The piecewise smooth Fourier theorem, or direct coefficients of the displayed linear function, gives H'' on $0 < x < 2\pi$. At a join the series equals zero, the average of the limits, where the classical periodic second derivative does not exist.

Step 4: Rule out uniform second differentiation. Every finite sum $-\sum_{n=1}^N \sin(nx)/n$ tends to zero as $x \downarrow 0$, but the interior target tends to $-\pi/2$. Thus its error supremum on the full open period is at least $\pi/2$ for every N . The derivative series for the first differentiation had summable bounds n^{-2} ; the next series has no such global uniform convergence. There is no contradiction between a valid first differentiation and failure of the stronger second-derivative conclusion.

Question 6

For $N \geq 1$, consider real trigonometric polynomials $p(x) = A_0 + \sum_{n=1}^N (A_n \cos nx + B_n \sin nx)$ on $[-\pi, \pi]$. Use $\|p\|_2^2 = \int_{-\pi}^{\pi} p^2$ and define $D_N(x) = 1 + 2 \sum_{n=1}^N \cos nx$.

Tasks

1. Prove $\|p\|_{\infty} \leq \sqrt{(2N+1)/(2\pi)} \|p\|_2$ by an orthonormal-basis argument.
2. Prove the constant is sharp and characterize all nonzero equality cases at a prescribed point x_0 .
3. For $p_N = D_N/(2N+1)$, compute both norms exactly and determine its pointwise limit. Explain what this says about point evaluation under mean-square convergence.
4. Derive the sine-quotient form of D_N and sketch p_2, p_8 over a period. Explain why this sequence is not the sequence of Fourier partial sums of one fixed function.

Question 6 – Solution

Strategy. Point evaluation on a finite-dimensional space has a norm that grows with the number of available modes.

Step 1: Apply Cauchy–Schwarz in coefficient space. An orthonormal basis is $1/\sqrt{2\pi}$, $\cos nx/\sqrt{\pi}$, $\sin nx/\sqrt{\pi}$. The sum of their squared values at any point is

$$\frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^N (\cos^2 nx + \sin^2 nx) = \frac{2N+1}{2\pi}.$$

Cauchy–Schwarz against the orthonormal coefficient vector proves

$$\|p\|_{\infty} \leq \sqrt{\frac{2N+1}{2\pi}} \|p\|_2.$$

Step 2: Identify equality cases. Equality at x_0 holds exactly when the coefficient vector is a real nonzero multiple of the basis-value vector at x_0 . Equivalently,

$$p(x) = CD_N(x - x_0), \quad C \in \mathbb{R} \setminus \{0\}.$$

This exhibits the sharp constant, not merely an upper estimate.

Step 3: Build a vanishing-energy sequence with a fixed peak. The triangle inequality and the value at zero give $\|D_N\|_{\infty} = 2N+1$. Orthogonality gives $\|D_N\|_2^2 = 2\pi(2N+1)$. Consequently

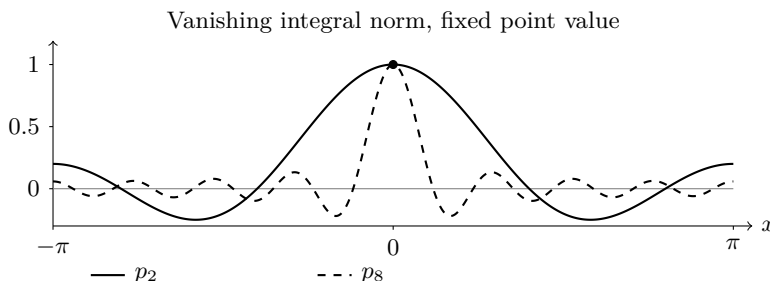
$$\|p_N\|_{\infty} = 1, \quad \|p_N\|_2^2 = \frac{2\pi}{2N+1} \rightarrow 0.$$

At 0 modulo 2π , $p_N = 1$. At every other fixed point it tends to zero by the quotient below. Point evaluation is therefore not continuous in the L^2 norm on the union of these polynomial spaces.

Step 4: Explain the oscillatory peak. Summing $\sum_{n=-N}^N e^{inx}$ yields

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)} \quad (x \notin 2\pi\mathbb{Z}).$$

The removable value at zero is $2N+1$. The normalized peak narrows with N . These p_N cannot be successive partial sums of one fixed function: their constant coefficients $1/(2N+1)$ change with N , whereas truncation preserves every already-included Fourier coefficient.



Question 7

For real bounded piecewise continuous 2π -periodic f , its degree-one Fourier projection is

$$(S_1 f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)(1+2\cos t) dt.$$

Use the essential supremum norm, which ignores changes on sets of measure zero.

Tasks

1. Prove $\|S_1 f\|_{\infty} \leq \Lambda \|f\|_{\infty}$, where $\Lambda = (2\pi)^{-1} \int_{-\pi}^{\pi} |1+2\cos t| dt$.
2. Compute Λ exactly, locating every sign change of the kernel.
3. Construct an admissible f with $\|f\|_{\infty} = 1$ for which equality is attained at $x = 0$. Verify its mean and first cosine coefficient directly.
4. Use the result to give a sharp worst-case amplification factor for a bounded input error under S_1 . Explain why orthogonal projection being a contraction in L^2 does not imply contraction in the supremum norm.

Question 7 – Solution

Strategy. The signed Fourier kernel controls integral projection, while its absolute integral controls worst-case pointwise error.

Step 1: Bound the projection in the supremum norm. Taking absolute values in the given integral gives

$$|S_1 f(x)| \leq \frac{\|f\|_\infty}{2\pi} \int_{-\pi}^{\pi} |1 + 2 \cos t| dt.$$

Null-set changes do not affect this estimate or the projected polynomial. Taking the supremum over x proves the claimed inequality.

Step 2: Integrate the absolute kernel. The kernel is nonnegative for $|t| \leq 2\pi/3$ and negative outside. Its full signed integral is 2π . Its integral on the two negative intervals is

$$2 \int_{2\pi/3}^{\pi} (1 + 2 \cos t) dt = \frac{2\pi}{3} - 2\sqrt{3}.$$

Changing the sign of that contribution gives

$$\Lambda = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} > 1.$$

Step 3: Attain the bound with an explicit input. Choose $f(t) = \operatorname{sgn}(1 + 2 \cos t)$, with value zero at its two zeros. It is even and piecewise constant with essential supremum one. Its positive region has length $4\pi/3$, so its mean is $1/3$. Direct integration gives

$$a_1 = \frac{2}{\pi} \left(\int_0^{2\pi/3} \cos t dt - \int_{2\pi/3}^{\pi} \cos t dt \right) = \frac{2\sqrt{3}}{\pi}, \quad b_1 = 0.$$

Hence $S_1 f(0) = 1/3 + 2\sqrt{3}/\pi = \Lambda$, proving sharpness.

Step 4: Interpret the stability statement. For input error e with $\|e\|_\infty \leq \varepsilon$, linearity gives $\|S_1(f + e) - S_1 f\|_\infty \leq \Lambda \varepsilon$. The choice $e = \varepsilon \operatorname{sgn}(1 + 2 \cos t)$ attains this bound. Orthogonal projection still satisfies $\|S_1 e\|_2 \leq \|e\|_2$ by the Pythagorean identity. That statement uses an integral norm, and does not prevent a larger pointwise peak. Contraction is a property of a specified norm.

Question 8

Let real coefficient sequences satisfy

$$\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \leq M, \quad M > 0.$$

Define the formal series $f = m + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$ and let S_N be its degree- N truncation, $N \geq 1$. Write $T_N = \sum_{n > N} n^{-2}$.

Tasks

1. Prove that the series defines a continuous periodic function by establishing uniform convergence.
2. Prove $\|f - S_N\|_{\infty} \leq \sqrt{MT_N} < \sqrt{M/N}$.
3. For a fixed N and prescribed point x_0 , construct coefficients satisfying the constraint for which the bound $\sqrt{MT_N}$ is attained.
4. Prove $\|f - S_N\|_2^2 \leq \pi M/(N+1)^2$ and construct an equality case for this bound. Explain why the two optimizing coefficient patterns differ.

Question 8 – Solution

Strategy. Use the weighted coefficient budget with two different dual estimates: one for a point value and one for energy.

Step 1: Prove uniform convergence from the budget. For any finite tail from p to q , Cauchy–Schwarz gives

$$\left| \sum_{n=p}^q (a_n \cos nx + b_n \sin nx) \right| \leq \left(\sum_{n=p}^q n^2 (a_n^2 + b_n^2) \right)^{1/2} \left(\sum_{n=p}^q \frac{\cos^2 nx + \sin^2 nx}{n^2} \right)^{1/2}.$$

It is at most $\sqrt{M}(\sum_{n \geq p} n^{-2})^{1/2}$, independently of x , and tends to zero as $p \rightarrow \infty$. The uniform Cauchy criterion applies. The limit of these continuous periodic partial sums is continuous and periodic.

Step 2: Bound the infinite tail. Passing to the uniform limit of the finite-tail inequality gives

$$\|f - S_N\|_\infty \leq \sqrt{MT_N} < \sqrt{M/N} \quad (N \geq 1).$$

The weighted hypothesis controls the full collection of omitted modes, not just a finite list of observed coefficients.

Step 3: Attain the exact pointwise bound. Set every coefficient with $n \leq N$ to zero and, for $n > N$, choose

$$a_n = \sqrt{\frac{M}{T_N}} \frac{\cos(nx_0)}{n^2}, \quad b_n = \sqrt{\frac{M}{T_N}} \frac{\sin(nx_0)}{n^2}.$$

The weighted sum equals M , and at x_0 the tail equals $\sqrt{MT_N}$. This proves sharpness of the first bound over all admissible sequences. The mean m may be arbitrary because it cancels from the error.

Step 4: Optimize the integral error. Orthogonality, justified by the uniform convergence, gives

$$\|f - S_N\|_2^2 = \pi \sum_{n > N} (a_n^2 + b_n^2) \leq \frac{\pi}{(N+1)^2} \sum_{n > N} n^2 (a_n^2 + b_n^2) \leq \boxed{\frac{\pi M}{(N+1)^2}}.$$

Equality is attained by $a_{N+1} = \sqrt{M}/(N+1)$ and all other coefficients zero. The integral optimizer concentrates its budget in the cheapest omitted mode; the pointwise optimizer distributes it across all omitted modes with phases aligned at the chosen point.

Question 9

For $p > 0$, study $F_{p,N}(x) = \sum_{n=1}^N \cos(nx)/n^p$ on $[-\pi, \pi]$. You may use completeness of L^2 : a Cauchy sequence in its norm has an L^2 limit; taking an inner product with any fixed L^2 function is continuous. Also use Dirichlet's test for a scalar series with bounded partial sums multiplied by decreasing positive weights tending to zero.

Tasks

1. Classify exactly those p for which the sequence converges in L^2 , proving both directions by orthogonality.
2. Determine convergence at $x = 0$ modulo 2π , and at every other fixed point.
3. Classify uniform convergence on the full period. Prove uniform convergence on each closed subinterval avoiding $2\pi\mathbb{Z}$ for every $p > 0$.
4. For $1/2 < p \leq 1$, explain rigorously how an L^2 function can have these Fourier partial sums although they diverge at zero. Can redefining the limit's value at zero change that divergence?

Question 9 – Solution

Strategy. The thresholds for square summability, pointwise convergence at a common phase, and uniform convergence need not coincide.

Step 1: Classify mean-square convergence. For $M > N$, orthogonality gives

$$\|F_{p,M} - F_{p,N}\|_2^2 = \pi \sum_{n=N+1}^M n^{-2p}.$$

This tends to zero uniformly in $M > N$ exactly when $2p > 1$. Completeness proves existence of an L^2 limit in that case. For $p \leq 1/2$, the norms $\pi \sum_{n=1}^N n^{-2p}$ are unbounded, so no L^2 limit is possible. Thus $p > 1/2$ is necessary and sufficient.

Step 2: Classify each pointwise series. At zero modulo 2π the series is $\sum n^{-p}$, converging exactly when $p > 1$; otherwise its partial sums tend to $+\infty$. At any other fixed x , the geometric-sum bound gives bounded partial sums of $\cos nx$. The weights n^{-p} decrease to zero for every $p > 0$, so Dirichlet's test proves convergence at every such point.

Step 3: Distinguish global and local uniform convergence. For $p > 1$, the Weierstrass test proves absolute uniform convergence globally. For $0 < p \leq 1$, divergence at zero precludes uniform convergence on the full period. Away from $2\pi\mathbb{Z}$, the cosine block sums are bounded by $1/\sin(\delta/2)$. Summation by parts with weights n^{-p} gives

$$\sup_{\delta \leq x \leq 2\pi - \delta} \left| \sum_{n > N} \frac{\cos nx}{n^p} \right| \leq \frac{(N+1)^{-p}}{\sin(\delta/2)}.$$

Hence local uniform convergence holds for every $p > 0$.

Step 4: Interpret the L^2 limit. For $1/2 < p \leq 1$, continuity of inner products shows that the L^2 limit has mean zero, cosine coefficients n^{-p} and zero sine coefficients. Thus these really are its Fourier partial sums, not arbitrary approximations. An L^2 function is defined only up to changes on sets of measure zero; a single divergent point does not contradict convergence in the integral norm. Assigning any value to the limit at zero changes neither coefficients nor partial sums, and therefore cannot remove the divergence there.

Question 10

Let $f \in L^2(-\pi, \pi)$ be real and periodic with mean m and Fourier coefficients a_n, b_n . For $t > 0$, define the damped expansion

$$U_t(x) = m + \sum_{n=1}^{\infty} e^{-n^2 t} (a_n \cos nx + b_n \sin nx).$$

Use Parseval and L^2 convergence of Fourier partial sums. This problem concerns the series itself; no partial differential equation is assumed.

Tasks

1. Prove that for each fixed $t > 0$ the series, and every series obtained by differentiating in x finitely many times, converge uniformly.
2. Prove $U_t \rightarrow f$ in L^2 as $t \downarrow 0$ using a finite-head and small-tail argument, with an exact coefficient formula for the error.
3. If $\sum n^2(a_n^2 + b_n^2) \leq M$, prove $\|U_t - f\|_2 \leq \sqrt{\pi M t}$.
4. Show that the mean is preserved and that $\|U_t - m\|_2 \leq \|f - m\|_2$. Explain why the general L^2 assumption alone cannot guarantee uniform convergence to the assigned function f as $t \downarrow 0$.

Question 10 – Solution

Strategy. Damping controls all high frequencies for positive t ; removing it requires a separate convergence argument.

Step 1: Prove smoothness for positive damping. Parseval gives $E = \sum (a_n^2 + b_n^2) < \infty$. After k derivatives, the absolute value of the n th harmonic is at most $n^k e^{-n^2 t} \sqrt{a_n^2 + b_n^2}$. Cauchy-Schwarz bounds the sum by

$$\sqrt{E} \left(\sum_{n \geq 1} n^{2k} e^{-2n^2 t} \right)^{1/2} < \infty.$$

The exponential dominates every power, so the Weierstrass test applies for each fixed $t > 0$ and each integer $k \geq 0$. Repeated use of the uniform derivative theorem justifies differentiation; U_t is smooth and periodic.

Step 2: Remove damping in the integral norm. The Fourier coefficients of the difference give

$$\|U_t - f\|_2^2 = \pi \sum_{n \geq 1} (1 - e^{-n^2 t})^2 (a_n^2 + b_n^2).$$

Given $\varepsilon > 0$, choose K so that $\pi \sum_{n > K} (a_n^2 + b_n^2) < \varepsilon/2$. Since $0 \leq 1 - e^{-n^2 t} \leq 1$, this bounds the error tail for every t . The remaining finite sum tends to zero, hence is below $\varepsilon/2$ for sufficiently small t . This proves the limit.

Step 3: Obtain a quantitative rate under extra information. For $z \geq 0$, $1 - e^{-z} \leq \min(z, 1)$, so $(1 - e^{-z})^2 \leq z$. Applying this with $z = n^2 t$ gives

$$\|U_t - f\|_2^2 \leq \pi t \sum n^2 (a_n^2 + b_n^2) \leq \pi M t.$$

Taking square roots yields the requested rate; this uses a stronger hypothesis than mere square summability.

Step 4: Separate smoothing from uniform recovery. Uniform integration of the damped series preserves the mean m . Parseval gives $\|U_t - m\|_2^2 = \pi \sum e^{-2n^2 t} (a_n^2 + b_n^2) \leq \|f - m\|_2^2$. Every U_t is continuous. Choose an L^2 periodic step function with a jump: it cannot be a uniform limit of these continuous functions. Even isolated redefinitions of f are invisible to the coefficients. Thus smooth approximants and mean-square recovery do not alone guarantee uniform recovery of an arbitrarily assigned representative.