

Question 1

Consider the real differential equation

$$(x^2 - 1)y^{(4)} + 2xy''' - 3y' = \ln x.$$

A leading coefficient that vanishes at one point requires more care than simply counting derivatives.

Tasks

1. Classify the equation by order, linearity and homogeneity. Write its normalized form wherever that form is defined over the reals.
2. At $x_0 = 1/2$, prescribe $y = 1$, $y' = 0$, $y'' = -2$ and $y''' = 3$. State the largest interval containing x_0 on which the standard linear IVP theorem guarantees a unique solution, and explain why four data are needed.
3. A proposed C^4 solution near $x = 1$ has $y(1) = 0$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 1$. Test these data directly in the original equation.
4. Replace only the last datum by $y'''(1) = 3$. Explain what this compatibility check establishes and what it does not establish. Does a vanishing leading coefficient make the original equation third order on a neighborhood of 1?

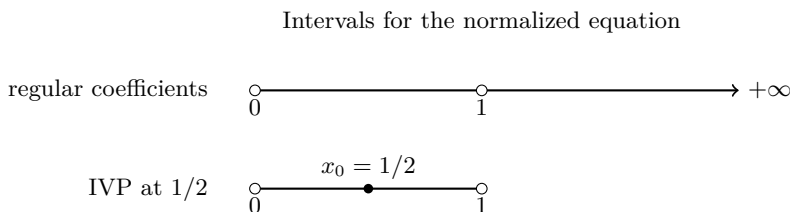
Strategy. Separate the equation's order from the intervals on which its highest derivative can be isolated.

$$y^{(4)} + \frac{2x}{x^2 - 1}y''' - \frac{3}{x^2 - 1}y' = \frac{\ln x}{x^2 - 1}.$$

Step 2: Apply the theorem on its proper interval. All normalized coefficients and the forcing are continuous on $\boxed{(0,1)}$, the largest such interval through $1/2$. The linear IVP theorem gives one solution for arbitrary values of y, y', y'', y''' at that ordinary point. These four data specify the fourth-order initial state. The theorem does not assert that this particular solution cannot extend beyond the interval; it simply gives no guarantee across 1.

$$2y'''(1) - 3y'(1) = 0.$$

Step 4: Interpret compatibility without overclaiming. With $y'''(1) = 3$, the same test gives $6 - 6 = 0$. This removes that contradiction, but alone proves neither existence nor uniqueness; the normalized IVP theorem still does not apply at 1. The coefficient of $y^{(4)}$ is nonzero at nearby points other than 1, so the equation remains fourth order on a neighborhood. At the isolated point, its value imposes a lower-derivative constraint instead of determining $y^{(4)}(1)$.



Question 2

Let $L[y] = y^{(4)}$ and consider $L[y] = 24x$ on $\mathbb{R}$. Define

$$p(x) = \frac{x^5}{5}, \quad u(x) = p(x) + 1 + x^2, \quad v(x) = p(x) + 2x - x^3.$$

For a nonhomogeneous equation, the phrase “superposition of solutions” needs a precise qualification.

Tasks

1. Verify that p, u, v solve the equation. Determine whether $u + v$, $u - v$ and $(u + v)/2$ solve the same equation or its homogeneous counterpart.
2. Find every triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ for which $\alpha u + \beta v + \gamma p$ solves $L[y] = 24x$.
3. Prove that the complete solution set is p plus the space of cubic polynomials. Explain why it is an affine set but not a vector space.
4. Find the unique solution with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 6$, $y'''(0) = 0$, and verify the forcing and all four data.

Question 2 – Solution

Strategy. Apply the linear operator before deciding which combinations preserve the nonzero forcing.

Step 1: Track the right-hand side. Since $p^{(4)} = 24x$ and fourth derivatives of cubics vanish, $L[p] = L[u] = L[v] = 24x$. Linearity gives

$$L[u + v] = 48x, \quad L[u - v] = 0, \quad L[(u + v)/2] = 24x.$$

Thus the sum solves neither of the two stated equations, the difference solves the homogeneous equation, and the average solves the original one.

Step 2: Characterize every admissible combination. The proposed combination has image $24x(\alpha + \beta + \gamma)$. Because $24x$ is not identically zero on $\mathbb{R}$,

$$\boxed{\alpha u + \beta v + \gamma p \text{ solves the equation} \iff \alpha + \beta + \gamma = 1.}$$

The forcing's isolated zero at 0 does not remove this identity requirement.

Step 3: Describe the complete solution set. If y is any solution, $(y - p)^{(4)} = 0$. Four integrations show that $y - p$ is a polynomial of degree at most three. Conversely, adding any such polynomial to p preserves the equation. Hence

$$\boxed{y = p + c_0 + c_1x + c_2x^2 + c_3x^3.}$$

This is a translate of a four-dimensional vector space, hence affine. It is not itself a vector space: the zero function does not satisfy the nonzero forcing, and sums generally double the forcing.

Step 4: Fit and check the initial state. The data give $c_0 = 1$, $c_1 = -2$, $2c_2 = 6$, $6c_3 = 0$, so

$$\boxed{y = \frac{x^5}{5} + 1 - 2x + 3x^2.}$$

The added cubic has fourth derivative zero. Its value and first three derivatives at 0 are $1, -2, 6, 0$; p contributes zero to all four. The coefficients are uniquely fixed.

Question 3

For $y^{(4)} = 0$ on $\mathbb{R}$, consider the shifted functions

$$f_0 = 1, \quad f_1 = x - 1, \quad f_2 = \frac{(x-1)^2}{2}, \quad f_3 = \frac{(x-1)^3}{6}.$$

Initial derivatives and polynomial coefficients coincide only for an appropriately normalized basis.

Tasks

1. Verify that the four functions solve the equation and compute their Wronskian for arbitrary x .
2. Construct the solution with $y(1) = 2$, $y'(1) = -1$, $y''(1) = 3$ and $y'''(1) = 4$. Explain why the coefficients in this basis can be read directly from the data.
3. Express the same solution in the unshifted basis $1, x, x^2, x^3$ and verify the four data in that form.
4. Prove that the map sending a solution to $(y(1), y'(1), y''(1), y'''(1))$ is a linear isomorphism onto $\mathbb{R}^4$. What does this say about the number of free constants and the effect of changing the basis?

Question 3 – Solution

Strategy. Build the derivative matrix and use its normalization at the initial point.

Step 1: Verify solutions and a nonzero determinant. Every function is cubic or lower, so its fourth derivative is zero. With $h = x - 1$, their derivative matrix is

$$F(x) = \begin{pmatrix} 1 & h & h^2/2 & h^3/6 \\ 0 & 1 & h & h^2/2 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is triangular with unit diagonal, giving $\boxed{W = \det F = 1}$ at every x .

Step 2: Use the identity matrix at the center. Since $F(1) = I$, the coefficient vector is exactly the prescribed derivative vector. Therefore

$$\boxed{y = 2 - h + \frac{3}{2}h^2 + \frac{2}{3}h^3, \quad h = x - 1.}$$

The factorials in the basis make its initial derivative matrix the identity; without them, derivative values would not equal coefficients.

Step 3: Change coordinates without changing the solution. Expansion gives

$$\boxed{y = \frac{23}{6} - 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3.}$$

Here $y' = -2 - x + 2x^2$, $y'' = -1 + 4x$, $y''' = 4$. At 1, these are $-1, 3, 4$, while $y(1) = 2$, verifying all data independently of the shifted expression.

Step 4: Establish the initial-data isomorphism. Differentiation and evaluation are linear. For any vector in $\mathbb{R}^4$, the corresponding combination of f_0, f_1, f_2, f_3 realizes it, proving surjectivity. If all four derivatives at 1 vanish, the coefficient vector in this basis is zero, proving injectivity. Thus the solution space has dimension four. A change of basis changes coordinates, not the number of free constants, the initial data themselves or the resulting function.

Question 4

Four functions $y_1, \dots, y_4$ solve the same homogeneous equation

$$y^{(4)} - \frac{2x}{1+x^2}y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

on $\mathbb{R}$, where p_0, p_1, p_2 are continuous. Their Wronskian at zero is $W(0) = -6$.

Tasks

1. Derive the Wronskian identity $W' = -p_3W$ for a normalized fourth-order homogeneous equation, by differentiating the determinant row by row.
2. Determine $W(x)$ for the stated equation and decide whether the four functions form a fundamental set on the whole real line.
3. Explain why every four-component initial derivative vector at any finite x_0 is realized by exactly one linear combination of these functions.
4. A computation reports $\widetilde{W}(x) = -6(1+x^2)(1-x)$. It has the correct value at 0. Give two independent reasons why it cannot be the Wronskian of this common solution family.

Question 4 – Solution

Strategy. The coefficient of the next-to-highest derivative controls the entire Wronskian, not just its value at one point.

Step 1: Differentiate the determinant. The Wronskian rows contain derivative orders 0, 1, 2, 3. Differentiating any of the first three rows duplicates the following row, giving determinant zero. In the last-row term, substitute $y_j^{(4)} = -p_3 y_j''' - p_2 y_j'' - p_1 y_j' - p_0 y_j$ for every column. All terms except the multiple of the original last row again duplicate a row. Therefore

$$W' = -p_3 W.$$

Step 2: Determine the Wronskian globally. Here $p_3 = -2x/(1+x^2)$, so

$$W(x) = -6 \exp\left(\int_0^x \frac{2s}{1+s^2} ds\right) = -6(1+x^2).$$

It never vanishes. The four solutions are linearly independent, and the regular fourth-order homogeneous solution space has dimension four; hence they form a fundamental set on $\mathbb{R}$.

Step 3: Interpret invertibility of the derivative matrix. At any finite x_0 , the matrix with columns $(y_j, y_j', y_j'', y_j''')(x_0)$ has determinant $W(x_0) \neq 0$. It therefore maps one and only one coefficient vector to any prescribed initial derivative vector. The resulting combination solves the equation; the linear IVP theorem shows it is the unique solution for those data on $\mathbb{R}$.

Step 4: Reject a plausible-looking candidate. First, $\widetilde{W}(1) = 0$ contradicts the nonvanishing formula forced by $W(0) \neq 0$ on this regular connected interval. Second, it fails the differential identity even at the initial point: $\widetilde{W}'(0) = 6$, whereas $-p_3(0)\widetilde{W}(0) = 0$. Matching one determinant value is insufficient; a valid Wronskian must obey the identity throughout the interval.

Question 5

You may use the fact that the function

$$\phi(s) = \begin{cases} \exp\left(-\frac{1}{1-4s^2}\right), & |s| < 1/2, \\ 0, & |s| \geq 1/2 \end{cases}$$

is C^∞ on $\mathbb{R}$, with every derivative zero at $s = \pm 1/2$. Define $f_1(x) = \phi(x+2)$, $f_2(x) = \phi(x)$ and $f_3(x) = \phi(x-2)$.

Tasks

1. Locate the supports of the three functions and prove that they are linearly independent on $\mathbb{R}$.
2. Prove that their three-by-three Wronskian is identically zero. Use their derivative supports, not a long symbolic expansion.
3. Could these functions all solve one equation $y''' + p_2y'' + p_1y' + p_0y = 0$ with continuous coefficients on $\mathbb{R}$? Prove your answer using uniqueness of an initial-value problem.
4. State the extra hypothesis that makes a zero Wronskian imply dependence for solutions of a linear equation. Explain why the example does not contradict that theorem.

Question 5 – Solution

Strategy. Compare independence of arbitrary smooth functions with independence inside one regular homogeneous solution space.

Step 1: Prove independence by separated values. The supports are $[-5/2, -3/2]$, $[-1/2, 1/2]$ and $[3/2, 5/2]$, respectively. They are disjoint, and each function has value e^{-1} at its center $-2, 0, 2$. If $c_1f_1 + c_2f_2 + c_3f_3 = 0$ identically, evaluation at those three centers gives $c_1 = c_2 = c_3 = 0$. The functions are therefore linearly independent.

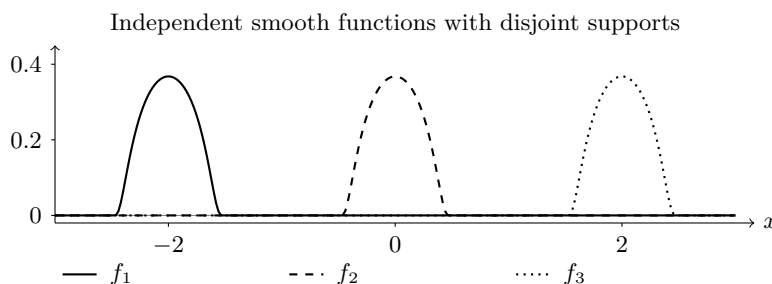
Step 2: Compute the determinant structurally. Outside a function's support, it and its derivatives vanish; at the support boundary, the supplied smoothness facts give the same conclusion. At every x , at most one column of

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}$$

can be nonzero. Thus $\boxed{W(f_1, f_2, f_3)(x) = 0 \text{ for all } x}$.

Step 3: Test the common-equation hypothesis. Suppose all three solved the stated regular homogeneous equation. The function f_1 has $f_1(0) = f_1'(0) = f_1''(0) = 0$. The zero solution has the same data, so uniqueness on $\mathbb{R}$ would force $f_1 \equiv 0$. But $f_1(-2) = e^{-1} > 0$, a contradiction. Such continuous coefficients cannot exist.

Step 4: State the theorem with its hypotheses. For three solutions of the same normalized third-order homogeneous linear equation with continuous coefficients on a connected interval, a zero Wronskian at one point gives a nonzero coefficient combination with zero initial derivative vector. Uniqueness then makes that combination identically zero, proving dependence. Our functions are arbitrary smooth functions, not solutions of one such equation. Their identically zero Wronskian alone therefore does not imply dependence.



Question 6

For $y^{(4)} = 0$ on $[0, 1]$, prescribe four scalar endpoint conditions:

$$y(0) = 0, \quad y(1) = 0, \quad y'(0) + y'(1) = 0, \quad y''(1) - y''(0) = \eta,$$

where η is a real parameter. Counting four conditions is not the same as verifying that they determine four constants independently.

Tasks

1. Write the general solution and translate all four conditions into a linear system for its coefficients. Determine the rank of the coefficient matrix.
2. Classify all η according to existence and uniqueness, and give every solution whenever solutions exist.
3. Replace the fourth condition by $y'(0) = 1$, keeping the first three. Find the resulting solution and prove it is unique.
4. Explain why none of these conclusions contradicts the standard fourth-order IVP theorem. Describe how the family in the compatible original problem varies while retaining all four endpoint conditions.

Question 6 – Solution

Strategy. Test the endpoint functionals on the cubic solution space instead of relying on their number.

Step 1: Form the coefficient system. Write $y = a + bx + cx^2 + dx^3$. The conditions become

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}.$$

The fourth row is six times the third minus twelve times the second plus twelve times the first. The first three rows are independent, so the rank is $\boxed{3}$.

Step 2: Classify the parameter completely. The first three equations imply $a = 0$, $b + c + d = 0$ and $2b + 2c + 3d = 0$; subtracting twice the middle equation gives $d = 0$. Thus the last equation can hold only if $\eta = 0$. For that value, $c = -b$ and

$$y = Kx(1 - x), \quad K \in \mathbb{R}.$$

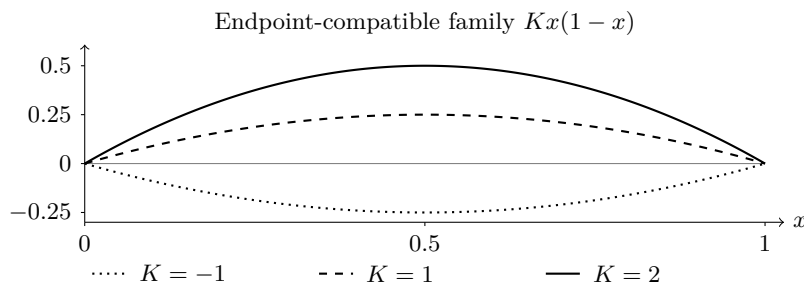
There are infinitely many solutions if $\eta = 0$ and none if $\eta \neq 0$. No value of η gives a unique solution under the original conditions.

Step 3: Replace the redundant information. The condition $y'(0) = 1$ gives $b = 1$. The first three conditions already imply $a = d = 0$ and $c = -b$, so

$$y = x(1 - x)$$

is the unique solution after replacement. The new condition fixes the one remaining free parameter.

Step 4: Distinguish endpoint data from a full initial state. The IVP theorem concerns y, y', y'', y''' specified at one ordinary point. These endpoint conditions mix values at two points and can be dependent or inconsistent on the solution space. In the compatible family, changing K changes height and curvature while preserving zero endpoint values, opposite endpoint slopes, and equal endpoint curvatures. The figure shows three such members; their different initial derivative vectors explain why IVP uniqueness is not violated.



Question 7

Consider the singular third-order equation

$$xy''' - 2y'' = 0$$

on the whole real line. A classical solution here means a C^3 function satisfying the original equation at every point, including 0.

Tasks

1. Solve on each half-line by first setting $z = y''$. Display all constants before matching across zero.
2. Classify every global C^3 solution and determine the dimension of this solution space.
3. For prescribed $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$, determine exactly when solutions exist and how many free parameters remain.
4. Determine which global solutions are C^4 . Does demanding this extra derivative restore uniqueness for compatible initial data at zero? Explain why the regular third-order dimension theorem cannot be used across this singular point.

Question 7 – Solution

Strategy. Solve separately where the leading coefficient is nonzero, then impose the actual regularity required at the join.

Step 1: Solve on both regular components. For $x \neq 0$, $xz' - 2z = 0$ gives $z = Cx^2$. Two integrations, with the constant absorbed into the quartic coefficient, yield

$$y = \begin{cases} A_+x^4 + B_+x + C_+, & x > 0, \\ A_-x^4 + B_-x + C_-, & x < 0. \end{cases}$$

The constants on the two components are initially independent.

Step 2: Impose a classical join. Continuity of y and y' forces $C_+ = C_- = C$ and $B_+ = B_- = B$. The second and third derivatives are $12A_{\pm}x^2$ and $24A_{\pm}x$, both tending to zero, so C^3 regularity imposes no relation between A_+ and A_- . Every global solution is therefore

$$y(x) = C + Bx + \begin{cases} A_+x^4, & x \geq 0, \\ A_-x^4, & x < 0, \end{cases}$$

with four independent constants. At zero the equation reduces to $-2y''(0) = 0$, which these functions satisfy. This is a four-dimensional solution space.

Step 3: Classify the initial data. Every solution has $y(0) = C$, $y'(0) = B$ and $y''(0) = 0$. Thus $\boxed{\gamma = 0}$ is necessary and sufficient. When it holds, $C = \alpha$, $B = \beta$ and both A_+, A_- remain arbitrary. When $\gamma \neq 0$, no classical solution exists.

Step 4: Check the extra derivative. The one-sided fourth derivatives are $24A_+$ and $24A_-$. A C^4 join requires and is guaranteed by $A_+ = A_- = A$, giving the polynomial $C + Bx + Ax^4$. One free parameter A still survives for the compatible initial data, so uniqueness is not restored. The leading coefficient vanishes at 0, and normalization introduces $-2/x$. The hypotheses guaranteeing dimension three and arbitrary unique initial data hold only on the separate half-lines.

Question 8

For the homogeneous equation $y''' + y' = 0$ on $\mathbb{R}$, consider the three functions $1, \sin x, \cos x$. A student believes that a nonzero solution cannot have both its value and its first derivative zero at the same point.

Tasks

1. Verify the three functions as solutions and compute their Wronskian. Explain why they form a fundamental set on $\mathbb{R}$.
2. Find the solution with $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$, and verify the full initial state.
3. Determine all zeros of this solution and their exact multiplicities. Decide whether finitely or infinitely many such zeros are possible here.
4. Correct the student's claim for a regular homogeneous equation of general order n . State which consecutive derivatives must vanish at a point to force the entire solution to be zero, and explain why the present example does not satisfy that condition.

Question 8 – Solution

Strategy. Use the full initial derivative vector, rather than only the value and slope, to apply uniqueness.

Step 1: Verify a fundamental set. Each candidate satisfies $y''' + y' = 0$. Their Wronskian is

$$W = \det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} = \boxed{-1}.$$

Thus the three solutions are independent. The regular third-order equation has a three-dimensional homogeneous solution space, so they form a fundamental set.

Step 2: Determine the unique IVP solution. Write $y = A + B \sin x + C \cos x$. The data give $A + C = 0$, $B = 0$ and $-C = 1$. Hence

$$\boxed{y = 1 - \cos x.}$$

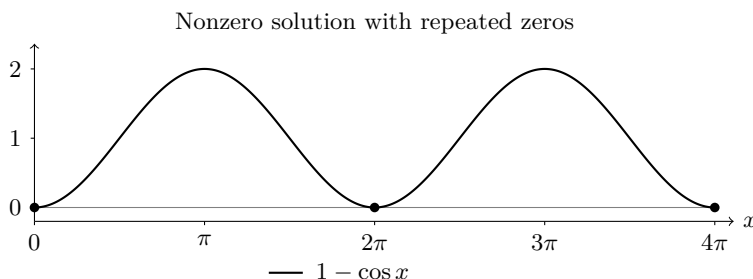
Its first two derivatives are $\sin x$ and $\cos x$, giving initial vector $(0, 0, 1)$; its third derivative $-\sin x$ verifies the equation.

Step 3: Classify all zero contacts. The equation $1 - \cos x = 0$ holds exactly at $x = 2k\pi$, $k \in \mathbb{Z}$. At each such point, $y' = 0$ but $y'' = 1 \neq 0$, so every zero has multiplicity exactly two. There are infinitely many, with no finite accumulation point. Their repeated tangency does not make the function identically zero.

Step 4: State the correct uniqueness consequence. For a normalized homogeneous linear equation of order n with continuous coefficients on a connected interval, if

$$\boxed{y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0,}$$

then uniqueness of the zero-data IVP forces $y \equiv 0$ throughout that interval. Here $n = 3$, and the second derivative at every zero is 1, so the required three-component vector is not zero. Value and slope alone are insufficient; an arbitrary number of separated double zeros is not excluded by the IVP theorem.



Question 9

Consider the nonlinear third-order IVP

$$y''' = \sqrt{|y|}, \quad y(0) = y'(0) = y''(0) = 0.$$

For each delay $\tau \geq 0$, investigate a candidate that is zero for $x \leq \tau$ and has the form $C(x - \tau)^p$ for $x > \tau$, with $C > 0$ and $p > 3$.

Tasks

1. Explain why the standard linear IVP theorem does not apply, despite a nonzero leading coefficient and a continuous right-hand side.
2. Find the exponent p and coefficient C that make the positive branch satisfy the equation.
3. Verify the joined function as a classical solution on $\mathbb{R}$ and check all three initial data for every $\tau \geq 0$. Determine its exact smoothness at the joining point.
4. Explain how this family proves nonuniqueness. Contrast it with the linear equation $y''' = y$ under the same initial data, and examine the ratio $\sqrt{h}/h$ as $h \rightarrow 0^+$ to identify a regularity issue in the nonlinear dependence on y .

Question 9 – Solution

Strategy. Construct actual delayed solutions; continuity of a nonlinear right-hand side alone is not a uniqueness proof.

Step 1: Check linearity, not just the leading coefficient. The equation is third order but nonlinear because $\sqrt{|y|}$ is not a linear function of the unknown. A leading coefficient of 1 and continuity do not make the linear existence-and-uniqueness theorem applicable.

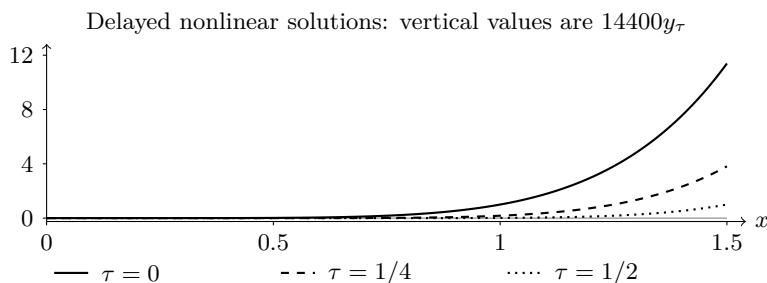
Step 2: Balance exponents and amplitudes. For $s = x - \tau > 0$, the proposed branch has $y''' = Cp(p-1)(p-2)s^{p-3}$ and $\sqrt{|y|} = \sqrt{C}s^{p/2}$. Since $p > 3$ and $C > 0$, equality requires $p-3 = p/2$, giving $p = 6$. Then $120C = \sqrt{C}$, so $C = 1/14400$.

Step 3: Verify the join and the initial state. Define

$$y_\tau(x) = \begin{cases} 0, & x \leq \tau, \\ (x - \tau)^6/14400, & x > \tau. \end{cases}$$

Derivatives through order five approach zero from the right and equal zero on the left; difference quotients at the join give the same derivatives. Thus $y_\tau \in C^5$, and the equation holds at the join too, since $y''' = \sqrt{|y|} = 0$ there. The sixth derivative jumps from 0 to $6!/14400 = 1/20$, so it is not C^6 at τ . Because $0 \leq \tau$, all three initial data vanish, including when $\tau = 0$.

Step 4: Identify the precise contrast. Distinct delays give distinct solutions with identical initial data; the identically zero function is another solution. This proves nonuniqueness constructively. For the linear equation $y''' - y = 0$, continuous normalized coefficients and the same zero initial vector force the unique zero solution. Here $\sqrt{h}/h = 1/\sqrt{h} \rightarrow \infty$, so the nonlinear right-hand side is not locally Lipschitz in y at zero. Nonlinearity alone does not always cause nonuniqueness; the displayed family establishes the failure in this specific case.



Question 10

Two solutions of $y^{(4)} = 0$ on $\mathbb{R}$ have initial derivative errors

$$\delta_j = \tilde{y}^{(j)}(0) - y^{(j)}(0), \quad |\delta_j| \leq \varepsilon \quad (j = 0, 1, 2, 3),$$

where $\varepsilon \geq 0$. Let $e = \tilde{y} - y$. We want a guaranteed error bound over a whole interval, not just at the initial point.

Tasks

1. Find $e(x)$ exactly in terms of the four initial errors.
2. For $T > 0$, prove the sharp uniform bound

$$\max_{|x| \leq T} |e(x)| \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Show that the constant cannot be improved when only the stated error information is available.

3. At $T = 2$, find the largest ε guaranteeing solution error at most 10^{-2} for every admissible initial error vector. Evaluate the guarantee for $\varepsilon = 10^{-3}$.
4. If only $\delta_0, \delta_1, \delta_2$ are controlled, can any finite uniform guarantee depending on those three errors alone hold on $[-2, 2]$? Construct a decisive example and interpret what the result says about a complete initial state.

Question 10 – Solution

Strategy. The linear initial-data map is explicit here, so its worst-case amplification can be determined exactly.

Step 1: Propagate all four initial errors. The difference satisfies $e^{(4)} = 0$ and $e^{(j)}(0) = \delta_j$. Therefore

$$e(x) = \delta_0 + \delta_1 x + \frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}.$$

The factorials are required because the data are derivatives, not monomial coefficients.

Step 2: Prove and attain the uniform bound. For $|x| \leq T$, the triangle inequality gives

$$|e(x)| \leq \varepsilon \left(1 + |x| + \frac{|x|^2}{2} + \frac{|x|^3}{6} \right) \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Choose all four $\delta_j = \varepsilon$. At $x = T$, every term has the same nonnegative sign, attaining equality. Thus the stated worst-case constant is exact, including the trivial case $\varepsilon = 0$.

Step 3: Translate the certificate into a tolerance. At $T = 2$, the amplification factor is $1+2+2+8/6 = 19/3$. Consequently the largest admissible tolerance is

$$\varepsilon_{\max} = \frac{10^{-2}}{19/3} = \frac{3}{1900}.$$

Any larger value fails for the equal-sign error vector from Step 2. For $\varepsilon = 10^{-3}$, the exact worst-case bound is $19/3000 \approx 0.0063333 < 0.01$.

Step 4: Show why one uncontrolled component matters. Take $y = 0$ and $\tilde{y} = Mx^3/6$ for arbitrary real M . Both solve the equation and have the same value, first derivative and second derivative at zero, but their third-derivative difference is M . At $x = 2$, the solution error is $4|M|/3$, which is unbounded as $|M| \rightarrow \infty$. Thus controlling only three components cannot give the proposed finite guarantee. For the fourth-order IVP, the complete initial derivative vector is needed for this quantitative stability bound.

Question 1

Write $D = d/dx$. Consider the constant-coefficient homogeneous equation

$$D^2(D-1)(D+2)((D+1)^2+4)y=0$$

on $\mathbb{R}$. Long-term behavior depends on both the locations and the multiplicities of its characteristic roots.

Tasks

1. List every characteristic root with its multiplicity, state the order, and write the complete real general solution.
2. Characterize exactly which solutions are bounded on $[0, \infty)$. Give the dimension of that subspace.
3. Characterize exactly which solutions tend to zero as $x \rightarrow +\infty$. Give a bounded nonconstant solution that does not tend to zero.
4. Determine every solution bounded on the whole real line. Justify why terms cannot cancel to create additional bounded solutions.

Question 1 – Solution

Strategy. Separate exponential growth, polynomial drift, decay and oscillation before imposing boundedness.

Step 1: Read all multiplicities. The roots are 0 (twice), 1, -2 , and $-1 \pm 2i$. Their multiplicities total six, so a real fundamental family gives

$$y = c_0 + c_1 x + Ae^x + Be^{-2x} + e^{-x}(C \cos 2x + S \sin 2x).$$

The six constants are independent. In particular, the double zero root requires both 1 and x .

Step 2: Remove forward growth. If $A \neq 0$, the e^x term dominates as $x \rightarrow +\infty$. Once $A = 0$, the remaining exponential terms decay, so boundedness also requires $c_1 = 0$. Conversely those two conditions suffice:

$$\text{bounded on } [0, \infty) \iff A = c_1 = 0.$$

This is a four-dimensional subspace, with free constants c_0, B, C, S .

Step 3: Distinguish a limit from zero. Every forward-bounded solution tends to c_0 . Thus the decaying subspace has $A = c_1 = c_0 = 0$ and dimension three. For example, $1 + e^{-2x}$ is bounded and nonconstant on $[0, \infty)$ but tends to 1.

Step 4: Impose the backward restriction. Start with a forward-bounded solution. Multiplication by e^{2x} and passage to $x \rightarrow -\infty$ shows $B = 0$, since bounded y would give a zero limit. If $(C, S) \neq (0, 0)$, write $C \cos 2x + S \sin 2x = R \cos(2x - \phi)$ with $R > 0$. Along a sequence tending to $-\infty$ at cosine maxima, Re^{-x} is unbounded. Hence $C = S = 0$ as well, leaving

$$y(x) = c_0 \quad \text{as the entire-line bounded family.}$$

Different growth rates and the explicit oscillatory subsequence rule out hidden cancellation.

Question 2

Consider the fourth-order initial-value problem

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 2, \quad y'''(0) = -6.$$

All characteristic roots can lie strictly in the left half-plane while a particular response first grows in magnitude.

Tasks

1. Factor the characteristic polynomial and write a complete real general solution.
2. Set $y = e^{-x}v$. Derive the equation and all four initial data for v , then solve the IVP without a four-by-four coefficient elimination.
3. Find every critical point of the IVP solution on $[0, \infty)$ and its exact global maximum there. Determine its limit at infinity.
4. Prove that every solution of the differential equation tends to zero as $x \rightarrow +\infty$. Explain why this does not require each solution's magnitude to decrease from its initial value.

Question 2 – Solution

Strategy. Remove the repeated exponential factor, then analyze the resulting polynomial times exponential.

Step 1: Include the entire repeated-root chain. The polynomial is $(r + 1)^4$, so

$$y = e^{-x}(c_0 + c_1x + c_2x^2 + c_3x^3).$$

Four distinct powers are needed for a root of multiplicity four.

Step 2: Transfer the initial derivatives. Since $(D + 1)(e^{-x}v) = e^{-x}v'$, four applications give $v^{(4)} = 0$. Also $v = e^x y$, so the product rule gives

$$v(0) = 0, \quad v'(0) = y'(0) + y(0) = 0, \quad v''(0) = y''(0) + 2y'(0) + y(0) = 2,$$

$$v'''(0) = y'''(0) + 3y''(0) + 3y'(0) + y(0) = 0.$$

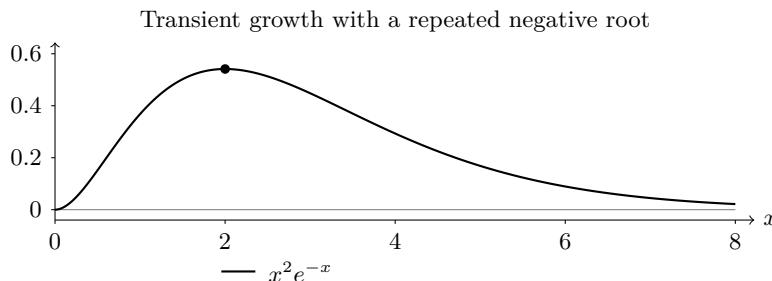
Therefore $v = x^2$ and $y = x^2 e^{-x}$. Its initial value and first three derivatives are 0, 0, 2, -6 as required; the shift identity verifies the equation.

Step 3: Locate the transient maximum. Here $y' = e^{-x}x(2 - x)$. It vanishes at 0 and 2, is positive on $(0, 2)$ and negative on $(2, \infty)$. Thus

$$\max_{x \geq 0} y(x) = 4e^{-2} \text{ at } x = 2, \quad \lim_{x \rightarrow \infty} y(x) = 0.$$

The endpoint 0 is a zero of the derivative and a minimum on the half-line.

Step 4: Separate eventual decay from monotonicity. Each $x^k e^{-x}$, $0 \leq k \leq 3$, tends to zero; hence every linear combination does too. Decay at infinity is an asymptotic statement. This IVP solution starts at zero and increases before decreasing, so neither its value nor its magnitude must decrease throughout the half-line.



Question 3

Consider

$$y^{(4)} + 8y'' + 16y = 0, \quad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 8.$$

A repeated imaginary root differs substantially from a simple imaginary root.

Tasks

1. Find the characteristic roots and write the full real general solution, taking account of multiplicities.
2. Determine the unique IVP solution and verify its four initial data.
3. Evaluate the solution and its first derivative at $x_n = n\pi/2$, $n = 1, 2, \dots$. Use these values to prove unboundedness on $[0, \infty)$.
4. Characterize all solutions of the equation that are bounded on $[0, \infty)$. Prove that no nonzero solution tends to zero at infinity.

Question 3 – Solution

Strategy. Retain the polynomial factors attached to repeated conjugate roots and test growth at suitable phases.

Step 1: Build the real basis. The polynomial is $(r^2 + 4)^2$, with roots $\pm 2i$, each twice. Thus

$$y = (A + Bx) \cos 2x + (C + Dx) \sin 2x.$$

Omitting the factors $x \cos 2x$ and $x \sin 2x$ would lose half the solution space.

Step 2: Fit the initial vector. At zero the four derivatives are

$$A, \quad B + 2C, \quad -4A + 4D, \quad -12B - 8C.$$

The data give $A = D = 0$, $B = -1$ and $C = 1/2$, hence

$$y = \frac{1}{2} \sin 2x - x \cos 2x.$$

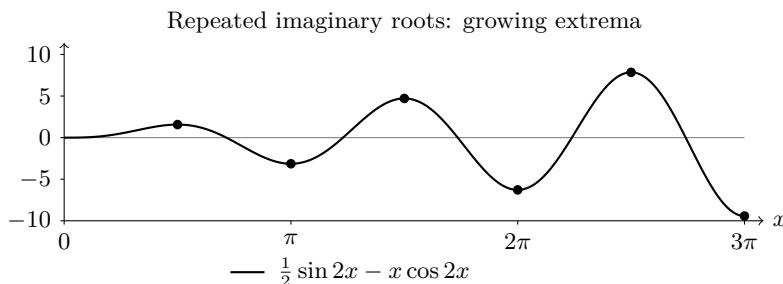
Indeed $y' = 2x \sin 2x$, $y'' = 2 \sin 2x + 4x \cos 2x$, and $y''' = 8 \cos 2x - 8x \sin 2x$, verifying all four data. Each basis term satisfies the factored equation.

Step 3: Exhibit growing extrema. For $x_n = n\pi/2$, we have

$$y'(x_n) = 0, \quad y(x_n) = (-1)^{n+1} \frac{n\pi}{2}, \quad y''(x_n) = 4x_n(-1)^n \neq 0.$$

These are alternating strict extrema with unbounded magnitude. In particular, purely imaginary roots alone do not imply boundedness when they are repeated.

Step 4: Classify the bounded family. If $(B, D) \neq (0, 0)$, choose a sequence $x \rightarrow \infty$ on which $B \cos 2x + D \sin 2x = \sqrt{B^2 + D^2} > 0$. The linear-in- x part grows, whereas $A \cos 2x + C \sin 2x$ is bounded. Therefore boundedness is equivalent to $B = D = 0$, a two-dimensional subspace. A nonzero member of that subspace attains its positive amplitude infinitely often, so it cannot tend to zero. An unbounded member cannot tend to zero either.



Question 4

An observed signal is known exactly on $\mathbb{R}$:

$$f(x) = e^{2x}(1 + x^2) + x \sin 3x.$$

Seek a real, monic, constant-coefficient homogeneous differential equation of the smallest possible order having this signal as a solution. Here monic means that the coefficient of the highest derivative is 1.

Tasks

1. Find a factored characteristic polynomial that annihilates f , carefully determining the multiplicity of every root.
2. Prove that no nonzero constant-coefficient differential operator of smaller order can annihilate f . State which independence fact prevents cancellation between its different exponential modes.
3. Write the complete real general solution of the resulting minimal equation and explain why its monic characteristic polynomial is unique.
4. Must this same equation also admit xe^{2x} and $\cos 3x$ as solutions? Explain why the two displayed summands of f do not mean that a second-order equation suffices. Restrict the conclusion to constant coefficients.

Question 4 – Solution

Strategy. Read the highest polynomial degree at each distinct exponential frequency, including complex conjugates.

Step 1: Find sufficient root multiplicities. The shift identity $P(D)(e^{\lambda x}q) = e^{\lambda x}P(D + \lambda)q$ shows that a quadratic polynomial multiplying e^{2x} is killed by $(D - 2)^3$. Also

$$x \sin 3x = \frac{x}{2i}(e^{3ix} - e^{-3ix}),$$

so each of $\pm 3i$ requires multiplicity two. A sufficient real monic polynomial is

$$P(r) = (r - 2)^3(r^2 + 9)^2, \quad \deg P = 7.$$

Step 2: Prove minimality, not just sufficiency. Suppose a polynomial Q has a root of multiplicity m at λ ; write $Q(r) = (r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. Applied to $e^{\lambda x}q(x)$, the polynomial $R(D + \lambda)q$ retains the degree of q because its leading coefficient is multiplied by $R(\lambda)$. Thus D^m kills it exactly when $m > \deg q$.

Exponential-polynomial modes with distinct complex exponents are linearly independent on $\mathbb{R}$; this is the generalized characteristic-root independence theorem. Therefore the contributions at $2, 3i, -3i$ cannot cancel each other. Every annihilating Q must have multiplicities at least $3, 2, 2$, so must be divisible by P and have degree at least seven.

Step 3: Recover the entire solution space. The real general solution is

$$y = e^{2x}(a_0 + a_1x + a_2x^2) + (b_0 + b_1x) \cos 3x + (c_0 + c_1x) \sin 3x.$$

There are seven free real constants. A monic degree-seven annihilator divisible by the monic degree-seven P must equal P , proving uniqueness at minimal order.

Step 4: Interpret the forced companion modes. Both xe^{2x} and $\cos 3x$ belong to this solution space. The two written summands package several repeated-root modes: their number is not the differential order. The lower bound concerns nonzero constant-coefficient operators only; it does not assert a minimum order among variable-coefficient equations.

Question 5

For the fourth-order equation

$$y^{(4)} - 5y'' + 4y = 0,$$

write the initial derivatives as $(y(0), y'(0), y''(0), y'''(0)) = (a, b, c, d)$. For each simple characteristic root λ , define the polynomial

$$\Pi_\lambda(r) = \prod_{\mu \neq \lambda} \frac{r - \mu}{\lambda - \mu},$$

where the product runs over the other characteristic roots.

Tasks

1. Find the roots and prove that $\Pi_\lambda(D)y$ isolates the single mode $A_\lambda e^{\lambda x}$ in the general solution.
2. Compute the coefficients of e^x and e^{2x} explicitly from a, b, c, d using these polynomials.
3. Give necessary and sufficient conditions on the initial derivatives for boundedness on $[0, \infty)$. Express them also as two relations obtained from $(D+1)(D+2)y = 0$.
4. Start from $(a, b, c, d) = (1, -1, 1, -1)$ and perturb only d by a nonzero amount ε . Find the limit

$$\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x).$$

Explain the long-term consequence of arbitrarily small such errors.

Question 5 – Solution

Strategy. Use polynomial interpolation at the roots to extract growing modes directly from the initial data.

Step 1: Construct mode selectors. The polynomial factors as $(r-1)(r+1)(r-2)(r+2)$. Thus y is a linear combination of $e^{\lambda x}$ for $\lambda = -2, -1, 1, 2$. Since $Q(D)e^{\mu x} = Q(\mu)e^{\mu x}$ and $\Pi_\lambda(\mu)$ is 1 if $\mu = \lambda$ and 0 otherwise,

$$\Pi_\lambda(D)y = A_\lambda e^{\lambda x}.$$

Evaluating at zero reads off A_λ without solving all four coefficients simultaneously.

Step 2: Extract the unstable coefficients. The relevant polynomials are

$$\begin{aligned}\Pi_1(r) &= -\frac{(r+2)(r+1)(r-2)}{6} = \frac{-r^3 - r^2 + 4r + 4}{6}, \\ \Pi_2(r) &= \frac{(r+2)(r+1)(r-1)}{12} = \frac{r^3 + 2r^2 - r - 2}{12}.\end{aligned}$$

Hence

$$\boxed{A_1 = \frac{4a + 4b - c - d}{6}, \quad A_2 = \frac{d + 2c - b - 2a}{12}.$$

Step 3: Describe exactly the decaying initial states. Forward boundedness requires $A_2 = A_1 = 0$; distinct positive exponential rates cannot cancel. The remaining modes decay. Equivalently,

$$\boxed{c + 3b + 2a = 0, \quad d + 3c + 2b = 0.}$$

These follow from $y'' + 3y' + 2y = 0$ and its derivative for the two negative-root modes. Conversely, substituting these relations into A_1, A_2 makes both zero, so the relations are sufficient as well.

Step 4: Quantify the loss of cancellation. The unperturbed solution is e^{-x} . Replacing $d = -1$ by $-1 + \varepsilon$ produces $A_1 = -\varepsilon/6$ and $A_2 = \varepsilon/12$. All modes other than e^{2x} disappear after multiplication by e^{-2x} in the limit, giving

$$\boxed{\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x) = \varepsilon/12.}$$

Every nonzero error of this form eventually yields exponential growth in magnitude. Exact decay requires precise cancellation of both positive-root modes.

Question 6

Let $\alpha \in \mathbb{R}$ and consider the third-order homogeneous equation

$$(D + 1)(D^2 + 2\alpha D + 1)y = 0$$

on $\mathbb{R}$. Call the equation *forward bounded* if every solution is bounded on $[0, \infty)$, and *forward decaying* if every solution tends to zero as $x \rightarrow +\infty$.

Tasks

1. Find the characteristic roots and identify all values of α where roots coincide, including coincidence with the fixed root -1 .
2. Write a complete real general solution in every parameter regime. Treat $\alpha = 1$ and $\alpha = -1$ explicitly.
3. Classify exactly the parameters for forward boundedness and forward decay. Prove the boundary cases rather than deciding from root real parts alone.
4. At $\alpha = 0, 1, -1$, respectively, give a nondecaying bounded solution, a decaying solution with a nonconstant polynomial factor, and an unbounded solution. Explain the different effects of repeated negative and repeated positive roots.

Question 6 – Solution

Strategy. Track the quadratic roots and separately check when either merges with the fixed root.

Step 1: Locate the collisions. Besides $r = -1$, the roots are $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - 1}$. They coincide when $\alpha = \pm 1$. Substituting $r = -1$ in the quadratic gives $2 - 2\alpha$, so coincidence with the fixed root occurs only at $\alpha = 1$, producing a triple root -1 .

Step 2: State a complete basis in each case. For $|\alpha| > 1$, all three roots are real and distinct, and

$$y = Ce^{-x} + Ae^{r_+x} + Be^{r_-x}.$$

For $|\alpha| < 1$, let $\beta = \sqrt{1 - \alpha^2} > 0$. Then

$$y = Ce^{-x} + e^{-\alpha x}(A \cos \beta x + B \sin \beta x).$$

The exceptional cases are

$$\begin{aligned}\alpha = 1 : \quad y &= e^{-x}(C + Ax + Bx^2), \\ \alpha = -1 : \quad y &= Ce^{-x} + (A + Bx)e^x.\end{aligned}$$

Each family has three free constants.

Step 3: Classify all forward behavior. If $\alpha > 1$, both quadratic roots are negative since $\sqrt{\alpha^2 - 1} < \alpha$. If $0 < \alpha < 1$, their real part is $-\alpha < 0$. At $\alpha = 1$, even the quadratic polynomial factor is dominated by e^{-x} . Thus every solution decays for every $\alpha > 0$.

At $\alpha = 0$, the modes are $e^{-x}, \cos x, \sin x$: all are bounded, but not all decay. If $-1 < \alpha < 0$, the oscillatory amplitude grows exponentially; choosing a cosine and its phase maxima proves unboundedness. At $\alpha = -1$ there are positive-root modes, and for $\alpha < -1$ both quadratic roots are positive. Consequently,

forward bounded $\iff \alpha \geq 0$,	forward decaying $\iff \alpha > 0$.
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Step 4: Test the special parameters concretely. The requested examples are $\cos x$ at $\alpha = 0$, x^2e^{-x} at $\alpha = 1$, and xe^x at $\alpha = -1$. A finite polynomial factor does not defeat exponential decay at a negative root. A positive root already permits growth, and repetition adds further polynomial amplification.

Question 7

A real solution of

$$y^{(4)} + 5y'' + 4y = 0$$

is sampled at all times $x = n\pi$, $n = 0, 1, 2, \dots$. The measured values are 3 for even n and 1 for odd n .

Tasks

1. Find the complete real general solution and determine exactly what the value samples reveal about its four coefficients.
2. Describe the full subspace of solutions invisible to these samples, meaning $y(n\pi) = 0$ for every non-negative integer n . Prove that infinitely many value samples still leave nonuniqueness.
3. Two additional measurements give $y'(0) = 1$ and $y'(\pi) = 3$. Recover the unique solution and verify every type of measurement.
4. Prove that, for any solution of this equation, the four measurements $y(0), y(\pi), y'(0), y'(\pi)$ uniquely determine it. Explain why the equally spaced value samples alone miss actual oscillations between sampling times.

Question 7 – Solution

Strategy. Evaluate the real modes at the sampling grid before interpreting the amount of information in the data.

Step 1: Read the visible coefficients. The polynomial is $(r^2 + 1)(r^2 + 4)$, so

$$y = A \cos x + B \sin x + C \cos 2x + D \sin 2x.$$

At the sample times, $y(n\pi) = A(-1)^n + C$. Thus $A + C = 3$ and $-A + C = 1$, giving $A = 1$ and $C = 2$. These samples impose no restriction on B, D .

Step 2: Find the invisible subspace. Zero values at both even and odd sample indices force $A = C = 0$, while the sine modes vanish at every $n\pi$. Therefore

$$\{y : y(n\pi) = 0 \ \forall n \geq 0\} = \text{span}\{\sin x, \sin 2x\}.$$

It has dimension two. Adding any member to one fitting signal leaves every recorded value unchanged, even though the functions differ between samples.

Step 3: Recover the missing oscillations. The derivative samples give $B + 2D = 1$ and $-B + 2D = 3$, hence $B = -1, D = 1$. The unique signal is

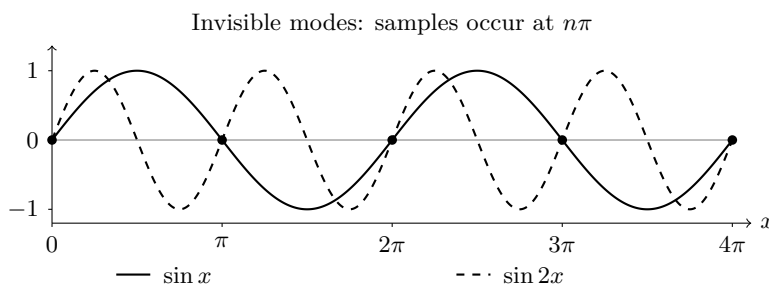
$$y = \cos x - \sin x + 2 \cos 2x + \sin 2x.$$

Its value samples are $(-1)^n + 2$ and its derivative samples at $0, \pi$ are $1, 3$, respectively. Every term solves the equation.

Step 4: Prove general identifiability. For arbitrary measurements $u = y(0)$, $v = y(\pi)$, $p = y'(0)$ and $q = y'(\pi)$,

$$A = \frac{u - v}{2}, \quad C = \frac{u + v}{2}, \quad B = \frac{p - q}{2}, \quad D = \frac{p + q}{4}.$$

These formulas exist and are unique for every real measurement vector. Value samples alone repeat only two linear constraints; their number does not restore the sine information. The figure displays the two independent invisible modes and their common sample zeros.



Question 8

Consider two homogeneous constant-coefficient operators

$$P(D) = (D - 1)^2(D + 2)(D^2 + 1), \quad Q(D) = (D - 1)(D + 2)^2(D^2 + 4).$$

A *common solution* must satisfy both $P(D)y = 0$ and $Q(D)y = 0$ on $\mathbb{R}$.

Tasks

1. Write the complete real general solution of each equation separately.
2. Determine the full common solution space and its dimension. Justify why the smaller multiplicity at each shared root is the relevant one.
3. Find the common solution with $y(0) = 3$ and $y'(0) = 0$. Compute the higher initial derivatives $y''(0)$ and $y'''(0)$ that are then forced.
4. Find the monic real constant-coefficient operator of smallest order that annihilates every solution of either equation. Explain why this last requirement uses larger multiplicities and differs from finding common solutions.

Question 8 – Solution

Strategy. Compare each exponent and its polynomial degree, distinguishing an intersection of solution spaces from a space containing both.

Step 1: List the two five-dimensional families. For P , the roots are 1 twice, -2 , and $\pm i$, giving

$$y_P = (a + bx)e^x + ce^{-2x} + d \cos x + e \sin x.$$

For Q , the roots are 1, -2 twice, and $\pm 2i$, giving

$$y_Q = Ae^x + (B + Cx)e^{-2x} + E \cos 2x + F \sin 2x.$$

All constants in each display are independent.

Step 2: Retain only common modes. Exponential-polynomial modes at different exponents are independent. Equality of the two displays therefore removes both oscillatory pairs, xe^x , and xe^{-2x} . Precisely the simple shared modes survive:

$$y = Ae^x + Be^{-2x}.$$

They are independent, so the common space has dimension two. In polynomial language, the common characteristic factor is the greatest common divisor $(r - 1)(r + 2)$: a mode must satisfy both multiplicity restrictions and hence the smaller one.

Step 3: Fit the common initial data. We need $A + B = 3$ and $A - 2B = 0$, so $A = 2, B = 1$. Thus

$$y = 2e^x + e^{-2x}, \quad y''(0) = 6, \quad y'''(0) = -6.$$

Each exponential is a shared mode, proving both equations directly. The first two data fix the solution within the common space; higher data cannot be chosen independently there.

Step 4: Build the smallest operator containing both kernels. An operator annihilating every solution of either equation must include both repeated real-root chains and both oscillatory pairs. Its characteristic polynomial must be divisible by both P and Q . Their monic least common multiple is

$$R(r) = (r - 1)^2(r + 2)^2(r^2 + 1)(r^2 + 4), \quad \deg R = 8.$$

The multiplicity of each root is now the larger of its two multiplicities. This polynomial is sufficient by the characteristic-root rule and necessary because each listed mode must be killed. Thus its order is minimal and its monic choice unique. Its kernel contains both five-dimensional spaces, whereas the common space is their two-dimensional intersection.

Question 9

Multiplication by x is an operator: $(D - x)u = u' - xu$. On $\mathbb{R}$, consider

$$(D - x)D^2y = 0.$$

Unlike constant-coefficient factors, these factors need not commute.

Tasks

1. Expand the equation. Set $z = y''$ and find its complete real solution space, allowing definite integrals rather than elementary antiderivatives.
2. Define $F(x) = \int_0^x (x - t)e^{t^2/2} dt$. Prove that $1, x, F$ form a fundamental set by computing their Wronskian.
3. Solve the IVP $y(0) = y'(0) = 0, y''(0) = 1$. Prove that its solution is even and satisfies $y(x) \geq x^2/2$ for every real x , with strict inequality when $x \neq 0$.
4. Expand the reversed product $D^2(D - x)y$. Determine all functions that solve both the original and reversed equations, and explain why a characteristic polynomial cannot be used here as if x were constant.

Question 9 – Solution

Strategy. Solve the factors in their given order and use the product rule when reversing them.

Step 1: Reduce the original equation. Expansion gives $y''' - xy'' = 0$. Thus $z' - xz = 0$ and $z = Ce^{x^2/2}$. Integrating twice from zero yields

$$y = A + Bx + CF(x), \quad F(x) = \int_0^x (x-t)e^{t^2/2} dt.$$

This formula holds for negative x as well, with the usual oriented integral.

Step 2: Verify the fundamental set. Differentiating the integral gives $F' = \int_0^x e^{t^2/2} dt$ and $F'' = e^{x^2/2}$, so $F''' = xF''$. The derivative matrix is triangular:

$$W(1, x, F) = \det \begin{pmatrix} 1 & x & F \\ 0 & 1 & F' \\ 0 & 0 & e^{x^2/2} \end{pmatrix} = \boxed{e^{x^2/2} > 0}.$$

These are three independent solutions of a regular third-order linear equation, hence a fundamental set on $\mathbb{R}$.

Step 3: Fit and analyze the initial data. Since $F(0) = F'(0) = 0$ and $F''(0) = 1$, the IVP gives $\boxed{y = F}$. Its second derivative is even, and its value and slope at zero vanish; integrating shows that F is even. For $x > 0$,

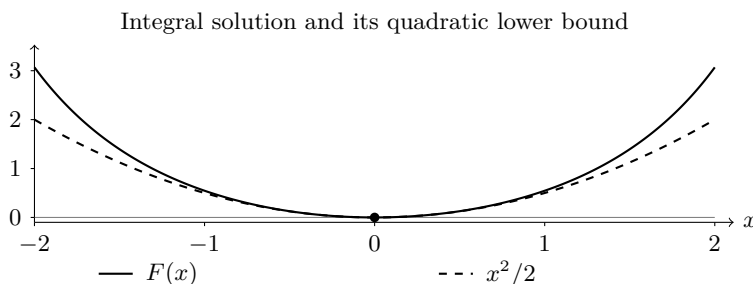
$$F(x) - \frac{x^2}{2} = \int_0^x (x-t)(e^{t^2/2} - 1) dt > 0.$$

The inequality follows because the integrand is positive for $0 < t < x$. Evenness proves the same strict inequality for $x < 0$; equality holds at zero.

Step 4: Respect the order of the factors. The product rule gives

$$D^2(y' - xy) = y''' - xy'' - 2y'.$$

Subtracting the two equations forces $y' = 0$ for any common solution. Conversely, every constant solves both, so their common space consists exactly of constants. The variable coefficient creates the extra $-2y'$ term; treating x as a fixed characteristic root would discard it and give an invalid method.



Question 10

For the fourth-order equation

$$y^{(4)} + 4y = 0,$$

define along a real solution

$$E(x) = y'''(x)y'(x) - \frac{1}{2}(y''(x))^2 + 2y(x)^2.$$

A quantity that is constant along solutions need not control their size.

Tasks

1. Find the characteristic roots and complete real general solution.
2. Differentiate E and prove that it is constant along every solution.
3. Verify that $y = e^x \cos x$ is an unbounded solution with $E = 0$. Give a specific sequence of times proving its unboundedness.
4. Characterize all solutions bounded on $[0, \infty)$ and all solutions bounded on $\mathbb{R}$. Explain why conservation of E does not contradict these classifications and cannot yield a bound of the form $|y(x)|^2 \leq CE(0)$ with a fixed positive C for all solutions.

Question 10 – Solution

Strategy. Compare the root-based behavior with what the conserved expression actually measures.

Step 1: Find growing and decaying oscillatory modes. Since $r^4 + 4 = (r^2 - 2r + 2)(r^2 + 2r + 2)$, the roots are $1 \pm i$ and $-1 \pm i$. Thus

$$y = e^x(A \cos x + B \sin x) + e^{-x}(C \cos x + D \sin x).$$

All roots are simple and the four real modes are independent.

Step 2: Verify the conservation law. The product rule gives exact cancellation:

$$E' = y^{(4)}y' + y'''y'' - y''y''' + 4yy' = y'(y^{(4)} + 4y) = 0.$$

Hence $E(x) = E(0)$ for every x .

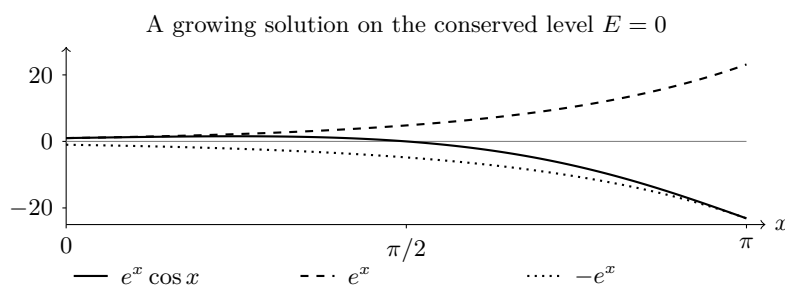
Step 3: Test a zero-level growing solution. For $y = e^x \cos x$, the successive derivatives are

$$y' = e^x(\cos x - \sin x), \quad y'' = -2e^x \sin x, \quad y''' = -2e^x(\sin x + \cos x), \quad y^{(4)} = -4y.$$

At zero its initial vector is $(1, 1, 0, -2)$, so $E(0) = -2 + 2 = 0$. Conservation yields $E \equiv 0$. Yet $y(2\pi n) = e^{2\pi n} \rightarrow \infty$, proving unboundedness.

Step 4: Identify the limits of the invariant. On $[0, \infty)$ boundedness requires $A = B = 0$: any nonzero growing sinusoid attains its amplitude along an unbounded sequence, while the other term decays. These conditions also suffice, giving a two-dimensional decaying family. Requiring boundedness backward as well forces $C = D = 0$ by the same phase argument, so the only entire-line bounded solution is zero.

The expression E is not positive definite in the four initial derivatives: it has a mixed product and a negative square. In particular a nonzero state can have $E = 0$. The displayed solution has $|y(0)|^2 = 1$ and $E(0) = 0$, immediately refuting the proposed bound for every positive C . Constancy of an indefinite expression does not guarantee bounded motion.



Question 1

Write $D = d/dx$ and let

$$L = D^2(D - 1)^3(D^2 + 4)^2.$$

Consider on $\mathbb{R}$ the equation

$$Ly = 3x^2 + (x + 1)e^x + x \cos 2x + e^{-x} \sin x.$$

This question concerns designing and justifying the coefficient trial, not carrying out a large elimination.

Tasks

1. State the order and complete real homogeneous solution, with all root multiplicities.
2. Write the standard undetermined-coefficient trial for a particular solution, including all polynomial degrees, both trigonometric partners, and every resonance factor. Count its unknown coefficients.
3. Explain why a trial containing only cosine terms for $x \cos 2x$, or only sine terms for $e^{-x} \sin x$, is not the appropriate full trial. Explain why one common resonance factor for all four forcing terms is inappropriate.
4. Prove that substitution determines exactly one member of your trial space. Use the effect of L on each polynomial-exponential family and its intersection with the homogeneous solution space. You need not compute the eleven coefficients.

Question 1 – Solution

Strategy. Treat each distinct exponential frequency separately and remove only its own homogeneous overlap.

Step 1: List the nine homogeneous modes. The roots are 0 twice, 1 three times, and $\pm 2i$ twice each. Thus L has order nine and

$$y_h = c_0 + c_1x + e^x(c_2 + c_3x + c_4x^2) + (c_5 + c_6x)\cos 2x + (c_7 + c_8x)\sin 2x.$$

Step 2: Build the trial block by block. A standard trial is

$$\begin{aligned} y_p = & x^2(a_2x^2 + a_1x + a_0) + x^3e^x(b_1x + b_0) \\ & + x^2((d_1x + d_0)\cos 2x + (e_1x + e_0)\sin 2x) \\ & + e^{-x}(f\cos x + g\sin x). \end{aligned}$$

There are $3 + 2 + 4 + 2 = \boxed{11}$ unknowns. The resonance powers are respectively 2, 3, 2, 0, because $-1 \pm i$ are not roots of L .

Step 3: Keep the derivative families complete. Differentiation mixes sine and cosine, and differentiating polynomial factors also generates lower powers of x . Therefore the full paired families must be allowed; coefficients that ultimately vanish should emerge from the equations. Resonance is tied to the particular exponent, not to the equation's total order. A common factor would either retain homogeneous overlap in some blocks or unnecessarily alter others.

Step 4: Establish solvability and uniqueness in the trial. For a root λ of multiplicity m , write the characteristic polynomial as $(r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. The shift identity gives

$$L(e^{\lambda x} x^m q) = e^{\lambda x} D^m R(D + \lambda)(x^m q).$$

For $\deg q \leq d$, the resulting polynomial has degree at most d ; its top coefficient is the top coefficient of q times $R(\lambda)(m + d)!/d!$, which is nonzero when that coefficient is nonzero. Successive coefficient matching is therefore invertible. Real and imaginary parts give the same conclusion for each paired block.

Equivalently, the eleven-dimensional trial maps into the eleven-dimensional forcing family and contains no nonzero homogeneous member: each block starts beyond its root's allowed homogeneous degree. Its kernel is zero, so this map is bijective. Hence the stated forcing determines exactly one trial member, although other particular solutions can differ by y_h .

Question 2

Consider the resonant third-order equation

$$(D - 1)^3 y = e^x(x^2 + 1)$$

on $\mathbb{R}$.

Tasks

1. Explain why $e^x(Ax^2 + Bx + C)$ cannot be a particular-solution trial. Give a complete resonance-corrected trial.
2. Use $y = e^x v$ to determine a particular solution by coefficient matching and write the full real general solution.
3. Solve the IVP $y(0) = y'(0) = y''(0) = 0$. Verify all three initial data and the original forcing.
4. For an arbitrary solution, calculate $\lim_{x \rightarrow \infty} e^{-x} y(x)/x^5$. Prove that no choice of homogeneous constants makes a solution bounded on $[0, \infty)$.

Question 2 – Solution

Strategy. Shift the triple root to zero, where the coefficient calculation becomes polynomial differentiation.

Step 1: Remove the homogeneous overlap. The characteristic root 1 has multiplicity three. Every proposed term $e^x(Ax^2 + Bx + C)$ is already homogeneous and gives zero under $(D-1)^3$. The corrected trial is

$$y_p = x^3 e^x (Ax^2 + Bx + C).$$

Step 2: Solve the shifted coefficient equations. Since $(D-1)(e^x v) = e^x v'$, the equation becomes $v''' = x^2 + 1$. For $v_p = Ax^5 + Bx^4 + Cx^3$, matching gives

$$60Ax^2 + 24Bx + 6C = x^2 + 1, \quad A = \frac{1}{60}, \quad B = 0, \quad C = \frac{1}{6}.$$

The complete solution is consequently

$$y = e^x \left(c_0 + c_1 x + c_2 x^2 + \frac{x^3}{6} + \frac{x^5}{60} \right).$$

Step 3: Impose and verify the initial data. With $v = e^{-x}y$, the initial values are

$$v(0) = y(0) = 0, \quad v'(0) = y'(0) - y(0) = 0, \quad v''(0) = y''(0) - 2y'(0) + y(0) = 0.$$

Thus $c_0 = c_1 = c_2 = 0$. The polynomial $v = x^3/6 + x^5/60$ and its first two derivatives vanish at zero, so the corresponding initial data for $y = e^x v$ vanish as well. Its third derivative is $v''' = 1 + x^2$, and the shift identity verifies the forcing exactly.

Step 4: Identify unavoidable resonant growth. For every choice of c_0, c_1, c_2 ,

$$\lim_{x \rightarrow \infty} \frac{e^{-x} y(x)}{x^5} = \frac{1}{60}.$$

The lower-degree terms disappear after division by x^5 . Hence $y(x)$ is eventually positive and asymptotic to $e^x x^5/60$, so it is unbounded. Homogeneous terms have polynomial degree at most two after removing e^x and cannot cancel the forced degree-five term.

Question 3

Consider the initially resting response of a fourth-order equation:

$$(D^2 + 1)^2 y = \cos x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Tasks

1. Identify the resonance multiplicity and choose a full real particular-solution trial. Explain why multiplying the ordinary sine-cosine trial by only x still fails.
2. Determine a particular solution by direct substitution, then add the complete homogeneous family.
3. Impose the four initial data and verify the resulting solution explicitly.
4. Evaluate the IVP solution at $x = 2\pi n$ and prove quadratic growth along this sequence. Explain why no homogeneous correction can make any solution bounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Account for the double conjugate roots before imposing the initial state.

Step 1: Use the full resonance multiplicity. The roots $\pm i$ both have multiplicity two. The functions $\cos x, \sin x, x \cos x, x \sin x$ are all homogeneous, so an x -weighted trial still gives zero. Use $y_p = x^2(A \cos x + B \sin x)$.

Step 2: Match the forcing directly. A first application gives $(D^2 + 1)(x^2 \cos x) = 2 \cos x - 4x \sin x$, and a second gives $-8 \cos x$. Likewise, $(D^2 + 1)^2(x^2 \sin x) = -8 \sin x$. Thus $A = -1/8$, $B = 0$, and

$$y = -\frac{x^2 \cos x}{8} + (a + bx) \cos x + (c + dx) \sin x.$$

Step 3: Recover the initially resting solution. The particular term has initial vector $(0, 0, -1/4, 0)$. The homogeneous term has vector $(a, b + c, -a + 2d, -3b - c)$. Setting their sum to zero gives $a = b = c = 0, d = 1/8$, hence

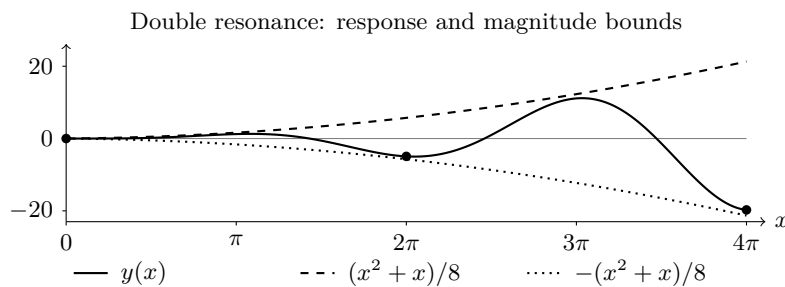
$$y = \frac{x \sin x - x^2 \cos x}{8}.$$

The two numerator terms cancel to order two: $x \sin x - x^2 \cos x = x^4/3 + O(x^6)$ near zero. Thus y and its first three derivatives vanish there. The term $x \sin x/8$ is homogeneous, while the already checked particular term produces $\cos x$.

Step 4: Separate quadratic forcing from linear freedom. At $x = 2\pi n$,

$$y(2\pi n) = -\frac{(2\pi n)^2}{8}.$$

This proves unbounded quadratic growth in magnitude along that sequence. Every homogeneous term is $O(x)$ on the positive half-line, so no correction can cancel these quadratic values. The figure shows the response and the valid envelopes $\pm(x^2 + x)/8$.



Question 4

On $\mathbb{R}$, consider

$$y^{(4)} - y = 6x^2 + 8e^{-x} + 10 \cos x.$$

Its three forcing terms require different coefficient calculations.

Tasks

1. Factor the characteristic polynomial, write the full real homogeneous solution, and select a separate particular-solution trial for each forcing term.
2. Determine all coefficients and assemble one particular solution.
3. Verify each forcing contribution directly using $L = D^4 - 1$, rather than relying only on the trial-selection rule. Write the complete general solution.
4. Decide whether any solution is bounded on $[0, \infty)$. Prove your answer even when homogeneous constants are chosen to cancel all removable growing terms.

Question 4 – Solution

Strategy. Use linearity to separate polynomial forcing from two distinct resonant frequencies.

Step 1: Identify roots and trials. The characteristic roots are $1, -1, i, -i$, all simple, so

$$y_h = Ae^x + Be^{-x} + C \cos x + D \sin x.$$

Use $ax^2 + bx + c$ for $6x^2$, kxe^{-x} for $8e^{-x}$, and $x(m \cos x + n \sin x)$ for $10 \cos x$. Only the last two blocks are resonant.

Step 2: Compute the coefficients. The fourth derivative of a quadratic is zero, giving $a = -6, b = c = 0$. For a simple root λ of $P(r) = r^4 - 1$, the shift identity gives $P(D)(xe^{\lambda x}) = P'(\lambda)e^{\lambda x}$. Since $P'(-1) = -4$, we obtain $k = -2$.

For the trigonometric terms, $L(x \cos x) = 4 \sin x$ and $L(x \sin x) = -4 \cos x$. Therefore $m = 0, n = -5/2$, and

$$y_p = -6x^2 - 2xe^{-x} - \frac{5}{2}x \sin x.$$

Step 3: Verify all three residuals. Direct differentiation gives

$$L(-6x^2) = 6x^2, \quad D^4(xe^{-x}) = e^{-x}(x - 4),$$

$$D^4(x \sin x) = x \sin x - 4 \cos x.$$

Thus the second and third terms produce $8e^{-x}$ and $10 \cos x$ exactly. The full solution is $y = y_h + y_p$; no homogeneous constants are determined without additional data.

Step 4: Rule out bounded solutions. If $A \neq 0$, the Ae^x term dominates every other term, so boundedness is impossible. If $A = 0$, division by x^2 gives

$$\lim_{x \rightarrow \infty} \frac{y(x)}{x^2} = -6,$$

because the remaining oscillatory growth is only $O(x)$ and the other exponential terms decay. Hence even after removing the positive exponential, quadratic forcing remains. No solution is bounded on the positive half-line.

Question 5

Consider

$$(D + 1)^2(D - 2)y = e^{-x}.$$

An annihilator can simplify the forcing while introducing extra solutions that must later be rejected.

Tasks

1. Apply a minimal constant-coefficient annihilator of the forcing and write the full solution of the resulting fourth-order homogeneous equation.
2. Substitute that enlarged family into the original third-order equation. Find the constraint that removes the extra freedom and write the actual general solution.
3. Find the solution with $y(0) = y'(0) = y''(0) = 0$. Determine the coefficient of its growing exponential and its limit after multiplication by e^{-2x} .
4. Instead require $y(0) = y'(0) = 0$ and decay as $x \rightarrow +\infty$. Find this solution and the initial value $y''(0)$ it requires. Explain why decaying forcing does not guarantee that the zero-data response decays.

Question 5 – Solution

Strategy. Retain the original equation as a constraint after annihilating its right-hand side.

Step 1: Solve the enlarged equation. The forcing is annihilated by $D + 1$. Applying it gives $(D + 1)^3(D - 2)y = 0$, whose general solution is

$$y = e^{-x}(A + Bx + Cx^2) + Ee^{2x}.$$

This family has four constants, although the original equation has order three.

Step 2: Restore the original forcing. Under $y = e^{-x}v$, the original operator becomes $e^{-x}D^2(D - 3)v$. For $v = A + Bx + Cx^2$ its value is $-6Ce^{-x}$; the Ee^{2x} term is homogeneous. Thus $C = -1/6$, leaving

$$y = e^{-x}(A + Bx - x^2/6) + Ee^{2x}.$$

Annihilation alone only ensures that the original residual is a multiple of e^{-x} , not that its coefficient equals one.

Step 3: Fit the zero initial vector. The three equations are $A + E = 0$, $B - A + 2E = 0$, and $A - 2B - 1/3 + 4E = 0$. They yield $E = 1/27$, $A = -1/27$, $B = -1/9$, so

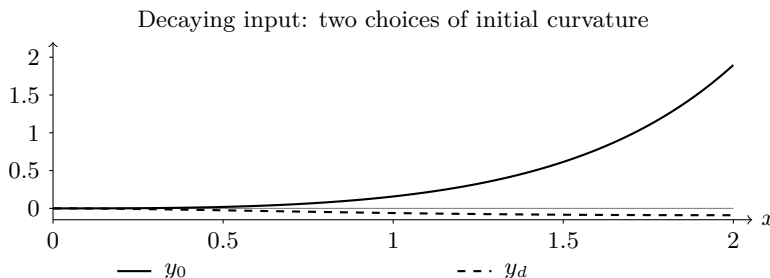
$$y_0 = \frac{e^{2x}}{27} - e^{-x} \left(\frac{1}{27} + \frac{x}{9} + \frac{x^2}{6} \right), \quad \lim_{x \rightarrow \infty} e^{-2x} y_0 = \frac{1}{27}.$$

The displayed coefficient equations verify the initial data, and Step 2 verifies the forcing.

Step 4: Replace curvature data by a future condition. Decay requires $E = 0$; with $y(0) = y'(0) = 0$, this gives $A = B = 0$. Hence

$$y_d = -x^2 e^{-x}/6, \quad y_d''(0) = -1/3.$$

This is the unique decaying response with those first two data. The zero-data response must add a growing homogeneous mode to cancel the particular solution's initial curvature. Decay of the input alone does not suppress that mode.



Question 6

Let $\omega \geq 0$ and consider the zero-data response

$$(D^2 + 1)(D^2 + 4)y_\omega = \cos(\omega x), \quad y_\omega(0) = y'_\omega(0) = y''_\omega(0) = y'''_\omega(0) = 0.$$

Tasks

1. For $\omega \neq 1, 2$, find a particular solution and then the full zero-data response by adding homogeneous modes.
2. Solve the IVP separately at each resonant frequency $\omega = 1$ and $\omega = 2$, using resonance-corrected trials.
3. Rewrite the nonresonant answer using differences of cosines and prove that it tends to the resonant answer as $\omega \rightarrow 1$ for every fixed x .
4. Classify which zero-data responses are bounded on $[0, \infty)$. Explain why the fixed- x limit in the previous task does not provide a bound uniform over all $x \geq 0$ as $\omega \rightarrow 1$.

Question 6 – Solution

Strategy. Enforce the initial data before taking a resonant limit; separate divergent terms can cancel.

Step 1: Solve away from resonance. Set $K = [(1 - \omega^2)(4 - \omega^2)]^{-1}$. The trial $K \cos \omega x$ has the required residual. Odd initial derivatives vanish; adding $A \cos x + C \cos 2x$ and matching the value and second derivative gives

$$y_\omega = K \cos \omega x - \frac{\cos x}{3(1 - \omega^2)} + \frac{\cos 2x}{3(4 - \omega^2)}, \quad \omega \neq 1, 2.$$

Indeed $A + C = -K$ and $A + 4C = -K\omega^2$, which verify the remaining two data.

Step 2: Solve at the two simple resonances. For $L = (D^2 + 1)(D^2 + 4)$, direct differentiation gives $L(x \sin x) = 6 \cos x$ and $L(x \sin 2x) = -12 \cos 2x$. Adding homogeneous cosine terms to enforce zero value and curvature yields

$$y_1 = \frac{x \sin x}{6} + \frac{\cos 2x - \cos x}{9}, \quad y_2 = -\frac{x \sin 2x}{12} + \frac{\cos x - \cos 2x}{9}.$$

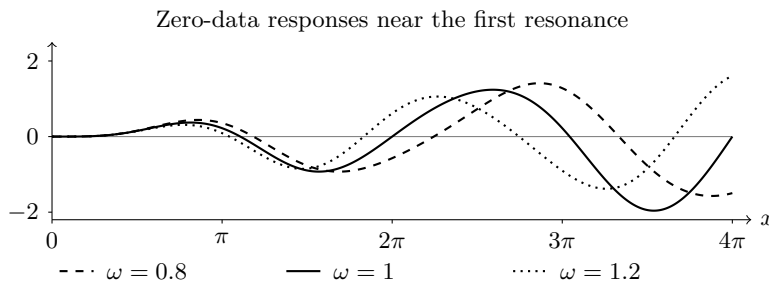
Both are even and vanish at zero. Their second derivatives at zero vanish by cancellation of $1/3$ with $-1/3$ in each case, so all four initial data hold.

Step 3: Take the limit of the combined response. Partial fractions give

$$y_\omega = \frac{1}{3} \left(\frac{\cos \omega x - \cos x}{1 - \omega^2} - \frac{\cos \omega x - \cos 2x}{4 - \omega^2} \right).$$

For fixed x , the first ratio tends to $x \sin x/2$ by differentiating numerator and denominator with respect to ω . The second tends to $(\cos x - \cos 2x)/3$. The resulting limit is exactly y_1 .

Step 4: Distinguish the two limit questions. Every fixed nonresonant response is a bounded sum of cosines. At resonance, the linear sine term has unbounded values along its phase maxima, and the added cosines remain bounded. Thus boundedness holds exactly when $\omega \notin \{1, 2\}$. A common bound for all $x \geq 0$ and all nonresonant ω sufficiently near 1 would pass to the fixed- x limit and bound y_1 everywhere, a contradiction.



Question 7

For real parameters a, b, c , consider

$$y^{(4)} + y'' = a + b \cos x + c \cos 2x.$$

A forcing with zero average over a period need not admit a periodic response.

Tasks

1. Use undetermined coefficients to find a particular solution for all a, b, c , and write the full real general solution.
2. Give necessary and sufficient conditions on the parameters and homogeneous constants for boundedness on $[0, \infty)$.
3. Determine exactly when the equation admits a 2π -periodic solution. Verify the necessary restrictions independently by integrating the equation against 1 and $\cos x$ over one period.
4. Give a forcing in this family with zero average but no periodic solution, and another nonzero forcing that does admit periodic solutions. Explain the different obstructions caused by a nonzero mean and by a resonant harmonic.

Question 7 – Solution

Strategy. Track the zero-frequency resonance and the oscillatory resonance separately.

Step 1: Calculate the resonant and nonresonant pieces. The characteristic polynomial is $r^2(r^2 + 1)$. Direct substitution gives

$$L(x^2/2) = 1, \quad L(x \sin x) = -2 \cos x, \quad L(\cos 2x) = 12 \cos 2x.$$

Consequently,

$$y = \frac{ax^2}{2} - \frac{b}{2}x \sin x + \frac{c}{12} \cos 2x + k_0 + k_1x + A \cos x + B \sin x.$$

The factors x^2 and x reflect the double root at zero and simple roots at $\pm i$, respectively.

Step 2: Classify boundedness exactly. If $a \neq 0$, division by x^2 gives a nonzero limit, so boundedness fails. Once $a = 0$, a bounded solution would satisfy

$$0 = \lim_{x \rightarrow \infty} \frac{y(x)}{x}, \quad \frac{y(x)}{x} = k_1 - \frac{b}{2} \sin x + o(1).$$

Along sequences with sine equal to 1 and -1 , this requires both $k_1 - b/2 = 0$ and $k_1 + b/2 = 0$. Hence

$$\text{bounded on } [0, \infty) \iff a = b = k_1 = 0.$$

These conditions suffice because all remaining terms are bounded.

Step 3: Check periodic compatibility independently. With $a = b = 0$, choosing $k_1 = 0$ gives a 2π -periodic solution for every c . Conversely, for a 2π -periodic C^4 solution, integration gives

$$\int_0^{2\pi} (y^{(4)} + y'') dx = 0 = 2\pi a.$$

Integration by parts transfers the even derivatives onto $\cos x$ without boundary terms. Since $(D^4 + D^2) \cos x = 0$, orthogonality then gives $0 = b\pi$. Thus $a = b = 0$ is necessary and sufficient for existence of a 2π -periodic response.

Step 4: Explain why zero mean is insufficient. The forcing $\cos x$ has zero average but produces the unavoidable term $-x \sin x/2$, so no solution is periodic. In contrast, $\cos 2x$ admits $y = \cos 2x/12$. A nonzero mean excites quadratic drift through the double zero root; a first harmonic excites linear oscillatory growth. Removing only the mean addresses just one of these two resonances.

Question 8

Consider the initially resting response

$$(D^2 + 1)(D^2 + 9)y = \cos^3 x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

A nonlinear expression in the independent variable can conceal several resonant harmonics while the differential equation remains linear in y .

Tasks

1. Expand the forcing into elementary harmonics. Identify all resonances and select a complete real particular-solution trial.
2. Determine a particular solution by coefficient matching, and write the full homogeneous correction.
3. Impose the initial data and verify the resulting response, including the forcing and the initial curvature.
4. Evaluate the IVP solution at $x_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, to prove unboundedness. Derive an explicit envelope for its magnitude valid for every $x \geq 0$.

Question 8 – Solution

Strategy. Expand the trigonometric product before choosing resonance factors.

Step 1: Expose both forcing frequencies. The identity $\cos^3 x = (3 \cos x + \cos 3x)/4$ reveals simple resonance at both $\pm i$ and $\pm 3i$. A full trial is

$$x(A \cos x + B \sin x + C \cos 3x + D \sin 3x).$$

There is no nonlinear dependence on the unknown y ; the right-hand side is a known function of x .

Step 2: Match each harmonic. For $L = (D^2 + 1)(D^2 + 9)$, differentiation gives $L(x \sin x) = 16 \cos x$ and $L(x \sin 3x) = -48 \cos 3x$. The cosine-weighted partners generate sine forcing and therefore have zero coefficients. Thus

$$y_p = \frac{3x \sin x}{64} - \frac{x \sin 3x}{192}, \quad y_h = a \cos x + b \sin x + c \cos 3x + d \sin 3x.$$

The residual is $3 \cos x/4 + \cos 3x/4 = \cos^3 x$.

Step 3: Correct the initial curvature. The particular term is even, has value zero, and has $y_p''(0) = 3/32 - 1/32 = 1/16$. The zero data require $a + c = 0$, $-a - 9c = -1/16$, and $b + 3d = -b - 27d = 0$. Hence

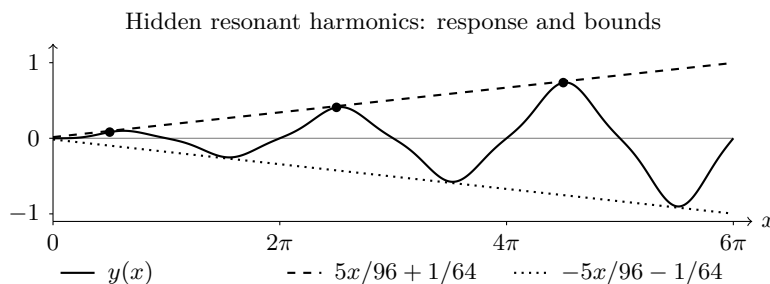
$$y = \frac{x(9 \sin x - \sin 3x)}{192} + \frac{\cos 3x - \cos x}{128}.$$

Here $a = -1/128, c = 1/128, b = d = 0$. The cosine correction has second derivative $-1/16$ at zero and cancels the particular curvature; evenness verifies the odd initial derivatives.

Step 4: Prove growth and a global envelope. At $x_n = \pi/2 + 2\pi n$, the sine values are 1, -1 and both cosines vanish, so $y(x_n) = 5x_n/96 \rightarrow \infty$. Also $9 \sin x - \sin 3x = 6 \sin x + 4 \sin^3 x$, whose absolute value is at most 10. Therefore

$$|y(x)| \leq \frac{5x}{96} + \frac{1}{64} \quad (x \geq 0).$$

The displayed envelopes are bounds, not assertions that equality holds at every point.



Question 9

For the operator $L = (D + 1)^3$, consider three separate forcing functions

$$f_1(x) = x^2 e^{-x}, \quad f_2(x) = \sin^2 x, \quad f_3(x) = e^{x^2}.$$

The standard finite undetermined-coefficient method uses polynomial-exponential-trigonometric families closed under differentiation, with resonance corrections when required.

Tasks

1. Decide which forcings admit this standard finite method. Give complete real trials for each admissible forcing, first simplifying trigonometric products.
2. Compute and verify one particular solution for each admissible forcing. State the common homogeneous family to be added in both cases.
3. Prove that the derivatives of e^{x^2} span an infinite-dimensional space: write $D^k e^{x^2} = P_k(x) e^{x^2}$ and determine the degree and leading coefficient of P_k .
4. Explain why this rules out the standard finite trial method for f_3 , while not ruling out existence or uniqueness of the IVP with any prescribed $y(0), y'(0), y''(0)$. Do not use variation of parameters to solve it here.

Question 9 – Solution

Strategy. Test closure under differentiation, then distinguish a method's scope from the equation's solvability.

Step 1: Select the admissible trial families. The root -1 has multiplicity three. For f_1 , use $y_{p,1} = x^3 e^{-x}(Ax^2 + Bx + C)$. For $f_2 = (1 - \cos 2x)/2$, none of the frequencies $0, \pm 2i$ is a characteristic root, so use $y_{p,2} = a + b \cos 2x + c \sin 2x$. Both have finite derivative families. The third forcing does not, as shown below.

Step 2: Match and check the admissible cases. For $y = e^{-x}v$, the equation becomes $v''' = x^2$ in the first case. Thus

$$y_{p,1} = x^5 e^{-x} / 60.$$

Its shifted third derivative is exactly x^2 . In the second case,

$$L(1) = 1, \quad L(\cos 2x) = -11 \cos 2x + 2 \sin 2x, \quad L(\sin 2x) = -2 \cos 2x - 11 \sin 2x.$$

The coefficient equations are $a = 1/2$, $-11b - 2c = -1/2$, $2b - 11c = 0$. They give

$$y_{p,2} = \frac{1}{2} + \frac{11}{250} \cos 2x + \frac{1}{125} \sin 2x.$$

Each complete solution adds $e^{-x}(c_0 + c_1 x + c_2 x^2)$.

Step 3: Prove infinite derivative dimension. Start with $P_0 = 1$. Differentiation gives $P_{k+1} = P'_k + 2xP_k$. If P_k has degree k and leading coefficient 2^k , the second term has degree $k+1$ and leading coefficient 2^{k+1} , whereas the derivative has lower degree. Induction proves these claims for every k . Polynomials of distinct degrees are linearly independent; multiplying by the never-zero e^{x^2} preserves independence. Hence no nonzero constant-coefficient operator annihilates f_3 .

Step 4: Interpret the obstruction correctly. Every standard finite trial, even after multiplication by a resonance power, lies in a finite-dimensional differentiation-invariant family. Applying L keeps it in that family. It therefore cannot produce e^{x^2} , whose derivative family is infinite-dimensional. This is a limitation of the specified method. The normalized equation has constant coefficients and smooth forcing on $\mathbb{R}$, so the linear IVP theorem still gives a unique global solution for any three initial data.

Question 10

A forcing pulse acts on a third-order equation for $x \geq 0$:

$$(D+1)^3 y = \begin{cases} e^{-x} p(x), & 0 < x < 1, \\ 0, & x > 1, \end{cases} \quad p(x) = 120x^3 + ax^2 + bx + c.$$

Initially $y(0) = y'(0) = y''(0) = 0$. Require y, y', y'' to be continuous at $x = 1$; the equation is imposed on the two open intervals, not at the switch itself.

Tasks

1. Set $y = e^{-x}v$. Write the shifted equations and identify the conditions at $x = 1$ equivalent to having $y(x) = 0$ for every $x \geq 1$.
2. Use a polynomial undetermined-coefficient solution on $[0, 1]$ to find the unique a, b, c producing this complete cancellation. Give y on both intervals.
3. Prove that no nonzero polynomial pulse of degree at most two can achieve the same cancellation under the same zero initial data.
4. Verify all initial and joining conditions and the forcing. Determine whether the resulting solution is C^3 across $x = 1$, and locate the minimum of the scaled response $e^x y$ on $[0, 1]$.

Question 10 – Solution

Strategy. Convert the pulse into a polynomial third derivative and enforce triple zeros at both ends.

Step 1: Translate the initial and terminal states. The shift gives $v''' = p$ on $(0, 1)$ and $v''' = 0$ on $(1, \infty)$. Multiplication by the smooth, nonzero e^{-x} preserves matching of the first three state components. Thus $v(0) = v'(0) = v''(0) = 0$, and complete cancellation requires $v(1) = v'(1) = v''(1) = 0$. These terminal conditions also suffice: the post-pulse quadratic solution is then zero.

Step 2: Determine the polynomial exactly. The zero-data polynomial solution of $v''' = p$ has degree six and leading coefficient 1, since $D^3x^6 = 120x^3$. The three zero conditions at each endpoint require divisibility by $x^3(x-1)^3$. Equal degrees and leading coefficients therefore force

$$v = x^3(x-1)^3 = x^6 - 3x^5 + 3x^4 - x^3.$$

Differentiating three times gives $p = 120x^3 - 180x^2 + 72x - 6$. Hence

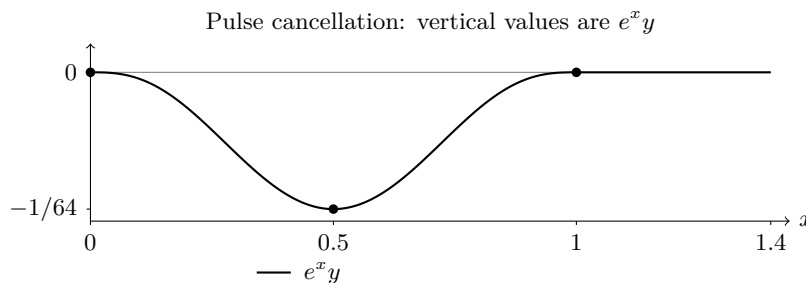
$$(a, b, c) = (-180, 72, -6), \quad y(x) = \begin{cases} e^{-x}x^3(x-1)^3, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases}$$

The formulas agree at the shared endpoint and uniquely determine the pulse.

Step 3: Prove the degree is necessary. For a polynomial p of degree at most two, the zero-data solution v has degree at most five. The terminal requirements would still make it divisible by the degree-six polynomial $x^3(x-1)^3$. It must therefore be zero, and so must $p = v'''$. No nonzero lower-degree polynomial pulse works.

Step 4: Check the join and the scaled shape. The triple endpoint zeros verify all six state conditions. The computed third derivative verifies the forcing before the switch; the zero branch verifies it afterward. At 1, the left third derivative of y is $e^{-1}v'''(1) = 6/e$, while the right third derivative is zero. Thus y is C^2 but not C^3 across the switch, exactly as allowed.

On $[0, 1]$, $e^xy = -[x(1-x)]^3$. Since $x(1-x)$ has maximum $1/4$ at $x = 1/2$, the scaled response has its unique minimum $-1/64$ there.



Question 1

Consider the fourth-order IVP

$$(1 + x^2)y^{(4)} = 1, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the normalized homogeneous basis $1, x, x^2/2, x^3/6$.

Tasks

1. Normalize the equation and verify that the stated basis is fundamental. Write the four equations for the parameter derivatives in variation of parameters.
2. Solve those equations. Integrate each parameter from zero and combine the result into one definite integral for $y(x)$.
3. Verify the equation and all initial data by differentiating the combined integral. Explain exactly what goes wrong if the leading coefficient is not divided out.
4. Prove upper and lower bounds for $y(x)$ when $x \geq 0$ using the largest and smallest values of $1/(1 + t^2)$ on $[0, x]$. State when equality holds.

Question 1 – Solution

Strategy. Normalize first, then combine the four parameter integrals into one positive kernel.

Step 1: Form the normalized parameter system. Put $g(x) = 1/(1+x^2)$. The derivative matrix of the given basis is triangular with unit diagonal, so its Wronskian is 1. The parameter derivatives satisfy

$$\begin{pmatrix} 1 & x & x^2/2 & x^3/6 \\ 0 & 1 & x & x^2/2 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u'_0 \\ u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g \end{pmatrix}.$$

The first three zero rows remove derivative terms introduced by varying the constants; the last row supplies the normalized forcing.

Step 2: Solve and combine. Back substitution gives $(u'_0, u'_1, u'_2, u'_3) = g(-x^3/6, x^2/2, -x, 1)$. Choose each $u_j(0) = 0$. In $y = \sum u_j f_j$, the combined numerator is $(x-t)^3$, yielding

$$y(x) = \frac{1}{6} \int_0^x \frac{(x-t)^3}{1+t^2} dt.$$

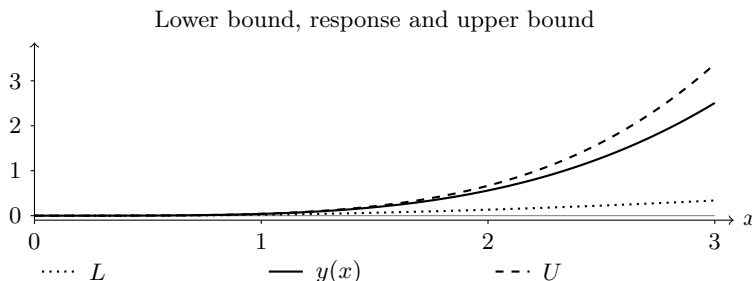
Oriented integration also defines the solution for negative x .

Step 3: Verify the initial state and residual. Successive derivatives have kernels $(x-t)^2/2$, $x-t$, and 1, all divided by $1+t^2$. The fourth derivative is $1/(1+x^2)$. The first four state components vanish at zero, and multiplication by $1+x^2$ gives the original forcing. If one incorrectly uses $g = 1$, the same construction gives $x^4/24$, whose original residual is $1+x^2$, not 1.

Step 4: Bound the positive integral. For $0 \leq t \leq x$, $1/(1+x^2) \leq 1/(1+t^2) \leq 1$. Hence

$$\frac{x^4}{24(1+x^2)} \leq y(x) \leq \frac{x^4}{24} \quad (x \geq 0).$$

Both inequalities are strict for $x > 0$, since their integrands differ on an interval of positive length where $(x-t)^3 > 0$. Both become equalities at $x = 0$. The figure labels the lower and upper bounds L and U , respectively.



Question 2

For $x \geq 0$, consider

$$y''' - y' = e^{-x^2}.$$

Use the homogeneous basis $1, e^x, e^{-x}$; definite integrals may be left unevaluated.

Tasks

1. Verify the basis and derive the three parameter derivatives by variation of parameters.
2. Find the solution y_0 with $y_0(0) = y_0'(0) = y_0''(0) = 0$, expressed as one integral. Verify its kernel and initial data.
3. Determine $\lim_{x \rightarrow \infty} e^{-x} y_0(x)$ and prove that it is positive. Justify convergence of every improper integral used.
4. Keep $y(0) = y'(0) = 0$ but choose $y''(0)$ so that the response is bounded on $[0, \infty)$. Prove uniqueness of this choice and determine the finite limit of the resulting solution.

Question 2 – Solution

Strategy. Use parameter integrals to isolate the growing homogeneous mode and the initial curvature needed to cancel it.

Step 1: Solve the derivative system. The functions solve $y''' - y' = 0$ and have Wronskian 2. With $g = e^{-x^2}$, the equations are

$$u'_0 + e^x u'_+ + e^{-x} u'_- = 0, \quad e^x u'_+ - e^{-x} u'_- = 0, \quad e^x u'_+ + e^{-x} u'_- = g.$$

Thus $u'_0 = -g$, $u'_+ = e^{-x}g/2$, and $u'_- = e^xg/2$.

Step 2: Construct the zero-data response. Integrating from zero and combining gives

$$y_0(x) = \int_0^x [\cosh(x-t) - 1] e^{-t^2} dt.$$

For $G(s) = \cosh s - 1$, we have $G(0) = G'(0) = 0$, $G''(0) = 1$, and $G''' - G' = 0$. Leibniz differentiation therefore gives $y_0''' - y_0' = g$ and the three zero initial data.

Step 3: Identify the growing coefficient. Define

$$A(x) = \int_0^x e^{-t^2-t} dt, \quad B(x) = \int_0^x e^{-t^2+t} dt, \quad C(x) = \int_0^x e^{-t^2} dt.$$

Their limits $A_\infty, B_\infty, C_\infty$ are finite: for $t \geq 2$, each integrand is at most $e^{-t^2/2} \leq e^{-t}$. All are positive. Since $y_0 = e^x A(x)/2 + e^{-x} B(x)/2 - C(x)$,

$$\lim_{x \rightarrow \infty} e^{-x} y_0(x) = A_\infty/2 > 0.$$

Step 4: Cancel growth without losing the first two data. Any other solution with the same value and slope is $y = y_0 + k(\cosh x - 1)$, where $k = y''(0)$. Boundedness requires $k = -A_\infty$, and no other choice cancels the growing coefficient. For this choice,

$$y = -\frac{1}{2}e^x(A_\infty - A(x)) + \frac{1}{2}e^{-x}(B(x) - A_\infty) + A_\infty - C(x).$$

The first term tends to zero: for $x > 0$, $e^x \int_x^\infty e^{-t^2-t} dt \leq e^{-x^2}$. The second also tends to zero. Therefore the choice is sufficient, and

$$y''(0) = -A_\infty, \quad \lim_{x \rightarrow \infty} y = A_\infty - C_\infty.$$

Question 3

On $x > 0$, consider the Euler-type equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x.$$

Use the proposed homogeneous basis x, x^2, x^3 and variation of parameters, rather than a guessed logarithmic trial.

Tasks

1. Verify the three homogeneous solutions and their Wronskian. Normalize the equation and write the parameter system with the correct right-hand side.
2. Solve for the parameter derivatives and integrate to obtain one particular solution. Write the complete general solution.
3. Impose $y(1) = y'(1) = y''(1) = 0$. Verify all three initial data and the forcing directly.
4. State the interval containing 1 on which this construction and the standard linear IVP theorem apply. Explain why evaluating the same parameter system at $x = 0$ is invalid, even if some expressions have finite limits there.

Question 3 – Solution

Strategy. Normalize the Euler operator before solving the derivative equations, and retain its positive-domain restriction.

Step 1: Verify the basis and normalization. For $y = x^m$, the left side is $(m-1)(m-2)(m-3)x^m$. Thus the proposed functions are homogeneous. Their Wronskian is $2x^3 \neq 0$ for $x > 0$. Division by x^3 gives normalized forcing $x \ln x$, so

$$\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \ln x \end{pmatrix}.$$

Step 2: Integrate the solved parameter derivatives. Elimination gives $u'_1 = x^2 \ln x / 2$, $u'_2 = -x \ln x$, $u'_3 = \ln x / 2$. Convenient antiderivatives are

$$u_1 = \frac{x^3 \ln x}{6} - \frac{x^3}{18}, \quad u_2 = -\frac{x^2 \ln x}{2} + \frac{x^2}{4}, \quad u_3 = \frac{x \ln x}{2} - \frac{x}{2}.$$

Combining $xu_1 + x^2u_2 + x^3u_3$ gives

$$y_p = x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right), \quad y = y_p + c_1x + c_2x^2 + c_3x^3.$$

Integration constants are already represented by the homogeneous terms.

Step 3: Fit and check the initial vector. The particular solution has vector $(-11/36, -19/18, -5/2)$ at 1. The three linear equations for its cancellation give

$$y = \frac{x}{18} - \frac{x^2}{4} + \frac{x^3}{2} + x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right).$$

The added terms have vector $(11/36, 19/18, 5/2)$, verifying the data. For a direct residual check, let $\theta = xD$. The operator is $(\theta-1)(\theta-2)(\theta-3)$; on $x^4(a \ln x + b)$ it gives $x^4(6a \ln x + 11a + 6b)$. With $a = 1/6$, $b = -11/36$, this is exactly $x^4 \ln x$.

Step 4: Respect the regular interval. The normalized coefficients, forcing and nonsingular derivative matrix are valid throughout $(0, \infty)$, the largest such interval containing 1. At zero the leading coefficient and Wronskian vanish and $\ln x$ is undefined. Finite limits of selected terms do not make that parameter system invertible or extend the theorem across zero.

Question 4

Let g be continuous on $\mathbb{R}$ and consider $y''' = g(x)$. Two homogeneous bases are

$$f = (1, x, x^2/2), \quad \tilde{f} = (1 + x, x + x^2/2, 1 + x^2/2).$$

Regard these as row vectors of functions and write $y_p = fu = \tilde{f}v$, with column vectors of parameters.

Tasks

1. Find the constant matrix M such that $\tilde{f} = fM$. Verify that both bases are fundamental.
2. Derive u' by variation of parameters, then obtain v' from the change of basis. Explain why the lower-limit-zero constructions produce exactly the same particular solution.
3. Replace the lower integration limit 0 by an arbitrary real b . Find the difference between the two resulting particular solutions and prove that it is homogeneous.
4. Explain why changing basis or choosing separate integration constants does not create more than three free constants in the general solution. Distinguish unique parameter coordinates in a chosen basis from the nonuniqueness of a particular solution.

Question 4 – Solution

Strategy. Track the constant coordinate transformation and isolate the effect of the integration base point.

Step 1: Identify the basis transformation. The coefficient columns give

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \det M = 2.$$

All six displayed functions are quadratic or lower and solve $y''' = 0$. The Wronskian of f is 1, and that of $\tilde{f}$ is $\det M = 2$, so both triples are fundamental.

Step 2: Transform the coordinates. The standard derivative system gives $u' = g(x)(x^2/2, -x, 1)^T$. Since the derivative matrices satisfy $\tilde{F} = FM$, their parameter equations imply $Mv' = u'$, hence

$$v' = g(x) \begin{pmatrix} x^2/4 - x/2 - 1/2 \\ -x^2/4 - x/2 + 1/2 \\ x^2/4 + x/2 + 1/2 \end{pmatrix}.$$

With $u(0) = v(0) = 0$, integration gives $u = Mv$ at every x . Thus $fu = fMv = \tilde{f}v$ exactly, not merely up to a homogeneous term. Both equal

$$Y_0(x) = \frac{1}{2} \int_0^x (x-t)^2 g(t) dt.$$

Step 3: Change the integration base point. The construction based at b gives $Y_b(x) = \frac{1}{2} \int_b^x (x-t)^2 g(t) dt$. Therefore

$$Y_b - Y_0 = -\frac{x^2}{2} \int_0^b g(t) dt + x \int_0^b tg(t) dt - \frac{1}{2} \int_0^b t^2 g(t) dt.$$

The three integrals are constants with respect to x , even when $b < 0$. The difference is a quadratic polynomial and hence homogeneous.

Step 4: Count genuine solution freedom. Once a basis and lower limit are fixed, the invertible derivative matrix uniquely fixes the parameter derivatives, and the specified parameter values fix their integrals. Allowing arbitrary integration constants adds a constant linear combination of the three homogeneous basis functions. Adding a separate homogeneous triple afterward repeats that same freedom rather than creating six independent constants. Particular solutions need not coincide, but their differences are homogeneous; the full solution space of $y''' = g$ is one particular solution plus the three-dimensional homogeneous space.

Question 5

On the positive half-line, consider

$$y''' = \frac{1}{x}, \quad y(1) = y'(1) = y''(1) = 0.$$

Use the homogeneous basis $1, x, x^2/2$ and integrate the variation parameters from 1.

Tasks

1. Derive the parameter derivatives and compute the complete IVP solution.
2. Verify the original equation and all initial data. Find the one-sided limits of y, y', y'' as $x \rightarrow 0^+$.
3. Determine whether the solution admits a C^1 or C^2 extension to the endpoint 0. Explain why a divergent parameter does not necessarily force y itself to diverge.
4. Test the attempted lower-limit-zero formula $\frac{1}{2} \int_0^x (x-t)^2/t dt$. Decide whether it converges for fixed $x > 0$, and explain why finite endpoint values of y, y' do not justify using zero as a regular IVP base point.

Question 5 – Solution

Strategy. Separate parameter divergence, solution limits and the regularity needed for initial data.

Step 1: Integrate from the regular point. The triangular parameter system for $y''' = g$ gives $(u'_0, u'_1, u'_2) = (x^2 g/2, -xg, g)$. Here $g = 1/x$, so

$$u_0 = (x^2 - 1)/4, \quad u_1 = 1 - x, \quad u_2 = \ln x.$$

Combining $u_0 + xu_1 + x^2 u_2/2$ yields

$$y = \frac{x^2 \ln x}{2} - \frac{3x^2}{4} + x - \frac{1}{4} \quad (x > 0).$$

Step 2: Check derivatives and endpoint behavior. Direct differentiation gives

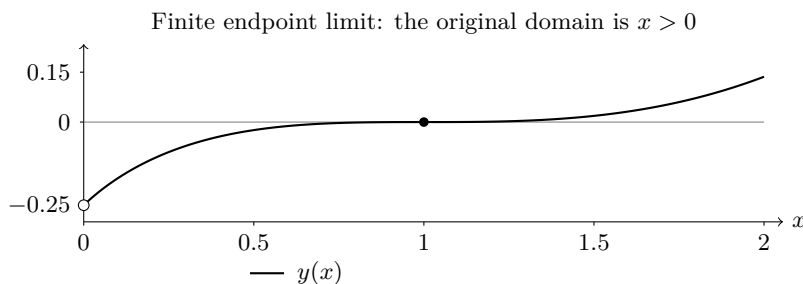
$$y' = x \ln x - x + 1, \quad y'' = \ln x, \quad y''' = 1/x.$$

These formulas and $\ln 1 = 0$ verify all three data at 1. Since $x \ln x$ and $x^2 \ln x$ tend to zero at the positive endpoint,

$$\lim_{x \rightarrow 0^+} y = -1/4, \quad \lim_{x \rightarrow 0^+} y' = 1, \quad \lim_{x \rightarrow 0^+} y'' = -\infty.$$

Step 3: Classify the extension. Defining $y(0) = -1/4$ gives right derivative 1, because $(y(x) + 1/4)/x \rightarrow 1$. This derivative is continuous from the right, so the solution extends as a C^1 function to $[0, \infty)$; it may also be continued to the left with matching value and slope. No C^2 extension is possible because y'' diverges. Although $u_2 = \ln x$ diverges, it is multiplied by $x^2/2$ in y , whose contribution tends to zero.

Step 4: Reject the singular base point. For fixed $x > 0$ and $0 < t < x/2$, $(x-t)^2/(2t) \geq x^2/(8t)$. The proposed improper integral therefore diverges to $+\infty$. Its lower limit cannot simply be moved to zero. The equation is singular there, and even the solution's second derivative lacks finite initial data. Two finite endpoint limits do not supply a regular third-order initial state.



Question 6

For continuous f on $[0, \infty)$, consider

$$(D + 1)^4 y = f(x), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the factorial-normalized homogeneous basis $e^{-x}, xe^{-x}, x^2e^{-x}/2, x^3e^{-x}/6$.

Tasks

1. Derive the variation-parameter equations or an equivalent triangular system, and combine their integrals into one response kernel.
2. Verify that the kernel gives the original equation and all four zero initial data for every continuous forcing.
3. If $|f(x)| \leq M$ for $M \geq 0$, derive an explicit bound on $|y(x)|$ for each $x \geq 0$. Prove its sharpness and deduce an all-time bound.
4. Apply the same reasoning to the difference of two responses with identical initial data and forcing discrepancy at most ε . Explain why bounded forcing need not produce a response tending to zero.

Question 6 – Solution

Strategy. Use the repeated-root basis to produce a positive kernel whose integral is explicit.

Step 1: Solve the shifted parameter system. Write $y = e^{-x}v$. The equation becomes $v^{(4)} = e^x f(x)$. The original basis becomes $1, x, x^2/2, x^3/6$ for v , with parameter derivatives

$$(u'_0, u'_1, u'_2, u'_3) = e^x f(x)(-x^3/6, x^2/2, -x, 1).$$

The homogeneous basis is fundamental: its Wronskian is $e^{-4x} \neq 0$. Integrating from zero gives

$$y(x) = \int_0^x G(x-t)f(t) dt, \quad G(s) = \frac{s^3 e^{-s}}{6}.$$

Step 2: Verify the response kernel. The function G satisfies $(D+1)^4 G = 0$ for $s > 0$, and $G(0) = G'(0) = G''(0) = 0$, $G'''(0) = 1$. Repeated Leibniz differentiation therefore adds no endpoint terms until the fourth derivative, where it adds exactly $f(x)$. Combining derivatives in $(D+1)^4 y$ cancels the integral terms. All four initial data vanish because the relevant integrals have zero length.

Step 3: Integrate the positive kernel sharply. For $s \geq 0$, $G(s) \geq 0$. Repeated integration by parts yields

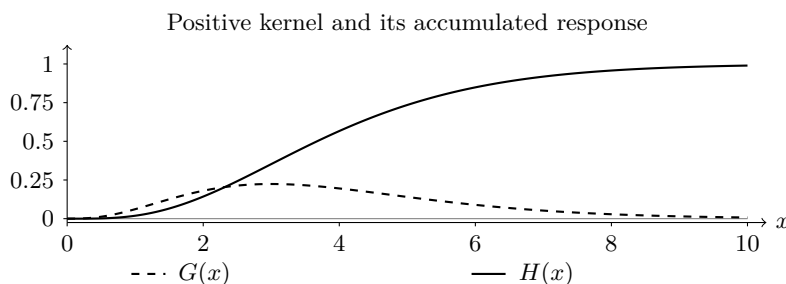
$$\int_0^x G(s) ds = 1 - e^{-x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} \right) = H(x).$$

Consequently,

$$|y(x)| \leq MH(x) \leq M \quad (x \geq 0).$$

The forcing $f \equiv M$ gives $y = MH$ and attains the pointwise bound. Also $H(x) \rightarrow 1$, so no smaller all-time constant than M works for all admissible forcings when $M > 0$. The case $M = 0$ gives the zero solution.

Step 4: Bound input error without claiming decay. For identical initial data, the response difference has zero initial vector and forcing difference bounded by ε . Thus $|y_1 - y_2| \leq \varepsilon H(x) \leq \varepsilon$. But $f \equiv 1$ gives $y = H(x) \rightarrow 1$, not zero. A positive integrable kernel controls size; it does not force the output to vanish for a persistent input.



Question 7

For the variable-coefficient IVP

$$y''' - xy'' = 1, \quad y(0) = y'(0) = y''(0) = 0,$$

define $F(x) = \int_0^x (x-t)e^{t^2/2} dt$ and use the homogeneous basis $1, x, F(x)$.

Tasks

1. Verify the basis and its Wronskian, then derive the three variation-parameter derivatives.
2. Combine the lower-limit-zero parameter integrals into $y(x) = \int_0^x K(x,t) dt$. Write K both using F, F' and as a definite integral for $x \geq t$.
3. Differentiate to verify the IVP. Prove that y is strictly convex and $y(x) > x^3/6$ for every $x > 0$.
4. Determine whether $K(x,t)$ can depend only on $x-t$. Use a derivative of the kernel to give a precise obstruction to treating this variable-coefficient response as a translation-invariant convolution.

Question 7 – Solution

Strategy. Exploit the triangular derivative matrix while retaining both kernel variables.

Step 1: Solve the parameter equations. We have $F(0) = F'(0) = 0$, $F'' = e^{x^2/2}$ and $F''' = xF''$. Thus $1, x, F$ solve the homogeneous equation and have Wronskian $e^{x^2/2} > 0$. The triangular system gives

$$u_2' = e^{-x^2/2}, \quad u_1' = -F'e^{-x^2/2}, \quad u_0' = (xF' - F)e^{-x^2/2}.$$

Step 2: Combine the parameter integrals. Multiplying by $1, x, F(x)$ and integrating from zero yields

$$K(x, t) = e^{-t^2/2} [F(x) - F(t) - (x - t)F'(t)].$$

Taylor's integral identity, obtained by integrating F'' twice, gives for $x \geq t$

$$K(x, t) = e^{-t^2/2} \int_t^x (x - s)e^{s^2/2} ds, \quad y(x) = \int_0^x K(x, t) dt.$$

The factor involving t is essential; it comes from the inverse derivative matrix evaluated at the integration variable.

Step 3: Verify the equation and inequalities. At the diagonal, $K(t, t) = K_x(t, t) = 0$, while

$$K_{xx}(x, t) = e^{(x^2-t^2)/2}, \quad y''(x) = e^{x^2/2} \int_0^x e^{-t^2/2} dt.$$

Hence $y''' = xy'' + 1$ and all three initial data vanish. For $x > 0$, $y'' > 0$, proving strict convexity. Also $y''' = 1 + xy'' > 1$ there. Integrating this strict inequality three times from the zero initial vector gives $y(x) > x^3/6$ for every $x > 0$.

Step 4: Test translation invariance. If $K(x, t) = H(x - t)$, then $K_{xx} = H''(x - t)$ would have the same value at all pairs with the same difference. But for any fixed $h > 0$,

$$K_{xx}(t + h, t) = e^{th+h^2/2},$$

which varies with t . Therefore this kernel is not a function of $x - t$ alone. The variable coefficient distinguishes the absolute source time from the elapsed time.

Question 8

Let g be continuous and nonnegative on $[0, \infty)$, and consider

$$y''' + y' = g(x), \quad y(0) = y'(0) = y''(0) = 0.$$

Use the homogeneous basis $1, \cos x, \sin x$.

Tasks

1. Derive the variation parameters and combine them into a single response integral.
2. Prove that $y(x) \geq 0$ for every $x \geq 0$. Give a sufficient condition for strict positivity at a specified $x > 0$ and justify it analytically.
3. For the continuous pulse $g(x) = \sin^2 x$ on $[0, \pi]$ and $g(x) = 0$ for $x \geq \pi$, compute $y(x)$ for all $x \geq \pi$ using the integrals of g , $g \cos x$ and $g \sin x$.
4. Determine the exact minimum and maximum of this post-pulse response. Prove that a nonnegative forcing need not make y monotone, even though the solution stays nonnegative.

Question 8 – Solution

Strategy. Distinguish a nonnegative response kernel from the sign of its derivative.

Step 1: Compute the response kernel. The derivative matrix of $1, \cos x, \sin x$ has Wronskian 1. Its parameter equations give

$$u'_0 = g, \quad u'_c = -g \cos x, \quad u'_s = -g \sin x.$$

Integrating from zero and combining by the cosine difference identity yields

$$y(x) = \int_0^x [1 - \cos(x-t)]g(t) dt.$$

The kernel and its first derivative vanish at zero, its second derivative is 1 there, and its third plus first derivative vanishes. These facts verify the equation and zero data.

Step 2: Prove positivity without a monotonicity assumption. Both factors in the integrand are nonnegative. If $g(t_0) > 0$ for some $0 \leq t_0 < x$, continuity gives an interval of positive forcing inside $[0, x)$. The zeros of $1 - \cos(x-t)$ are isolated, so the integral is strictly positive. In particular, a continuous nonnegative forcing that is not identically zero on $[0, x)$ suffices.

Step 3: Evaluate the post-pulse moments. For the stated pulse,

$$\int_0^\pi \sin^2 t dt = \frac{\pi}{2}, \quad \int_0^\pi \sin^2 t \cos t dt = 0, \quad \int_0^\pi \sin^3 t dt = \frac{4}{3}.$$

The first follows from the double-angle identity, the second from the derivative of $\sin^3 t/3$, and the third by substituting $u = \cos t$. Thus

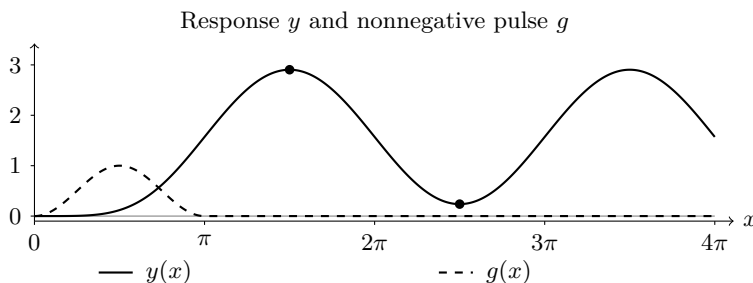
$$y(x) = \frac{\pi}{2} - \frac{4}{3} \sin x \quad (x \geq \pi).$$

For plotting the initial interval, evaluating the same kernel gives $y = x/2 - 2 \sin x/3 + \sin 2x/12$ on $[0, \pi]$; its state matches the displayed branch at π .

Step 4: Separate positivity from monotonicity. The post-pulse range is

$$\frac{\pi}{2} - \frac{4}{3} \leq y(x) \leq \frac{\pi}{2} + \frac{4}{3}.$$

Both bounds are attained infinitely often; the lower bound is positive. Yet $y' = -4 \cos x/3$ changes sign, and $y'(2\pi) = -4/3 < 0$. Thus the nonnegative input gives a nonnegative response that continues oscillating after the pulse. Positivity of $1 - \cos s$ does not imply positivity of its derivative $\sin s$.



Question 9

On $x > 0$, consider

$$x^4 y^{(4)} = 1, \quad y(1) = y'(1) = y''(1) = y'''(1) = 0.$$

A proposed response kernel for $x \geq t > 0$ is

$$\hat{K}(x, t) = \frac{(x-t)^3}{6x^4}.$$

The placement of the leading-coefficient factor must be checked, not guessed.

Tasks

1. Normalize the equation and derive the correct kernel from variation of parameters using $1, x, x^2/2, x^3/6$.
2. For fixed t , verify the correct kernel's homogeneous equation when $x > t$ and its first four diagonal data, through the third x -derivative.
3. Integrate the correct kernel from 1 to x to find the IVP solution in elementary form. Verify its initial data and forcing.
4. Show that $\hat{K}$ has the same diagonal data but is still wrong. Compute the response it would predict and evaluate $x^4 \hat{y}^{(4)}$ at $x = 2$. Explain which additional kernel requirement fails.

Question 9 – Solution

Strategy. Evaluate the normalized forcing at the integration variable and check the kernel away from its diagonal.

Step 1: Derive the correctly normalized kernel. The equation is $y^{(4)} = x^{-4}$. For the given basis, variation of parameters gives $u' = x^{-4}(-x^3/6, x^2/2, -x, 1)^T$. Integrating from 1 and combining yields

$$K(x, t) = \frac{(x-t)^3}{6t^4}, \quad y(x) = \int_1^x K(x, t) dt.$$

The denominator is evaluated at the source variable t . The same polynomial expression and oriented integral apply for $0 < x < 1$.

Step 2: Verify both kernel requirements. For fixed $t > 0$, K is a cubic polynomial in x , so $x^4 K_{xxxx} = 0$ for $x > t$. Its diagonal data are

$$K(t, t) = K_x(t, t) = K_{xx}(t, t) = 0, \quad K_{xxx}(t, t) = 1/t^4.$$

Leibniz differentiation gives $y^{(4)}(x) = 1/x^4$, because the fourth kernel derivative is zero and the third diagonal derivative supplies exactly that term.

Step 3: Evaluate and check the elementary response. Expanding $(x-t)^3/t^4$ and integrating its four powers of t gives

$$y = \frac{x^3}{18} - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln x}{6} - \frac{11}{36}.$$

Its derivatives are $y' = x^2/6 - x/2 + 1/2 - 1/(6x)$, $y'' = x/3 - 1/2 + 1/(6x^2)$, and $y''' = 1/3 - 1/(3x^3)$. These and y vanish at 1; one further derivative gives $y^{(4)} = x^{-4}$.

Step 4: Expose the misleading diagonal agreement. The factor $(x-t)^3$ ensures that $\hat{K}$ has the same three zero diagonal values and $\hat{K}_{xxx}(t, t) = 1/t^4$. But its proposed response is

$$\hat{y} = \frac{1}{6x^4} \int_1^x (x-t)^3 dt = \frac{(x-1)^4}{24x^4}.$$

It satisfies the zero initial data, yet

$$x^4 \hat{y}^{(4)} = -\frac{4}{x} + \frac{30}{x^2} - \frac{60}{x^3} + \frac{35}{x^4}, \quad x^4 \hat{y}^{(4)} \Big|_{x=2} = \boxed{3/16 \neq 1}.$$

The missing requirement is the homogeneous equation in x away from $x = t$. Correct diagonal data alone do not make a valid response kernel.

Question 10

For the zero-data IVP

$$y''' + y' = e^x, \quad y(0) = y'(0) = y''(0) = 0,$$

a proposed approximation on $[0, 1]$ is $q(x) = x^3/6 + x^4/24$.

Tasks

1. Compute the residual $r = q''' + q' - e^x$ and check the approximation's initial data. Express the residual using the remainder after the cubic Taylor polynomial of e^x .
2. Use variation of parameters to represent the error $E = y - q$ by a definite integral with zero initial data. Prove whether the approximation lies above or below the true solution on $(0, 1]$.
3. Use Taylor's remainder theorem and $1 - \cos s \leq s^2/2$ to prove $0 \leq E(x) \leq ex^7/5040$ for $0 \leq x \leq 1$. Decide whether an absolute-error tolerance of 10^{-3} is certified throughout this interval.
4. Independently verify $Y(x) = (e^x + \cos x - \sin x)/2 - 1$ as the exact IVP solution. Compare its error at $x = 1$ with the certified upper bound, and explain why direct agreement at finitely many sample points is weaker than the integral certificate.

Question 10 – Solution

Strategy. Treat the residual as an error forcing and bound its full response, not just sampled values.

Step 1: Compute the forcing error. We have $q''' = 1 + x$ and $q' = x^2/2 + x^3/6$, so

$$r = 1 + x + x^2/2 + x^3/6 - e^x = -R_4(x),$$

where $R_4 = e^x - (1 + x + x^2/2 + x^3/6)$. The values $q(0), q'(0), q''(0)$ are zero. Thus $E = y - q$ has a zero initial vector and satisfies $E''' + E' = R_4$.

Step 2: Derive the positive error integral. For the basis $1, \cos x, \sin x$, the variation derivatives for this error equation are $(R_4, -R_4 \cos x, -R_4 \sin x)$. Integrating from zero gives

$$E(x) = \int_0^x [1 - \cos(x-t)] R_4(t) dt.$$

For $0 < t < x \leq 1$, both factors are strictly positive. Hence $E(x) > 0$ on $(0, 1]$: the approximation underestimates the true solution.

Step 3: Certify a uniform error bound. Taylor's theorem gives $0 \leq R_4(t) \leq et^4/24$ on $[0, 1]$. Therefore

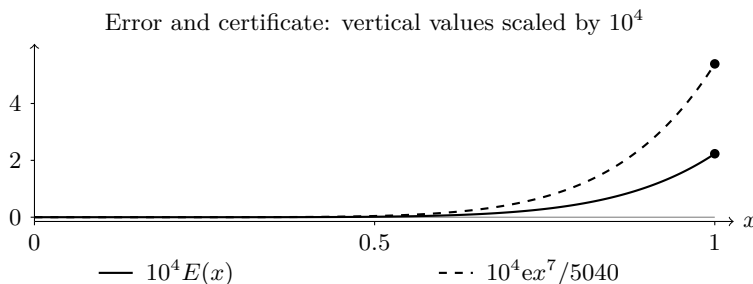
$$0 \leq E(x) \leq \frac{e}{48} \int_0^x (x-t)^2 t^4 dt = \boxed{\frac{ex^7}{5040}},$$

where the polynomial integral equals $x^7/105$. The maximum certified error is $e/5040 \approx 0.00053934 < 10^{-3}$, so the tolerance holds throughout $[0, 1]$.

Step 4: Compare with an exact independent check. For $Y = (e^x + \cos x - \sin x)/2 - 1$, direct differentiation gives $Y''' + Y' = e^x$ and its initial vector is zero. Uniqueness therefore implies $Y = y$. At 1, $q(1) = 5/24$, so the actual error is

$$E(1) = \frac{e + \cos 1 - \sin 1}{2} - \frac{29}{24}.$$

Numerically, $E(1) \approx 0.00022324$, positive and smaller than the certified bound. Exact evaluation or sample agreement is useful corroboration, but finitely many values cannot exclude larger errors between them. The integral inequality proves the bound for every point of the interval.



Question 1

Consider the initial-value problem, for $t \geq 0$,

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \quad (y, y', y'', y''')(0) = (1, 0, 0, 0).$$

A student writes $(s+1)^4 Y(s) = s^3$ and concludes that the initial displacement can be represented by a single term from the fourth derivative.

Tasks

1. Derive the correct transformed equation, displaying the initial terms contributed by every derivative. Identify the student's error.
2. Find $Y(s)$ and invert it by organizing the numerator in powers of $s + 1$.
3. Verify the differential equation and all four initial values directly. Determine whether the response ever increases or changes sign.
4. Find a half-plane of convergence and the initial and final limits. Explain why a quadruple pole does not prevent decay, and sketch the response.

Question 1 – Solution

Strategy. Collect every initial contribution before exploiting the repeated shifted pole.

Step 1: Transform all derivatives. The derivative transforms are $s^4Y - s^3$, $s^3Y - s^2$, $s^2Y - s$, and $sY - 1$. Thus

$$(s+1)^4Y = s^3 + 4s^2 + 6s + 4.$$

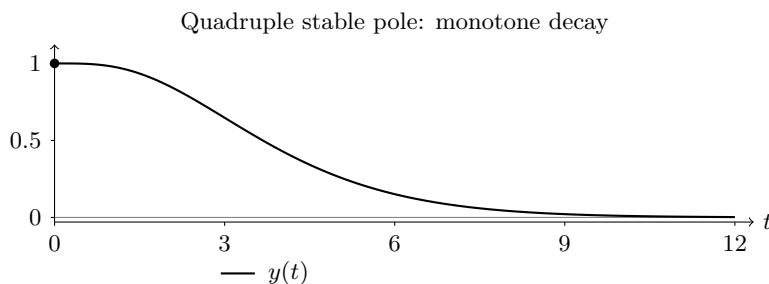
The student omitted the initial terms from $4y''' + 6y'' + 4y'$.

Step 2: Use shifted powers. Writing $z = s + 1$ makes the numerator $z^3 + z^2 + z + 1$. Therefore

$$Y = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}, \quad \boxed{y(t) = e^{-t} \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} \right)}.$$

Step 3: Check the state and shape. For $y = e^{-t}p(t)$, $(D+1)^4y = e^{-t}p^{(4)} = 0$. Moreover $y' = -t^3e^{-t}/6$, so $y'(0) = y''(0) = y'''(0) = 0$ and $y(0) = 1$. Every term in p is positive on $t \geq 0$; hence $y > 0$ and $y' < 0$ for $t > 0$.

Step 4: Check convergence and limits. The transform converges absolutely for $\operatorname{Re} s > -1$. Also $\lim_{s \rightarrow \infty} sY = 1$ and $\lim_{s \rightarrow 0} sY = 0$. The final-value theorem applies because all poles of sY lie at -1 . Polynomial growth from multiplicity is dominated by e^{-t} , so $y(t) \rightarrow 0$.



Question 2

A sixth-order initial-value problem is

$$(D^2 + 1)^3 y = 0, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 4), \quad y^{(5)}(0) = 1,$$

where $D = d/dt$ and $t \geq 0$. To invert a repeated quadratic pole, introduce $H_a(s) = 1/(s^2 + a^2)$ for $a > 0$.

Tasks

1. Expand the differential operator and transform the problem. Explain why only one initial term survives.
2. Starting with $\mathcal{L}^{-1}H_a = \sin(at)/a$, use differentiation with respect to a twice to invert $(s^2 + a^2)^{-3}$. Then set $a = 1$.
3. Verify the equation and initial state without relying on the transform calculation. Identify the first nonzero Taylor term.
4. Describe the pole multiplicities and a convergence half-plane. Decide whether the solution is bounded and whether the final-value theorem can be used, giving a decisive argument for each claim.

Question 2 – Solution

Strategy. Parameter differentiation resolves repeated quadratic poles while a Taylor check tests the high-order initial data.

Step 1: Transform the sixth-order problem. The equation is $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$. Only $\mathcal{L}(y^{(6)}) = s^6 Y - 1$ has a nonzero initial subtraction, so

$$Y(s) = \frac{1}{(s^2 + 1)^3}.$$

Step 2: Differentiate a transform family. Since $\partial_a(s^2 + a^2)^{-m} = -2ma(s^2 + a^2)^{-m-1}$,

$$h_2(t) = \frac{\sin(at) - at \cos(at)}{2a^3}, \quad h_3(t) = -\frac{1}{4a} \partial_a h_2(t) = \frac{3 \sin(at) - 3at \cos(at) - a^2 t^2 \sin(at)}{8a^5}.$$

Differentiation under the integral is valid for a in a compact positive interval and $\operatorname{Re} s > 0$, since the derivatives have polynomial growth. Hence

$$y(t) = \frac{3 \sin t - 3t \cos t - t^2 \sin t}{8}.$$

Step 3: Verify independently. Each term is the real or imaginary part of e^{it} times a polynomial of degree at most two; $(D - i)^3$ annihilates these complex modes, and the conjugate factor annihilates their conjugates. Thus $(D^2 + 1)^3 y = 0$. Expanding the displayed trigonometric expression gives

$$y(t) = \frac{t^5}{120} - \frac{t^7}{1680} + O(t^9).$$

This proves the five zero initial derivatives and $y^{(5)}(0) = 1$.

Step 4: Distinguish oscillation from boundedness. There are triple poles at $s = \pm i$; $\operatorname{Re} s > 0$ gives absolute convergence. Along $t_n = \pi/2 + 2\pi n$, $y(t_n) = (3 - t_n^2)/8 \rightarrow -\infty$, so the response is unbounded. The poles of sY on the imaginary axis invalidate the final-value theorem. The formal value $\lim_{s \rightarrow 0} sY = 0$ is not a time limit.

Question 3

Let $u(t - a)$ denote the unit step at a . Solve the delayed-ramp problem

$$(D + 2)^3 y = u(t - 1)(t - 1), \quad (y, y', y'')(0) = (0, 0, 0), \quad t \geq 0.$$

The value assigned to $u(0)$ does not affect an ordinary Laplace transform.

Tasks

1. Transform the initial-value problem, taking care to shift the entire ramp.
2. Obtain a closed form for the response on each side of $t = 1$. Display a partial-fraction decomposition for the unshifted transform.
3. Check the equation on the two open time intervals and determine exactly which derivative first jumps at $t = 1$. State the global smoothness of y .
4. Prove the response is positive after switching, find its large-time linear asymptote, and sketch it together with that asymptote on $t \geq 1$.

Question 3 – Solution

Strategy. Solve a zero-state ramp response, then translate time and check the switching jet.

Step 1: Transform the shifted input. All initial terms vanish. The second shifting theorem gives

$$Y(s) = \frac{e^{-s}}{s^2(s+2)^3}, \quad \operatorname{Re} s > 0.$$

Step 2: Invert before translating. The unshifted fraction decomposes as

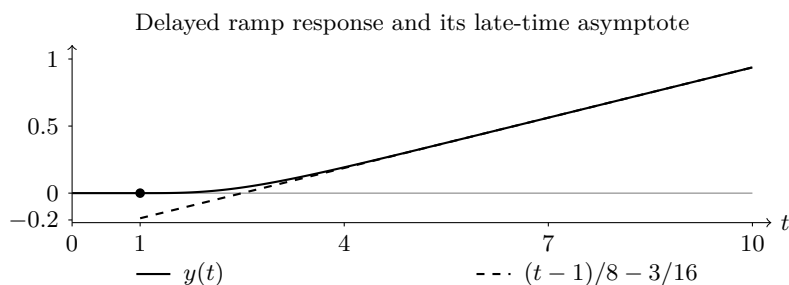
$$\frac{1}{s^2(s+2)^3} = -\frac{3}{16s} + \frac{1}{8s^2} + \frac{3}{16(s+2)} + \frac{1}{4(s+2)^2} + \frac{1}{4(s+2)^3}.$$

Thus, with $r = t - 1$,

$$g(r) = \frac{r}{8} - \frac{3}{16} + e^{-2r} \left(\frac{3}{16} + \frac{r}{4} + \frac{r^2}{8} \right), \quad y(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ g(t-1), & t > 1. \end{cases}$$

Step 3: Verify forcing and regularity. The exponential term is annihilated by $(D+2)^3$; applying it to $r/8 - 3/16$ gives r . Also $g(r) = r^4/24 + O(r^5)$ as $r \downarrow 0$. Consequently y, y', y'', y''' match zero at switching, while $y^{(4)}(1+) - y^{(4)}(1-) = 1$. The solution is C^3 but not C^4 ; the original equation also holds at $t = 1$.

Step 4: Establish shape and asymptote. Convolution gives $g(r) = \int_0^r (r-v)v^2 e^{-2v}/2 dv > 0$ for $r > 0$. The exponential correction tends to zero, so the asymptote is $y = (t-1)/8 - 3/16$, approached from above. It is drawn only for $t \geq 1$.



Question 4

Consider the causal, distributional equation

$$y^{(4)} + 2y''' + y'' = 3\delta(t-1) + \delta'(t-1).$$

The response is zero for $t < 0$, with no impulse at the origin. Here δ' is the distributional derivative of δ , and $[v]_1 = v(1+) - v(1-)$ denotes a jump.

Tasks

1. Find the transformed response, using the zero prehistory and both delayed distribution transforms.
2. First invert $H(s) = 1/[s^2(s+1)^2]$, then use differentiation to obtain a real-time formula for y .
3. Derive all four jumps $[y]_1, [y']_1, [y'']_1, [y''']_1$ by matching the coefficients of distributional derivatives. Verify them from your formula.
4. Determine the motion before the impulse, the eventual linear asymptote and the smoothness at switching. Explain why assigning an isolated value at $t = 1$ cannot remove the derivative jump. Sketch the response and its asymptote.

Question 4 – Solution

Strategy. Separate the impulse kernel from the input derivative, then match distributions rather than imposing false continuity conditions.

Step 1: Transform the distributions. For positive delay, $\mathcal{L}\delta(t-1) = e^{-s}$ and $\mathcal{L}\delta'(t-1) = se^{-s}$. Zero prehistory therefore gives

$$Y = e^{-s}(s+3)H(s), \quad H(s) = \frac{1}{s^2(s+1)^2}, \quad \operatorname{Re} s > 0.$$

Step 2: Invert the kernel. Since $H = -2/s + 1/s^2 + 2/(s+1) + 1/(s+1)^2$, $h(r) = r - 2 + (r+2)e^{-r}$ and $h(0) = 0$. Thus sH transforms h' , without an additional origin impulse. For $r = t - 1$,

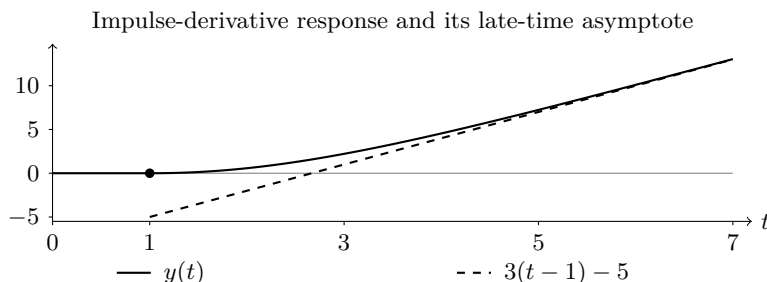
$$y(t) = u(t-1)k(r), \quad k(r) = 3r - 5 + (2r+5)e^{-r}.$$

Step 3: Match every singular coefficient. Put $J_j = [y^{(j)}]_1$. The coefficients of $\delta''', \delta'', \delta', \delta$ in the left side are, respectively,

$$J_0, \quad J_1 + 2J_0, \quad J_2 + 2J_1 + J_0, \quad J_3 + 2J_2 + J_1.$$

Matching 0, 0, 1, 3 gives $(J_0, J_1, J_2, J_3) = (0, 0, 1, 1)$. Directly, $(k, k', k'', k''')(0) = (0, 0, 1, 1)$; away from $r = 0$, $D^2(D+1)^2k = 0$, completing the distributional verification.

Step 4: Interpret the motion. The response vanishes before $t = 1$ and approaches $3(t-1) - 5$ afterward. It is C^1 , but its one-sided second derivatives differ. Changing a function at one point cannot change those one-sided limits or its distribution.



Question 5

An unstable homogeneous mode may or may not appear in a forced response. For real parameters a, b , consider

$$y''' + y'' - 2y' = e^{-t}, \quad y(0) = 0, \quad y'(0) = a, \quad y''(0) = b, \quad t \geq 0.$$

Tasks

1. Derive $Y(s)$ and identify all candidate poles before cancellation.
2. Find the necessary and sufficient relation between a, b for a bounded response. Give a complete inverse transform that proves both necessity and sufficiency.
3. Among the bounded responses, find the unique initial pair for which $y(t) \rightarrow 0$. Compute that solution and verify its equation and initial data.
4. An experiment perturbs only b from this special pair to $b + \varepsilon$. Find the exact response error. Explain how an arbitrarily small nonzero error can invalidate boundedness, and state precisely when a final-value calculation is justified.

Question 5 – Solution

Strategy. Inspect the residue of the positive pole before using a final-value theorem.

Step 1: Keep the initial-data numerator. Transformation yields

$$P(s)Y = a(s+1) + b + \frac{1}{s+1}, \quad P(s) = s(s-1)(s+2).$$

The candidate poles are $1, 0, -1, -2$; $\operatorname{Re} s > 1$ is a common convergence half-plane before any cancellation.

Step 2: Classify bounded responses. Residues give

$$y(t) = -\frac{a+b+1}{2} + \frac{2a+b+1/2}{3}e^t + \frac{1}{2}e^{-t} + \frac{-a+b-1}{6}e^{-2t}.$$

Only e^t is unbounded, and no other mode can cancel it asymptotically. Therefore

$$\boxed{y \text{ is bounded} \iff b = -2a - \frac{1}{2}.$$

For this line of data, $\lim y = (a - 1/2)/2$.

Step 3: Remove the constant mode as well. Zero limit requires $a = 1/2$, $b = -3/2$, giving

$$\boxed{y_*(t) = \frac{1}{2}(e^{-t} - e^{-2t}).}$$

Its first three initial values are $(0, 1/2, -3/2)$. Since $P(-1) = 2$ and $P(-2) = 0$, $P(D)y_* = e^{-t}$, directly checking the equation. Its transform converges for $\operatorname{Re} s > -1$.

Step 4: Quantify sensitivity and theorem use. A change $b \mapsto b + \varepsilon$ gives

$$\boxed{y - y_* = \varepsilon \left(-\frac{1}{2} + \frac{e^t}{3} + \frac{e^{-2t}}{6} \right).$$

This error has initial state $(0, 0, \varepsilon)$ and is asymptotic to $\varepsilon e^t/3$. Every nonzero perturbation is eventually unbounded. Only after the positive pole cancels do all poles of sY lie strictly left of the imaginary axis; then $\lim y = \lim_{s \rightarrow 0} sY$ is valid. For unstable data, the finite algebraic value at $s = 0$ does not represent a time limit.

Question 6

A variable coefficient makes the transform satisfy a differential equation. Seek a real-analytic solution at $t = 0$ of

$$ty''' + 3y'' + ty' + y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

for $t \geq 0$, with a Laplace transform. A third initial value is not specified.

Tasks

1. Determine the only compatible value of $y''(0)$. Explain why the usual three-free-data theorem for a regular third-order equation does not apply at the origin.
2. Use $\mathcal{L}\{tf(t)\} = -F'(s)$ to derive and solve an equation for $Y(s)$ on real $s > 0$. Fix the integration constant from the behavior as $s \rightarrow \infty$.
3. Invert the result by writing it as an integral of elementary cosine transforms. Extend the answer to $t = 0$ and verify the original differential equation.
4. Derive the power-series coefficient recurrence and prove uniqueness among analytic solutions with the prescribed data. Check the first three terms of the large- s expansion of Y against the initial derivatives.

Question 6 – Solution

Strategy. Use the singular-endpoint constraint before transforming; the transform's boundary condition replaces its integration constant.

Step 1: Enforce compatibility. At $t = 0$, continuity of the equation gives $3y''(0) + 1 = 0$, so $y''(0) = -1/3$. The leading coefficient t vanishes at the origin; normalizing by it would produce singular coefficients there.

Step 2: Transform carefully. Writing $c = y''(0)$, the four transforms are

$$-\frac{d}{ds}(s^3Y - s^2 - c), \quad 3(s^2Y - s), \quad -\frac{d}{ds}(sY - 1), \quad Y.$$

Their sum is $-(s^3 + s)Y' - s = 0$. Thus $Y' = -1/(1 + s^2)$ for $s > 0$. Because the transform of a continuous exponential-order function tends to zero as $s \rightarrow \infty$, the constant is fixed:

$$Y(s) = \int_s^\infty \frac{dv}{1 + v^2} = \arctan(1/s), \quad s > 0.$$

Step 3: Invert an integral representation. Since $Y(s) = \int_0^1 s/(s^2 + u^2) du$, absolute convergence for $\operatorname{Re} s > 0$ permits interchange of integrals. Therefore

$$y(t) = \int_0^1 \cos(ut) du = \frac{\sin t}{t} \quad (t > 0), \quad y(0) = 1.$$

For $t > 0$, this function satisfies $ty'' + 2y' + ty = 0$. Differentiating gives the required third-order equation; its analytic extension satisfies the equation at zero too.

Step 4: Prove analytic uniqueness. For $y = \sum_{n \geq 0} c_n t^n$, coefficient matching gives

$$(n+1)((n+2)(n+3)c_{n+2} + c_n) = 0 \quad (n \geq 0).$$

The seeds $c_0 = 1, c_1 = 0$ uniquely give $c_{2k} = (-1)^k/(2k+1)!$ and $c_{2k+1} = 0$, an everywhere convergent series. Ordinary uniqueness on $t > 0$ extends local uniqueness to the whole positive axis. Finally $Y = s^{-1} - (3s^3)^{-1} + (5s^5)^{-1} + O(s^{-7})$, agreeing with $y(0) = 1, y''(0) = -1/3$, and $y^{(4)}(0) = 1/5$ in $Y \sim \sum y^{(j)}(0)/s^{j+1}$.

Question 7

A third-order equation has an exponentially fading memory:

$$(D + 1)^3 y(t) = \int_0^t e^{-(t-v)} y(v) dv + e^{-t}, \quad (y, y', y'')(0) = (0, 0, 0).$$

All functions are defined for $t \geq 0$.

Tasks

1. Transform the convolution equation and solve algebraically for $Y(s)$. Factor the resulting fourth-degree denominator.
2. Invert the transform in a real form, displaying a useful decomposition in $z = s + 1$.
3. Derive an equivalent ordinary fourth-order initial-value problem. Explain why differentiation requires an additional initial condition, and prove that your fourth-order solution really satisfies the original memory equation.
4. Determine the long-time limit and justify any final-value theorem used. Explain how the feedback changes the zero-state response from what it would be if the memory integral were omitted.

Question 7 – Solution

Strategy. Convert memory into a rational factor, then guard against the extra solutions that differentiation can introduce.

Step 1: Transform and factor. The convolution theorem gives

$$(s+1)^3 Y = \frac{Y}{s+1} + \frac{1}{s+1}, \quad \boxed{Y = \frac{1}{(s+1)^4 - 1}}.$$

The denominator is $s(s+2)((s+1)^2+1)$, with roots $0, -2, -1 \pm i$. A sufficient convergence half-plane is $\operatorname{Re} s > 0$.

Step 2: Invert real quadratic factors. For $z = s+1$,

$$\frac{1}{z^4 - 1} = \frac{1}{2} \left(\frac{1}{z^2 - 1} - \frac{1}{z^2 + 1} \right), \quad \boxed{y(t) = \frac{1}{2} e^{-t} (\sinh t - \sin t)}.$$

Equivalently, $y = (1 - e^{-2t})/4 - e^{-t} \sin t/2$.

Step 3: Prove equivalence, including the lost condition. If $m(t) = \int_0^t e^{-(t-v)} y(v) dv$, then $(D+1)m = y$ and $m(0) = 0$. Applying $D+1$ to the original equation gives

$$((D+1)^4 - 1)y = 0, \quad (y, y', y'', y''')(0) = (0, 0, 0, 1).$$

The fourth value follows from the original equation at zero. The displayed solution has $y = t^3/6 + O(t^4)$ and lies in the four characteristic modes, so it satisfies this fourth-order IVP. Conversely its residual $R = (D+1)^3 y - m - e^{-t}$ obeys $R' + R = 0$ and $R(0) = 0$. Hence $R \equiv 0$, proving equivalence rather than just necessity.

Step 4: Interpret the surviving mode. The explicit formula gives $\boxed{\lim_{t \rightarrow \infty} y(t) = 1/4}$. This also equals $\lim_{s \rightarrow 0} sY$; after cancellation, the poles of sY are $-2, -1 \pm i$, all strictly stable. Without memory, the transform is $1/(s+1)^4$ and the response is $t^3 e^{-t}/6 \rightarrow 0$. Feedback introduces a zero characteristic root and thus a nonzero constant component, although the external forcing itself decays.

Question 8

A stable third-order filter receives identical unit impulses at times $T, 2T, 3T, \dots$, where $T > 0$:

$$(D + 1)^3 y = \sum_{n=1}^{\infty} \delta(t - nT).$$

Assume zero prehistory and zero initial state. Interpret the equation in the sense of distributions; there is no impulse at $t = 0$.

Tasks

1. Find $Y(s)$, stating a convergence half-plane. Give the exact finite-sum response at any finite time.
2. For $t = NT + x$ with $0 \leq x < T$, find the limiting periodic profile as $N \rightarrow \infty$. Sum it explicitly using $q = e^{-T}$.
3. Determine the continuity and derivative jumps across each impulse, including the join in the limiting profile. Compute the mean of the limiting profile over a period.
4. Decide whether $y(t)$ has a pointwise limit as $t \rightarrow \infty$. Explain the failure or validity of a final-value argument, and sketch the transient and limiting profile for $T = 2$.

Question 8 – Solution

Strategy. Sum delayed impulse kernels and distinguish a periodic limiting profile from a constant final value.

Step 1: Sum the transformed impulses. For $\operatorname{Re} s > 0$,

$$Y = \frac{e^{-sT}}{(1 - e^{-sT})(s + 1)^3}, \quad y(t) = \sum_{1 \leq n \leq \lfloor t/T \rfloor} h(t - nT), \quad h(v) = \frac{1}{2}v^2e^{-v}.$$

The sum is empty before T ; including the current impulse adds $h(0) = 0$.

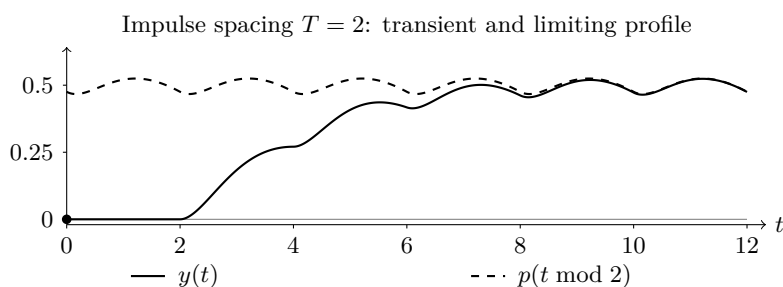
Step 2: Sum the limiting profile. At $t = NT + x$ the sum is $\sum_{k=0}^{N-1} h(x + kT)$. Geometric-series identities give

$$p(x) = \frac{e^{-x}}{2} \left[\frac{x^2}{1 - q} + \frac{2xTq}{(1 - q)^2} + \frac{T^2q(1 + q)}{(1 - q)^3} \right], \quad q = e^{-T}.$$

The omitted tail is uniformly $O(N^2q^N)$ for $0 \leq x \leq T$ and fixed T , so $y(NT + x) \rightarrow p(x)$ uniformly on a period.

Step 3: Check joins and mean. Since $(h, h', h'')(0) = (0, 0, 1)$, y, y' are continuous and y'' jumps upward by one. Reindexing the convergent profile series gives the same periodic joins: $p(0) = p(T-)$, $p'(0) = p'(T-)$, $p''(0) - p''(T-) = 1$. Also $\int_0^T p(x) dx = \int_0^\infty h(v) dv = 1$, so its mean is $1/T$.

Step 4: Reject a constant final value. The profile is nonconstant: on each open gap $(D+1)^3p = 0$, so a constant profile would be zero, contradicting its integral. Distinct phases therefore have distinct subsequential limits. Poles at $s = 2\pi ik/T$, $k \neq 0$, remain in sY ; the formal limit $\lim_{s \rightarrow 0} sY = 1/T$ is a mean, not a final value.



Question 9

An experiment supplies the exact transform

$$Y(s) = \frac{s^4 + 3s^3 + 7s^2 + 11s + 18}{(s+3)(s^2+1)(s^2+4)}$$

of a solution to

$$y^{(4)} + 5y'' + 4y = Ae^{-3t}, \quad t \geq 0.$$

The forcing amplitude A and all four initial values are unknown real numbers.

Tasks

1. Recover A and $(y, y', y'', y''')(0)$ by polynomial division of $(s^2+1)(s^2+4)Y$. Prove that this recovery is unique.
2. Invert Y completely using real sine, cosine and exponential terms.
3. Verify the recovered forcing and all four initial values directly from your answer, and state an absolute-convergence half-plane.
4. Find the monic constant-coefficient homogeneous differential operator of least order that annihilates this particular response. Explain why its order differs from that of the forced equation, and prove minimality.

Question 9 – Solution

Strategy. Separate the polynomial encoding of initial data from the proper rational forcing term.

Step 1: Recover data uniquely. Write $P(s) = s^4 + 5s^2 + 4$. Polynomial division gives

$$P(s)Y = s^3 + 7s - 10 + \frac{48}{s+3}.$$

For initial values $c_j = y^{(j)}(0)$, transformation of the equation gives

$$PY = c_0s^3 + c_1s^2 + (c_2 + 5c_0)s + (c_3 + 5c_1) + \frac{A}{s+3}.$$

Thus $\boxed{A = 48, \quad (c_0, c_1, c_2, c_3) = (1, 0, 2, -10)}$. Polynomial division has a unique polynomial part and remainder; the displayed triangular relation then determines every c_j uniquely.

Step 2: Resolve the real modes. Real partial fractions yield

$$Y = \frac{(2/5)s + 22/15}{s^2 + 1} + \frac{(3/13)s - 14/39}{s^2 + 4} + \frac{24/65}{s + 3},$$

so

$$\boxed{y = \frac{2}{5} \cos t + \frac{22}{15} \sin t + \frac{3}{13} \cos(2t) - \frac{7}{39} \sin(2t) + \frac{24}{65} e^{-3t}.$$

Step 3: Check the recovered experiment. The oscillatory terms are annihilated by $P(D)$ and $P(-3) = 130$, so the exponential contributes $130(24/65)e^{-3t} = 48e^{-3t}$. At zero the four derivatives are

$$\begin{aligned} y(0) &= 2/5 + 3/13 + 24/65 = 1, \\ y'(0) &= 22/15 - 14/39 - 72/65 = 0, \\ y''(0) &= -2/5 - 12/13 + 216/65 = 2, \\ y'''(0) &= -22/15 + 56/39 - 648/65 = -10. \end{aligned}$$

Absolute convergence holds for $\operatorname{Re} s > 0$.

Step 4: Prove the minimum homogeneous order. All five complex modes at $i, -i, 2i, -2i, -3$ have nonzero coefficients. Their linear independence forces every annihilating polynomial to vanish at all five distinct roots. Hence the least monic operator is $\boxed{(D+3)(D^2+1)(D^2+4)}$, of order five. The fourth-order equation leaves the exponential as a nonzero forcing; annihilating that forcing adds the fifth factor.

Question 10

For $\varepsilon \geq 0$, let h_ε be the causal impulse response of

$$(D + 1)^2(D + 1 + \varepsilon)h_\varepsilon = \delta(t),$$

with zero prehistory. The impulse is at the origin; use the causal transform convention $\mathcal{L}\delta(t) = 1$. Ordinary formulas below describe $t \geq 0$.

Tasks

1. Find the transform and invert it for $\varepsilon > 0$ by partial fractions in $z = s + 1$.
2. Derive a convolution formula with a nonnegative integrand. Use it to obtain h_0 and prove $h_\varepsilon \rightarrow h_0$ without subtracting divergent coefficients.
3. Prove two-sided bounds for $h_\varepsilon(t)$ and an explicit bound of the form $\sup_{t \geq 0} |h_\varepsilon(t) - h_0(t)| \leq C\varepsilon$ with a numerical constant C .
4. Explain the change in pole multiplicities and verify the four right-hand initial derivatives required by a unit impulse. Discuss reliable evaluation for small εt , and sketch the kernels for $\varepsilon = 0, 1/2, 2$.

Question 10 – Solution

Strategy. Use partial fractions for a closed form and convolution for a stable coalescing-pole limit.

Step 1: Invert distinct repeated poles. For $\varepsilon > 0$, $H_\varepsilon = 1/[z^2(z + \varepsilon)^2]$ and

$$H_\varepsilon = \frac{1}{\varepsilon^2} \left(\frac{1}{z^2} + \frac{1}{(z + \varepsilon)^2} \right) + \frac{2}{\varepsilon^3} \left(\frac{1}{z + \varepsilon} - \frac{1}{z} \right).$$

Hence $h_\varepsilon = e^{-t} \{t(1 + e^{-\varepsilon t})/\varepsilon^2 - 2(1 - e^{-\varepsilon t})/\varepsilon^3\}$.

Step 2: Take the limit through convolution. Convolving te^{-t} with $te^{-(1+\varepsilon)t}$ gives

$$h_\varepsilon(t) = e^{-t} \int_0^t v(t-v)e^{-\varepsilon v} dv, \quad h_0(t) = \frac{t^3 e^{-t}}{6}.$$

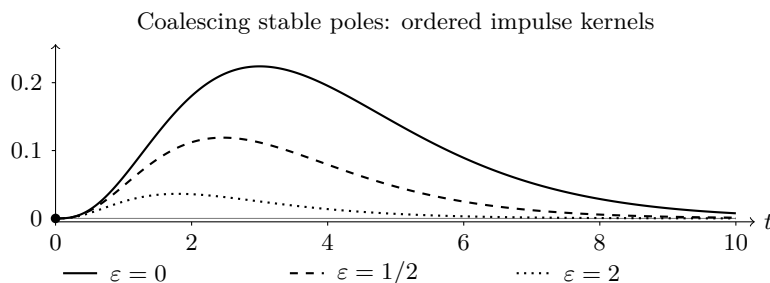
For fixed t , dominated convergence justifies the limit, including $t = 0$.

Step 3: Prove global error bounds. Bounding the integrand and using $1 - e^{-\varepsilon v} \leq \varepsilon v$ yields

$$e^{-(1+\varepsilon)t} \frac{t^3}{6} \leq h_\varepsilon(t) \leq e^{-t} \frac{t^3}{6}, \quad 0 \leq h_0 - h_\varepsilon \leq \frac{\varepsilon t^4 e^{-t}}{12}.$$

The maximum of $t^4 e^{-t}$ is $256e^{-4}$ at $t = 4$. Thus $\sup_{t \geq 0} |h_\varepsilon - h_0| \leq 64\varepsilon/(3e^4)$.

Step 4: Check poles, impulses and evaluation. Two double poles merge into a quadruple pole at -1 ; absolute convergence holds for $\operatorname{Re} s > -1$. Convolution gives $h_\varepsilon = t^3/6 + O(t^4)$, so its right-hand jet is $(0, 0, 0, 1)$, producing exactly a unit δ . For small εt , use the integral or its exponential series; the closed form subtracts large nearly equal terms and can lose significant digits.



Question 1

Consider the variable-coefficient initial-value problem on the real line,

$$(1 + t^2)y''' - 2ty'' + 2y' = 0, \quad (y, y', y'')(0) = (1, 0, 2).$$

A state representation must preserve the scalar equation's leading coefficient and all three initial data.

Tasks

1. Set $X = (y, y', y'')^T$ and derive $X' = A(t)X$. State the initial vector and the largest real interval on which the conversion is regular.
2. Find the solution by testing a polynomial of degree at most two. Verify every component of the system and explain why this trajectory is unique.
3. Let $\Phi' = A\Phi$, $\Phi(0) = I$. Determine $\det \Phi(t)$ without finding the full matrix. Interpret the result as a change in three-dimensional state volume.
4. Decide whether the increasing volume factor forces every individual trajectory to grow. Give an explicit counterexample or proof, and sketch all three components of the prescribed trajectory on $[-2, 2]$.

Question 1 – Solution

Strategy. Normalize first, and distinguish the evolution of a family of states from that of one trajectory.

Step 1: Preserve the derivative chain. Since $1 + t^2 > 0$ for every real t ,

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2/(1+t^2) & 2t/(1+t^2) \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

The conversion is regular on $\mathbb{R}$, and its first two rows enforce $X_2 = X_1'$, $X_3 = X_1''$, so it is reversible.

Step 2: Verify the complete state. The initial data select $y = 1 + t^2$ among quadratic polynomials. Its derivatives are $2t, 2, 0$, and the scalar residual is $-4t + 4t = 0$. Thus

$$X(t) = (1 + t^2, 2t, 2)^T.$$

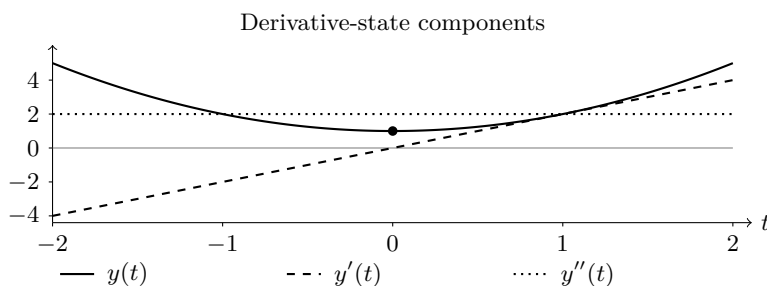
The system's last row gives $(-4t + 4t)/(1 + t^2) = 0 = X_3'$; the other rows and initial vector also agree. Continuity of $A(t)$ gives uniqueness on $\mathbb{R}$.

Step 3: Compute the state-volume factor. Liouville's formula gives

$$\det \Phi(t) = \exp\left(\int_0^t \frac{2v}{1+v^2} dv\right) = 1 + t^2.$$

An initial parallelepiped of states has its oriented volume multiplied by this positive factor. The factor increases with $|t|$, not with t on the entire line.

Step 4: Test the individual-growth claim. For any constant c , $X = (c, 0, 0)^T$ is a constant solution. Hence expansion of state volume does not force every trajectory to grow. The prescribed components are shown below; their different behaviors also caution against that inference.



Question 2

A scalar measurement of a three-state system is

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \quad y = X_1 + 2X_2.$$

The measured initial derivatives are $y(0) = 1$, $y'(0) = 0$, $y''(0) = -12$. All coefficients are constant and $t \in \mathbb{R}$.

Tasks

1. Form the matrix O such that $(y, y', y'')^T = OX$. Determine whether the three measured derivatives uniquely determine the state.
2. Derive a scalar third-order equation for y and explicit reconstruction formulas for all three state components in terms of y, y', y'' .
3. Recover $X(0)$ and find the measured response $y(t)$ explicitly.
4. Prove that every solution of your scalar equation reconstructs a solution of the original system. Verify the recovered trajectory and explain why differentiating the measurement has lost no state information.

Question 2 – Solution

Strategy. Invert the map from state coordinates to the measurement's derivative coordinates.

Step 1: Test the measurement map. With $c = (1, 2, 0)$, the rows of O are c, cA, cA^2 . Hence

$$O = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -12 & -22 & -11 \end{pmatrix}, \quad \det O = -15 \neq 0.$$

The derivative triple determines the state uniquely.

Step 2: Convert and reconstruct. The identity $A^3 + 6A^2 + 11A + 6I = 0$ gives

$$y''' + 6y'' + 11y' + 6y = 0.$$

Solving $OX = (y, y', y'')^T$ gives

$$\begin{aligned} X_1 &= (-33y - 22y' - 4y'')/15, \\ X_2 &= (24y + 11y' + 2y'')/15, \\ X_3 &= (-12y + 2y' - y'')/15. \end{aligned}$$

Step 3: Solve the recovered IVP. The measured data give $X(0) = (1, 0, 0)^T$. The scalar roots are $-1, -2, -3$; imposing the measured derivative triple yields

$$y = -3e^{-t} + 9e^{-2t} - 5e^{-3t}.$$

The state is $X = (f, f', f'')^T$, where $f = 3e^{-t} - 3e^{-2t} + e^{-3t}$. In particular, $f + 2f' = y$ and $(f, f', f'')(0) = (1, 0, 0)$.

Step 4: Prove full equivalence. If $Z = (y, y', y'')^T$, then $Z' = BZ$ with last row of B equal to $(-6, -11, -6)$. Direct multiplication gives $OA = BO$, so for any scalar solution, $X = O^{-1}Z$ satisfies $X' = O^{-1}BZ = AX$. Its measurement is y by the first row of O . Conversely every state solution produces this scalar equation. The recovered f satisfies $f''' + 6f'' + 11f' + 6f = 0$, checking the original last row directly. Invertibility of O proves that no hidden state was discarded.

Question 3

Only one coordinate of the following system is observed:

$$X' = AX, \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad y = X_1, \quad t \geq 0.$$

A proposed scalar model is obtained by replacing the characteristic polynomial of A with a differential operator.

Tasks

1. Derive the lowest-order monic constant-coefficient homogeneous scalar equation satisfied by every possible measurement y . Prove that this order is minimal.
2. For a prescribed measurement, reconstruct all compatible states. Determine exactly which initial states are bounded on $[0, \infty)$.
3. Write the third-order scalar equation suggested by the characteristic polynomial. Identify its extra scalar mode, and find the initial constraint that excludes that mode.
4. Decide whether an arbitrarily long, exact record of y can certify boundedness of the full state. Exhibit two states with identical measurements and different boundedness behavior.

Question 3 – Solution

Strategy. Separate the measured Jordan block from the invisible unstable coordinate.

Step 1: Find the true measurement equation. From $y' = -y + X_2$ and $X_2' = -X_2$,

$$y'' + 2y' + y = 0, \quad y = (a + bt)e^{-t}.$$

Both e^{-t} and te^{-t} occur as measurements and are independent. Any constant-coefficient annihilator must have a double root at -1 , so order two is minimal.

Step 2: Recover the visible and hidden states. Every compatible state is

$$X(t) = ((a + bt)e^{-t}, be^{-t}, ce^{2t})^T, \quad c \in \mathbb{R}.$$

The first two coordinates always decay. Therefore the full state is bounded if and only if $c = X_3(0) = 0$. The measurement does not determine c .

Step 3: Remove the spurious scalar mode. The characteristic polynomial gives

$$(D - 2)(D + 1)^2 y = 0, \quad \text{equivalently} \quad y''' - 3y' - 2y = 0.$$

Its general solution adds de^{2t} to the actual measurement family. Put $r = (D + 1)^2 y$. Then $r' = 2r$ and $r = 9de^{2t}$, so the precise exclusion condition is

$$y''(0) + 2y'(0) + y(0) = 0.$$

The characteristic-polynomial equation is necessary, but without this constraint it is not sufficient for a scalar function to be a measurement.

Step 4: Test what observation can certify. The initial states $(1, 0, 0)^T$ and $(1, 0, 1)^T$ both produce $y = e^{-t}$ forever. Their full trajectories are $(e^{-t}, 0, 0)^T$ and $(e^{-t}, 0, e^{2t})^T$; one is bounded and the other is not. Thus even an exact infinite measurement record cannot certify boundedness without additional information about $X_3(0)$.

Question 4

A smooth system on the entire real line is

$$x' = z, \quad z' = tw, \quad w' = 0.$$

Eliminating z, w appears to produce a third-order scalar equation for x . The elimination must be examined at $t = 0$.

Tasks

1. Eliminate z, w for $t \neq 0$. Explain why the resulting scalar equation is singular at the origin although the original system is regular there.
2. Solve the original system for arbitrary data $(x, z, w)(0) = (a, b, c)$. Determine which derivatives of x at zero encode these three constants.
3. Classify all $C^3(\mathbb{R})$ solutions of the scalar equation. Prove that each reconstructs a unique smooth system solution, explicitly treating the origin.
4. Decide whether $x(0), x'(0), x''(0)$ alone determine a unique scalar solution. Give a one-parameter counterexample if needed, and sketch three members sharing the zero derivative triple on $[-1, 1]$.

Question 4 – Solution

Strategy. Retain the state coordinate that division by t conceals at the singular point.

Step 1: Eliminate on the regular subintervals. Since $x'' = tw$, for $t \neq 0$ we have $w = x''/t$. Differentiating gives

$$tx''' - x'' = 0.$$

The leading coefficient vanishes at zero. This singularity came from the chosen scalar coordinate; the original system matrix is continuous everywhere.

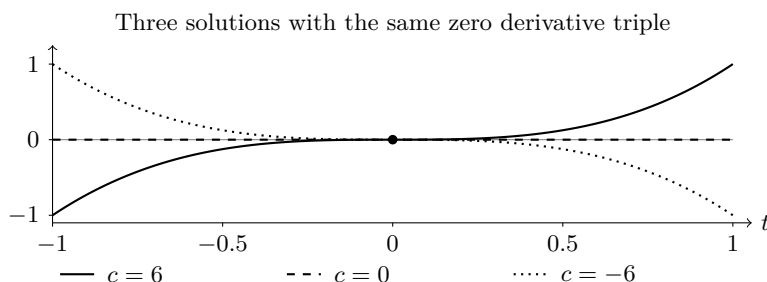
Step 2: Solve before dividing. Integrating successively gives

$$w = c, \quad z = b + ct^2/2, \quad x = a + bt + ct^3/6.$$

Thus $(x, x', x'')(0) = (a, b, 0)$, while $x'''(0) = c$. The missing state coordinate is encoded in the third derivative, not in the usual second derivative.

Step 3: Check global equivalence. On either side of zero, $(x''/t)' = 0$, so $x = a_{\pm} + b_{\pm}t + c_{\pm}t^3/6$. Continuity of x, x' matches $a_{\pm}, b_{\pm}$, and continuity of x''' matches $c_{\pm}$. All C^3 solutions therefore have the single global polynomial form above. Reconstruct $z = x'$ and $w = x''/t$ off zero, setting $w(0) = x'''(0)$. These are smooth and satisfy all three system equations. At zero, the scalar equation itself requires $x''(0) = 0$.

Step 4: Explain the apparent nonuniqueness. The family $x = ct^3/6$ has $(x, x', x'')(0) = (0, 0, 0)$ for every c . It corresponds to distinct system data $(0, 0, c)$, so it does not contradict uniqueness for the regular system. The plotted members use $c = -6, 0, 6$.



Question 5

Two coupled displacements satisfy

$$x'' + 2x - z = 0, \quad z'' + 2z - x = 0,$$

with $(x, x', z, z')(0) = (1, 0, 0, 0)$. Time is measured in units that make the coefficients dimensionless.

Tasks

1. Write a four-state first-order system using the order (x, x', z, z') . Eliminate z to obtain a fourth-order scalar equation for x and translate all four initial values.
2. Introduce $u = x + z$, $v = x - z$. Solve the resulting independent equations and recover x, z .
3. Prove that every solution of the scalar fourth-order equation reconstructs a unique solution of the coupled equations. Verify the prescribed solution and its initial data.
4. Derive a positive conserved energy, express it in u, v , and use it to prove boundedness. Sketch both displacements for $0 \leq t \leq 12$ and explain how two different frequencies appear.

Question 5 – Solution

Strategy. Compare the derivative-state, scalar-elimination and normal-mode descriptions of the same four-dimensional motion.

Step 1: Convert and transfer the initial state. For $X = (x, x', z, z')^T$,

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} X.$$

Since $z = x'' + 2x$, substitution into the second equation gives

$$x^{(4)} + 4x'' + 3x = 0, \quad (x, x', x'', x''')(0) = (1, 0, -2, 0).$$

Step 2: Solve the normal modes. Adding and subtracting give $u'' + u = 0$, $v'' + 3v = 0$, with initial displacements one and velocities zero. Thus

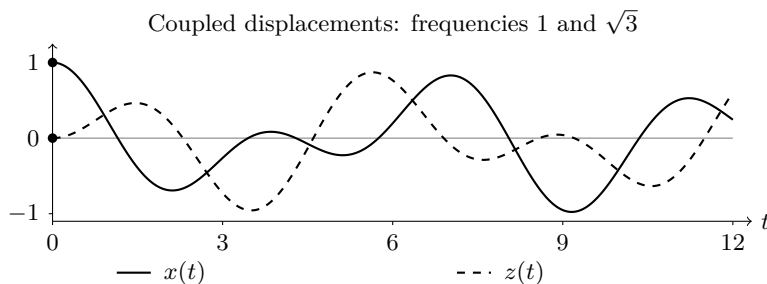
$$x = \frac{1}{2}(\cos t + \cos(\sqrt{3}t)), \quad z = \frac{1}{2}(\cos t - \cos(\sqrt{3}t)).$$

Step 3: Prove reversibility. For any scalar solution, define $z = x'' + 2x$. The first coupled equation is then an identity, and the second has residual $x^{(4)} + 4x'' + 3x = 0$. This formula also proves uniqueness of z . The displayed cosines satisfy the scalar factors $(D^2 + 1)(D^2 + 3)$ and give $z(0) = z'(0) = 0$ as required.

Step 4: Check energy and boundedness. Multiplying the equations by x', z' and adding proves conservation of

$$E = \frac{1}{2}(x'^2 + z'^2) + x^2 + z^2 - xz = \frac{1}{4}(u'^2 + u'^2 + v'^2 + 3v'^2) = 1.$$

Since $x^2 + z^2 - xz \geq (x^2 + z^2)/2$, both displacements are bounded; the explicit formulas sharpen this to $|x|, |z| \leq 1$. Each displacement mixes the frequencies 1 and $\sqrt{3}$.



Question 6

Let $D = d/dt$. Ordered differential factors act from right to left in

$$(D + 1)(D - t)(D - 1)y = 0, \quad (y, y', y'')(0) = (0, 0, 1).$$

Here $(D - t)f = f' - tf$. Multiplication by t does not commute with differentiation.

Tasks

1. Define $q_1 = y$, $q_2 = (D - 1)y$, $q_3 = (D - t)q_2$. Derive the first-order system for q and its initial vector.
2. Expand the scalar third-order equation and show that the change between (y, y', y'') and q is invertible for every real t .
3. Solve the triangular system by successive integrating factors. Give an explicit nested-integral formula for y and verify the initial data and equation through the state relations.
4. Compare the given operator with $(D - t)(D + 1)(D - 1)$. Compute their difference, and exhibit a function that solves the reordered equation but not the original one.

Question 6 – Solution

Strategy. Keep the factors in their stated order and regard intermediate derivatives as state coordinates.

Step 1: Form the ordered cascade. The definitions give

$$\boxed{q'_1 = q_1 + q_2, \quad q'_2 = tq_2 + q_3, \quad q'_3 = -q_3, \quad q(0) = (0, 0, 1)^T.}$$

In particular, $q_3 = y'' - (t+1)y' + ty$; using only y'' as this coordinate would lose the variable-coefficient terms.

Step 2: Expand and invert the coordinate map. Differentiating the expression for q_3 and adding q_3 yields

$$\boxed{y''' - ty'' - 2y' + (t+1)y = 0.}$$

The coordinate matrix is

$$q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ t & -(t+1) & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \quad \det = 1.$$

Its inverse relations are $y = q_1$, $y' = q_1 + q_2$ and $y'' = q_1 + (t+1)q_2 + q_3$, so the conversion is globally reversible.

Step 3: Integrate in triangular order. Successive integrating factors give

$$q_3 = e^{-t}, \quad q_2 = e^{t^2/2} \int_0^t e^{-v-v^2/2} dv,$$

$$\boxed{y(t) = e^t \int_0^t e^{-u+u^2/2} \left(\int_0^u e^{-v-v^2/2} dv \right) du.}$$

These oriented integrals apply to every real t . Their derivatives give the three cascade equations. At zero the state is $(0, 0, 1)$, and the inverse relations give $(y, y', y'') = (0, 0, 1)$. Finally $q'_3 + q_3 = 0$ verifies the original ordered product directly.

Step 4: Quantify the order error. The commutator $(D+1)(D-t) - (D-t)(D+1)$ is multiplication by -1 . Thus the original operator minus the reordered one equals $-(D-1)$. The reordered operator is $D^3 - tD^2 - D + t$, whereas the original has the extra $-D + 1$. For $y = e^{-t}$ the reordered product vanishes, but the original gives $2e^{-t} \neq 0$. Reordering these factors changes the solution space.

Question 7

A controlled third derivative moves a state from rest to rest in unit time:

$$y''' = u(t), \quad (y, y', y'')(0) = (0, 0, 0), \quad (y, y', y'')(1) = (1, 0, 0).$$

For square-integrable controls u on $[0, 1]$, interpret the derivative chain through its integral solution. The effort is $J(u) = \int_0^1 u(t)^2 dt$.

Tasks

1. Write the first-order state system and its transition matrix. Derive the three terminal constraints as integrals of u .
2. Find the unique quadratic control satisfying the terminal conditions. Integrate it to obtain the displacement.
3. Prove this quadratic is the unique minimum-effort control among all square-integrable admissible controls, up to equality almost everywhere. Compute the minimum effort.
4. Verify both endpoint states and explain why the minimization proof covers controls that are not polynomials. Sketch the displacement and $u/60$, explicitly labeling the scaling.

Question 7 – Solution

Strategy. Express the terminal state as three moments, then use orthogonality to prove optimality over the full control class.

Step 1: Derive the terminal moments. For $X = (y, y', y'')^T$, $X' = NX + e_3u$, where N has ones just above the diagonal and zeros elsewhere. Since $N^3 = 0$,

$$e^{Nr} = \begin{pmatrix} 1 & r & r^2/2 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad X(1) = \int_0^1 \begin{pmatrix} (1-t)^2/2 \\ 1-t \\ 1 \end{pmatrix} u(t) dt.$$

The required three moments are $(1, 0, 0)^T$.

Step 2: Solve within the quadratic space. For $u = a + bt + ct^2$, the constraints are

$$a/6 + b/24 + c/60 = 1, \quad a/2 + b/6 + c/12 = 0, \quad a + b/2 + c/3 = 0.$$

Their solution is

$$u_* = 60 - 360t + 360t^2, \quad y_* = 10t^3 - 15t^4 + 6t^5.$$

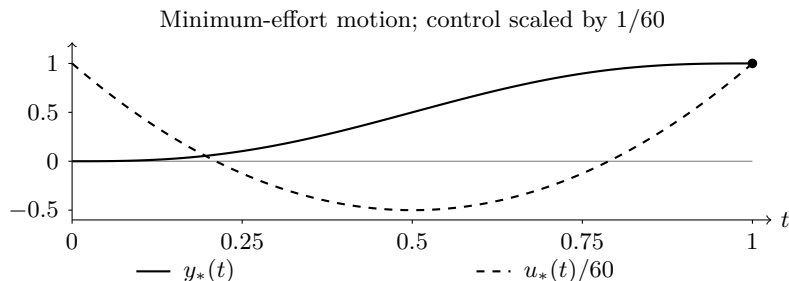
The moment kernels span all quadratics. A quadratic in the homogeneous null space is orthogonal to itself and hence zero, proving uniqueness here.

Step 3: Prove global minimum effort. For any admissible $u = u_* + v$, all three terminal moments of v vanish. Because u_* is in their span, $\int_0^1 u_* v = 0$. Therefore

$$J(u) = J(u_*) + \int_0^1 v^2 dt \geq J(u_*) = 720.$$

Equality holds exactly when $v = 0$ almost everywhere. This argument only uses square integrability and moment constraints, not polynomial form for v .

Step 4: Verify the endpoint motion. We have $y'_* = 30t^2(1-t)^2$ and $y''_* = 60t(1-t)(1-2t)$, so the two endpoint states are exactly $(0, 0, 0)$ and $(1, 0, 0)$; differentiating again gives u_* . The graph uses $u_*/60$ so the control and displacement are both readable.



Question 8

The functions $1, t, e^t$ form a scalar basis for $y''' - y'' = 0$. For the derivative state $X = (y, y', y'')^T$, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A fundamental matrix need not equal the identity at the chosen base time.

Tasks

1. Form the fundamental matrix $F(t)$ from the given scalar basis and its derivatives. Check $F' = AF$ and compute its determinant.
2. Derive the transition matrix $\Phi(t, s) = F(t)F(s)^{-1}$ explicitly. Verify its normalization, composition law and determinant.
3. Use this transition matrix to solve $y''' - y'' = 1$ with $(y, y', y'')(1) = (0, 0, 0)$ for every real t . Treat the integral orientation when $t < 1$.
4. Explain why using $F(t)$ directly as the transition from time zero gives wrong initial data in general. Prove that replacing F by FC , for any constant invertible C , leaves the true transition matrix unchanged.

Question 8 – Solution

Strategy. Normalize a scalar-derived fundamental matrix at the actual base time before transporting states or forcing.

Step 1: Lift the scalar basis. The derivative columns give

$$F(t) = \begin{pmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{pmatrix}, \quad \det F = e^t \neq 0.$$

Differentiating the rows gives $F' = AF$, including the last row because $(e^t)' = e^t$. Thus F is fundamental on the entire real line.

Step 2: Normalize and compose. With $r = t - s$, direct multiplication gives

$$\Phi(t, s) = \begin{pmatrix} 1 & r & e^r - 1 - r \\ 0 & 1 & e^r - 1 \\ 0 & 0 & e^r \end{pmatrix}.$$

It equals I at $t = s$, and $\det \Phi = e^{t-s}$. The identity $F(t)F(s)^{-1}F(s)F(a)^{-1} = F(t)F(a)^{-1}$ proves $\Phi(t, s)\Phi(s, a) = \Phi(t, a)$ for all real base times.

Step 3: Transport the forcing in both directions. The zero initial vector at time one gives

$$X(t) = \int_1^t \Phi(t, v)e_3 dv, \quad \boxed{y(t) = e^{t-1} - 1 - (t-1) - \frac{1}{2}(t-1)^2}.$$

Indeed its first component integrates $e^{t-v} - 1 - (t-v)$. For $t < 1$ the integral is oriented: it is minus the integral from t to 1. Writing $r = t - 1$, we obtain $y' = e^r - 1 - r$, $y'' = e^r - 1$, $y''' = e^r$. These give the zero state at $t = 1$ and residual $y''' - y'' = 1$ on all of $\mathbb{R}$.

Step 4: Separate basis from transition. Here $F(0) \neq I$. For example, $F(t)e_3 = (e^t, e^t, e^t)^T$ starts at $(1, 1, 1)^T$, not at e_3 . The correct expression is $F(t)F(0)^{-1}e_3$. For any constant invertible C ,

$$(F(t)C)(F(s)C)^{-1} = F(t)CC^{-1}F(s)^{-1} = \Phi(t, s).$$

A basis change alters coordinates of solution columns, but not the transport of a specified physical state.

Question 9

A fourth-order oscillator has a repeated quadratic factor:

$$y^{(4)} + 2y'' + y = 0, \quad t \geq 0.$$

Introduce the state

$$W = (y, y', y'' + y, y''' + y')^T.$$

All four scalar initial derivatives are arbitrary real numbers.

Tasks

1. Derive the system for W and show that this coordinate change is invertible. Identify the autonomous oscillator inside the system.
2. Write the full solution in terms of $a = y(0)$, $b = y'(0)$, $A = y''(0) + y(0)$ and $B = y'''(0) + y'(0)$.
3. Give necessary and sufficient conditions on the four initial derivatives for boundedness. Prove necessity even when the secular terms oscillate and have infinitely many zeros.
4. Find a conserved quadratic expression involving only the last two state coordinates. Decide whether the full four-state system can have a conserved positive-definite quadratic form, and justify your answer.

Question 9 – Solution

Strategy. Expose an internal oscillator that resonantly forces the displacement oscillator.

Step 1: Reveal the oscillator cascade. The original equation gives

$$W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} W.$$

The coordinate map is triangular with determinant one; its inverse is $(y, y', y'', y''') = (W_1, W_2, W_3 - W_1, W_4 - W_2)$. The last pair satisfies $W_3' = W_4$, $W_4' = -W_3$ independently.

Step 2: Solve the resonant forcing. We have $W_3 = A \cos t + B \sin t$ and $y'' + y = W_3$. A resonant particular solution is $(A/2)t \sin t - (B/2)t \cos t$. Enforcing the first two data gives

$$y = a \cos t + (b + B/2) \sin t + (A/2)t \sin t - (B/2)t \cos t.$$

It has $y''(0) = A - a$ and $y'''(0) = B - b$, checking all four data. Applying $D^2 + 1$ twice verifies the fourth-order equation.

Step 3: Classify bounded data. If $A = B = 0$, only bounded sine and cosine remain. If $(A, B) \neq (0, 0)$, choose a phase θ with $A \sin \theta - B \cos \theta = \sqrt{A^2 + B^2}$. Along $t_n = \theta + 2\pi n \geq 0$, the secular part grows as $t_n \sqrt{A^2 + B^2}/2$, while the other terms remain bounded. Consequently

$$y \text{ bounded} \iff y''(0) = -y(0), \quad y'''(0) = -y'(0).$$

The whole state is bounded exactly on this same two-dimensional subspace.

Step 4: Distinguish two kinds of conserved energy. The internal oscillator has

$$E_{\text{int}} = \frac{1}{2}(W_3^2 + W_4^2) = \frac{1}{2}(A^2 + B^2), \quad E'_{\text{int}} = W_3 W_4 - W_4 W_3 = 0.$$

This is not positive definite on the full state space. If a conserved positive-definite form $W^T P W$ existed, its smallest eigenvalue would bound $\|W(t)\|^2$ by a constant for every initial state. The unbounded solutions just constructed contradict that conclusion. Purely imaginary characteristic roots with repeated factors therefore do not imply boundedness of all trajectories.

Question 10

For the stable scalar equation $(D + 1)^3 y = 0$, introduce scaled state coordinates

$$q_1 = y, \quad q_2 = (y' + y)/4, \quad q_3 = (y'' + 2y' + y)/16.$$

Let $\|\cdot\|_2$ be the Euclidean norm and consider $t \geq 0$.

Tasks

1. Derive the constant system for q and the inverse map to (y, y', y'') . Explain why all characteristic roots are still -1 .
2. Solve the system for $q(0) = (0, 0, 1)^T$. Show explicitly that its Euclidean norm can exceed its initial value even though every solution tends to zero.
3. Set $w = (q_1, 4q_2, 16q_3)^T$. Prove a quadratic norm estimate that is strictly decreasing for nonzero states, using

$$2w_2(w_1 + w_3) \leq \sqrt{2}\|w\|_2^2.$$

Obtain an exponential bound for $\|q(t)\|_2$ valid for every initial state.

4. Explain why transient growth is consistent with asymptotic stability. For the prescribed initial state, sketch $\|q(t)\|_2$ and $\|w(t)\|_2/16$, labeling the normalization, and distinguish the two quantities.

Question 10 – Solution

Strategy. Compare Euclidean growth in scaled coordinates with decay in a suitable weighted norm.

Step 1: Derive the scaled cascade. With N having ones just above the diagonal,

$$q' = (-I + 4N)q, \quad y = q_1, \quad y' = 4q_2 - q_1, \quad y'' = 16q_3 - 8q_2 + q_1.$$

The coordinate map is invertible, so this is a similarity transformation of the scalar companion system. Its triangular matrix has three eigenvalues -1 .

Step 2: Compute an actual transient. Since $N^3 = 0$, $e^{(-I+4N)t} = e^{-t}(I + 4tN + 8t^2N^2)$. Therefore

$$q = e^{-t}(8t^2, 4t, 1)^T, \quad \|q(t)\|_2 = e^{-t}\sqrt{64t^4 + 16t^2 + 1}.$$

The initial norm is one, but $\|q(1)\|_2 = 9/e > 1$. Every state still tends to zero because its components are polynomials times e^{-t} .

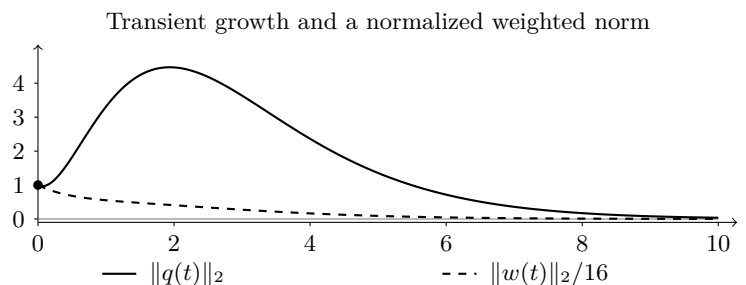
Step 3: Prove a uniform weighted estimate. The new state satisfies $w' = (-I + N)w$, so

$$\frac{d}{dt}\|w\|_2^2 = -2\|w\|_2^2 + 2w_2(w_1 + w_3) \leq -(2 - \sqrt{2})\|w\|_2^2.$$

The stated inequality follows from Cauchy–Schwarz and $2\sqrt{2}|w_2|\sqrt{w_1^2 + w_3^2} \leq \sqrt{2}\|w\|_2^2$. With $\alpha = 1 - 1/\sqrt{2} > 0$, integration and $\|q\|_2 \leq \|w\|_2 \leq 16\|q\|_2$ give

$$\|q(t)\|_2 \leq 16e^{-\alpha t}\|q(0)\|_2.$$

Step 4: Interpret coordinate-dependent growth. Asymptotic stability allows a finite transient; it does not require every norm to decrease at every instant. For this trajectory, $\|w\|_2/16 = e^{-t}\sqrt{t^4/4 + t^2 + 1}$ decreases strictly, whereas $\|q\|_2$ has a pronounced transient. They measure the same state with different weights.



Question 1

Consider the entire-coefficient problem

$$y^{(4)} = x^2 y'' + 2xy', \quad (y, y', y'', y''')(0) = (0, 1, 0, 0).$$

Write $y = \sum_{n \geq 0} a_n x^n$ using ordinary power-series coefficients.

Tasks

1. Derive the recurrence, including its exceptional first index. Identify all potentially nonzero coefficient classes.
2. Find the terms through degree thirteen and verify the initial data and the recurrence relations determining those terms.
3. Prove that the solution series is entire. For the degree-nine truncation p_9 , give a rigorous uniform error bound below 10^{-6} on $[-1, 1]$.
4. Determine the sign of $y - p_9$ on either side of zero. Plot the error on $[0, 1]$ together with a pointwise certificate, explicitly scaling the vertical values by 10^6 .

Question 1 – Solution

Strategy. Follow the surviving coefficient chain and bound its tail by a geometric majorant.

Step 1: Match every coefficient. The coefficient of x^n on the right is $n(n+1)a_n$, so

$$a_{n+4} = \frac{n a_n}{(n+4)(n+3)(n+2)} \quad (n \geq 0).$$

In particular $a_4 = 0$, not a freely chosen coefficient. The initial coefficients are $(a_0, a_1, a_2, a_3) = (0, 1, 0, 0)$; only indices $1 \pmod{4}$ survive.

Step 2: Compute the first terms. Successive use of $n = 1, 5, 9$ gives

$$y = x + \frac{x^5}{60} + \frac{x^9}{6048} + \frac{x^{13}}{1153152} + \cdots$$

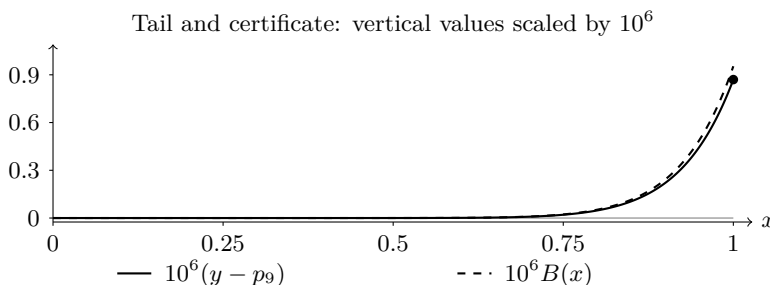
The missing coefficients are zero. The first four derivatives at zero match the data, and substitution into the recurrence verifies the displayed chain.

Step 3: Prove convergence and certify the tail. For fixed x , successive nonzero terms have ratio in magnitude $|x|^4 n / [(n+4)(n+3)(n+2)] \rightarrow 0$, so the series is entire and may be differentiated termwise. For $n \geq 9$, the coefficient ratio is at most $1/(n+2)^2 < 1/100$. Consequently, for $|x| \leq 1$,

$$|y - p_9| \leq \frac{100|x|^{13}}{99 \cdot 1153152} \leq B(x) := \frac{11|x|^{13}}{10 \cdot 1153152} \leq \frac{11}{10 \cdot 1153152} < 10^{-6}.$$

This bound concerns the full infinite tail, not just its first omitted term.

Step 4: Check the error's sign and graph. Every surviving coefficient is positive and every surviving degree is odd. Thus $y - p_9$ is positive for $x > 0$, negative for $x < 0$, and zero at zero. The figure compares $10^6(y - p_9)$ and $10^6 B(x)$ on $[0, 1]$.



Question 2

For the fourth-order problem

$$y^{(4)} + y'' + y = e^x, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 3),$$

let $b_n = y^{(n)}(0)$ and $a_n = b_n/n!$. The ordinary generating function $B(z) = \sum_{n \geq 0} b_n z^n$ is not the solution's Taylor series.

Tasks

1. Derive a recurrence for b_n and prove that the derivative sequence is periodic. Give one full period.
2. Find $B(z)$ and its radius of convergence. Write the actual solution series using a_n and determine its radius separately.
3. Verify the differential equation directly from the solution series and explain why treating b_n as ordinary Taylor coefficients would give a wrong answer.
4. Determine $\lim_{x \rightarrow +\infty} e^{-x}y(x)$ without explicitly solving for all homogeneous constants. Justify why the homogeneous contribution disappears in this limit.

Question 2 – Solution

Strategy. Keep derivative values separate from coefficients divided by factorials.

Step 1: Derive periodic derivative data. Differentiating the equation n times at zero gives

$$b_{n+4} + b_{n+2} + b_n = 1 \quad (n \geq 0).$$

The first six values are $(0, 0, 0, 0, 1, 1)$. Subtracting this equation from its version at $n + 2$ gives $b_{n+6} = b_n$. Thus $b_n = 1$ precisely when $n \equiv 4, 5 \pmod{6}$, and otherwise $b_n = 0$.

Step 2: Compare the two generating functions. The ordinary generating function of the derivatives is

$$B(z) = \frac{z^4 + z^5}{1 - z^6}, \quad R_B = 1.$$

There is a genuine pole at $z = 1$. In contrast, the solution is

$$y(x) = \sum_{k=0}^{\infty} \left(\frac{x^{6k+4}}{(6k+4)!} + \frac{x^{6k+5}}{(6k+5)!} \right), \quad R_y = \infty.$$

Absolute convergence follows by comparison with $e^{|x|}$.

Step 3: Verify the equation and normalization. The coefficient of $x^n/n!$ in $y^{(4)} + y'' + y$ is $b_{n+4} + b_{n+2} + b_n = 1$, giving e^x . The first four derivative values vanish. Using b_n as ordinary coefficients would make the fourth derivative at zero 24, whereas the equation and initial values require $y^{(4)}(0) = 1$.

Step 4: Find the dominant exponential. A particular solution is $e^x/3$, since $1^4 + 1^2 + 1 = 3$. The characteristic polynomial factors as

$$r^4 + r^2 + 1 = (r^2 + r + 1)(r^2 - r + 1).$$

Its distinct roots have real parts $-1/2$ or $1/2$. Every homogeneous term is therefore $O(e^{x/2})$ as $x \rightarrow +\infty$, regardless of its constants. It follows that

$$\lim_{x \rightarrow +\infty} e^{-x} y(x) = 1/3.$$

Question 3

For a real parameter λ , investigate polynomial solutions of

$$y^{(4)} - xy' + \lambda y = 0.$$

A polynomial solution is required to be nonzero; its degree is not prescribed.

Tasks

1. Derive the ordinary coefficient recurrence and use the highest-degree term to restrict λ .
2. Prove that your restriction is sufficient. For each allowed parameter, determine the dimension of the polynomial solution space and give a monic formula.
3. Find the monic solution when $\lambda = 8$ and verify it directly in the differential equation.
4. Explain why the other three coefficient chains cannot supply additional independent polynomials for the same parameter. Decide what remains of the polynomial classification when λ is negative or nonintegral.

Question 3 – Solution

Strategy. Combine a highest-degree argument with the exact termination mechanism in the recurrence.

Step 1: Identify the only possible parameters. Coefficient matching gives

$$a_{n+4} = \frac{(n-\lambda)a_n}{(n+4)(n+3)(n+2)(n+1)}.$$

If a nonzero polynomial has degree m , its leading term in $-xy' + \lambda y$ is $(\lambda - m)a_mx^m$; $y^{(4)}$ has smaller degree. Hence $\lambda = m$ must be a nonnegative integer.

Step 2: Construct and count the terminating chain. For $\lambda = m$, normalize $a_m = 1$. Descending the recurrence gives

$$p_m(x) = \sum_{j=0}^{\lfloor m/4 \rfloor} \frac{(-1)^j m!}{4^j j! (m-4j)!} x^{m-4j}.$$

At $n = m$ the next coefficient is zero and the chain terminates. The displayed finite sum satisfies every coefficient equation, so it is a solution. All its coefficients are fixed by the leading coefficient; thus the polynomial solution space, including zero, is one-dimensional.

Step 3: Check a concrete eighth-degree solution. For $m = 8$,

$$p_8 = x^8 - 420x^4 + 1260.$$

We have $p_8^{(4)} = 1680x^4 - 10080$ and $-xp_8' + 8p_8 = -1680x^4 + 10080$, so the residual is identically zero.

Step 4: Rule out hidden terminating chains. The recurrence preserves the index modulo four. A nonzero chain can terminate only when its numerator $n - m$ vanishes, which occurs solely in the class containing m . Any nonzero seed in another class continues forever and cannot be part of a polynomial. Equivalently, a second polynomial of lower degree would require that degree to equal m , a contradiction. For negative or nonintegral λ , only the zero polynomial solves the equation.

Question 4

Near $x = 0$, consider

$$(1 - x^2)y''' - 6xy'' - 6y' = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad y''(0) = 2a_2.$$

The leading coefficient vanishes at $x = \pm 1$, but this does not by itself prove that every solution is singular there.

Tasks

1. Derive the recurrence for the Taylor coefficients and sum the resulting series on $|x| < 1$.
2. Classify exactly which initial triples give analytic continuation through $x = 1$, through $x = -1$, and through both points.
3. Determine the Taylor radius at zero for every initial triple. Explain how an entire solution can coexist with singular coefficients in the normalized equation.
4. For the data $(a_0, a_1, a_2) = (0, 1, -1)$, find the continued value and first two derivatives at $x = 1$. Verify the original equation at that point, and give the maximal real interval on which this solution is analytic.

Question 4 – Solution

Strategy. Sum the two coefficient tails, then check cancellation of each possible pole separately.

Step 1: Derive and sum the recurrence. The coefficient equation simplifies to

$$(n+3)(n+2)(n+1)(a_{n+3} - a_{n+1}) = 0 \quad (n \geq 0).$$

Thus the odd tail repeats a_1 and the positive even tail repeats a_2 :

$$y = a_0 + \frac{a_1x + a_2x^2}{1 - x^2}, \quad |x| < 1.$$

As an independent check, the differential equation is exactly $((1 - x^2)y)''' = 0$.

Step 2: Classify pole cancellation. The numerator of the fractional term vanishes at 1 exactly when $a_1 + a_2 = 0$, and at -1 exactly when $-a_1 + a_2 = 0$. Because the denominator's zeros are simple, these are also sufficient for analytic continuation through the respective point. Continuation through both requires $a_1 = a_2 = 0$, yielding the constant solution.

Step 3: Determine every convergence radius. If $a_1 = a_2 = 0$, the Taylor series is constant and its radius is infinite. Otherwise at least one genuine pole remains at distance one, so the radius is exactly one, even if the other pole cancels. Singularities of the normalized coefficients constrain the general theory; they need not be singularities of a particular solution after cancellation.

Step 4: Verify passage through a singular point. For the stated data, $y = x/(1+x)$ after cancellation. Therefore

$$y(1) = \frac{1}{2}, \quad y'(1) = \frac{1}{4}, \quad y''(1) = -\frac{1}{4}.$$

At $x = 1$, the original equation reduces to $-6y'' - 6y' = 0$, which these values satisfy. The continued solution is analytic on $(-1, \infty)$, its maximal real analytic interval containing zero. Its Taylor series about zero still has radius one because the pole at -1 remains.

Question 5

A formal power series is an algebraic coefficient expansion without an assumed positive radius of convergence. Investigate

$$x^2 y''' - y'' = -x, \quad y(0) = y'(0) = y''(0) = 0.$$

For a possible one-sided smooth solution, you may use

$$z(x) = \int_0^\infty e^{-t} \frac{x}{1 - xt} dt, \quad x \leq 0.$$

Tasks

1. Put $z = y''$ and derive its formal coefficient recurrence. Find the first four nonzero terms of the formal series for y .
2. Prove that the formal series for y has radius zero. Deduce whether an analytic solution at zero can satisfy the equation and data.
3. Verify that the given integral defines a smooth function from the left at zero and satisfies $x^2 z' - z = -x$. Recover a one-sided smooth solution y with the required initial data.
4. Explain how this smooth solution can have the divergent formal series as its Taylor series. Identify the hypothesis that prevents application of the ordinary-point analytic existence theorem at zero.

Question 5 – Solution

Strategy. Separate formal coefficient solvability from convergence and from one-sided smooth existence.

Step 1: Compute the formal coefficients. For $z = \sum b_n x^n$, coefficient matching gives $b_0 = 0$, $b_1 = 1$, and $b_n = (n-1)b_{n-1}$ for $n \geq 2$. Thus $b_n = (n-1)!$ for $n \geq 1$. Two integrations with zero constants give

$$y \sim \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)(n+2)} x^{n+2} = \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{10} + \frac{x^6}{5} + \cdots.$$

Step 2: Prove divergence. The ratio of consecutive displayed coefficients is $n(n+1)/(n+3) \rightarrow \infty$. For every fixed nonzero x , the terms eventually fail to tend to zero. The radius is zero. Any analytic solution would have exactly these forced coefficients, so no analytic solution at zero exists.

Step 3: Construct and verify a smooth solution. For $x \leq 0$, $1 - xt \geq 1$. For $n \geq 1$,

$$\partial_x^n \frac{x}{1 - xt} = \frac{n! t^{n-1}}{(1 - xt)^{n+1}}.$$

An integrable majorant permits differentiation of every order up to the left endpoint, giving $z(0) = 0$ and $z^{(n)}(0-) = n!(n-1)!$. Also

$$x^2 z' - z = x \int_0^\infty \frac{d}{dt} \left(\frac{e^{-t}}{1 - xt} \right) dt = -x.$$

Hence $y(x) = \int_0^x (x-v)z(v) dv$, with oriented integration, is smooth on $(-\infty, 0]$, satisfies the equation there and has the zero initial triple. The equality at zero follows by continuity.

Step 4: Resolve the apparent contradiction. Its derivatives give exactly the formal coefficients above, but smoothness does not require the Taylor series to converge or represent the function. The leading coefficient x^2 vanishes at zero, so this is not an ordinary point. The analytic existence theorem for a normalized regular equation does not apply.

Question 6

The series solution of $y''' + y' = 0$ with $(y, y', y'')(0) = (0, 1, 0)$ describes an oscillator derivative pair (y', y'') . A proposed rational approximation is

$$U(x) = \frac{1 - x^2/4}{1 + x^2/4}, \quad V(x) = -\frac{x}{1 + x^2/4}.$$

Its designer claims that correct initial values and exact conservation of $U^2 + V^2$ are enough to make it the true derivative pair.

Tasks

1. Derive the series recurrence and identify the exact solution. Give its terms through degree seven.
2. Expand U, V through the first coefficient at which each differs from the true pair. Check their initial values and quadratic invariant.
3. Test the necessary derivative relation $U' = V$. Compute its defect exactly and decide whether any function can have both $y' = U$ and $y'' = V$ on an interval containing zero.
4. Find the phase $\tau(x)$ for which $(U, V) = (\cos \tau, -\sin \tau)$. Explain the actual evolution law and plot the phase lag $x - \tau(x)$ beside its leading cubic approximation on $[0, 2]$.

Question 6 – Solution

Strategy. Test derivative compatibility in addition to an invariant and a few Taylor coefficients.

Step 1: Solve the series IVP. Coefficient matching gives $a_{n+3} = -a_{n+1}/[(n+3)(n+2)]$. The data $(a_0, a_1, a_2) = (0, 1, 0)$ give the entire series

$$y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

The true pair is $(\cos x, -\sin x)$.

Step 2: Check local accuracy and the invariant. Near zero,

$$U = 1 - x^2/2 + x^4/8 + O(x^6), \quad V = -x + x^3/4 + O(x^5).$$

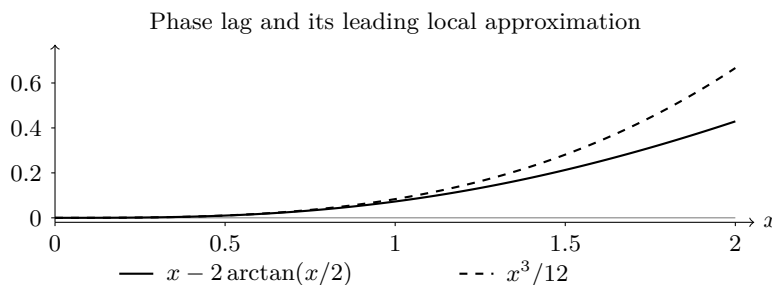
The correct coefficients are $x^4/24$ and $x^3/6$, respectively. Nevertheless $(U, V)(0) = (1, 0)$, and direct squaring gives $U^2 + V^2 = 1$ exactly.

Step 3: Expose the derivative defect. Writing $d = 1 + x^2/4$, we obtain

$$U' - V = \frac{x^3}{4d^2}.$$

This is nonzero for $x \neq 0$. If $y' = U$, differentiation forces $y'' = U'$, so no such function has both proposed derivatives on a nontrivial interval containing zero. Conservation alone does not enforce the differential equation.

Step 4: Identify the altered clock. The half-angle identities give $\tau = 2 \arctan(x/2)$ and $\tau' = 1/d$. Thus $U' = V/d$, $V' = -U/d$: the pair traverses the correct circle with a different speed. The phase lag is $x - \tau = x^3/12 + O(x^5)$ and is positive for $x > 0$ because its derivative is $x^2/(4 + x^2) > 0$.



Question 7

Use series to investigate the directional boundedness of solutions to

$$y''' - 2xy'' - 4y' = 0, \quad y(0) = a, \quad y'(0) = c, \quad y''(0) = 2b.$$

You may use $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Tasks

1. Derive the coefficient recurrence and construct three entire basis functions normalized by the three initial coefficients.
2. Sum the even chain and express the odd chain as an integral involving e^{-t^2} . Verify the resulting general solution and its initial values.
3. Give necessary and sufficient conditions for boundedness as $x \rightarrow +\infty$, as $x \rightarrow -\infty$, and in both directions. Justify cancellation limits with an explicit Gaussian-tail bound.
4. For $a = b = -\sqrt{\pi}/2$, $c = 1$, determine the limit at $+\infty$ and the sign of the response for $x \geq 0$. Sketch that one-sided bounded response on $[0, 4]$.

Question 7 – Solution

Strategy. Build an entire basis from the recurrence, then examine its exponentially amplified coefficients at each end of the real line.

Step 1: Follow the even and odd chains. Matching x^n gives $a_{n+3} = 2a_{n+1}/(n+3)$ for $n \geq 0$. The independent seeds are $a_0 = a$, $a_1 = c$, $a_2 = b$. Ratio tests along each chain give infinite radius. A normalized basis is

$$1, \quad e^{x^2} - 1, \quad H(x) = x + \frac{2x^3}{3} + \frac{4x^5}{15} + \cdots.$$

Step 2: Identify and verify the odd basis. The function $H = e^{x^2} \int_0^x e^{-t^2} dt$ satisfies $H' = 2xH + 1$; differentiating twice proves the third-order equation and its initial triple $(0, 1, 0)$. The other two basis functions also satisfy the equation. Thus

$$y = a + b(e^{x^2} - 1) + ce^{x^2} \int_0^x e^{-t^2} dt.$$

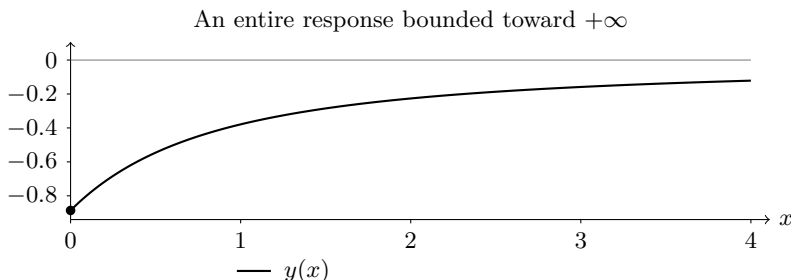
Its initial coefficients are exactly (a, c, b) in degrees zero, one and two.

Step 3: Classify both ends. Put $K = \sqrt{\pi}/2$. At $+\infty$, a nonzero $b + cK$ forces exponential growth. If $b + cK = 0$, the remaining tail tends to zero because, for $x > 0$,

$$0 \leq e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{2x}.$$

The inequality follows by replacing 1 with t/x in the integral. Therefore boundedness at $+\infty$ is equivalent to $b + cK = 0$, with limit $a - b$. Oddness of the integral gives the condition $b - cK = 0$ at $-\infty$, also with limit $a - b$. Both conditions hold exactly when $b = c = 0$, leaving constants.

Step 4: Check the selected response. Here $y = -e^{x^2} \int_x^\infty e^{-t^2} dt < 0$ for $x \geq 0$, and the tail estimate gives $y \rightarrow 0$ as $x \rightarrow +\infty$.



Question 8

Investigate the forced singular equation

$$xy''' - y'' = x, \quad x > 0.$$

One wishes to continue a solution to $x = 0$. Analytic continuation, twice continuous differentiability and three times continuous differentiability are different requirements here.

Tasks

1. Attempt an ordinary power series at zero. Identify the precise coefficient equation that obstructs a formal Taylor solution.
2. Solve the equation on $x > 0$ by first setting $z = y''$. Give the complete three-constant family.
3. Determine the highest integer k for which the solutions extend as C^k functions to $[0, \infty)$. Explain why no choice of the homogeneous constants removes the obstruction.
4. Select the constants giving $y = x^3 \log x/6$. Find its endpoint limit, its minimum on $(0, 1]$ and its value at one. Sketch it, showing zero as an excluded endpoint of the original domain.

Question 8 – Solution

Strategy. Let the failed coefficient equation identify a logarithmic term forced by resonance.

Step 1: Locate the formal obstruction. For $y = \sum a_n x^n$, matching x^n gives

$$(n+2)(n+1)(n-1)a_{n+2} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

At $n = 0$ this requires $a_2 = 0$, but at $n = 1$ it requires $0 = 1$. No formal ordinary power series, and hence no analytic solution, exists at zero.

Step 2: Solve the reduced equation. For $z = y''$, $(z/x)' = 1/x$, giving $z = x \log x + Cx$. Integrating twice yields

$$y = A + Bx + \frac{x^3 \log x}{6} + \left(\frac{C}{6} - \frac{5}{36}\right)x^3.$$

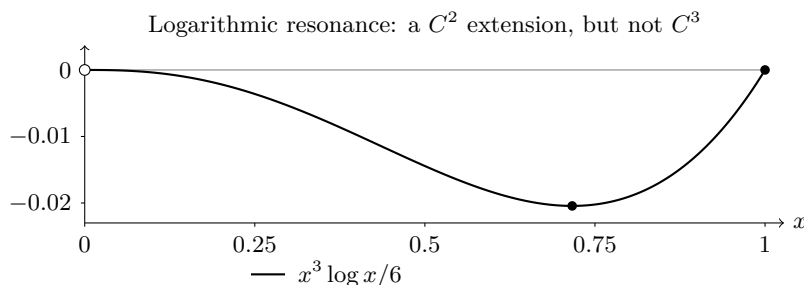
The three homogeneous modes are $1, x, x^3$; the forced logarithm is additional.

Step 3: Classify endpoint smoothness. As $x \downarrow 0$, $y \rightarrow A$, $y' \rightarrow B$ and $y'' = x \log x + Cx \rightarrow 0$. These limits give a C^2 extension. But $y''' = \log x + 1 + C \rightarrow -\infty$, so no C^3 extension exists. Constants can change finite terms, not cancel the coefficient of $\log x$. This is a formal inconsistency, unlike a solvable but divergent coefficient recurrence.

Step 4: Verify the illustrated member. Set $A = B = 0$, $C = 5/6$. Then $y' = x^2(3 \log x + 1)/6$, so

$$x_{\min} = e^{-1/3}, \quad y_{\min} = -\frac{1}{18e}, \quad y(1) = 0, \quad \lim_{x \downarrow 0} y(x) = 0.$$

The derivative changes from negative to positive at this unique minimum. The open marker records the original domain $x > 0$, even though a C^2 extension to zero is possible.



Question 9

A solution is known to satisfy a constant-coefficient homogeneous equation

$$y''' = Ay'' + By' + Cy, \quad (y, y', y'')(0) = (0, 0, 1),$$

where A, B, C are unknown real constants. Its measured Taylor expansion is

$$y(x) = \frac{x^2}{2} + \frac{x^3}{3} + \frac{7x^4}{24} + \frac{x^5}{5} + O(x^6).$$

Tasks

1. Convert the measured coefficients into derivatives and recover A, B, C uniquely. Show the triangular recovery equations.
2. Explain why degree five is the lowest Taylor degree that suffices in general. Predict the coefficient of x^6 from the recovered equation.
3. Derive the full coefficient recurrence and justify that it determines an entire solution with the given data. Rule out any lower-order monic constant-coefficient homogeneous equation for this same solution.
4. Explain why the stated model assumption is essential. Construct analytic functions with the same measured expansion through degree five that do not solve the recovered equation.

Question 9 – Solution

Strategy. Recover constant coefficients from the earliest derivatives at which they enter, while keeping the scope of identification explicit.

Step 1: Recover the constants. The measured derivatives are $b_2 = 1$, $b_3 = 2$, $b_4 = 7$, $b_5 = 24$, where $b_n = y^{(n)}(0)$. The equation and its first two derivatives at zero give

$$b_3 = A, \quad b_4 = A^2 + B, \quad b_5 = A^3 + 2AB + C.$$

Thus $\boxed{A = 2, B = 3, C = 4}$. The triangular equations prove uniqueness.

Step 2: Check the information threshold. Through degree four, only A, B are fixed; arbitrary C gives the same earlier derivatives. Degree five first reveals C , so it is necessary and sufficient in general. The next derivative is

$$b_6 = 2b_5 + 3b_4 + 4b_3 = 77, \quad \boxed{a_6 = 77/720}.$$

Step 3: Extend the series and check minimal order. For ordinary coefficients,

$$a_{n+3} = \frac{2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + 4a_n}{(n+3)(n+2)(n+1)} \quad (n \geq 0).$$

The three seeds determine every coefficient. Equivalently, its constant companion system has the everywhere-convergent matrix exponential, so the series is entire. A homogeneous regular equation of order one or two with $y(0) = y'(0) = 0$ would have only the zero solution by uniqueness, contradicting $y''(0) = 1$. Thus the least possible monic homogeneous order is three.

Step 4: Show the model assumption matters. Let y be the recovered solution and $y_\varepsilon = y + \varepsilon x^6$. For any nonzero ε , this is entire and has the same measured coefficients through degree five. Yet for $L = D^3 - 2D^2 - 3D - 4$,

$$Ly_\varepsilon = \varepsilon(120x^3 - 60x^4 - 18x^5 - 4x^6) \neq 0.$$

A finite Taylor record identifies the equation within the stipulated model class; it does not establish that an arbitrary analytic function belongs to that class.

Question 10

For $\varepsilon > 0$, consider the fourth-order initial-value problem

$$\varepsilon y^{(4)} + y'' = 0, \quad (y, y', y'', y''')(0) = (0, 0, 1, 0).$$

A Taylor approximation accurate for each fixed parameter need not be accurate uniformly as the parameter tends to zero.

Tasks

1. Derive the Taylor recurrence, sum the solution series, and determine its radius for each fixed $\varepsilon > 0$.
2. Prove uniform convergence of y_ε and y'_ε on the whole real line as $\varepsilon \downarrow 0$. Show that y''_ε has no limit at any fixed nonzero x .
3. At fixed $x \neq 0$, investigate the limit of every fixed even-degree Taylor truncation of degree at least two. Explain why taking a parameter limit and truncating the series give incompatible conclusions.
4. Use $\xi = x/\sqrt{\varepsilon}$ to obtain a parameter-independent profile. Give a uniform Taylor error bound for bounded ξ , and plot the profile and its degree-six polynomial on $0 \leq \xi \leq \pi$.

Question 10 – Solution

Strategy. Resolve the shrinking oscillation scale before judging a series truncation.

Step 1: Derive and sum the series. The recurrence is $a_{n+4} = -a_{n+2}/[\varepsilon(n+4)(n+3)]$, with $a_0 = a_1 = a_3 = 0$, $a_2 = 1/2$. Therefore

$$y_\varepsilon = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j}}{(2j)! \varepsilon^{j-1}} = \varepsilon (1 - \cos(x/\sqrt{\varepsilon})).$$

The radius is infinite for each fixed positive parameter. The formula verifies all four initial derivatives and the equation directly.

Step 2: Distinguish levels of derivative convergence. The bounds $|y_\varepsilon| \leq 2\varepsilon$ and $|y'_\varepsilon| \leq \sqrt{\varepsilon}$ prove uniform convergence to zero on $\mathbb{R}$. But $y''_\varepsilon = \cos(x/\sqrt{\varepsilon})$. For fixed $x \neq 0$, the sequences $\varepsilon_n = x^2/(2\pi n)^2$ and $\tilde{\varepsilon}_n = x^2/((2n+1)\pi)^2$ give values 1 and -1 . No pointwise limit exists there. The reduced equation $y'' = 0$ cannot retain the original second-derivative datum.

Step 3: Test fixed truncations. The degree-two polynomial is $x^2/2$, which fails to approach zero at fixed nonzero x . For any fixed degree $2m \geq 4$, its highest term is a nonzero constant times ε^{1-m} and dominates the lower powers as $\varepsilon \downarrow 0$. Thus the truncation diverges while the full sum tends to zero. The Taylor remainder is not uniform in this parameter at fixed x .

Step 4: Rescale and certify. With $y_\varepsilon/\varepsilon = 1 - \cos \xi$, Taylor's theorem gives, for $|\xi| \leq R$,

$$\left| 1 - \cos \xi - \sum_{j=1}^m \frac{(-1)^{j+1} \xi^{2j}}{(2j)!} \right| \leq \frac{R^{2m+2}}{(2m+2)!}.$$

Multiplying by ε bounds the original displacement error. For the graph, $m = 3$, $R = \pi$, so the profile error is at most $\pi^8/40320$.

