

Question 1

Consider the real power series

$$F(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

A student reports a radius of convergence and assumes that this alone specifies where the series represents a finite function.

Tasks

1. Identify the center and radius of convergence. Explain what the ratio test says inside and outside the radius, and what it leaves undecided.
2. Test both boundary points separately. Give the exact interval of convergence and distinguish absolute from conditional convergence.
3. Derive a closed formula for the sum inside the open interval by integrating a geometric series. Include the constant fixed by the value at the center.
4. Determine the sum at the included boundary point without substituting into an unjustified endpoint identity. Sketch the convergence interval and explain why the excluded endpoint behaves differently.

Question 1 – Solution

Strategy. Normalize the displacement from the center, test the endpoints as numerical series, then justify any boundary value independently.

Step 1: Apply the ratio test away from the center. Write $z = (x - 2)/3$. For $z \neq 0$, the ratio of successive term magnitudes is $|z|n/(n+1) \rightarrow |z|$. Thus the center is 2 and $R = 3$. The series converges absolutely for $|z| < 1$ and diverges for $|z| > 1$. At the center all terms vanish. The ratio test is inconclusive at $z = \pm 1$.

Step 2: Resolve the two endpoints. At $x = -1$, the series is $\sum_{n \geq 1} (-1)^n/n$, which converges by alternation but not absolutely. At $x = 5$, it is the divergent harmonic series. Therefore

$$I = [-1, 5), \quad \text{absolute on } (-1, 5), \quad \text{conditional only at } -1.$$

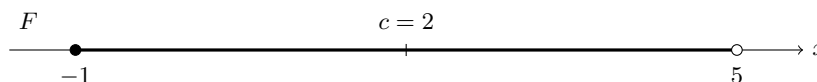
A radius specifies distances, not whether either boundary point is included.

Step 3: Integrate the geometric series. On any compact subinterval of $(-1, 1)$, the geometric series converges uniformly, so integration from 0 to z gives

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \frac{du}{1-u} = -\log(1-z).$$

This also holds for negative z with the usual oriented integral. Both sides vanish at $z = 0$, fixing the constant. Hence $F(x) = -\log(1 - (x - 2)/3)$ for $-1 < x < 5$.

Step 4: Establish the boundary value. The finite geometric identity gives $\sum_{k=0}^{N-1} (-u)^k = (1 - (-u)^N)/(1 + u)$. Integrating over $[0, 1]$, its difference from $\log 2$ has magnitude at most $\int_0^1 u^N du = 1/(N+1) \rightarrow 0$. Negating the resulting alternating harmonic sum proves $F(-1) = -\log 2$. At $x = 5$ there is no alternation and harmonic divergence persists; the interior logarithm also tends to $+\infty$ there.



Question 2

A power series with large gaps between nonzero coefficients is written as

$$L(x) = \sum_{k=0}^{\infty} 2^k x^{2^k}.$$

The first terms are $x + 2x^2 + 4x^4 + 8x^8 + \cdots$. Use the ordinary definition of a power series even though many coefficients vanish.

Tasks

1. Write its coefficient rule in the form $\sum_{n=0}^{\infty} a_n x^n$. Explain why applying $|a_{n+1}/a_n|$ to all consecutive coefficients is inappropriate.
2. Prove the exact radius using comparison when $|x| < 1$ and the necessary term test when $|x| > 1$.
3. Test both endpoints, paying attention to the parity of 2^k . Does the endpoint $x = -1$ produce an alternating tail?
4. Let $x_m = e^{-1/2^m}$ for integers $m \geq 0$. Use a single term to bound $L(x_m)$ from below, and decide whether the sparse exponents make L bounded as $x \rightarrow 1^-$.

Question 2 – Solution

Strategy. Treat the missing coefficients explicitly. A convergent positive comparison series controls the interior, while selected large terms control boundary growth.

Step 1: Identify the coefficients. We have $a_n = n$ when $n = 2^k$ for an integer $k \geq 0$, and $a_n = 0$ otherwise, including $n = 0$. Consecutive coefficient ratios repeatedly have zero denominators, so the usual coefficient-ratio shortcut is unavailable. This does not prevent other convergence tests from determining the radius.

Step 2: Establish the radius directly. For $r = |x| < 1$,

$$\sum_{k=0}^{\infty} 2^k r^{2^k} \leq \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} < \infty.$$

Thus the series converges absolutely throughout $|x| < 1$. If $|x| > 1$, the magnitudes $2^k |x|^{2^k}$ do not tend to zero; indeed they grow without bound. Therefore $\boxed{R = 1}$. The comparison sum follows by differentiating the geometric series inside its radius.

Step 3: Test parity at the endpoints. At $x = 1$, the terms are 2^k and fail the term test. At $x = -1$, the first term is -1 , but every term with $k \geq 1$ is $+2^k$, since 2^k is even. There is no alternating tail. Both endpoint series diverge, so $\boxed{I = (-1, 1)}$, with absolute convergence at every point of this interval.

Step 4: Prove unbounded growth despite the gaps. For $0 < x_m < 1$, every term is positive. Keeping just the term indexed by $k = m$ gives

$$\boxed{L(x_m) \geq 2^m x_m^{2^m} = 2^m / e \rightarrow \infty.}$$

Also $x_m \uparrow 1$. Thus L is not bounded near 1. In fact the sum is increasing on $[0, 1)$ because each term is increasing, so this sequence of lower bounds proves $L(x) \rightarrow +\infty$ as $x \rightarrow 1^-$. Sparse powers do not offset the growth of the coefficients and their near-boundary terms.

Question 3

Let

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad G(x) = \int_0^x F(t) dt.$$

Use termwise differentiation and integration only where their hypotheses hold. No closed formula for F itself is required.

Tasks

1. Determine the radius and exact convergence interval of the series for F , including the type of convergence at each endpoint.
2. Obtain power series for F' and G . Determine each radius and interval, testing the endpoints afresh rather than copying them from F .
3. Sum the derivative series for $|x| < 1$ using the logarithmic series, treating $x = 0$ separately. Find its limit as $x \rightarrow -1^+$ and as $x \rightarrow 1^-$.
4. Prove F is continuous on $[-1, 1]$. Determine its one-sided endpoint slopes and explain why a finite endpoint value need not have a finite slope. Compare the three convergence intervals in a diagram.

Question 3 – Solution

Strategy. Differentiation and integration preserve the radius but can alter endpoints; continuity and the mean value theorem settle endpoint slopes.

Step 1: Check the original series. The ratio test gives $R = 1$. At $x = 1$ and $x = -1$, the absolute values sum to $\sum 1/n^2 < \infty$. Thus $I_F = [-1, 1]$, absolutely throughout. For $|x| > 1$ the terms do not tend to zero.

Step 2: Differentiate and integrate inside the radius. For $|x| < 1$,

$$F'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad G(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)}.$$

The constant in G is zero because $G(0) = 0$. Both new radii are 1. For the derivative series, $x = 1$ gives harmonic divergence; $x = -1$ gives conditional alternating convergence. Hence $I_{F'} = [-1, 1)$. The integrated series converges absolutely at both endpoints, so $I_G = [-1, 1]$. Here these interval labels refer to the displayed series, not an assumed endpoint differentiation rule.

Step 3: Sum the derivative in the open interval. Using $\sum_{n \geq 1} x^n/n = -\log(1-x)$ gives

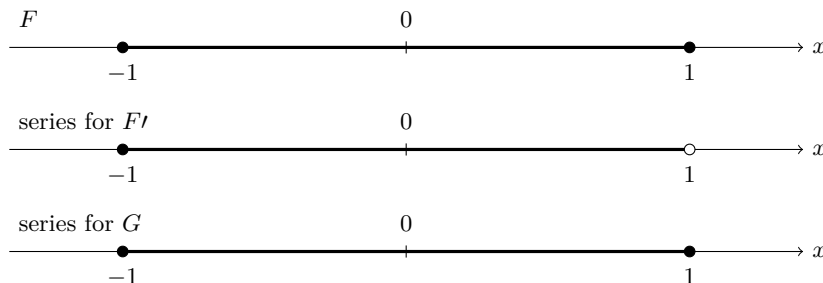
$$F'(x) = \frac{-\log(1-x)}{x} \quad (0 < |x| < 1), \quad F'(0) = 1.$$

The value at zero follows from the constant term and also from the limit. Consequently $\lim_{x \rightarrow -1+} F'(x) = \log 2$, whereas $\lim_{x \rightarrow 1-} F'(x) = +\infty$.

Step 4: Justify what happens at the boundary. Since $|x^n/n^2| \leq 1/n^2$ on $[-1, 1]$, the uniform majorant test gives uniform convergence there and continuity of F . Apply the mean value theorem between an interior point and either endpoint. The intervening derivative then tends to the corresponding limit found above. Therefore

$$F'_+(-1) = \log 2, \quad \lim_{x \rightarrow 1-} \frac{F(1) - F(x)}{1 - x} = +\infty.$$

So the finite value $F(1)$ has no finite left derivative. Endpoint behavior must be proved separately even when the power-series radius is unchanged.



Question 4

For an integer $N \geq 0$, define

$$S_N(x) = \sum_{n=0}^N x^n, \quad f(x) = \frac{1}{1-x}, \quad |x| < 1.$$

A student says that choosing a sufficiently large N makes this approximation uniformly accurate throughout the entire convergence interval.

Tasks

1. Derive the exact remainder $f - S_N$ from a finite geometric identity. State its sign for positive and negative x .
2. Find the least N guaranteeing absolute error at most 10^{-3} for every $x \in [-1/2, 1/2]$. State both the largest exponent and the number of retained terms.
3. Express the relative error $|f - S_N|/|f|$. For any fixed N , find a point in $(0, 1)$ where this error is exactly $1/2$.
4. Decide whether convergence is uniform on $(-1, 1)$, for either absolute or relative error, and justify the distinction from convergence at each fixed point. Plot S_2, S_5, S_{10} and f on $[0, 0.85]$.

Question 4 – Solution

Strategy. An exact remainder supplies a sharp interval bound and exposes what fails when points approach the boundary as the degree changes.

Step 1: Retain the exact geometric remainder. Multiplication gives $(1-x)S_N = 1 - x^{N+1}$. Therefore

$$f(x) - S_N(x) = \frac{x^{N+1}}{1-x}.$$

For $0 < x < 1$ it is positive. For $-1 < x < 0$, its sign is $(-1)^{N+1}$; at $x = 0$ it is zero. The denominator is positive throughout the interval.

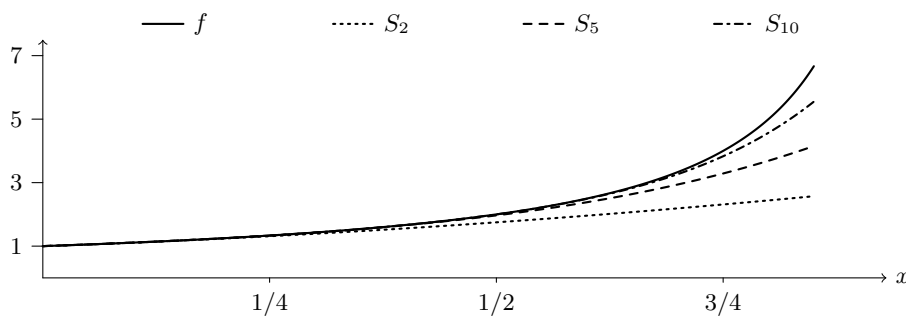
Step 2: Find the least degree on the compact interval. For $|x| \leq 1/2$,

$$|f - S_N| \leq \frac{(1/2)^{N+1}}{1 - 1/2} = 2^{-N}.$$

Equality holds at $x = 1/2$, so this is the exact maximum, not merely a convenient estimate. Since $2^{-9} > 10^{-3}$ and $2^{-10} < 10^{-3}$, $N = 10$ is least. This retains 11 terms, including the constant.

Step 3: Select a moving point with fixed relative error. The relative error simplifies to $|x|^{N+1}$. Taking $x_N = 2^{-1/(N+1)}$ gives a point in $(0, 1)$ with relative error exactly $1/2$ for every $N \geq 0$. These points approach 1 as N increases.

Step 4: Distinguish fixed-point from uniform accuracy. At each fixed $|x| < 1$, both errors tend to zero. Yet for every fixed N , $x^{N+1}/(1-x) \rightarrow +\infty$ as $x \rightarrow 1^-$, so the supremum of the absolute error on $(-1, 1)$ is infinite. The supremum of relative error is 1. Neither error converges uniformly to zero on that whole interval. The plot shows slower approximation near the right boundary; on any fixed $[-r, r]$, $r < 1$, the bound $r^{N+1}/(1-r) \rightarrow 0$ does give uniform convergence.



Question 5

Consider two series centered at zero:

$$A(x) = \sum_{n=0}^{\infty} x^n, \quad B(x) = 1 - \sum_{n=1}^{\infty} \frac{x^n}{2^n}.$$

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the series obtained by their Cauchy product, using finite coefficient convolutions.

Tasks

1. Find the sums and exact convergence intervals of A and B separately.
2. Compute c_n for every $n \geq 0$, including the constant coefficient. Determine the radius and interval of the resulting series C .
3. Identify the interval where the Cauchy-product theorem directly proves $C = AB$. Explain the rational cancellation that permits the new series to converge on a larger interval.
4. Evaluate the series C at $x = 1$ and $x = 3/2$. At each point, decide whether this value can legitimately be called the product of the two original series sums.

Question 5 – Solution

Strategy. Compute the product coefficients before interpreting the enlarged domain; a new convergent series does not make a divergent factor converge.

Step 1: Sum and test the two factors. Geometric summation gives

$$A(x) = \frac{1}{1-x} \quad (|x| < 1), \quad B(x) = 1 - \frac{x/2}{1-x/2} = \frac{1-x}{1-x/2} \quad (|x| < 2).$$

At each respective endpoint the terms fail to tend to zero. Thus the exact intervals are $(-1, 1)$ and $(-2, 2)$, both absolutely convergent inside.

Step 2: Compute the finite coefficient sums. The coefficients of A are all 1; those of B are $b_0 = 1$ and $b_n = -2^{-n}$ for $n \geq 1$. Hence

$$c_n = \sum_{k=0}^n b_k = 1 - \sum_{k=1}^n 2^{-k} = 2^{-n} \quad (n \geq 0).$$

In particular $c_0 = 1$. Therefore $C(x) = \sum_{n \geq 0} (x/2)^n = 1/(1-x/2)$ for $|x| < 2$. Its radius is 2 and both endpoints diverge.

Step 3: State where multiplication is justified. Both factors converge absolutely on $(-1, 1)$, so the Cauchy-product theorem proves equality there. Their rational formulas give

$$\frac{1}{1-x} \frac{1-x}{1-x/2} = \frac{1}{1-x/2}.$$

Cancellation removes the factor responsible for A 's boundary at 1. The coefficient series C consequently has radius 2, exceeding the smaller input radius. The theorem guarantees convergence on the common interior; it does not require the resulting radius to equal the smaller one.

Step 4: Interpret the additional points carefully. The new series gives $C(1) = 2$, $C(3/2) = 4$. At $x = 1$, A diverges and B sums to zero; an undefined series sum cannot be multiplied by zero to produce 2. At $x = 3/2$, A again diverges, although B converges to -2 . Thus neither value is a product of the two original series sums. They are valid values of the independently convergent product-coefficient series and its rational formula.

Question 6

Let $f(x) = 1/(4 - x)$, defined for real $x \neq 4$. Represent this same function by geometric power series with two different centers, $c_1 = 1$ and $c_2 = -1$. Do not use a Taylor-coefficient formula.

Tasks

1. Rewrite the denominator about each center and obtain the two power series explicitly.
2. Find both radii and exact real convergence intervals, testing their endpoints. Mark centers and intervals in a diagram.
3. At $x = -4$, decide which series converges and find its sum. At $x = 5$, decide whether either series converges even though $f(5)$ exists.
4. Explain why the two series agree wherever both converge, why changing the center changes the radius here, and why the domain of the rational function is not the convergence interval of either series.

Question 6 – Solution

Strategy. Factor out the distance from each center to the denominator's zero, then apply the geometric-series condition to the normalized displacement.

Step 1: Expand about each center. Using $4 - x = 3 - (x - 1)$ and $4 - x = 5 - (x + 1)$ gives

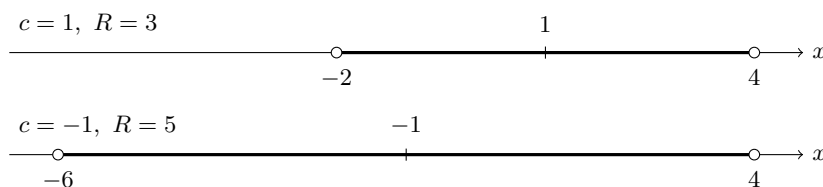
$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^{n+1}}, \quad f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{5^{n+1}}.$$

Each equality is initially asserted only when its geometric ratio has magnitude less than one.

Step 2: Determine the two intervals. The first ratio is $(x-1)/3$, so $R_1 = 3$ and $I_1 = (-2, 4)$. The second is $(x+1)/5$, so $R_2 = 5$ and $I_2 = (-6, 4)$. At each endpoint the terms have constant nonzero magnitude ($1/3$ or $1/5$), so all four endpoint tests give divergence. Both series converge absolutely inside their respective intervals.

Step 3: Test the specified points. At $x = -4$, the first ratio is $-5/3$, so its terms do not tend to zero. The second ratio is $-3/5$, so it converges and sums to $\boxed{(1/5)/(1+3/5) = 1/8 = f(-4)}$. At $x = 5$, the ratios are $4/3$ and $6/5$, so both series diverge even though $\boxed{f(5) = -1}$ is a valid rational-function value.

Step 4: Compare representations and domains. On the common interval $(-2, 4)$, each convergent geometric identity gives the same function $1/(4-x)$, so their sums agree. Their right endpoints coincide with the denominator's zero, but this point is distance 3 from one center and distance 5 from the other. The left boundaries follow from the same absolute-ratio conditions, even though f is finite there. The rational function has domain $\mathbb{R} \setminus \{4\}$, while each series represents it on only its own convergence interval. Evaluation of the rational formula outside that interval is not evaluation of the series.



Question 7

Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ has positive radius R . Consider $H(x) = xy'(x) + y(x)$ only for $|x-2| < R$. A student claims that $H(x) = \sum_{n=0}^{\infty} (n+1)a_n(x-2)^n$.

Tasks

1. Derive the correct coefficient of $(x-2)^n$ in H for every $n \geq 0$, reindexing the derivative carefully.
2. Write the constant, linear, quadratic and cubic coefficients explicitly in terms of the a_n .
3. Identify the student's missing contribution and characterize exactly which series y make the claimed identity valid throughout the open interval.
4. Apply your formula to $a_n = 3^{-n-1}$. Sum y geometrically, compute H directly from this closed form, and verify the coefficients agree. State the resulting series' radius.

Question 7 – Solution

Strategy. Work in $h = x - 2$ and remember that multiplication by x means multiplication by $h + 2$, not merely by h .

Step 1: Reindex before multiplying. Inside the radius, $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}h^n$. Now $hy' = \sum_{n=1}^{\infty} na_nh^n$; its constant coefficient is zero. Adding $2y'$ and y yields

$$H = \sum_{n=0}^{\infty} ((n+1)a_n + 2(n+1)a_{n+1})h^n.$$

The formula also works for $n = 0$, where the constant term of xy' comes entirely from $2y'$.

Step 2: Display the first four coefficients. The requested coefficients are

$$a_0 + 2a_1, \quad 2a_1 + 4a_2, \quad 3a_2 + 6a_3, \quad 4a_3 + 8a_4.$$

As a direct constant-term check, $H(2) = 2y'(2) + y(2) = 2a_1 + a_0$. Keeping the same summation index through a differentiation would miss the shift from a_n to a_{n+1} .

Step 3: Determine exactly when the shortcut works. The claimed series actually represents $hy' + y$. Its difference from H is $2y'$. The identity therefore holds throughout the interval exactly when $y' \equiv 0$, or equivalently $a_n = 0$ for every $n \geq 1$. One justification of the equivalence is uniqueness of power-series coefficients, obtained by successive differentiation at the center. Any constant a_0 is allowed. Equality at a single isolated point would not imply this identity.

Step 4: Check against a geometric closed form. For $a_n = 3^{-n-1}$, $y = (1/3) \sum (h/3)^n = 1/(3-h) = 1/(5-x)$ when $|h| < 3$. Hence

$$H = \frac{x}{(5-x)^2} + \frac{1}{5-x} = \frac{5}{(5-x)^2}.$$

The coefficient formula gives $(n+1)/3^{n+1} + 2(n+1)/3^{n+2} = 5(n+1)/3^{n+2}$. This agrees with the differentiated geometric identity $5/(3-h)^2 = \sum_{n \geq 0} 5(n+1)h^n/3^{n+2}$. The radius is $\boxed{3}$; the factors growing linearly in n do not change it.

Question 8

Consider a series of polynomials,

$$P(x) = \sum_{n=0}^{\infty} \left(\frac{x^2 - 4}{2} \right)^n.$$

It is geometric in $(x^2 - 4)/2$, but its displayed terms are not single monomials in x . Investigate how its convergence set compares with that of an ordinary power series representing the same sum.

Tasks

1. Determine the exact real convergence set of the displayed series, including all boundary tests. Explain why it need not be one interval centered at zero.
2. Sum the series wherever it converges. Compare the domain of that rational formula with the convergence set just found.
3. Starting afresh from the rational formula, derive an ordinary power series centered at zero. Find its radius and endpoint behavior.
4. Compare the two series at $x = 0$ and $x = 1$, and state exactly where their sums are known to agree. Diagram both convergence sets and explain why one should not silently substitute the new domain for the original one.

Question 8 – Solution

Strategy. Apply the geometric test to the substituted variable first; then derive and test a separate series in powers of x .

Step 1: Pull back the geometric convergence condition. The displayed series converges exactly when $|(x^2 - 4)/2| < 1$, or $2 < x^2 < 6$. Thus its convergence set is

$$(-\sqrt{6}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{6}).$$

At $x = \pm\sqrt{2}$ the ratio is -1 ; at $x = \pm\sqrt{6}$ it is 1 . In each case the terms fail to approach zero. Elsewhere outside the two intervals the ratio has magnitude greater than one. These polynomial terms do not have the ordinary form $a_n(x - c)^n$ for a fixed center.

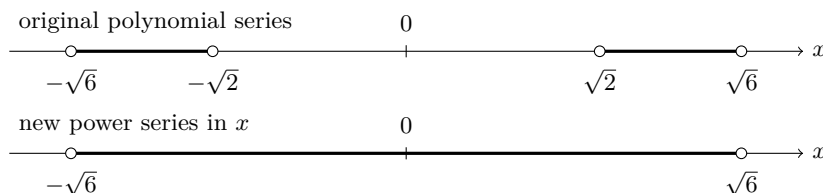
Step 2: Sum on the justified set. Geometric summation gives $P(x) = 2/(6 - x^2)$ on the two intervals above. This rational formula is defined for every real $x \neq \pm\sqrt{6}$, which is a much larger set. Its being finite at other points says nothing by itself about convergence of the displayed polynomial series there.

Step 3: Construct a distinct centered power series. Factoring the denominator differently gives

$$\frac{2}{6 - x^2} = \frac{1/3}{1 - x^2/6} = \sum_{n=0}^{\infty} \frac{x^{2n}}{3 \cdot 6^n}, \quad |x| < \sqrt{6}.$$

Its odd coefficients vanish. Its radius is $\sqrt{6}$ and both endpoints diverge because the terms there equal $1/3$. The new convergence interval is $(-\sqrt{6}, \sqrt{6})$.

Step 4: Compare actual sums, not just formulas. At $x = 0$, the original terms are $(-2)^n$, so the original series diverges; the new series sums to $1/3$. At $x = 1$, the original ratio is $-3/2$ and its series diverges, while the new series sums to $2/5$. On $\sqrt{2} < |x| < \sqrt{6}$, both converge and both equal $2/(6 - x^2)$. The new representation adds valid points; it does not retroactively change the convergence set of the original sequence of partial sums.



Question 9

For a real parameter p , consider

$$F_p(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^p}.$$

Use this family to investigate which endpoint patterns can be designed by changing coefficient decay, and which require changing coefficient signs.

Tasks

1. Find the center and radius for every real p , including $p \leq 0$.
2. Give a complete classification of convergence and absolute convergence at both endpoints as p varies. Include the threshold cases $p = 0$ and $p = 1$.
3. Determine all p yielding each of the intervals $(-2, 4)$, $[-2, 4)$ and $[-2, 4]$. Supply one concrete value for each pattern.
4. Can any member of this family have convergence interval $(-2, 4]$? Prove your answer, then construct another power series centered at 1 with radius 3 and exactly that interval.

Question 9 – Solution

Strategy. The exponential coefficient scale fixes the radius, while power-law decay and signs decide the endpoints.

Step 1: Determine the radius independently of p . For $x \neq 1$, the successive absolute-term ratio is

$$\frac{|x-1|}{3} \left(\frac{n}{n+1} \right)^p \rightarrow \frac{|x-1|}{3}.$$

Thus the center is 1 and $\boxed{R=3}$ for every finite real p . At the center the sum is zero. The open interval $(-2, 4)$ is always a region of absolute convergence, and outside the closed interval the series diverges.

Step 2: Classify the endpoints. At $x = 4$, the series is $\sum 1/n^p$, convergent exactly when $p > 1$. At $x = -2$, it is $\sum (-1)^n/n^p$: it converges for $p > 0$, since the magnitudes then decrease to zero, and fails the term test for $p \leq 0$. Absolute convergence at either endpoint occurs exactly for $p > 1$. Thus the left endpoint is conditional for $0 < p \leq 1$, including $p = 1$; at $p = 0$ neither endpoint's terms tend to zero.

Step 3: List all parameter regimes. Combining the tests gives

parameter range	convergence interval	example
$p \leq 0$	$(-2, 4)$	$p = 0$
$0 < p \leq 1$	$[-2, 4)$	$p = 1$
$p > 1$	$[-2, 4]$	$p = 2$

No other parameter cases remain. Changing p affects the boundary tests, not the radius or the open interval of absolute convergence.

Step 4: Reverse the endpoint pattern by signs. Within the given family, convergence at the right endpoint implies $p > 1$, which forces absolute convergence at the left endpoint too. Hence $(-2, 4]$ is impossible in this family. A different series that works is

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n 3^n}}.$$

Its radius is 3. At $x = 4$ it is alternating harmonic and converges conditionally; at $x = -2$ it is the divergent positive harmonic series. This explicitly separates a restriction of the chosen family from a restriction on power series in general.

Question 10

An unknown real power series $\sum_{n=0}^{\infty} a_n(x-2)^n$ converges at $x = 5$ and diverges at $x = -2$. No coefficient formula is supplied. Use only what these two observations imply; boundary convergence may be conditional.

Tasks

1. Find the strongest possible bounds on its radius of convergence R . Explain why either inequality may be an equality.
2. Classify the behavior forced at $x = 0, -1, 11/2, 6, 7$: absolute convergence, divergence, or not determined by the information. Give reasons using distances from the center.
3. Construct two explicit series satisfying both observations that establish all the undetermined cases in your table. Verify the observations and the disputed points for each example.
4. Show every radius in your proposed closed range can occur. Use explicit examples for the endpoints and a general construction for the intermediate radii.

Question 10 – Solution

Strategy. Convergence at a point controls smaller distances absolutely; divergence controls larger distances. Boundary tests cannot be transferred automatically to the opposite side.

Step 1: Bound the radius sharply. The convergent point is distance 3 from the center, so $R \geq 3$. The divergent point is distance 4, so $R \leq 4$: if $R > 4$ that point would be inside the region of absolute convergence. Thus $\boxed{3 \leq R \leq 4}$. Convergence at distance 3 could be boundary convergence, and divergence at distance 4 could be boundary divergence.

Step 2: Record only forced conclusions.

x	$ x - 2 $	conclusion
0	2	absolute convergence
-1	3	not determined
11/2	7/2	not determined
6	4	not determined
7	5	divergence

The absolute convergence at distance 2 follows from the convergent point at distance 3. The opposite point at distance 3 need not share an endpoint test, and distances between 3 and 4 are not settled by the bounds.

Step 3: Construct examples proving the uncertainty. Take, starting at $n = 1$ and setting $a_0 = 0$,

$$U(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 3^n}, \quad V(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 4^n}.$$

For U , $x = 5$ gives alternating harmonic convergence and $x = -2$ fails the term test; $R_U = 3$. At $x = -1$ it is positive harmonic and diverges, and at 11/2 and 6 its terms do not tend to zero. For V , $x = 5$ converges absolutely, while $x = -2$ is positive harmonic and diverges; $R_V = 4$. At -1 and 11/2 it converges absolutely, and at 6 it converges conditionally. These two examples resolve every “not determined” entry without changing the given observations.

Step 4: Realize the entire radius range. The examples above realize $R = 3$ and $R = 4$. For any

$3 < \rho < 4$, use $\boxed{\sum_{n=0}^{\infty} ((x - 2)/\rho)^n}$. Its radius is ρ ; at 5 its ratio has magnitude $3/\rho < 1$, and at -2 it has magnitude $4/\rho > 1$. Thus every radius in $[3, 4]$ occurs, and no narrower range follows from the two observations alone.

Question 1

Let $f(x) = \ln x$, and expand about $x = 2$. A student proposes

$$P_4(x) = \ln 2 + \sum_{n=1}^4 \frac{(-1)^{n-1}(n-1)!}{2^n} (x-2)^n.$$

A Taylor coefficient and a derivative value are different kinds of data.

Tasks

1. Derive $f^{(n)}(2)$ for $n \geq 1$, correct the proposed polynomial, and identify its first incorrect coefficient.
2. Find the full Taylor series and its interval of convergence, including both endpoints. Justify that its sum equals $\ln x$ on the open interval.
3. Use the Lagrange remainder to give a uniform absolute error bound for the corrected fourth-degree polynomial on $[1.8, 2.2]$.
4. Decide whether the corrected polynomial lies above or below $\ln x$ on each side of 2 in that interval. Support the conclusion with a signed remainder.

Question 1 – Solution

Strategy. Divide derivatives by factorials, then distinguish series convergence from finite-polynomial accuracy.

Step 1: Normalize the coefficients. For $n \geq 1$, $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$. Thus

$$T_4(2+h) = \ln 2 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{24} - \frac{h^4}{64}.$$

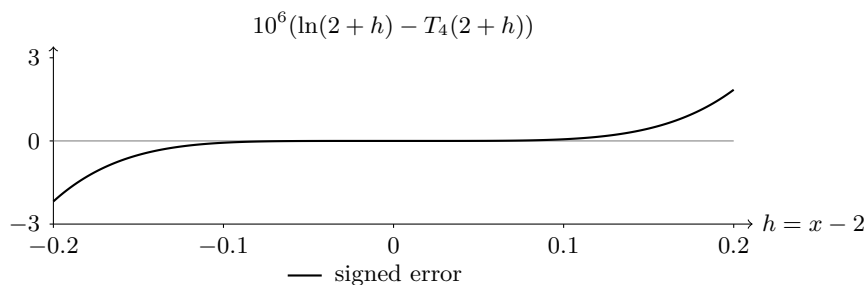
The student omitted division by $n!$; the first error is the h^2 coefficient, $-1/4$ instead of $-1/8$.

Step 2: Establish the represented function. Integrating the geometric series for $1/(2+h)$ from 0 to h gives $\ln(2+h) = \ln 2 + \sum_{n \geq 1} (-1)^{n-1} h^n / (n 2^n)$ for $|h| < 2$. Uniform convergence on the integration segment justifies this step. The term ratio tends to $|h|/2$, so the radius is 2; $h = -2$ gives the divergent negative harmonic series and $h = 2$ gives the alternating harmonic series. The convergence interval is $(0, 4]$ in x .

Step 3: Certify a uniform error. Since $f^{(5)}(x) = 24/x^5$, on $[1.8, 2.2]$,

$$|f(x) - T_4(x)| \leq \frac{24}{1.8^5} \frac{0.2^5}{120} = \frac{1}{5 \cdot 9^5} < 3.39 \times 10^{-6}.$$

Step 4: Determine the error sign. For $x \neq 2$, $f(x) - T_4(x) = h^5/(5\xi^5)$ for some positive ξ between 2 and x . Thus T_4 is above f on $[1.8, 2)$ and below it on $(2, 2.2]$; equality holds at 2. The plot magnifies this signed error.



Question 2

For $0 \leq x \leq 1$, let $T_n(x) = \sum_{k=0}^n (-x)^k / k!$ approximate e^{-x} . The number of terms in T_n is $n + 1$.

Tasks

1. Prove that the Maclaurin series represents e^{-x} on $[0, 1]$, using a remainder estimate that is uniform in x .
2. Find the smallest degree certified by the bound $|e^{-x} - T_n(x)| \leq 1/(n+1)!$ to achieve error at most 10^{-6} throughout $[0, 1]$.
3. A sufficient bound need not identify the true smallest degree. Prove in this case that the degree you found really is minimal, using the alternating tail at $x = 1$ to exclude every smaller degree.
4. Produce a rational enclosure for e^{-1} from consecutive partial sums at that degree, and explain which endpoint is the upper bound.

Question 2 – Solution

Strategy. Use Taylor's theorem for convergence and alternating tails for both upper and lower error bounds.

Step 1: Control all points simultaneously. Every derivative of e^{-x} has absolute value $e^{-x} \leq 1$ on $[0, 1]$. Taylor's theorem gives

$$|e^{-x} - T_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!} \rightarrow 0.$$

This proves equality to the series and uniform convergence on the stated interval.

Step 2: Find a certified degree. Because $9! = 362880 < 10^6 < 3628800 = 10!$, the smallest degree certified by this bound is $\boxed{n=9}$, using ten terms. Degree 8 is not certified by the upper bound alone.

Step 3: Prove actual minimality. At $x = 1$, the alternating tail has the sign of its first term, and its magnitude is strictly greater than the difference of its first two magnitudes:

$$|e^{-1} - T_n(1)| > \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{n+1}{(n+2)!} =: b_n.$$

Indeed, the remaining tail starts positive after factoring out its first sign and grouping consecutive pairs. Also $b_{n+1}/b_n = (n+2)/[(n+1)(n+3)] < 1$ for $n \geq 0$. Hence for every $0 \leq n \leq 8$, the error exceeds $b_8 = 9/10! > 10^{-6}$. No smaller degree works on all of $[0, 1]$.

Step 4: Enclose the value rationally. Odd partial sums lie below the limit and even ones above. Therefore

$$\boxed{\frac{16687}{45360} = T_9(1) < e^{-1} < T_{10}(1) = \frac{16481}{44800}}.$$

The interval width is exactly $1/10!$; the upper endpoint comes from the positive first omitted term after T_9 .

Question 3

For $x \neq 0$, define

$$H(x) = \frac{\sin x - x \cos x}{x^3}.$$

Near the origin the numerator suffers substantial cancellation.

Tasks

1. Derive the full Maclaurin series of H by combining the sine and cosine series before dividing. State its convergence domain.
2. Determine the unique continuous extension at 0, and find $H''(0)$ and $H^{(4)}(0)$ for that extension.
3. Construct its fourth-degree Taylor polynomial and certify its absolute error on $[-1, 1]$ by an alternating-series estimate.
4. Prove that the extended function is even and strictly positive on $[-1, 1]$. Explain why the series is useful for evaluating the original quotient near 0.

Question 3 – Solution

Strategy. Expose the first nonzero numerator term before removing the apparent singularity.

Step 1: Cancel and reindex. The coefficient of x^{2k+1} in the numerator is $(-1)^k[1/(2k+1)! - 1/(2k)!]$. It is zero when $k = 0$. Setting $j = k - 1$ for the surviving terms gives

$$H(x) = \sum_{j=0}^{\infty} \frac{(-1)^j 2(j+1)}{(2j+3)!} x^{2j}.$$

Sine and cosine converge absolutely for every real x ; the new series also has infinite radius by the ratio test. It agrees with the quotient when $x \neq 0$.

Step 2: Fill the removable singularity. The new series defines an analytic extension with

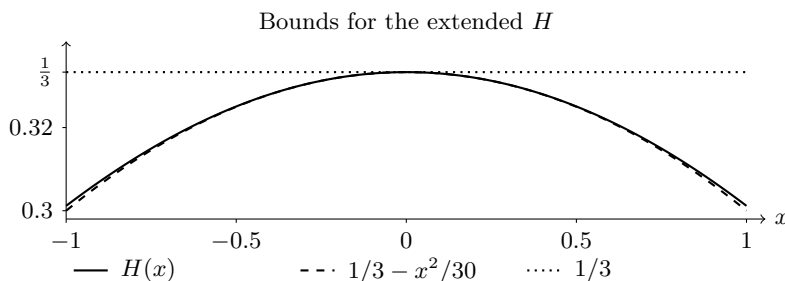
$$H(0) = \frac{1}{3}, \quad H''(0) = 2! \left(-\frac{1}{30} \right) = -\frac{1}{15}, \quad H^{(4)}(0) = 4! \frac{1}{840} = \frac{1}{35}.$$

Continuity forces the value $1/3$, so no other continuous extension is possible.

Step 3: Bound the truncated tail. The polynomial is $T_4(x) = 1/3 - x^2/30 + x^4/840$. For $|x| \leq 1$, successive term magnitudes have ratio $[(j+2)/(j+1)]x^2/[(2j+5)(2j+4)] \leq 1/10$. They decrease to zero, so

$$|H(x) - T_4(x)| \leq \frac{|x|^6}{45360} \leq \frac{1}{45360}.$$

Step 4: Interpret symmetry and stability. Only even powers occur. Alternating bounds give $H(x) \geq 1/3 - x^2/30 \geq 3/10 > 0$ on $[-1, 1]$. The numerator subtracts two quantities of order x to obtain one of order x^3 ; the series evaluates the finite limiting value without that subtraction. This is a cancellation concern, not a failure of the exact quotient.



Question 4

Let $F(x) = e^{x^2} \cos x$. A product of truncated series contains both useful terms and terms beyond the intended degree.

Tasks

1. Obtain the sixth-degree Maclaurin polynomial P_6 of F by explicit coefficient convolution.
2. Find $F^{(6)}(0)$ and explain why the remainder has order x^8 , rather than merely order x^7 .
3. Write $A = 1 + x^2 + x^4/2 + x^6/6$ and $B = 1 - x^2/2 + x^4/24 - x^6/720$. Compute $AB - P_6$ exactly.
4. On $|x| \leq 1/2$, prove the concrete bound $|F(x) - P_6(x)| < 0.0005$. Account separately for the exponential tail, cosine tail, and high-degree terms in AB .

Question 4 – Solution

Strategy. Convolve by degree, then bound the two truncation errors and the discarded product terms separately.

Step 1: Collect only the requested powers. Using the entire exponential and cosine series,

$$[x^2]F = 1 - \frac{1}{2} = \frac{1}{2}, \quad [x^4]F = \frac{1}{2} - \frac{1}{2} + \frac{1}{24} = \frac{1}{24},$$

$$[x^6]F = \frac{1}{6} - \frac{1}{4} + \frac{1}{24} - \frac{1}{720} = -\frac{31}{720}.$$

Thus $\boxed{P_6(x) = 1 + x^2/2 + x^4/24 - 31x^6/720}$.

Step 2: Recover a derivative and its parity. We have $F^{(6)}(0) = 6!(-31/720) = -31$. Both factors are even and analytic everywhere, so all odd coefficients vanish. The next possible power after degree six is eight, proving $F - P_6 = O(x^8)$ near 0.

Step 3: Retain the discarded product explicitly. Direct multiplication gives

$$AB - P_6 = -\frac{23}{360}x^8 + \frac{1}{160}x^{10} - \frac{1}{4320}x^{12}.$$

These terms belong to the product of polynomials, not to P_6 .

Step 4: Certify the error on the whole interval. Write $E = e^{x^2}$ and $C = \cos x$. Then $EC - P_6 = (E - A)C + A(C - B) + (AB - P_6)$. For $|x| \leq 1/2$, $|C| \leq 1$, $0 < A \leq e^{1/4} < 4/3$ (compare the exponential series with the geometric series). Taylor's theorem and the alternating cosine tail give

$$|E - A| \leq \frac{4}{3} \frac{|x|^8}{24}, \quad |C - B| \leq \frac{|x|^8}{8!}.$$

Consequently,

$$|F - P_6| \leq \frac{4/3}{24 \cdot 256} + \frac{4/3}{40320 \cdot 256} + \frac{23}{360 \cdot 256} + \frac{1}{160 \cdot 1024} + \frac{1}{4320 \cdot 4096} < 0.000473 < 0.0005.$$

The bound is conservative but rigorous; no unbounded O -term is used as a numerical certificate.

Question 5

Consider the nonelementary integral

$$I = \int_0^{1/2} e^{-t^2} dt, \quad S_m = \sum_{k=0}^m \frac{(-1)^k}{2^{2k+1} k! (2k+1)}.$$

An accuracy certificate is required; a decimal from a calculator is insufficient.

Tasks

1. Derive the series for I and justify integrating the Maclaurin series term by term on the whole integration interval.
2. Find the smallest m for which the first-omitted-term bound certifies $|I - S_m| \leq 10^{-6}$, and state the number of terms used.
3. Give an exact rational enclosure for I using that partial sum and the next one. Report an approximation with its certified absolute error.
4. Explain the sign of the error geometrically by comparing e^{-t^2} with the degree- $2m$ polynomial under the integral on $[0, 1/2]$.

Question 5 – Solution

Strategy. Integrate a uniformly convergent series, then use the integrated alternating tail as an error certificate.

Step 1: Justify the interchange. On $[0, 1/2]$, $|(-t^2)^k/k!| \leq (1/4)^k/k!$, whose sum is finite. The Weierstrass test gives uniform absolute convergence. Integrating yields

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}k!(2k+1)}.$$

The positive term magnitudes decrease strictly to zero.

Step 2: Choose a certified truncation. The first omitted magnitude is $b_m = 1/[2^{2m+3}(m+1)!(2m+3)]$. It decreases with m . For $m = 3$, $b_3 = 1/110592 > 10^{-6}$, whereas

$$b_4 = \frac{1}{2703360} < 3.70 \times 10^{-7} < 10^{-6}.$$

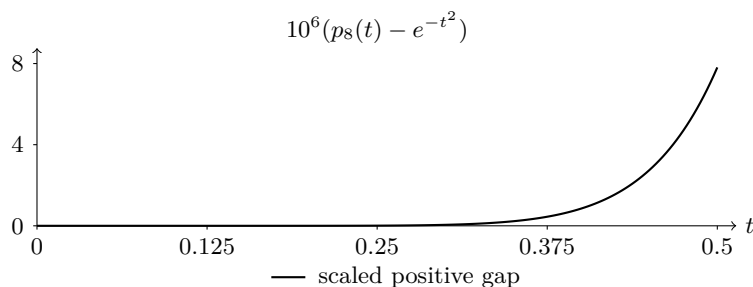
Thus $\boxed{m = 4}$ is the smallest index certified by this bound, using five terms.

Step 3: Supply an exact enclosure. The even partial sum is an upper bound and the next odd sum a lower bound:

$$S_4 - \frac{1}{2703360} = S_5 < I < S_4 = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \frac{1}{110592}.$$

Numerically, $S_4 \approx 0.4612813637$. Reporting $I \approx 0.46128136$ introduces less than 4×10^{-9} additional rounding error, so its total absolute error is less than 3.74×10^{-7} .

Step 4: Interpret the signed area. Set $p_8(t) = 1 - t^2 + t^4/2 - t^6/6 + t^8/24$. The alternating tail is strictly negative for $t > 0$ in this interval, so $p_8(t) > e^{-t^2}$. Hence $S_4 - I$ is the positive area between the curves. The scaled gap below makes the very small difference visible; it vanishes at $t = 0$.



Question 6

Define a function on the real line by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

A student claims: “Every infinitely differentiable function agrees with its Taylor series near the center.”

Tasks

1. Prove $|x|^{-m}e^{-1/x^2} \rightarrow 0$ as $x \rightarrow 0$ for every nonnegative integer m .
2. Prove that f is infinitely differentiable at 0 and that every derivative there vanishes. Use a polynomial-in- $1/x$ description away from 0 and difference quotients at 0.
3. Find the Maclaurin series, its radius of convergence, and the set of points where it equals f .
4. Explain precisely why Taylor’s theorem is still valid but fails to prove equality to the infinite series here. Identify the remainder $R_n(x)$ at any fixed nonzero x .

Question 6 – Solution

Strategy. Separate the existence of every derivative from the limiting behavior of the Taylor remainders.

Step 1: Exponential decay beats every power. Put $u = 1/x^2 \rightarrow \infty$. Choose an integer $k > m/2$. Since $e^u \geq u^k/k!$,

$$|x|^{-m} e^{-1/x^2} = u^{m/2} e^{-u} \leq k! u^{m/2-k} \rightarrow 0.$$

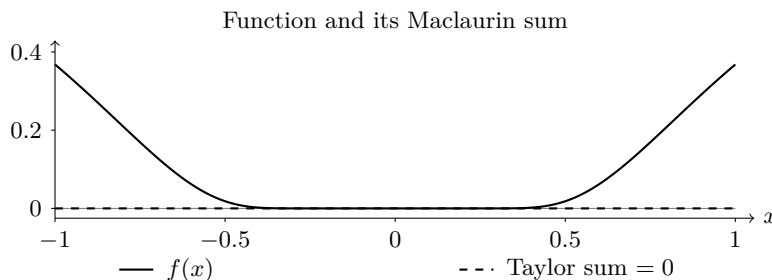
Step 2: Prove smoothness, including the center. For $x \neq 0$, induction gives $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where $P_0(z) = 1$ and

$$P_{n+1}(z) = -z^2 P'_n(z) + 2z^3 P_n(z).$$

Define each candidate derivative to be zero at 0. Step 1 proves its continuity there. Its difference quotient at 0 is $P_n(1/h)e^{-1/h^2}/h \rightarrow 0$, again by Step 1. Inductively each candidate is the derivative of the preceding one, so $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Step 3: Compare the series with the function. Every Taylor polynomial is zero. Thus the Maclaurin series is $\sum_{n=0}^{\infty} 0x^n = 0$, with infinite radius. It equals f only at $x = 0$, since $f(x) > 0$ for every nonzero x . Infinite radius does not by itself identify the represented function.

Step 4: Locate the missing hypothesis. Taylor's finite theorem remains valid on the segment from 0 to any fixed x . But $R_n(x) = f(x) - T_n(x) = e^{-1/x^2} > 0$ for $x \neq 0$, independently of n . Thus $R_n(x) \not\rightarrow 0$. Smoothness provides each finite remainder formula, not a derivative-growth bound forcing remainders to vanish. The graph and the zero Taylor sum touch to every order at 0 without agreeing nearby.



Question 7

The direct Maclaurin expansion of $\ln(1+x)$ converges slowly at $x=1$. Seek a faster Taylor-series evaluation of $\ln 2$ using

$$L(z) = \ln\left(\frac{1+z}{1-z}\right), \quad |z| < 1.$$

Count terms by the number of nonzero summands, not by polynomial degree.

Tasks

1. Derive the Maclaurin series for L by differentiating and integrating a geometric series, with a justification on $|z| < 1$.
2. Select a positive z for which $L(z) = \ln 2$. For the first N terms, prove the tail bound

$$\frac{2z^{2N+1}}{(2N+1)(1-z^2)}$$

and determine the error sign.

3. Find the smallest N for which this bound is strictly below 10^{-6} at the selected z . Give the resulting approximation and a rigorous enclosure.
4. For the direct alternating harmonic sum with N terms, how many terms does the next-term bound require to certify error strictly below 10^{-6} ? Compare the two certificates and explain the limits of this comparison.

Question 7 – Solution

Strategy. Change the evaluation point to shrink the geometric factor controlling the Taylor tail.

Step 1: Construct the odd-power series. Since $L'(z) = 2/(1 - z^2) = 2 \sum_{k \geq 0} z^{2k}$ and $L(0) = 0$,

$$L(z) = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

The geometric series converges uniformly on any closed segment inside $(-1, 1)$, so termwise integration is justified.

Step 2: Bound a positive tail. The equation $(1+z)/(1-z) = 2$ gives $z = 1/3$. With $A_N = 2 \sum_{k=0}^{N-1} z^{2k+1}/(2k+1)$ and $0 < z < 1$,

$$\begin{aligned} 0 < L(z) - A_N &= 2 \sum_{k=N}^{\infty} \frac{z^{2k+1}}{2k+1} \\ &< \frac{2z^{2N+1}}{(2N+1)(1-z^2)} =: B_N. \end{aligned}$$

Strictness follows because later denominators exceed $2N+1$.

Step 3: Count the accelerated terms. At $z = 1/3$, $B_N = 9/[4(2N+1)3^{2N+1}]$, strictly decreasing in N . Here $B_5 = 9/7794468 > 10^{-6}$ and $B_6 = 9/82904796 < 1.086 \times 10^{-7}$. Thus six terms are the smallest number certified by this bound, and

$$A_6 < \ln 2 < A_6 + \frac{9}{82904796}, \quad A_6 \approx 0.6931470738.$$

Step 4: Make a fair certificate comparison. For the direct alternating sum with N terms, the stated bound is $1/(N+1)$. To make that bound strictly less than 10^{-6} requires $N \geq 1,000,000$. The comparison is six versus one million terms under these particular certificates. It does not assert that a million terms are necessary for the true direct-series error, nor that a loose bound is an optimal algorithm. The transformed series profits from the factor $z^2 = 1/9$.

Question 8

An unknown function belongs to the family $f(x) = e^{ax} \cos(bx)$, where a, b are real. Exact local measurements give

$$f'(0) = 1, \quad f''(0) = -3.$$

A separate report claims $f'''(0) = -10$. No measurement uncertainty is assumed.

Tasks

1. Recover every admissible parameter pair (a, b) from the first two measurements. Explain the remaining ambiguity and whether it changes f .
2. Find the fourth-degree Maclaurin polynomial of every compatible function.
3. Test the separate third-derivative report and decide whether all three measurements can arise from one member of the stated family.
4. Prove a uniform remainder bound below 5.2×10^{-6} for your polynomial on $[-0.1, 0.1]$. You may use $f(x) = \operatorname{Re} e^{(a+ib)x}$ and $|\operatorname{Re} z| \leq |z|$.

Question 8 – Solution

Strategy. Turn measured derivatives into algebraic constraints before predicting higher coefficients.

Step 1: Recover the parameters. Differentiation at 0 gives $f'(0) = a$ and $f''(0) = a^2 - b^2$. Hence $a = 1$ and $b^2 = 4$, so

$$(a, b) = (1, 2) \text{ or } (1, -2).$$

The sign cannot be recovered because cosine is even; both pairs produce the same function $e^x \cos 2x$.

Step 2: Predict the Taylor polynomial. Using real parts of $(1 + 2i)^n$, the derivative values for $n = 0, \dots, 4$ are $1, 1, -3, -11, -7$. Dividing by $n!$ yields

$$T_4(x) = 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3 - \frac{7}{24}x^4.$$

For example, $(1 + 2i)^2 = -3 + 4i$ and $(1 + 2i)^3 = -11 - 2i$.

Step 3: Reject inconsistent exact data. Every pair compatible with the first two measurements forces $f'''(0) = -11$. The reported value -10 is incompatible. Changing the sign of b conjugates the complex powers and leaves their real parts unchanged, so it cannot repair the discrepancy.

Step 4: Bound the fifth derivative uniformly. For real t ,

$$|f^{(5)}(t)| = \left| \operatorname{Re} \left((1 + 2i)^5 e^{(1+2i)t} \right) \right| \leq 5^{5/2} e^t.$$

Taylor's theorem on each segment from 0 to $x \in [-0.1, 0.1]$ therefore gives

$$|f(x) - T_4(x)| \leq \frac{5^{5/2} e^{0.1}}{120} 10^{-5} < 5.2 \times 10^{-6}.$$

For a purely rational check of the last inequality, use $\sqrt{5} < 9/4$ and $e^{0.1} < 1.106$: the latter follows by bounding the tail after $1 + 0.1 + 0.1^2/2$ by $(0.1^3/6)/(1 - 0.025)$. These bounds give less than 5.185×10^{-6} .

Question 9

Let $T_4(x) = 1 - x^2/2 + x^4/24$, the fourth-degree Maclaurin polynomial of $\cos x$. Consider a rational competitor

$$Q(x) = \frac{1 + ax^2}{1 + bx^2}, \quad a, b \in \mathbb{R}.$$

A student believes that replacing a Taylor polynomial by a rational expression with the same local coefficients must improve accuracy.

Tasks

1. Determine a, b so that Q and $\cos x$ agree through degree four at 0, and identify any real poles of Q .
2. Find the leading nonzero terms of $\cos x - T_4(x)$ and $\cos x - Q(x)$. Compute the limiting ratio of their absolute errors as $x \rightarrow 0, x \neq 0$.
3. Use an alternating Taylor enclosure for $\cos 1$ to prove which approximation is closer at $x = 1$, without relying on a calculator value of $\cos 1$.
4. Evaluate the student's claim and explain why matching coefficients alone cannot rank all approximations on a whole interval.

Question 9 – Solution

Strategy. Match coefficients, then examine the first coefficient on which the competing approximations disagree.

Step 1: Solve the matching equations. Near 0, $Q = 1 + (a-b)x^2 + (b^2-ab)x^4 + \dots$. Thus $a-b = -1/2$ and $-b(a-b) = 1/24$, giving

$$b = \frac{1}{12}, \quad a = -\frac{5}{12}, \quad Q(x) = \frac{12-5x^2}{12+x^2}.$$

Since $12+x^2 > 0$, there are no real poles.

Step 2: Compare the leading errors. The rational geometric expansion has x^6 coefficient $-1/288$. Hence

$$\cos x - T_4(x) = -\frac{x^6}{720} + O(x^8), \quad \cos x - Q(x) = \frac{x^6}{480} + O(x^8).$$

Therefore $\lim_{x \rightarrow 0} \frac{|\cos x - Q(x)|}{|\cos x - T_4(x)|} = 3/2$. In particular, the rational approximation is worse for all sufficiently small nonzero x .

Step 3: Make an exact comparison at one. Alternating bounds give

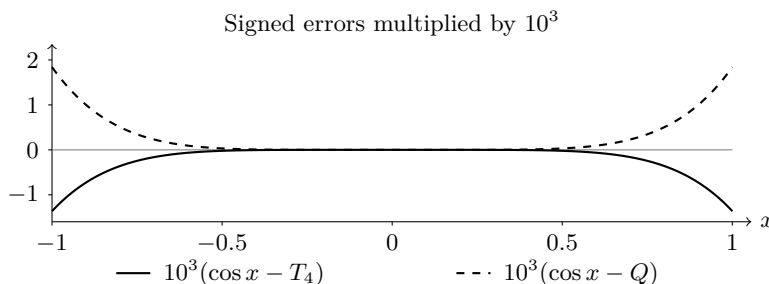
$$T_6(1) = \frac{389}{720} < \cos 1 < T_4(1) = \frac{13}{24}, \quad Q(1) = \frac{7}{13}.$$

Here $Q(1) < 389/720$, and the midpoint of the two approximations is $\frac{1}{2}(7/13 + 13/24) = 337/624 < 389/720$. Thus $\cos 1$ lies above their midpoint and between them: $T_4(1)$ is strictly closer than $Q(1)$.

Step 4: State the proper conclusion. The claim fails both locally and at $x = 1$. Matching four derivatives fixes contact at the center, not the next error coefficient or behavior far away. For $0 < |x| \leq 1$, alternating bounds give $\cos x - T_4(x) < 0$ and

$$\cos x - Q(x) \geq x^6 \left(\frac{1}{312} - \frac{1}{720} \right) > 0,$$

since $T_4 - Q = x^6/[288(1+x^2/12)]$. These signs explain the plotted errors.



Question 10

Let $f \in C^\infty([-r, r])$, with $r > 0$, and suppose constants $C, A > 0$ satisfy

$$|f^{(n)}(t)| \leq CA^n n! \quad (n \geq 0, |t| \leq r).$$

Write $T_N(x) = \sum_{n=0}^N f^{(n)}(0)x^n/n!$. For a boundary example you may use $g(x) = 1/(1+x^2)$ and the identity

$$g^{(n)}(x) = \frac{(-1)^n n!}{2i} \left((x-i)^{-n-1} - (x+i)^{-n-1} \right), \quad i^2 = -1.$$

Tasks

1. Prove that $T_N(x) \rightarrow f(x)$ for $|x| \leq r$ with $A|x| < 1$. Give a uniform error estimate on $[-\rho, \rho]$ when $0 < \rho \leq r$ and $A\rho < 1$.
2. Explain why this argument gives no conclusion at $A|x| = 1$. Show that both convergence and divergence there are possible under the stated derivative hypothesis, using $f = 1$ and g on $[-1, 1]$ with $C = A = 1$.
3. A function has all derivatives zero at 0 but is positive at every nearby nonzero point. Prove it cannot obey such a derivative bound on any neighborhood of 0.
4. Suppose the displayed bound is known only at $t = 0$. What can you conclude about convergence of the Taylor series, and what additional conclusion can you no longer justify? Supply a counterexample.

Question 10 – Solution

Strategy. A bound on derivatives throughout a segment controls Taylor's remainder; center-only data control coefficients.

Step 1: Force the remainders to vanish. Taylor's theorem uses some ξ between 0 and x . The hypothesis gives

$$|f(x) - T_N(x)| \leq \frac{CA^{N+1}(N+1)!}{(N+1)!}|x|^{N+1} = \boxed{C(A|x|)^{N+1}}.$$

This tends to zero whenever $|x| \leq r$ and $A|x| < 1$. On the specified smaller interval the uniform bound is $C(A\rho)^{N+1} \rightarrow 0$.

Step 2: Test the boundary honestly. At $A|x| = 1$ the bound is only C , which does not tend to zero. The constant function 1 satisfies the bound with $C = A = 1$ and its Taylor series converges at ± 1 . For real x , $|x \pm i| \geq 1$, so the given identity yields $|g^{(n)}(x)| \leq n!$. Yet the Maclaurin series of g is $\sum_{k \geq 0} (-1)^k x^{2k}$; at $x = \pm 1$ its terms fail to tend to zero. Thus neither boundary outcome is forced.

Step 3: Rule out the bound for a flat function. If such constants existed on some $[-r, r]$, choose $0 < |x| < \min(r, 1/A)$. All Taylor polynomials would be zero, but Step 1 would force $f(x) = 0$, contradicting positivity. Thus no finite positive C, A can work on any such neighborhood.

Step 4: Separate coefficients from function values. A bound only at 0 gives $|f^{(n)}(0)/n!| \leq CA^n$. Geometric comparison proves absolute convergence of the Taylor series for $|x| < 1/A$, but does not prove that its sum is $f(x)$. The smooth function e^{-1/x^2} extended by zero at 0 has every center derivative zero, so it satisfies every such center-only bound; its Taylor series is zero while the function is positive elsewhere.

Question 1

Consider the differential equation $y'' - xy = 0$ near $x = 0$. Write $y = \sum_{n=0}^{\infty} a_n x^n$; the coefficients are not the derivatives themselves.

Tasks

1. Derive the coefficient recurrence, stating the exceptional equation at degree zero. Explain how the indices split into three chains.
2. Construct the normalized solutions $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$. Give the first four nonzero terms of each.
3. Prove that both series converge for every real x and form a fundamental pair. State the solution with arbitrary initial values $y(0) = A$, $y'(0) = B$.
4. For the degree-nine truncation U_9 of u , prove a uniform error bound below 5.88×10^{-7} on $[-1, 1]$. A numerical sample of the error is not a proof of this bound.

Question 1 – Solution

Strategy. Keep the lowest index separate, then use the recurrence both to construct solutions and to bound their tails.

Step 1: Align the powers. The constant coefficient gives $a_2 = 0$. For $n \geq 1$, $(n+2)(n+1)a_{n+2} = a_{n-1}$, equivalently

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)} \quad (m \geq 0).$$

The chains start at a_0, a_1, a_2 ; the last is identically zero.

Step 2: Build the normalized series. With empty products interpreted as 1,

$$u(x) = \sum_{k \geq 0} \frac{x^{3k}}{\prod_{j=1}^k (3j)(3j-1)} = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots,$$

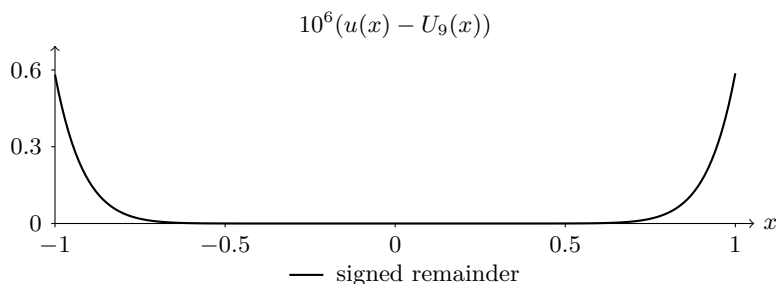
$$v(x) = \sum_{k \geq 0} \frac{x^{3k+1}}{\prod_{j=1}^k (3j+1)(3j)} = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots.$$

Step 3: Establish actual solutions and independence. For either series, the ratio of consecutive nonzero term magnitudes tends to zero at every fixed x . Thus both have infinite radius, may be differentiated termwise, and satisfy the equation and stated initial data. Their Wronskian satisfies $W' = uv'' - u''v = 0$ and $W(0) = 1$. Hence $y = Au + Bv$ is the unique solution for those initial values, on $\mathbb{R}$.

Step 4: Bound all omitted terms. The first omitted coefficient of u is $1/1710720$. On $|x| \leq 1$, ratios within the remaining tail are at most $1/(15 \cdot 14) = 1/210$. Therefore

$$|u(x) - U_9(x)| \leq \frac{|x|^{12}}{1710720} \frac{1}{1 - 1/210} < 5.88 \times 10^{-7}.$$

The plot magnifies the actual signed remainder; the geometric estimate covers both signs of x .



Question 2

The equation

$$y'' + 2xy' + 2y = 0$$

admits a useful relation between its power-series construction and a first-order identity. Let $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$.

Tasks

1. Derive the recurrence for a Maclaurin solution. Compute the first four nonzero terms of u and v , and identify their parity.
2. Find a quantity involving y, y' and x whose derivative is the left-hand side. Use it to express both normalized solutions through exponentials and a definite integral.
3. Prove convergence of both series on the whole real line directly from their coefficients. Explain why the series and integral constructions agree.
4. Compute the Wronskian of the normalized pair and determine whether any nontrivial linear combination can have both its value and derivative zero at a finite point.

Question 2 – Solution

Strategy. Compare the even and odd coefficient chains with an integrated form of the equation.

Step 1: Separate the two chains. Coefficient matching gives $(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$, so

$$a_{n+2} = -\frac{2}{n+2}a_n \quad (n \geq 0).$$

Thus $u = 1 - x^2 + x^4/2 - x^6/6 + \dots$ is even, while $v = x - 2x^3/3 + 4x^5/15 - 8x^7/105 + \dots$ is odd.

Step 2: Integrate a conserved quantity. Since $(y' + 2xy)' = 0$, $y' + 2xy = C$. At 0, $C = y'(0)$. Multiplying by e^{x^2} and integrating gives

$$y(x) = e^{-x^2} \left[y(0) + y'(0) \int_0^x e^{t^2} dt \right].$$

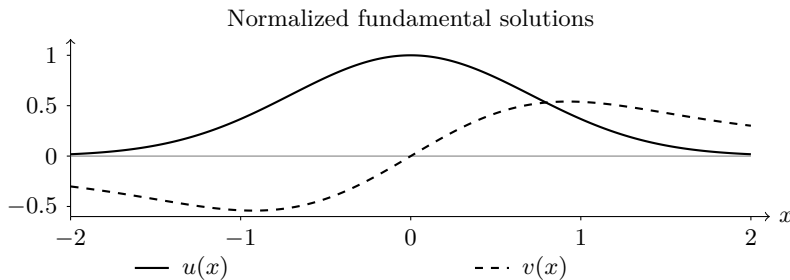
Consequently
$$u = e^{-x^2}, \quad v = e^{-x^2} \int_0^x e^{t^2} dt.$$

Step 3: Justify convergence and equality. The even coefficients are $(-1)^k/k!$, and the odd ones are $(-2)^k/(2k+1)!!$, where $(2k+1)!! = 1 \cdot 3 \cdots (2k+1)$. Successive nonzero term ratios are $x^2/(k+1)$ and $2x^2/(2k+3)$, respectively; both tend to zero. Thus the series define solutions for all real x . The integral formulas solve the same equation with the same initial data, so uniqueness identifies the constructions.

Step 4: Test independence at every point. Differentiating v gives $v' = -2xv + 1$, while $u' = -2xu$. Hence

$$W(u, v) = uv' - u'v = u = e^{-x^2} > 0.$$

At any finite point, the two-by-two system for the coefficients of a combination with zero value and derivative has nonzero determinant. Both coefficients must vanish. The plotted pair remains independent even where one curve crosses zero.



Question 3

Solve the initial-value problem

$$(1-x)y'' - y' = 0, \quad y(0) = 0, \quad y'(0) = 1$$

by a series about 0, then investigate what happens when that series is used far from its center.

Tasks

1. Derive the recurrence and determine every Maclaurin coefficient.
2. Identify the represented function and its maximal real interval containing 0. Independently determine the power series' radius and both endpoint outcomes.
3. Explain why the initial series fails at $x = -2$ even though the solution exists there. Is this a singularity of the solution at -2 ?
4. Reexpand about $x = -1$, deriving the new coefficient recurrence from the differential equation. Give the first four nonconstant terms, the new convergence interval including endpoints, and the value at $x = -2$.

Question 3 – Solution

Strategy. Distinguish a solution's real domain from the disk reached by a particular Taylor center.

Step 1: Determine the Maclaurin coefficients. For $n \geq 0$, coefficient matching gives $(n+2)(n+1)a_{n+2} - (n+1)^2a_{n+1} = 0$. With $a_0 = 0, a_1 = 1$, this yields

$$a_{n+2} = \frac{n+1}{n+2}a_{n+1}, \quad y(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Step 2: Identify two different domains. The differential equation says $[(1-x)y']' = 0$. Initial data give $y' = 1/(1-x)$ and $y = -\ln(1-x)$ on $(-\infty, 1)$, its maximal real interval through 0. The series has radius 1. It diverges at 1 (harmonic series) and converges at -1 (alternating series), so its interval is $[-1, 1)$. Integrating the geometric series identifies its sum on $(-1, 1)$; at -1 , the alternating harmonic value is $-\ln 2$.

Step 3: Diagnose the failed evaluation. At -2 , the terms $(-2)^n/n$ do not tend to zero. The series therefore diverges although $y(-2) = -\ln 3$ is finite. The obstruction is the center's distance to the singularity at 1, not a singularity at -2 .

Step 4: Move the expansion center. Let $h = x+1$ and $y = \sum b_n h^n$. The equation is $(2-h)y'' - y' = 0$, giving $b_{n+2} = (n+1)b_{n+1}/[2(n+2)]$. Since $b_0 = -\ln 2, b_1 = 1/2$,

$$y = -\ln 2 + \sum_{n \geq 1} \frac{h^n}{n2^n} = -\ln 2 + \frac{h}{2} + \frac{h^2}{8} + \frac{h^3}{24} + \frac{h^4}{64} + \cdots.$$

The radius in h is 2, with convergence at $h = -2$ and divergence at $h = 2$; in x the interval is $[-3, 1)$. Now $x = -2$ is interior, and the sum is $-\ln 2 - \ln(3/2) = -\ln 3$.

Question 4

Consider a nonhomogeneous equation centered away from the origin:

$$y'' + (x - 2)y' = e^{x-2}, \quad y(2) = 0, \quad y'(2) = 1.$$

You may use the analytic ordinary-point theorem: a linear equation with nonzero leading coefficient and analytic normalized coefficients and forcing in a complex disk has a convergent Taylor solution throughout that disk.

Tasks

1. Set $h = x - 2$ and derive the recurrence for $y = \sum_{n \geq 0} a_n h^n$, explicitly accounting for the constant coefficient of hy' .
2. Compute a_0 through a_6 and the resulting polynomial $P_6(h)$. Identify any coefficient that vanishes despite nonzero neighboring coefficients.
3. Define the residual $\mathcal{R} = P_6'' + hP_6' - e^h$. Find its leading nonzero term. Explain why a degree-six solution truncation need not satisfy the differential equation through degree six.
4. Produce a definite-integral representation of the exact solution and justify infinite radius of convergence for its series. Check both initial values in your representation.

Question 4 – Solution

Strategy. Expand every term in the shifted variable and keep the forcing coefficients in the same index convention.

Step 1: Match the forcing coefficient. Derivatives with respect to x and h agree. The coefficient of h^n in hy' is na_n for $n \geq 0$; it is zero when $n = 0$. Thus

$$a_{n+2} = \frac{1/n! - na_n}{(n+2)(n+1)} \quad (n \geq 0), \quad a_0 = 0, \quad a_1 = 1.$$

Step 2: Compute the initial segment. Successive applications give $a_2 = 1/2$, $a_3 = 0$, $a_4 = -1/24$, $a_5 = 1/120$, $a_6 = 1/144$. Therefore

$$P_6(h) = h + \frac{h^2}{2} - \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{144}.$$

The cubic coefficient vanishes because $1/1! - a_1 = 0$, not because the solution has a fixed parity.

Step 3: Interpret the residual correctly. Direct differentiation cancels the coefficients through h^4 . At degree five, hP'_6 contributes $1/24$ while e^h contributes $1/120$; hence

$$\mathcal{R}(h) = \frac{h^5}{30} + O(h^6).$$

The omitted term a_7h^7 would contribute $42a_7h^5$ to y'' . Indeed, the recurrence gives $a_7 = -1/1260$, cancelling that residual term. Two differentiations lower the degree of an omitted term by two.

Step 4: Recover and validate the exact solution. Writing $v = y'$ gives $v' + hv = e^h$. An integrating factor yields

$$y(2+h) = \int_0^h e^{-s^2/2} \left(1 + \int_0^s e^{t+t^2/2} dt \right) ds.$$

At $h = 0$ the outer integral is zero and its integrand is 1, verifying the initial values. Differentiating gives the first-order equation for v , hence the original equation. Its leading coefficient is 1 and all coefficients and forcing are entire, so the stated theorem applies to disks of every radius. The Taylor series is therefore entire.

Question 5

For real λ , consider

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

and seek analytic solutions near $x = 0$. A polynomial solution would remain meaningful even where the leading coefficient vanishes.

Tasks

1. Derive the recurrence and explain how it separates even and odd coefficients.
2. Classify all real λ for which a nonzero polynomial solution exists. Prove necessity from the highest-degree term and sufficiency from the recurrence; include degree zero.
3. For $\lambda = 20$, $y(0) = 3/8$ and $y'(0) = 0$, find the exact solution and verify the differential equation by substitution.
4. Find all its real zeros exactly, determine its values at $x = \pm 1$, and explain why this particular solution extends through those points although the normalized equation is singular there.

Question 5 – Solution

Strategy. Use coefficient termination to distinguish polynomial solutions from general local series.

Step 1: Derive the parity recurrence. Matching x^n gives

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n \quad (n \geq 0).$$

Even and odd indices evolve independently from a_0 and a_1 .

Step 2: Characterize termination completely. If a nonzero polynomial has degree m , its highest coefficient obeys $[\lambda - m(m+1)]a_m = 0$, so $\lambda = m(m+1)$ for an integer $m \geq 0$. Conversely, choose the starting coefficient of parity m nonzero and the other zero. For indices of that parity below m , no numerator vanishes; at $n = m$ it does, terminating the chain. Thus these and only these parameter values work, including $m = 0$, $\lambda = 0$ (constants).

Step 3: Construct and verify the requested polynomial. For $\lambda = 20$, the recurrence yields $a_2 = -15/4$, $a_4 = 35/8$, and $a_6 = 0$. Hence

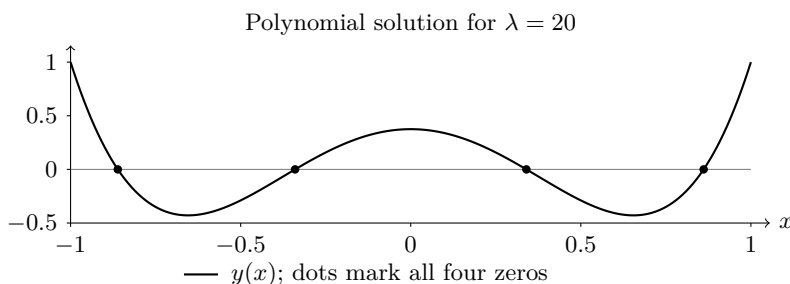
$$y(x) = \frac{35x^4 - 30x^2 + 3}{8}.$$

Here $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$. Substitution in $(1 - x^2)y'' - 2xy' + 20y$ cancels every coefficient. Initial data also hold, so uniqueness near 0 identifies the solution.

Step 4: Locate roots and extend the solution. Solving the quadratic in x^2 gives four distinct zeros:

$$x = \pm \sqrt{\frac{15 - 2\sqrt{30}}{35}}, \quad x = \pm \sqrt{\frac{15 + 2\sqrt{30}}{35}}.$$

They lie in $(-1, 1)$, and $y(-1) = y(1) = 1$. The polynomial satisfies the original equation identically for every real x , including ± 1 . Singular normalized coefficients do not forbid this special solution from extending through them.



Question 6

The leading coefficient vanishes at the proposed expansion center in

$$xy'' + 2y' + xy = 0.$$

A student therefore needs to check which initial data a Maclaurin solution can actually accommodate.

Tasks

1. Substitute $y = \sum_{n \geq 0} a_n x^n$. Find the exceptional constant-coefficient equation and the recurrence for all remaining coefficients.
2. Determine every analytic solution near 0 and identify its series in terms of a familiar function, including the value assigned at 0.
3. Decide whether $y(0) = 1$, $y'(0) = 1$ can be satisfied by an analytic solution or even by a twice continuously differentiable solution of the original equation near 0.
4. On $x > 0$, substitute $v = xy$ to find the full two-parameter solution. Determine which members extend boundedly to 0 and explain why only one free parameter survives in the analytic family.

Question 6 – Solution

Strategy. Apply the original equation at its singular center before invoking any ordinary-point intuition.

Step 1: Retain the exceptional equation. The coefficient of x^0 is $2a_1 = 0$. For $n \geq 1$, the coefficient of x^n is $(n+1)(n+2)a_{n+1} + a_{n-1} = 0$. Equivalently,

$$a_1 = 0, \quad a_{m+2} = -\frac{a_m}{(m+2)(m+3)} \quad (m \geq 0).$$

Thus a_0 is free but a_1 is not.

Step 2: Identify all analytic solutions. Every odd coefficient vanishes. The even chain gives

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 \frac{\sin x}{x} \quad (x \neq 0), \quad y(0) = a_0.$$

The ratio test gives infinite radius. Termwise substitution verifies the original equation, including $x = 0$. Conversely, every analytic solution must obey the recurrence, so the displayed family is exhaustive.

Step 3: Test the proposed initial values. The data $y(0) = 1$, $y'(0) = 1$ are incompatible with $a_1 = 0$. More generally, a C^2 solution satisfies the equation at 0, where it reduces to $2y'(0) = 0$. Even if the equation were initially imposed only at nearby nonzero points, continuity of y, y', y'' would give the same constraint by taking the limit. Such a C^2 solution cannot have the proposed data.

Step 4: Explain the lost second parameter. With $v = xy$, $v'' = xy'' + 2y'$, so $v'' + v = 0$ for $x > 0$. Consequently

$$y(x) = A \frac{\sin x}{x} + B \frac{\cos x}{x} \quad (x > 0).$$

The second term behaves like B/x as $x \rightarrow 0^+$; boundedness forces $B = 0$. The remaining term extends analytically with value A and slope 0. The ordinary-point theorem does not guarantee two arbitrary analytic initial coefficients at a center where the leading coefficient is zero.

Question 7

Consider the nonlinear initial-value problem

$$y' = 1 + y^2, \quad y(0) = 0.$$

Seek a Maclaurin solution $y = \sum_{n \geq 0} a_n x^n$. For the radius question, you may use these facts: sine and cosine are entire; the only complex zeros of cosine are $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$; and a Taylor series reaches its nearest nonremovable complex singularity.

Tasks

1. Derive the nonlinear coefficient recurrence using a Cauchy product. Compute all coefficients through degree seven.
2. Prove that the resulting solution is odd, and explain why the square y^2 cannot be handled by squaring individual coefficients separately.
3. Identify the exact solution by a method independent of the coefficient calculation. Determine its maximal real interval containing 0 and the exact Taylor radius.
4. For the degree-seven polynomial P_7 , prove $0 < P_7(x) < y(x)$ when $0 < x < \pi/2$. Explain why a fixed polynomial cannot approximate this solution uniformly on that whole open interval.

Question 7 – Solution

Strategy. Nonlinearity couples coefficients by convolution; an exact solution then validates convergence and reveals blow-up.

Step 1: Convolve rather than square termwise. The constant forcing contributes only at $n = 0$:

$$(n+1)a_{n+1} = \delta_{n0} + \sum_{k=0}^n a_k a_{n-k}, \quad a_0 = 0,$$

where $\delta_{n0} = 1$ for $n = 0$ and 0 otherwise. This gives

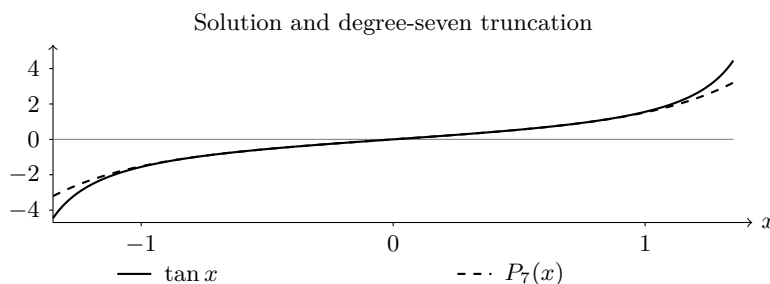
$$P_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315},$$

with $a_2 = a_4 = a_6 = 0$.

Step 2: Establish parity and coefficient signs. If y solves the IVP, $-y(-x)$ does too; local uniqueness gives oddness. The recurrence also proves by induction that all even coefficients vanish and all odd coefficients are positive: each required convolution contains positive odd-index products. For example, the coefficient of x^4 in y^2 is $2a_1a_3$, not a_2^2 . Cross terms are essential.

Step 3: Identify the function and its radius. Separation yields $\arctan y = x$ near 0, so $y = \tan x$. It satisfies the equation and initial condition on $(-\pi/2, \pi/2)$ and blows up at both endpoints, making this the maximal real interval through 0. Since $\sin(\pm\pi/2) \neq 0$, the nearest zeros of cosine are nonremovable poles of the quotient. The supplied facts give $R = \pi/2$. Within this radius, the analytic solution's coefficients obey the recurrence, which uniquely determines them; hence it is the constructed series. At $x = \pi/2$ the positive-term series diverges: a finite endpoint sum would bound $\tan x$ below that endpoint, contradicting blow-up. Oddness gives divergence at $-\pi/2$ as well.

Step 4: Interpret truncation and blow-up. For $0 < x < \pi/2$, every omitted odd term is positive, so $0 < P_7(x) < \tan x$. But P_7 stays bounded as $x \rightarrow (\pi/2)^-$ while $\tan x \rightarrow +\infty$. Their difference is unbounded on that interval. Every fixed Taylor truncation has this defect; convergence on smaller compact intervals does not imply uniform approximation up to a blow-up boundary.



Question 8

An unknown equation has the form

$$y'' + \alpha xy' + (\beta + \gamma x^2)y = 0,$$

where α, β, γ are real constants. Two analytic solutions have measured Taylor expansions

$$u(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + O(x^8), \quad v(x) = x - x^3 + \frac{1}{2}x^5 - \frac{1}{6}x^7 + O(x^9).$$

The displayed coefficients are exact.

Tasks

1. Determine α, β, γ by substituting only the coefficients needed from these data. Show the recovery is unique.
2. Derive the full recurrence for a generic series solution of the recovered equation, including its conventions at the lowest indices.
3. Use the substitution $y = e^{-x^2}w$ to identify every solution exactly. Verify that the two measured series are a fundamental pair and determine their radii of convergence.
4. Predict the coefficients of x^8 in u and x^9 in v . Can an additional exact measurement assigning the value $1/20$ to the former coefficient be consistent with the equation and the original data?

Question 8 – Solution

Strategy. Infer the differential operator from low-order data, then validate all orders by a transformation.

Step 1: Recover the three constants. The constant coefficient in the equation for u gives $-2 + \beta = 0$. The x coefficient for v gives $-6 + \alpha + \beta = 0$. The x^2 coefficient for u gives $6 - 2\alpha - \beta + \gamma = 0$. These successive equations uniquely force

$$\boxed{\beta = 2, \quad \alpha = 4, \quad \gamma = 4.}$$

Step 2: State a recurrence valid from the start. With $a_{-2} = a_{-1} = 0$ for the shifted contribution,

$$\boxed{a_{n+2} = -\frac{(4n+2)a_n + 4a_{n-2}}{(n+2)(n+1)} \quad (n \geq 0).}$$

The starting coefficients $a_0 = y(0)$ and $a_1 = y'(0)$ are free. In particular, $a_2 = -a_0$ and $a_3 = -a_1$; no negative-index coefficient is an additional parameter.

Step 3: Verify the recovered equation globally. Writing $E = e^{-x^2}$, differentiation gives

$$y' = E(w' - 2xw), \quad y'' = E(w'' - 4xw' + (4x^2 - 2)w).$$

Thus $y'' + 4xy' + (2 + 4x^2)y = Ew''$. The equation becomes $w'' = 0$, yielding

$$\boxed{y = e^{-x^2}(C_1 + C_2x), \quad u = e^{-x^2}, \quad v = xe^{-x^2}.}$$

Their initial data match the measured series, and their Wronskian is $E^2 = e^{-2x^2} > 0$. Both functions are entire, so both Taylor radii are infinite. The transformation also proves the entire measured series, not just the first displayed terms, are consistent.

Step 4: Test a new exact measurement. The exponential expansion gives $[x^8]u = 1/4! = 1/24$ and $[x^9]v = 1/24$. The recurrence produces the same values. Since the recovered parameters and each solution's initial data are fixed uniquely, $1/20$ is incompatible; it cannot be accommodated by choosing later coefficients independently.

Question 9

For

$$y'' + y = x, \quad y(0) = y'(0) = 0,$$

a power-series approximation is to be certified on $[0, 1]$. For a trial function p , define its differential residual by $\mathcal{R}_p = p'' + p - x$.

Tasks

1. Derive the coefficient recurrence and construct the degree-seven Taylor polynomial P_7 of the exact solution.
2. Compute $\mathcal{R}_{P_7}$ exactly. Derive a definite-integral formula for the error $e = y - P_7$ from its differential equation and initial values.
3. Prove the signed error enclosure $-x^9/362880 \leq e(x) \leq 0$ on $[0, 1]$. Identify the exact solution and compare this certificate with the alternating Taylor remainder.
4. Show that the residual alone cannot control approximation error if initial data are not also checked. Construct trial functions with the same residual as P_7 but arbitrarily large error at $x = 1$.

Question 9 – Solution

Strategy. Convert the residual into a forced error equation, keeping track of both initial errors.

Step 1: Construct the solution truncation. Matching powers gives $(n+2)(n+1)a_{n+2} + a_n = \delta_{n1}$, with $a_0 = a_1 = 0$. Thus all even coefficients vanish and

$$P_7(x) = \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040}.$$

Step 2: Derive the error equation. Direct differentiation yields $\mathcal{R}_{P_7} = x^7/5040$. Since P_7 satisfies both initial values, $e'' + e = -x^7/5040$ with $e(0) = e'(0) = 0$. The zero-data solution is

$$e(x) = -\frac{1}{5040} \int_0^x \sin(x-t)t^7 dt.$$

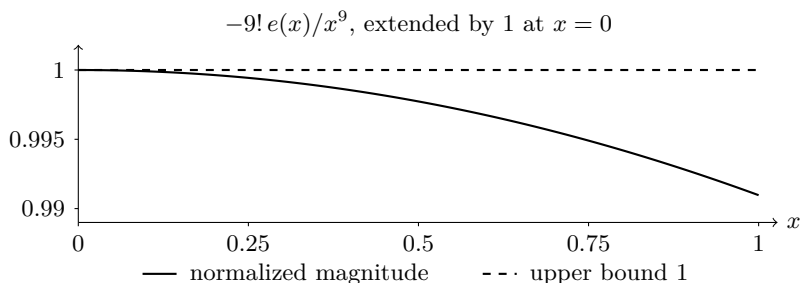
Differentiating this integral twice verifies the equation and the two initial conditions, so uniqueness proves the formula.

Step 3: Bound the signed integral. For $0 \leq t \leq x \leq 1$, $0 \leq \sin(x-t) \leq x-t$. Therefore

$$-\frac{1}{5040} \int_0^x (x-t)t^7 dt = -\frac{x^9}{362880} \leq e(x) \leq 0.$$

The exact solution is $y = x - \sin x$, as substitution and initial values verify. Its convergent sine expansion has first omitted term $-x^9/9!$, so the alternating estimate agrees with the integral certificate. For $x > 0$ the integral is strictly positive before its minus sign, giving $e(x) < 0$.

Step 4: Expose the role of initial data. Let $p_M = P_7 + M \sin x$ for any real M . Since $(\sin x)'' + \sin x = 0$, every p_M has exactly the same residual. But $p'_M(0) = M$, and $y(1) - p_M(1) = e(1) - M \sin 1$ is unbounded as $|M| \rightarrow \infty$. Residual size certifies error only together with suitable initial or boundary errors and a stability estimate.



Question 10

Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad y'' + y = f(x), \quad y(0) = y'(0) = 0.$$

You may use the established facts that f is infinitely differentiable, even, positive away from 0, and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Tasks

1. Determine every derivative of the solution at 0. What Taylor series does formal coefficient matching produce?
2. Derive a definite-integral representation of the actual solution and verify the equation and both initial values. Explain its existence and uniqueness on the whole real line.
3. Prove that the solution is even and strictly positive for $0 < |x| < \pi$. What does this imply about whether its Taylor series represents it near 0?
4. A student argues that constant coefficients make 0 an ordinary point, so the Taylor method must work. Identify the missing hypothesis and distinguish a solution of the formally expanded equation from a solution of the original equation.

Question 10 – Solution

Strategy. Test the forcing's analyticity before treating its Taylor coefficients as the forcing itself.

Step 1: Compute all center derivatives. The continuous forcing gives a C^2 solution; since f is smooth, repeated use of $y'' = f - y$ makes y smooth as well. Differentiation gives

$$y^{(n+2)}(0) = f^{(n)}(0) - y^{(n)}(0) = -y^{(n)}(0).$$

Starting with $y(0) = y'(0) = 0$, induction gives every derivative zero. Its Taylor series is therefore

$$\sum_{n \geq 0} 0x^n = 0, \text{ with infinite radius as a series.}$$

Step 2: Recover the actual forced solution. Variation of constants gives, for every real x ,

$$y(x) = \int_0^x \sin(x-t)f(t) dt.$$

Here $y'(x) = \int_0^x \cos(x-t)f(t) dt$ and $y''(x) = f(x) - y(x)$. At 0 both integrals vanish. The formula is defined on every finite real segment; any difference between two solutions satisfies $z'' + z = 0$ with zero initial values, hence is zero. This proves global existence and uniqueness.

Step 3: Prove nonzero values despite zero Taylor data. Because f is even, $y(-x)$ solves the same IVP; uniqueness gives $y(-x) = y(x)$. For $0 < x < \pi$ and $0 < t < x$, both $\sin(x-t)$ and $f(t)$ are strictly positive, so the integral is positive. Evenness extends this conclusion to $0 < |x| < \pi$. The zero Taylor series cannot equal y on any neighborhood of 0, even though it converges everywhere.

Step 4: Locate the missing analyticity assumption. The leading coefficient is nonzero and the homogeneous coefficients are analytic, but the forcing is not analytic at 0: its Taylor sum is zero while it is positive nearby. The analytic ordinary-point theorem for a nonhomogeneous equation requires analytic forcing too. Replacing f by its Taylor series changes the equation to $y'' + y = 0$. The formal zero series solves that changed equation, not the original equation away from 0.

Question 1

Consider the Euler initial-value problem

$$x^2y'' - 2xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 3.$$

A power x^m is a useful trial function, but the reason it works should also be understood.

Tasks

1. For $x > 0$, set $t = \ln x$ and $Y(t) = y(e^t)$. Derive the formulas for xy' and x^2y'' and transform the equation into one with constant coefficients.
2. Find a fundamental pair, compute its Wronskian, and solve the IVP exactly. Verify both initial values.
3. State the maximal real interval of this IVP and determine the behavior of its solution at each end of that interval.
4. Locate and classify every stationary point on that interval. Give the exact minimum value and explain how two monotone power modes can combine into a nonmonotone solution.

Question 1 – Solution

Strategy. Logarithmic coordinates turn powers into exponentials and make the characteristic equation systematic.

Step 1: Transform derivatives correctly. The chain rule gives $Y' = xy'$ and $Y'' = xy' + x^2y''$, hence

$$xy' = Y', \quad x^2y'' = Y'' - Y'.$$

The equation becomes $Y'' - 3Y' - 4Y = 0$, with characteristic polynomial $m^2 - 3m - 4 = (m - 4)(m + 1)$.

Step 2: Solve and verify the data. The powers x^4, x^{-1} form a pair with $W = x^4(-x^{-2}) - 4x^3x^{-1} = -5x^2 \neq 0$ for $x > 0$. Writing $y = C_1x^4 + C_2x^{-1}$ gives $C_1 + C_2 = 2$, $4C_1 - C_2 = 3$, so

$$y = x^4 + x^{-1}.$$

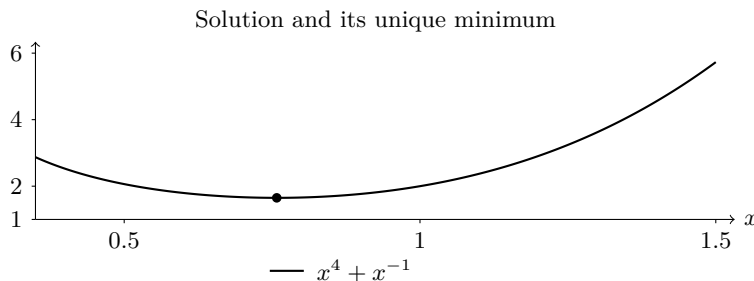
At 1, its value is 2 and derivative $4x^3 - x^{-2}$ is 3.

Step 3: Identify the maximal interval. The normalized coefficients are continuous on $(0, \infty)$, and the explicit solution exists throughout it. Since $y \rightarrow +\infty$ as $x \rightarrow 0^+$, no finite continuation through 0 is possible. Thus the maximal real interval through 1 is $(0, \infty)$. At infinity, $y \sim x^4 \rightarrow +\infty$; at zero, $y \sim x^{-1}$.

Step 4: Find the unique minimum. The equation $y' = 0$ is $4x^5 = 1$, giving $x_* = 4^{-1/5}$. Since $y'' = 12x^2 + 2x^{-3} > 0$, it is the unique global minimum. Using $x_*^{-1} = 4x_*^4$,

$$y(x_*) = 5 \cdot 4^{-4/5}.$$

One mode increases and the other decreases. Their derivatives cancel at exactly one point, although neither mode has a stationary point of its own.



Question 2

For $x > 0$, consider

$$x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

A repeated characteristic exponent needs a second independent solution and careful treatment at the singular point.

Tasks

1. Derive the repeated exponent and obtain a second independent solution using logarithmic coordinates. Verify independence rather than merely listing two functions.
2. Solve the IVP and find every positive zero of its solution.
3. Determine whether this IVP solution admits a continuous, C^1 , or C^2 extension to 0. Compute the relevant one-sided limits.
4. For the general positive-half-line solution, classify which choices permit a C^2 extension through 0 that solves the original equation. Explain whether the data $y(0) = y'(0) = 0$ determine that extension uniquely.

Question 2 – Solution

Strategy. A repeated exponential in logarithmic time produces a logarithmic factor in the original variable.

Step 1: Construct a genuine fundamental pair. With $t = \ln x$, the equation becomes $Y'' - 4Y' + 4Y = 0$, whose characteristic polynomial is $(m - 2)^2$. Thus

$$y = x^2(A + B \ln x).$$

For $u = x^2$, $v = x^2 \ln x$, the Wronskian is $W = x^2(2x \ln x + x) - 2x(x^2 \ln x) = x^3 \neq 0$ on $x > 0$.

Step 2: Use both initial values. At 1, $y(1) = A$ and $y'(1) = 2A + B$. Hence $A = B = 1$, giving

$$y = x^2(1 + \ln x).$$

Because $x^2 > 0$, the only positive zero is $x = e^{-1}$.

Step 3: Test successive derivatives at zero. For this solution,

$$y' = x(3 + 2 \ln x), \quad y'' = 5 + 2 \ln x.$$

As $x \rightarrow 0^+$, $y \rightarrow 0$ and $y' \rightarrow 0$, but $y'' \rightarrow -\infty$. Defining $y(0) = 0$ gives a right-hand C^1 extension with derivative 0: indeed $y(x)/x = x(1 + \ln x) \rightarrow 0$. It can be joined to zero on the negative side as a C^1 function, but no C^2 extension is possible.

Step 4: Classify classical extensions. In general, $y'' = 2A + B(2 \ln x + 3)$, so a finite second-derivative limit requires $B = 0$. On $x < 0$ the general solution is $x^2(C + D \ln |x|)$. A C^2 join similarly forces $D = 0$, and matching second derivatives forces $C = A$. Thus precisely $y = Ax^2$ extend through 0 as C^2 solutions of the original equation. Each satisfies $y(0) = y'(0) = 0$, so these data do not give uniqueness at the singular point.

Question 3

Study the logarithmically oscillating IVP

$$x^2 y'' + xy' + 4y = 0, \quad y(1) = 0, \quad y'(1) = 2, \quad x > 0.$$

Oscillations in $\ln x$ behave differently from oscillations in x .

Tasks

1. Find the real general solution and the specified IVP solution.
2. Determine all its positive zeros and the ratio of consecutive zeros in increasing order. Explain where the zeros accumulate.
3. Decide whether the IVP solution is bounded near 0 and whether it has a continuous extension there. Extend the conclusion about limits to every nonzero real solution of this equation.
4. Set $z = xy'/2$. Determine the curve traced by (y, z) and its direction as x increases. Locate the initial point and distinguish this curve from the graph of y against x .

Question 3 – Solution

Strategy. Use $t = \ln x$ to reveal a harmonic oscillator and then translate its time scale back to x .

Step 1: Solve the transformed oscillator. The transformed equation is $Y'' + 4Y = 0$, so $y = A \cos(2 \ln x) + B \sin(2 \ln x)$. At $x = 1$, $A = 0$ and $2B = y'(1) = 2$, hence

$$y(x) = \sin(2 \ln x), \quad x > 0.$$

The characteristic exponents are $m = \pm 2i$.

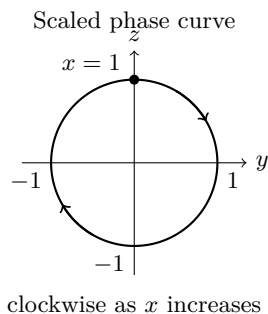
Step 2: Locate the geometric sequence of zeros. Zeros occur when $2 \ln x = k\pi$:

$$x_k = e^{k\pi/2}, \quad k \in \mathbb{Z}, \quad x_{k+1}/x_k = e^{\pi/2}.$$

They accumulate at 0 as $k \rightarrow -\infty$ and grow without bound as $k \rightarrow \infty$. There is no smallest positive zero.

Step 3: Separate boundedness from a limit. The IVP solution has $|y| \leq 1$, but sequences with $\ln x = \pi/4 - k\pi$ and $\ln x = 3\pi/4 - k\pi$ tend to 0 while giving values 1 and -1 . Thus no limit at zero exists. Any nonzero real solution is a nonzero-amplitude sinusoid in $\ln x$ and likewise attains two distinct limiting values along sequences approaching zero. Only the zero solution extends continuously there.

Step 4: Read the scaled phase curve. For the IVP, $z = \cos(2 \ln x)$, so $y^2 + z^2 = 1$. In logarithmic time, $\dot{y} = 2z$, $\dot{z} = -2y$; at $(y, z) = (0, 1)$ the motion is to the right. The circle is traversed clockwise as x increases, because $\ln x$ increases with x . The horizontal coordinate here is y , not x ; equal scales preserve the unit circle.



Question 4

Solve on the negative half-line:

$$x^2 y'' + xy' - \frac{1}{4}y = 0, \quad y(-4) = 3, \quad y'(-4) = -\frac{1}{8}.$$

Using fractional powers without specifying a real branch can invalidate a solution.

Tasks

1. Use $t = \ln(-x)$ to derive a real fundamental pair on $x < 0$. Explain why the positive-half-line expression $x^{1/2}$ is unsuitable as a real-valued basis there.
2. Translate both initial values into logarithmic coordinates and solve the IVP. Check the derivative sign directly in the original variable.
3. Give its maximal real interval and both endpoint behaviors.
4. Find its unique stationary point and minimum. Explain how increasing logarithmic time relates to increasing x on this negative interval.

Question 4 – Solution

Strategy. Work with positive $-x$, retaining the sign in the chain rule.

Step 1: Choose a real branch. For $x = -e^t$, $dx/dt = x$, so $Y' = xy'$ and $Y'' = xy' + x^2y''$ still hold. The equation becomes $Y'' - Y/4 = 0$. A real pair is $\boxed{(-x)^{1/2}, (-x)^{-1/2}}$ for $x < 0$. The expression $x^{1/2}$ has no real value on this interval; powers of $|x| = -x$ avoid that problem.

Step 2: Translate the slope correctly. At $t = \ln 4$, $Y = 3$ and $Y' = xy' = (-4)(-1/8) = 1/2$. Writing $y = A\sqrt{-x} + B/\sqrt{-x}$ yields $2A + B/2 = 3$ and $A - B/4 = 1/2$, hence $A = 1$, $B = 2$. Thus

$$\boxed{y = \sqrt{-x} + \frac{2}{\sqrt{-x}}, \quad y' = -\frac{1}{2\sqrt{-x}} + \frac{1}{(-x)^{3/2}}.}$$

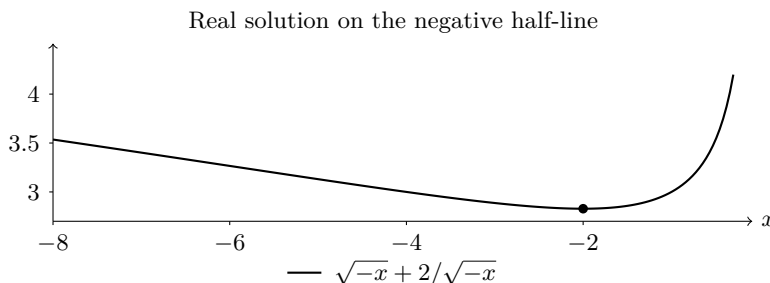
At -4 these give 3 and $-1/8$, as required.

Step 3: State the maximal interval. The solution is smooth throughout $(-\infty, 0)$, and its divergence at 0 prevents extension through that point. This is the maximal real interval containing -4 . As $x \rightarrow -\infty$, $y \sim \sqrt{-x} \rightarrow \infty$; as $x \rightarrow 0^-$, $y \sim 2/\sqrt{-x} \rightarrow \infty$.

Step 4: Locate the minimum and orient time. Put $r = -x > 0$. The derivative is $(2 - r)/(2r^{3/2})$, so it is negative for $x < -2$ and positive for $-2 < x < 0$. Therefore

$$\boxed{x_* = -2, \quad y(x_*) = 2\sqrt{2}}$$

is the unique global minimum. Since $dt/dx = 1/x < 0$, increasing x corresponds to decreasing t on this half-line. This reverses the relation between the two horizontal directions, not the derivative formulas.



Question 5

For $x > 0$, solve the resonantly forced Euler problem

$$x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad y(1) = y'(1) = 0.$$

A trial expression $x^2(A + B \ln x)$ resembles the forcing but may not produce it.

Tasks

1. Transform the equation using $t = \ln x$ and then $Y = e^{2t}Z$. Derive the resulting equation for Z .
2. Find the general solution. Explain exactly why the proposed trial expression fails and why a cubic logarithmic term appears.
3. Apply the initial conditions and verify the final solution by computing the transformed residual.
4. Determine whether the IVP solution extends continuously, as a C^1 function, or as a C^2 function to 0. Give the relevant limiting expressions.

Question 5 – Solution

Strategy. Remove the repeated characteristic exponential before integrating the forcing.

Step 1: Factor the logarithmic-time operator. The equation is $(D - 2)^2 Y = e^{2t}t$, where $D = d/dt$. Since $(D - 2)(e^{2t}Z) = e^{2t}Z'$, a second application gives

$$Z'' = t.$$

Step 2: Integrate through the resonance. Two integrations yield $Z = t^3/6 + C_1 + C_2t$, hence

$$y = x^2 \left(C_1 + C_2 \ln x + \frac{(\ln x)^3}{6} \right).$$

The proposed trial has $Z = A + Bt$ and therefore $Z'' = 0$: it belongs entirely to the homogeneous solution space. Integrating the degree-one forcing twice produces degree three, accounting for the cubic logarithm and its factor $1/6$.

Step 3: Impose and verify the initial data. At $x = 1$, $t = 0$, so $y(1) = C_1$ and $y'(1) = 2C_1 + C_2$. Thus both constants vanish and

$$y = \frac{x^2(\ln x)^3}{6}.$$

For this solution, $Z = t^3/6$ satisfies $Z'' = t$ exactly; therefore the original left-hand side is $e^{2t}t = x^2 \ln x$. The two initial values are zero.

Step 4: Inspect regularity at the singular point. Writing $t = \ln x$,

$$y' = x \left(\frac{t^3}{3} + \frac{t^2}{2} \right), \quad y'' = \frac{t^3}{3} + \frac{3t^2}{2} + t.$$

As $x \rightarrow 0^+$, the power of x dominates every logarithmic power, so $y \rightarrow 0$, $y' \rightarrow 0$, and $y(x)/x \rightarrow 0$. Assigning $y(0) = 0$ therefore gives a C^1 right extension, which may be joined to zero on the left as a C^1 function. But $y'' \rightarrow -\infty$, so no C^2 extension is possible.

Question 6

For $x > 0$, consider

$$x^2 y'' + xy' = \frac{x^2}{1+x^2}.$$

Instead of prescribing data at a positive point, seek the solution that is bounded near 0 and extends continuously with $y(0) = 0$.

Tasks

1. Integrate the equation to obtain its general solution in definite-integral form on $(0, \infty)$, displaying both arbitrary constants.
2. Impose boundedness and the specified value at zero. Prove that they select exactly one solution on the positive half-line.
3. Derive its power series about zero, give its first three nonzero terms, and determine the radius and both endpoint outcomes. Justify termwise integration.
4. Prove that the selected solution has a C^2 extension to zero, compute $y'(0)$ and $y''(0)$, and certify a two-term approximation on $[0, 1]$ using a rigorous remainder bound.

Question 6 – Solution

Strategy. Integrate the weighted derivative first, then use regularity at zero to remove the logarithmic homogeneous mode.

Step 1: Integrate the Euler operator. Dividing by x gives $(xy')' = x/(1+x^2)$. Hence

$$xy' = \frac{1}{2} \ln(1+x^2) + C, \quad y = \frac{1}{2} \int_1^x \frac{\ln(1+s^2)}{s} ds + C \ln x + D.$$

These formulas display both constants and solve the equation for every $x > 0$.

Step 2: Apply the singular-endpoint conditions. Since $\ln(1+s^2)/s \sim s$ at zero, its integral there converges. Boundedness therefore forces $C = 0$; the value $y(0) = 0$ fixes the remaining constant. The unique selected solution is

$$y(x) = \frac{1}{2} \int_0^x \frac{\ln(1+s^2)}{s} ds.$$

Any two bounded solutions differ by a constant, and their common value at zero makes that constant zero.

Step 3: Expand the integral with justification. The divided logarithmic series $\sum_{n \geq 1} (-1)^{n-1} s^{2n-1}/n$ is uniformly convergent on $|s| \leq \rho < 1$, by comparison with $\sum_{n \geq 1} \rho^{2n-1}/n$. Termwise integration therefore yields, for $|x| < 1$,

$$y(x) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n^2} = \frac{x^2}{4} - \frac{x^4}{16} + \frac{x^6}{36} - \dots$$

The ratio of successive nonzero terms tends to x^2 , so the radius is 1. At $x = \pm 1$ the series converges absolutely by comparison with $\sum 1/n^2$. The series and the even integral extension agree there by continuity: the series is uniformly convergent on $[-1, 1]$ by the same comparison.

Step 4: Recover derivatives and a certificate. The local power series supplies a smooth even extension with $y'(0) = 0$ and $y''(0) = 1/2$. With the forcing defined continuously as zero at 0, the original equation holds there as well. On $[0, 1]$, the alternating term magnitudes decrease, so for $P_4 = x^2/4 - x^4/16$,

$$0 \leq y(x) - P_4(x) \leq \frac{x^6}{36}.$$

The bound includes the endpoint $x = 1$; it is a certificate, not a claim that the two-term approximation is highly accurate there.

Question 7

Let $L > 0$ be fixed. Seek nonzero real solutions of the two-endpoint Euler problem

$$x^2 y'' + x y' + \lambda y = 0, \quad y(1) = y(e^L) = 0,$$

where λ is real. No general theory of boundary-value eigenproblems is required; solve the Euler equation directly.

Tasks

1. Transform the problem to $0 \leq t \leq L$ and prove that $\lambda \leq 0$ permits only the zero solution.
2. Find every $\lambda > 0$ permitting a nonzero solution, and give the corresponding solution families.
3. For each admissible value, impose $y'(1) = 1$. Find the normalized solution and all its zeros, including the number strictly inside $(1, e^L)$.
4. Explain why uniqueness of ordinary initial-value problems does not imply uniqueness for these two endpoint conditions. For $L = \pi$, describe the first three normalized profiles in the coordinate $t = \ln x$.

Question 7 – Solution

Strategy. In logarithmic coordinates, the two endpoint conditions reduce to elementary trigonometric or hyperbolic equations.

Step 1: Exclude nonpositive parameters. Set $Y(t) = y(e^t)$. Then $Y'' + \lambda Y = 0$, with $Y(0) = Y(L) = 0$. If $\lambda = 0$, $Y = A + Bt$ and both constants must vanish. If $\lambda = -\mu^2 < 0$, $Y = A \cosh(\mu t) + B \sinh(\mu t)$. The first condition gives $A = 0$; since $\sinh(\mu L) > 0$, the second gives $B = 0$.

Step 2: Find all nonzero families. For $\lambda = \omega^2 > 0$, the first condition leaves $Y = B \sin(\omega t)$. A nonzero B requires $\sin(\omega L) = 0$, so

$$\lambda_n = (n\pi/L)^2, \quad y = C \sin\left(\frac{n\pi \ln x}{L}\right), \quad n = 1, 2, \dots$$

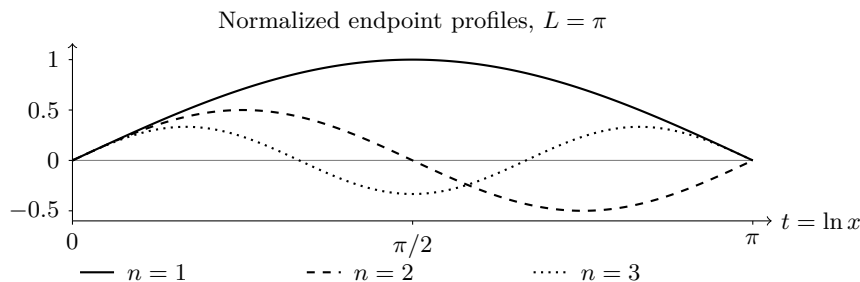
There are no other real parameter values admitting nonzero solutions.

Step 3: Normalize and locate the zeros. At $x = 1$, $y'(1) = Y'(0) = Cn\pi/L$. Thus

$$y_n(x) = \frac{L}{n\pi} \sin\left(\frac{n\pi \ln x}{L}\right).$$

Its zeros on the specified interval are $x_k = e^{kL/n}$ for $k = 0, \dots, n$. Exactly $n - 1$ lie strictly inside; they are equally spaced in $\ln x$, not in x .

Step 4: Distinguish endpoint and initial data. The original endpoint conditions do not specify the initial slope. At an admissible parameter, every amplitude satisfies both endpoints, including zero; hence those data are not unique. Once $y'(1) = 1$ is imposed, the usual IVP uniqueness applies. For $L = \pi$, the first three normalized profiles are $Y_n(t) = \sin(nt)/n$, $n = 1, 2, 3$, with 0, 1, 2 interior zeros, respectively. The figure uses logarithmic horizontal coordinates.



Question 8

For fixed $a \in \mathbb{R}$ and parameter $\varepsilon \geq 0$, consider on $x > 0$

$$x^2 y'' + (1 - 2a)xy' + (a^2 - \varepsilon^2)y = 0.$$

Two distinct power solutions become nearly identical when their characteristic exponents approach one another.

Tasks

1. Find the characteristic exponents for $\varepsilon > 0$ and explain why simply taking their two power solutions to the limit does not produce a fundamental pair at $\varepsilon = 0$.
2. Construct solutions $u_\varepsilon, v_\varepsilon$ satisfying $u_\varepsilon(1) = 1$, $u'_\varepsilon(1) = a$, $v_\varepsilon(1) = 0$, $v'_\varepsilon(1) = 1$.
3. Compute their Wronskian and prove independence for every $\varepsilon \geq 0$, using the limiting pair at zero.
4. Determine that limiting pair and justify uniform convergence of each basis function on every compact interval $[r, R] \subset (0, \infty)$ as $\varepsilon \rightarrow 0$. Explain how the logarithmic repeated-root solution emerges.

Question 8 – Solution

Strategy. Normalize a symmetric sum and a divided difference before allowing the characteristic roots to merge.

Step 1: Identify the degenerating pair. The power equation is $(m - a)^2 - \varepsilon^2 = 0$, giving $m = a \pm \varepsilon$. Both $x^{a+\varepsilon}$ and $x^{a-\varepsilon}$ tend to x^a as $\varepsilon \rightarrow 0$. Their separate limits are the same function, so they cannot remain a fundamental pair.

Step 2: Normalize before taking the limit. For $\varepsilon > 0$, set $t = \ln x$ and define

$$u_\varepsilon = x^a \cosh(\varepsilon t), \quad v_\varepsilon = x^a \frac{\sinh(\varepsilon t)}{\varepsilon}.$$

These are the half-sum and the difference divided by 2ε of the two powers. At $x = 1$ they have the four required value/slope data by direct differentiation.

Step 3: Check the determinant exactly. Differentiation and $\cosh^2 s - \sinh^2 s = 1$ give

$$W(u_\varepsilon, v_\varepsilon) = x^{2a-1} > 0.$$

At $\varepsilon = 0$, define $u_0 = x^a$ and $v_0 = x^a \ln x$. They solve the repeated-root equation, have the same normalized data, and their Wronskian is again x^{2a-1} . Independence holds for every parameter value.

Step 4: Justify the limit uniformly. On $[r, R]$, let $T = \max(|\ln r|, |\ln R|)$ and $M = \max_{[r, R]} x^a$. For $0 < \varepsilon \leq 1$, Taylor's theorem for $\cosh$ and $\sinh$ gives

$$|u_\varepsilon - u_0| \leq \frac{M\varepsilon^2 T^2 \cosh T}{2}, \quad |v_\varepsilon - v_0| \leq \frac{M\varepsilon^2 T^3 \cosh T}{6}.$$

Both bounds tend to zero uniformly. The divided difference tends to $\partial_m x^m|_{m=a} = x^a \ln x$, recovering the independent logarithmic solution that the two unnormalized limits lose.

Question 9

For real α and $\beta > 0$, consider a nonzero solution on $x > 0$ of

$$x^2 y'' + (1 - 2\alpha)xy' + (\alpha^2 + \beta^2)y = 0.$$

Its power envelope may tend to zero even while its derivatives keep oscillating.

Tasks

1. Find the real general solution and show that its j th derivative has the form $x^{\alpha-j}$ times a sinusoid in $\ln x$ with nonzero amplitude.
2. For each integer $k \geq 0$, prove a necessary and sufficient condition on α for a nonzero solution to have a C^k extension to $x = 0$. Treat the equality case explicitly.
3. Apply your criterion to $y = x^{5/2} \sin(\ln x)$. Compute y'' and determine exactly which integer orders of smoothness are possible at zero.
4. Join this particular solution to zero for $x \leq 0$. Verify that the joined function is a classical C^2 solution of the corresponding Euler equation on the whole real line. Explain the implication for initial data at the singular point.

Question 9 – Solution

Strategy. Track both the power envelope and the nonvanishing oscillation amplitude after every differentiation.

Step 1: Differentiate the complex powers. The exponents are $\alpha \pm i\beta$, so the real solutions are

$$y = x^\alpha (A \cos(\beta \ln x) + B \sin(\beta \ln x)).$$

The j th derivative of $x^{\alpha+i\beta}$ is $\prod_{\ell=0}^{j-1} (\alpha - \ell + i\beta) x^{\alpha-j+i\beta}$. Since $\beta > 0$, none of these factors vanishes. A nonzero real solution therefore retains a nonzero sinusoidal amplitude at every derivative order.

Step 2: Establish the exact regularity threshold. If $\alpha > k$, all derivatives through order k tend to zero. Their extensions are consistent derivatives: for $j < k$, $y^{(j)}(x)/x = O(x^{\alpha-j-1}) \rightarrow 0$. If $\alpha = k$, the k th derivative has constant nonzero amplitude and no limit. If $\alpha < k$, that derivative has an unbounded envelope and attains unbounded values along a sequence approaching zero. Hence

$$\boxed{\text{A nonzero solution extends as } C^k \text{ to } 0 \iff \alpha > k.}$$

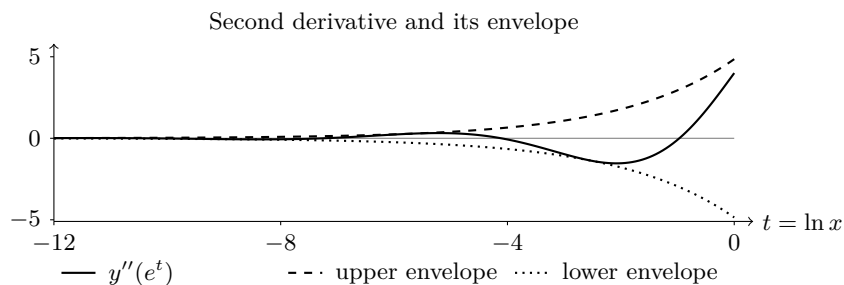
This includes $k = 0$; bounded oscillation at equality is not continuity.

Step 3: Inspect the concrete example. For $\alpha = 5/2$, $\beta = 1$,

$$\boxed{y'' = x^{1/2} \left(\frac{11}{4} \sin(\ln x) + 4 \cos(\ln x) \right).}$$

The solution extends as C^0, C^1, C^2 , with all three center values zero, but not as C^3 or any higher integer order. The plot shows $y''(e^t)$ and its decaying envelope as $t \rightarrow -\infty$.

Step 4: Verify the joined classical solution. The equation is $x^2 y'' - 4xy' + (29/4)y = 0$. On $x > 0$ the formula solves it; on $x < 0$ the zero function does. The zero limits through the second derivative make the join C^2 , and at 0 every term in the original equation is zero. Both this nonzero joined solution and the identically zero solution have $y(0) = y'(0) = 0$. Classical IVP uniqueness fails at this singular point.



Question 10

For $x \geq 1$, consider a finite interval of forcing in an Euler equation:

$$x^2 y'' - xy' + y = g(x), \quad y(1) = y'(1) = 0, \quad g(x) = \begin{cases} x, & 1 \leq x < e, \\ 0, & x \geq e. \end{cases}$$

Seek a solution that is C^1 across $x = e$ and solves the equation on each side. No classical second derivative at the switching point is assumed.

Tasks

1. Set $t = \ln x$ and $Y = e^t Z$. Derive a general definite-integral solution for zero initial data, valid for piecewise continuous forcing g .
2. Apply it to the specified forcing and give explicit formulas on $1 \leq x \leq e$ and $x \geq e$.
3. Verify continuity of y and y' at e . Compute the one-sided second derivatives and explain why a C^2 solution at that point is impossible.
4. Determine the large- x behavior after the forcing stops. Explain why the solution does not return to zero just because the right-hand side becomes zero.

Question 10 – Solution

Strategy. Remove the repeated exponent, integrate in logarithmic time, and match the finite-forcing transition.

Step 1: Derive the integral response. With $D = d/dt$, the transformed equation is $(D-1)^2 Y = g(e^t)$, so $Z'' = e^{-t}g(e^t)$. The data at $t = 0$ give $Z(0) = Z'(0) = 0$. Thus

$$Z(t) = \int_0^t (t-s)e^{-s}g(e^s)ds, \quad \boxed{y(x) = x \int_1^x \ln(x/r) \frac{g(r)}{r^2} dr.}$$

The second formula follows from $r = e^s$, including $ds = dr/r$.

Step 2: Evaluate the finite forcing interval. Here $Z'' = 1$ for $0 < t < 1$ and $Z'' = 0$ for $t > 1$. Integrating with the zero initial data gives $Z = t^2/2$ up to 1, then $Z = t - 1/2$. Therefore

$$\boxed{y(x) = \begin{cases} \frac{1}{2}x(\ln x)^2, & 1 \leq x \leq e, \\ x(\ln x - \frac{1}{2}), & x \geq e. \end{cases}}$$

Both formulas have the same value at e , and the first satisfies both initial conditions at 1.

Step 3: Check the join and the derivative jump. For $Y = e^t Z$, $y' = Z + Z'$ and $y'' = (Z' + Z'')/x$. At $t = 1$, both sides have $Z = 1/2$ and $Z' = 1$, so

$$y(e) = e/2, \quad y'(e) = 3/2, \quad y''(e^-) = 2/e, \quad y''(e^+) = 1/e.$$

The finite jump in forcing produces a jump $-1/e$ in y'' , while y, y' remain continuous. The unequal one-sided limits rule out C^2 regularity, irrespective of any value assigned to $g(e)$.

Step 4: Interpret the surviving homogeneous motion. After e , the solution is the nonzero homogeneous combination $x \ln x - x/2$. Thus $\boxed{y(x)/(x \ln x) \rightarrow 1}$ as $x \rightarrow \infty$. The completed forcing interval leaves nonzero value and slope at the switch. Those become initial data for the homogeneous equation; zero subsequent forcing does not reset them to zero.