

Question 1

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Consider the parameter-dependent system

$$x + y + z = 2, \quad 2x + 3y + z = 5, \quad 3x + 4y + 2z = k,$$

where k is real.

Tasks

1. Classify every k according to whether the system has no solution, one solution or infinitely many solutions.
2. For each consistent case, give the entire solution set with one free parameter and verify it in all three original equations.
3. Add the measurement $x - y = 0$. Determine all solutions of the enlarged system for every k .
4. Explain why three equations in three unknowns need not determine a unique answer. Compare the information supplied by the original third equation with that supplied by the new measurement.

Question 1 – Solution

Strategy. Inspect relationships between whole equations before committing to a long elimination.

Step 1: Test consistency. The third left-hand side is the sum of the first two. Any common solution must therefore satisfy $k = 2 + 5 = 7$. If $k \neq 7$, subtracting the first two equations from the third gives the contradiction $0 = k - 7$.

Step 2: Find and verify the full family. When $k = 7$, the third equation contributes no new restriction. Subtract twice the first equation from the second to get $y - z = 1$. Setting $z = t$ gives

$$(x, y, z) = (1 - 2t, 1 + t, t), \quad t \in \mathbb{R}.$$

The first left side is 2, the second is 5, and the third is 7 for every t . Conversely, every solution must have $y = 1 + z$ and $x = 1 - 2z$, so no solutions have been omitted. Thus $k = 7$ gives infinitely many solutions; no value of k gives exactly one solution in the original system.

Step 3: Use the additional measurement. On this family, $x - y = -3t$. The new measurement forces $t = 0$, giving

$$(x, y, z) = (1, 1, 0) \quad \text{if } k = 7.$$

If $k \neq 7$, the contradiction from the original equations remains, so adding a measurement cannot create a solution.

Step 4: Distinguish equation count from information. For $k = 7$, the third equation merely repeats a consequence already known. For $k \neq 7$, it contradicts that consequence. It never removes just the one remaining freedom. By contrast, $x - y$ varies as $-3t$ along the family, so its measured value selects exactly one member. Counting equations and unknowns alone cannot distinguish these situations; their relationships matter.

Question 2

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

For real λ , solve

$$(\lambda - 1)x + y = 1, \quad x + (\lambda - 1)y = 1.$$

Tasks

1. Use the sum and difference of the equations to classify the solution set for every λ .
2. Explain why dividing by λ or $\lambda - 2$ before checking their zeros can lose entire cases. Verify the exceptional cases directly.
3. Find the limits of the unique solution as $\lambda \rightarrow 2$ and as $\lambda \rightarrow 0$ from either side. Compare these limits with the solution sets at the exceptional values.
4. Classify all λ for which a solution with $x, y \geq 0$ exists, giving the full nonnegative solution set whenever it is not a single point.

Question 2 – Solution

Strategy. Adding and subtracting separates the sum of the unknowns from their difference, exposing both exceptional parameter values.

Step 1: Reduce reversibly. The sum and difference equations are

$$\lambda(x + y) = 2, \quad (\lambda - 2)(x - y) = 0.$$

They are equivalent to the originals: taking half their sum and half their difference recovers the two original equations. If $\lambda \neq 0, 2$, then

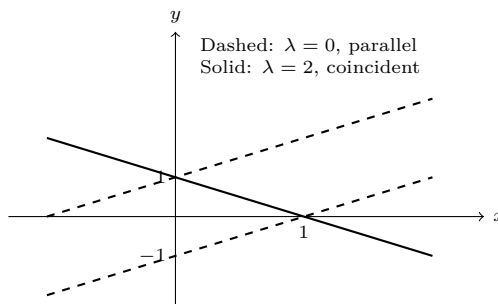
$$x = y = 1/\lambda.$$

At $\lambda = 0$, the sum is $0 = 2$, giving no solution. At $\lambda = 2$, only $x + y = 1$ remains, giving infinitely many solutions.

Step 2: Retain the exceptional equations. Directly, $\lambda = 0$ gives $-x + y = 1$ and $x - y = 1$, incompatible parallel lines. At $\lambda = 2$, both equations become $x + y = 1$. Dividing by a factor that can be zero discards these cases rather than solving them. The figure displays the two exceptional geometries.

Step 3: Compare limits with exact parameter values. As $\lambda \rightarrow 2$ through the unique-solution cases, $(x, y) \rightarrow (1/2, 1/2)$. This is one point of the entire line present at $\lambda = 2$, not the whole limiting parameter's solution set. As $\lambda \rightarrow 0^+$ both coordinates increase without bound; as $\lambda \rightarrow 0^-$ both decrease without bound. There is no finite limiting solution, consistent with the contradiction at zero.

Step 4: Enforce nonnegativity. For $\lambda \neq 0, 2$, the solution is nonnegative exactly when $\lambda > 0$. At $\lambda = 2$, the nonnegative solutions are $(x, y) = (u, 1 - u)$, $0 \leq u \leq 1$. Therefore a nonnegative solution exists exactly for $\lambda > 0$, with the entire stated segment at 2.



Question 3

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A system begins with an equation whose x coefficient is zero:

$$2y - z = 1, \quad x - y + 2z = 3, \quad 2x + y + z = 6 + \delta,$$

where δ is a real perturbation of the last measurement.

Tasks

1. Solve the system for $\delta = 0$ using valid elimination. Explain why the initial zero coefficient is not evidence of nonuniqueness.
2. Find the exact solution for arbitrary δ , keeping fractions exact.
3. Verify the perturbed solution in all original equations, and decide whether any δ makes the system inconsistent or nonunique.
4. Compute $x + 3y$ and recover δ from an exact measurement of z . Explain why measuring $x + 3y$ alone cannot detect the perturbation.

Question 3 – Solution

Strategy. Choose an equation with a nonzero coefficient for elimination. An inconvenient ordering does not change the information in a system.

Step 1: Eliminate without dividing by zero. The second equation gives $x = 3 + y - 2z$. Substituting into the third gives $3y - 3z = \delta$, or $y - z = \delta/3$. Together with $2y - z = 1$, this determines y, z , then x . For $\delta = 0$ the result is $(x, y, z) = (2, 1, 1)$. Swapping the first two equations would also provide an immediate nonzero x coefficient; neither approach requires a zero division.

Step 2: Solve for the perturbation exactly. Subtract $y - z = \delta/3$ from $2y - z = 1$ to obtain $y = 1 - \delta/3$. It follows that

$$x = 2 + \delta, \quad y = 1 - \frac{\delta}{3}, \quad z = 1 - \frac{2\delta}{3}.$$

Only the nonzero constants 1 and 3 were used as divisors, so this calculation applies to every real δ .

Step 3: Check the original measurements and uniqueness. Substitution gives

$$\begin{aligned} 2y - z &= 2 - 2\delta/3 - 1 + 2\delta/3 = 1, \\ x - y + 2z &= 2 + \delta - 1 + \delta/3 + 2 - 4\delta/3 = 3, \\ 2x + y + z &= 4 + 2\delta + 1 - \delta/3 + 1 - 2\delta/3 = 6 + \delta. \end{aligned}$$

Every solution was forced by reversible substitutions and nonzero divisions. Thus exactly one solution exists for every real δ .

Step 4: Identify a blind measurement. The combination $x + 3y = 2 + \delta + 3 - \delta = 5$ is independent of δ . In contrast, z determines $\delta = \frac{3}{2}(1 - z)$ uniquely. The first two equations fix certain combinations even when the last datum changes; a measurement that stays constant along that change cannot reveal it.

Question 4

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three unknown quantities are measured through pairwise sums and a total:

$$x + y = a, \quad y + z = b, \quad x + z = c, \quad x + y + z = d.$$

The recorded values are $(a, b, c, d) = (2, 3, 4, 5)$.

Tasks

1. Derive a necessary and sufficient condition on arbitrary real a, b, c, d for consistency. Give the unique solution when it exists.
2. For the recorded data, solve the first three equations and identify the conflict with the fourth.
3. Assume the total $d = 5$ is exact and exactly one of the three pairwise readings is wrong. Find every possible corrected reading and corresponding triple (x, y, z) .
4. Decide whether these measurements identify which pairwise reading is wrong, even if x, y, z must be nonnegative. Explain the difference between detecting an error and locating it.

Question 4 – Solution

Strategy. Redundant measurements can reveal a contradiction without identifying its source.

Step 1: Derive the consistency condition and solution. Adding the three pairwise equations gives $2(x + y + z) = a + b + c$. Thus consistency requires $a + b + c = 2d$. The pairwise equations alone uniquely give

$$x = \frac{a + c - b}{2}, \quad y = \frac{a + b - c}{2}, \quad z = \frac{b + c - a}{2}.$$

These values satisfy all three pairwise equations, and their total is $(a + b + c)/2$. Therefore the displayed condition is also sufficient.

Step 2: Diagnose the recorded discrepancy. For $(a, b, c) = (2, 3, 4)$, the unique triple is $(3/2, 1/2, 5/2)$. Its total is $9/2$, not 5. Hence the full recorded system has no solution; rounding the computed triple would not repair an exact contradiction.

Step 3: Enumerate every single-reading correction. The exact total requires the sum of the pairwise readings to be 10 rather than 9. If exactly one pairwise reading changes, its correction must be $+1$. There are precisely three possibilities:

Changed reading	(a, b, c)	(x, y, z)
$a : 2 \rightarrow 3$	$(3, 3, 4)$	$(2, 1, 2)$
$b : 3 \rightarrow 4$	$(2, 4, 4)$	$(1, 1, 3)$
$c : 4 \rightarrow 5$	$(2, 3, 5)$	$(2, 0, 3)$

Each triple satisfies its corrected pairwise readings and has total 5.

Step 4: State the identification limit. All three corrected triples are nonnegative. Thus even with nonnegativity, the measurements cannot select the faulty reading. The redundancy detects that the four recorded equations cannot all be true. Locating the error requires another independent measurement or an additional justified restriction; it does not follow merely from detecting the inconsistency.

Question 5

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Two nearly parallel measurement equations are

$$x + y = 2, \quad x + (1 + \varepsilon)y = 2 + \delta,$$

where $\varepsilon > 0$ is known and δ is an unknown measurement error satisfying $|\delta| \leq \eta$, with $\eta > 0$.

Tasks

1. Find the exact solution for every admissible δ and explain what would happen if $\varepsilon = 0$ were allowed.
2. Find the smallest uniform bound on the largest absolute coordinate error relative to the nominal solution at $\delta = 0$. Prove sharpness.
3. For $\varepsilon = \eta = 10^{-3}$, find the full range of possible solutions. Compare the small equation residual of the nominal answer with its possible coordinate error.
4. Find a necessary and sufficient condition on ε ensuring both coordinate errors are at most 0.1 for every $|\delta| \leq \eta$. Interpret the condition geometrically.

Question 5 – Solution

Strategy. Subtracting almost identical equations exposes the small coefficient that amplifies uncertainty.

Step 1: Solve and retain the excluded case. Subtract the first equation from the second to get $\varepsilon y = \delta$. For $\varepsilon > 0$,

$$(x, y) = \left(2 - \frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon} \right).$$

If $\varepsilon = 0$ and $\delta \neq 0$, there is no solution; if both are zero, the two equations coincide and every point on $x + y = 2$ is a solution.

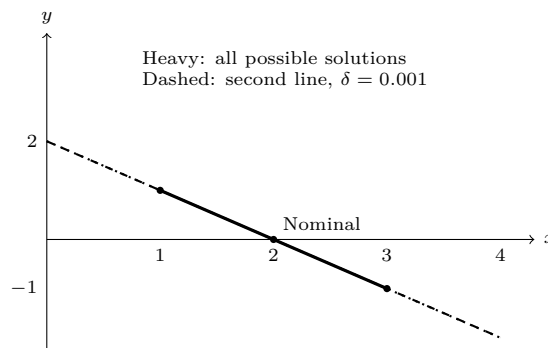
Step 2: Derive the exact uncertainty bound. The nominal solution is $(2, 0)$. Each coordinate changes in magnitude by $|\delta|/\varepsilon$. Therefore

$$\max(|x - 2|, |y|) \leq \eta/\varepsilon.$$

Equality occurs when $\delta = \eta$ or $\delta = -\eta$, so no smaller universal bound is possible.

Step 3: Quantify the nearly parallel example. For $\varepsilon = \eta = 10^{-3}$, let $u = \delta/\varepsilon \in [-1, 1]$. The full possible solution set is the segment $(x, y) = (2 - u, u)$, $-1 \leq u \leq 1$, from $(3, -1)$ to $(1, 1)$. The nominal answer satisfies the first equation exactly and has residual $-\delta$ in the second, of magnitude at most 0.001. Nevertheless its coordinate error can be 1. Small residuals do not always certify small solution errors. The graph uses true scales; the two measured lines are nearly indistinguishable, while the possible solution segment is visible.

Step 4: Give the exact design condition. The requested guarantee holds exactly when $\eta/\varepsilon \leq 0.1$, or $\varepsilon \geq 10\eta$. Necessity follows by taking an endpoint error; sufficiency follows from the bound. Increasing ε separates the line directions and reduces the effect of shifting one measurement's right-hand side.



Question 6

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A homogeneous system is

$$2x - y + z = 0, \quad x + y - 2z = 0.$$

Two proposed ways to choose a scale are $x + y + z = 9$ and $x^2 + y^2 + z^2 = 35$. These are alternative added conditions, not simultaneous unless explicitly stated.

Tasks

1. Find and verify the entire solution set of the homogeneous system, explaining the role of the zero solution.
2. Apply the linear normalization $x + y + z = 9$ and determine whether it selects a unique solution.
3. Apply the quadratic normalization $x^2 + y^2 + z^2 = 35$ instead. Explain its sign ambiguity and resolve it if nonnegative coordinates are additionally required.
4. For arbitrary real a, b, c, m , classify the effect of adding $ax + by + cz = m$: give exact conditions for no solution, one solution or infinitely many solutions.

Question 6 – Solution

Strategy. A homogeneous system can fix ratios while leaving the overall scale free. A normalization must actually measure that remaining freedom.

Step 1: Solve the homogeneous equations. From the second equation, $y = 2z - x$. Substitution into the first gives $3x - z = 0$, hence $z = 3x$ and $y = 5x$. Thus

$$(x, y, z) = t(1, 5, 3), \quad t \in \mathbb{R}.$$

Substitution gives $2t - 5t + 3t = 0$ and $t + 5t - 6t = 0$. Every solution has this form, including the zero solution at $t = 0$. Dividing by an unknown coordinate would risk losing that solution.

Step 2: Apply the linear normalization. The coordinate sum is $9t$. The condition $9t = 9$ fixes $t = 1$, giving $(1, 5, 3)$ uniquely.

Step 3: Compare the quadratic normalization. The squared length is $(1 + 25 + 9)t^2 = 35t^2$. Therefore the quadratic condition gives $t^2 = 1$ and exactly two solutions:

$$(1, 5, 3) \quad \text{and} \quad (-1, -5, -3).$$

Squaring loses the overall sign. If nonnegative coordinates are also required, only the positive triple remains. The zero triple does not satisfy either stated normalization.

Step 4: Classify an arbitrary linear measurement. On the homogeneous family, the extra left side is $(a + 5b + 3c)t$. Set $K = a + 5b + 3c$. Then

$$\begin{array}{ll} K \neq 0 : & t = m/K \text{ is the unique solution scale,} \\ K = 0, m \neq 0 : & \text{no solution,} \\ K = 0, m = 0 : & \text{every real } t \text{ remains possible.} \end{array}$$

A nonzero-looking added equation can still vanish on the entire family. What matters is its value on the remaining freedom, not the visual presence of three coefficients.

Question 7

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A sensor's response is assumed to be a quadratic polynomial $p(t) = a + bt + ct^2$. Exact readings at three times are

$$p(0) = 1, \quad p(1) = 2, \quad p(2) = 5.$$

Later, consider a perturbation of only the last reading to $5 + \epsilon$, where $|\epsilon| \leq \eta$ and $\eta > 0$.

Tasks

1. Find the unique polynomial from the exact readings using a system of equations. Predict $p(3)$.
2. Find the exact perturbed coefficients and the response error at an arbitrary time $T \geq 0$.
3. Identify all nonnegative times whose predicted value is insensitive to this perturbation. Find sharp uniform error bounds on $0 \leq T \leq 2$ and at $T = 10$.
4. Explain which conclusions depend on the quadratic model. Construct a nonquadratic smooth response with the same three original readings but a different value at 3.

Question 7 – Solution

Strategy. A model assumption converts function recovery into coefficient recovery, while extrapolation can amplify measurement errors.

Step 1: Solve the exact coefficient equations. The data give $a = 1$, $a + b + c = 2$ and $a + 2b + 4c = 5$. Thus $b + c = 1$ and $2b + 4c = 4$, yielding $c = 1$ and $b = 0$. Every coefficient is forced, so

$$p(t) = 1 + t^2, \quad p(3) = 10.$$

Step 2: Retain the measurement perturbation. The first two equations are unchanged, while $2b + 4c = 4 + \epsilon$. Solving gives

$$a = 1, \quad b = -\epsilon/2, \quad c = 1 + \epsilon/2, \quad p_\epsilon(T) - p(T) = \frac{\epsilon}{2}T(T-1).$$

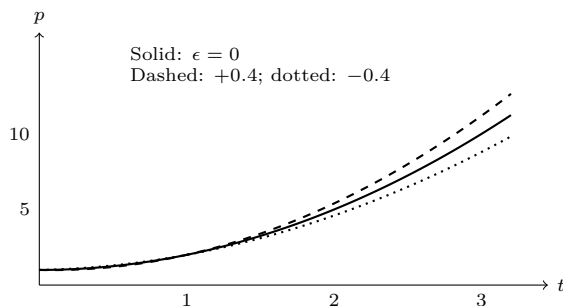
The error vanishes at both unchanged measurement times and equals ϵ at $T = 2$, as required.

Step 3: Compare interpolation and extrapolation. The error is independent of ϵ exactly when $T(T-1) = 0$, so the nonnegative insensitive times are 0 and 1. On $[0, 1]$, $|T(T-1)| \leq 1/4$; on $[1, 2]$, $T(T-1)$ increases from 0 to 2. Therefore

$$\sup_{0 \leq T \leq 2} |p_\epsilon - p| \leq \eta, \quad |p_\epsilon(10) - p(10)| \leq 45\eta.$$

Both bounds are sharp at $|\epsilon| = \eta$, the first at $T = 2$. The figure shows the true curves for $\epsilon = 0, \pm 0.4$.

Step 4: State the model's limitation. For any real K , the smooth function $q_K(t) = 1 + t^2 + Kt(t-1)(t-2)$ matches all three original readings. For $K \neq 0$ it is not quadratic, and $q_K(3) = 10 + 6K \neq 10$. Thus three readings uniquely determine a member of the quadratic family, not an arbitrary smooth response. Extrapolation bounds also rely on the quadratic family and on the stated location of the measurement error.



Question 8

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A steady flow network has directed internal flows p from A to B , q from B to C , and r from C to A . External flow enters A at rate 3; rates 1 and 2 leave B and C , respectively. There is no storage. All rates use the same units, and internal flows must be nonnegative.

Tasks

1. Write the three node-balance equations and identify any redundancy.
2. Find every nonnegative steady flow and explain why the external data do not determine the internal circulation uniquely.
3. Impose capacity bounds $p \leq 5$, $q \leq 4$ and $r \leq 3$. Find the entire feasible set and identify which capacities limit it.
4. Among feasible flows, minimize the cost rate $2p + q + 3r$. Prove uniqueness of the minimizer and interpret it in terms of circulation.

Question 8 – Solution

Strategy. Conservation fixes net transfers but can leave an arbitrary circulating flow around a closed loop.

Step 1: Balance every node. Inflow equals outflow, giving

$$A : r + 3 = p, \quad B : p = q + 1, \quad C : q = r + 2.$$

Equivalently $p - r = 3$, $q - p = -1$, $r - q = -2$. The three left sides sum to zero, as do their right sides. Any two balances imply the third because total external inflow equals total external outflow.

Step 2: Recover the free circulation. Set $r = t$. Then

$$(p, q, r) = (t + 3, t + 2, t), \quad t \geq 0.$$

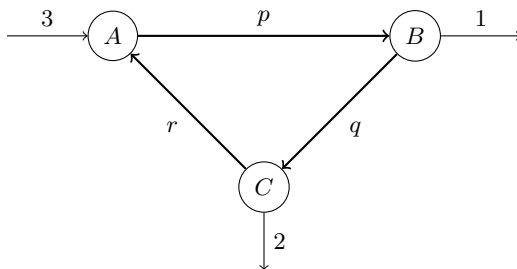
The condition $t \geq 0$ is necessary from $r \geq 0$ and sufficient for all three flows to be nonnegative. Adding the same amount to all three internal flows preserves every node balance: it is an additional circulation around the loop.

Step 3: Apply all capacities. The bounds give $t \leq 2$, $t \leq 2$ and $t \leq 3$, respectively. Thus the entire feasible family has $[0 \leq t \leq 2]$. The capacities on p and q both limit the upper endpoint; at $t = 2$ they are saturated, while $r = 2 < 3$ leaves unused capacity on the third link.

Step 4: Minimize the cost on the feasible family. Substituting the flow formulas yields

$$2p + q + 3r = 2(t + 3) + (t + 2) + 3t = 8 + 6t.$$

This increases strictly on $[0, 2]$. Therefore the unique minimizer is $(p, q, r) = (3, 2, 0)$, with cost rate $[8]$. It sends the required external transfers without an added loop circulation. The diagram shows directions, not arrows scaled in proportion to flow.



Question 9

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

You may design a second linear measurement to accompany $x + y = 1$:

$$ax + by = c.$$

It must select the target $(x, y) = (1/3, 2/3)$ uniquely. In a normalized design, require $a, b \geq 0$ and $a + b = 3$.

Tasks

1. Classify all real triples (a, b, c) that select the target uniquely. Explain the excluded case geometrically.
2. Under the normalization, describe every admissible design using one parameter and identify the excluded parameter value.
3. If the second right-hand side changes from c to $c + \delta$ while the first equation remains exact, derive the coordinate errors for an admissible design.
4. For $|\delta| \leq \eta$, where $\eta > 0$, find all normalized designs minimizing the worst possible absolute coordinate error. Give that minimum and verify the proposed designs directly.

Question 9 – Solution

Strategy. Passing through a target is necessary, but the measurement must also distinguish points along the first equation's line.

Step 1: Require both passage and uniqueness. The target lies on the second line exactly when $c = (a + 2b)/3$. Substituting $y = 1 - x$ gives $(a - b)x = c - b$. Hence unique selection requires

$$c = (a + 2b)/3, \quad a \neq b.$$

If $a = b$, the target condition forces $c = a$ and the second equation is a multiple of $x + y = 1$ (possibly $0 = 0$). The whole first line remains possible.

Step 2: List all normalized designs. Put $b = 3 - a$, with $0 \leq a \leq 3$. Then

$$b = 3 - a, \quad c = 2 - a/3, \quad a \in [0, 3] \setminus \{3/2\}.$$

The midpoint is excluded because both coefficients would equal $3/2$.

Step 3: Derive the exact measurement sensitivity. Subtract the target equations from the perturbed equations. With $\Delta x = x - 1/3$ and $\Delta y = y - 2/3$, one obtains $\Delta x + \Delta y = 0$ and $a\Delta x + b\Delta y = \delta$. Thus

$$\Delta x = \frac{\delta}{a - b}, \quad \Delta y = -\frac{\delta}{a - b}.$$

The worst coordinate error over $|\delta| \leq \eta$ is exactly $\eta/|a - b|$, attained at either endpoint of the allowed error interval.

Step 4: Optimize within the allowed normalization. Since $a + b = 3$ and both are nonnegative, $|a - b| \leq 3$, with equality exactly at $(a, b) = (3, 0)$ or $(0, 3)$. Therefore the minimum worst error is $\boxed{\eta/3}$, attained by exactly two designs:

$$3x = 1 \quad \text{or} \quad 3y = 2.$$

Perturbing these readings directly changes the measured coordinate by $\delta/3$ and the other by its negative. Both attain the claimed bound. Without the normalization, arbitrary rescaling would change the meaning of an absolute right-hand-side error, making this comparison ill-defined.

Question 10

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three nonnegative production amounts satisfy two exact resource totals:

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x, y, z \geq 0.$$

The return is $P = 4x + 2y + z$. Fractional production amounts are permitted.

Tasks

1. Find all solutions of the two equalities, then determine the exact parameter interval allowed by non-negativity.
2. Find the minimum and maximum return over the feasible set, giving every allocation at which either is attained.
3. Add the capacity constraint $x \leq 3$. Find the new feasible set and the new maximizing allocation and return.
4. Explain why solving the equalities alone does not solve the allocation problem. Verify the optimizing allocations against both original totals and all applicable inequalities.

Question 10 – Solution

Strategy. Solve the equalities first, then optimize over the remaining feasible interval rather than over an unrestricted algebraic family.

Step 1: Parameterize and restrict. Subtract the first equality from the second to obtain $y + 2z = 4$. Set $z = t$, giving

$$(x, y, z) = (2 + t, 4 - 2t, t).$$

Nonnegativity imposes $t \geq 0$, $t \leq 2$ and $t \geq -2$. The complete feasible set is therefore

$$(x, y, z) = (2 + t, 4 - 2t, t), \quad 0 \leq t \leq 2.$$

Each member satisfies both original totals; every feasible member must arise from this parameterization.

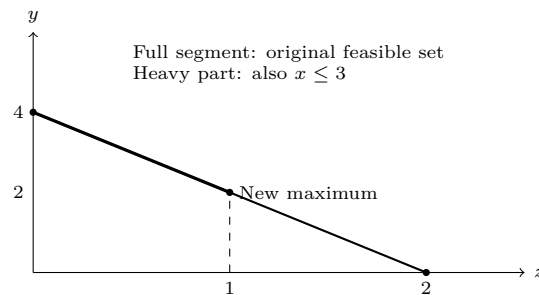
Step 2: Optimize on the feasible interval. The return becomes $P = 4(2 + t) + 2(4 - 2t) + t = 16 + t$. It is strictly increasing, so the unique minimum is $\boxed{16}$ at $(2, 4, 0)$, and the unique maximum is $\boxed{18}$ at $(4, 0, 2)$. Neither an interior stationary-point calculation nor an integer search is needed; the stated amounts may be fractional.

Step 3: Include the new capacity. The constraint $x \leq 3$ becomes $2 + t \leq 3$, or $t \leq 1$. The new feasible interval is $[0, 1]$, with unique maximizing allocation

$$(x, y, z) = (3, 2, 1), \quad P = 17.$$

The minimum remains at $t = 0$. In the figure, the feasible sets are shown in the (z, y) plane; $x = 2 + z$ is determined by the equalities.

Step 4: Verify the optimizers and the role of inequalities. The endpoint triples $(2, 4, 0)$ and $(4, 0, 2)$ have totals $(6, 10)$ and are nonnegative. The new optimizer $(3, 2, 1)$ also has totals $(6, 10)$ and satisfies $x = 3$. The former maximizer has $x = 4$, so it is no longer allowed. Over the unrestricted equality family, $P = 16 + t$ has no finite maximum or minimum. The inequalities make the allowed interval bounded and turn this into a well-posed allocation problem.



Question 1

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Consider the rectangular matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Tasks

1. Check the dimensions of AB and BA , compute both products, and identify which identity matrix occurs.
2. For an arbitrary $b = (b_1, b_2)^T$, describe every solution of $Ax = b$ and explain which solution Bb selects.
3. Prove that no matrix C can satisfy $CA = I_3$. Use a nonzero vector sent to zero by A , rather than only counting equations.
4. For $P = BA$, prove $P^2 = P$, describe all vectors fixed by P , and compute Pv for $v = (2, -1, 3)^T$. Explain what information this operation discards.

Question 1 – Solution

Strategy. A rectangular matrix can have a right inverse without having a left inverse. Check what happens to vectors, not just products.

Step 1: Multiply with dimensions attached. The sizes are $A : 2 \times 3$ and $B : 3 \times 2$. Thus

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad BA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

The two products even act on spaces of different dimensions.

Step 2: Recover the whole solution family. The equation $Ax = b$ says $x_1 + x_3 = b_1$ and $x_2 + x_3 = b_2$. Hence

$$x = (b_1 - t, b_2 - t, t)^T, \quad t \in \mathbb{R}.$$

Every such vector works, and every solution has this form. The vector $Bb = (b_1, b_2, 0)^T$ selects $t = 0$; it is a choice, not the only inverse image.

Step 3: Rule out a left inverse. The nonzero vector $n = (-1, -1, 1)^T$ satisfies $An = 0$. If $CA = I_3$, then $n = I_3 n = CAn = C0 = 0$, a contradiction. Thus no left inverse exists, even though $AB = I_2$.

Step 4: Interpret the repeated operation. Associativity gives $P^2 = B(AB)A = BI_2A = P$. For $x = (x_1, x_2, x_3)^T$, $Px = (x_1 + x_3, x_2 + x_3, 0)^T$. Consequently $Px = x$ exactly when $x_3 = 0$, and

$$P(2, -1, 3)^T = (5, 2, 0)^T.$$

The discarded difference is $x - Px = x_3(-1, -1, 1)^T$, invisible to A . Indeed $APx = Ax$. This is an idempotent projection onto the plane $x_3 = 0$, but it is not the perpendicular projection: it changes the first two coordinates.

Question 2

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In the plane, apply the shear and quarter-turn

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Composition acts on column vectors from right to left.

Tasks

1. Compute RS and SR . Apply each to $v = (1, 1)^T$ and explain the difference in operation order.
2. Find every vector w for which $RSw = SRw$. Prove that your list is complete.
3. Find $(RS)^{-1}$ by reversing the operations, and verify the inverse by multiplication.
4. Map the four vertices of the unit square in their cyclic order under both products. Compare areas and orientation, and explain why equal determinants do not imply equal transformations.

Question 2 – Solution

Strategy. Matrix multiplication records the order of geometric operations. Determinants retain only part of that information.

Step 1: Compute both compositions. Direct multiplication gives

$$RS = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad SR = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $RSv = (-1, 3)^T$ and $SRv = (1, 1)^T$. The first shears then rotates; the second rotates then shears.

Step 2: Test agreement on an arbitrary vector. Subtracting gives

$$(RS - SR)w = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

This is zero exactly when $w_1 = w_2 = 0$. Therefore the two compositions agree only on the zero vector, despite each being invertible.

Step 3: Undo the operations in reverse order. Since $S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $R^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

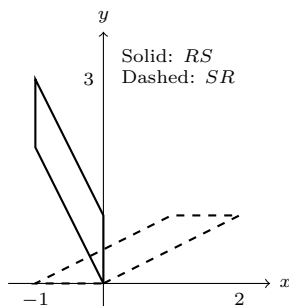
$$(RS)^{-1} = S^{-1}R^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Multiplication with RS on either side gives I_2 .

Step 4: Compare the images of the square. For vertices $(0, 0), (1, 0), (1, 1), (0, 1)$ in that order, the images are

$$\begin{array}{l|cccc} RS & (0, 0) & (0, 1) & (-1, 3) & (-1, 2) \\ SR & (0, 0) & (2, 1) & (1, 1) & (-1, 0) \end{array}$$

Both determinants are 1, so both images have area 1 and preserve the counterclockwise orientation. The parallelograms nevertheless differ. A determinant encodes signed area scaling, not the full image of each vector. The plot uses equal scales on both axes.



Question 3

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$A_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \tau & 1 \\ 1 & 1 & \tau \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \tau \in \mathbb{R}.$$

Tasks

1. Find $\det A_\tau$ by determinant-preserving row operations and classify invertibility.
2. For the invertible cases, derive A_τ^{-1} by solving $A_\tau x = c$ for a general column c . Verify the formula.
3. Solve $A_\tau x = b$ for every τ . At the singular parameter, classify consistency for an arbitrary right-hand side c .
4. Compare the behavior near $\tau = 1$ for right-hand sides $(1, 2, 3)^T$ and $(1, 1, 1)^T$. Explain why invertibility for each nearby parameter does not imply a uniform bound on recovered vectors.

Question 3 – Solution

Strategy. Subtracting the first row exposes the parameter that controls both inversion and the exceptional case.

Step 1: Compute the determinant. Replace rows two and three by themselves minus row one. These operations do not change the determinant and produce

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \tau - 1 & 0 \\ 0 & 0 & \tau - 1 \end{pmatrix}.$$

Therefore $\det A_\tau = (\tau - 1)^2$, and inversion is possible exactly when $\tau \neq 1$.

Step 2: Solve for a general right-hand side. Put $d = \tau - 1 \neq 0$. The subtracted equations give $x_2 = (c_2 - c_1)/d$, $x_3 = (c_3 - c_1)/d$, and $x_1 = c_1 - x_2 - x_3$. Thus

$$A_\tau^{-1} = \begin{pmatrix} 1 + 2/d & -1/d & -1/d \\ -1/d & 1/d & 0 \\ -1/d & 0 & 1/d \end{pmatrix}.$$

Substitution in the three original equations returns c_1, c_2, c_3 for every c , so multiplication $A_\tau A_\tau^{-1} = I_3$ is verified. The displayed matrices are symmetric, and transposing that equality also verifies multiplication in the opposite order.

Step 3: Handle the prescribed and singular systems. For $\tau \neq 1$,

$$x = \left(1 - \frac{3}{d}, \frac{1}{d}, \frac{2}{d} \right)^T.$$

At $\tau = 1$, all three left sides are $x_1 + x_2 + x_3$, while the entries of b differ, so there is no solution. For a general c , consistency at 1 is equivalent to $c_1 = c_2 = c_3$. If they coincide, the whole plane $x_1 + x_2 + x_3 = c_1$ is the solution set; otherwise it is empty.

Step 4: Distinguish invertibility from bounded recovery. For $b = (1, 2, 3)^T$, the coordinate $x_2 = 1/(\tau - 1)$ becomes unbounded near 1. For $c = (1, 1, 1)^T$, the unique nearby solution is always $(1, 0, 0)^T$. Thus some data remain harmless while other fixed data yield arbitrarily large solutions. The factors $1/d$ in the inverse prevent a uniform bound near the singular parameter, even though every matrix with $d \neq 0$ is invertible.

Question 4

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Use the ordered basis $u = (1, 1)^T$, $v = (1, -1)^T$ of the plane. Let P have columns u, v , so a coordinate column c represents the physical vector Pc . A transformation in standard coordinates is

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Tasks

1. Find P^{-1} and the coordinates of $w = (3, 1)^T$ in the ordered basis.
2. Derive the matrix representing T in this basis, explaining the order of all three operations.
3. Apply the coordinate matrix to your coordinate column and check the result against direct standard-coordinate multiplication.
4. For arbitrary coordinate columns, derive the relation between coordinate length and physical length. Compare the determinants of the two representations and interpret the physical area factor.

Question 4 – Solution

Strategy. A vector and its coordinate column describe the same object in different languages. Convert in the correct order.

Step 1: Build and invert the basis matrix. Here

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [w]_{u,v} = P^{-1}w = \boxed{(2, 1)^T}.$$

Thus $w = 2u + v$, as the figure illustrates.

Step 2: Express the transformation in the new coordinates. Start with a coordinate column c . Convert it to Pc , apply T , and convert the result back with P^{-1} . Consequently

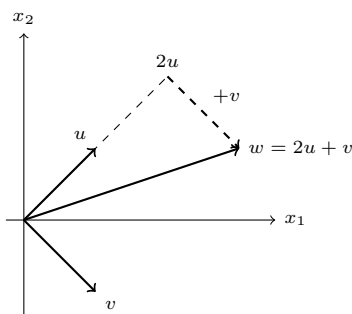
$$C = P^{-1}TP = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Step 3: Verify both routes. The coordinate route gives $C(2, 1)^T = (4, 3)^T$, representing $P(4, 3)^T = (7, 1)^T$. Directly, $T(3, 1)^T = (7, 1)^T$ as well. Equivalently, $PC = TP$ verifies the agreement for every input, not only w .

Step 4: Compare lengths and area factors. Since $P^T P = 2I_2$,

$$\boxed{\|Pc\|^2 = c^T P^T P c = 2\|c\|^2.}$$

The basis vectors are perpendicular but have length $\sqrt{2}$, so physical length is $\sqrt{2}$ times coordinate length. Also $\det C = \det(P^{-1}) \det T \det P = \det T = 2$. The physical transformation doubles areas and preserves orientation, regardless of which basis is used to describe it. The negative determinant of P describes the basis orientation; it does not change this area factor.



Question 5

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let $a = (1, 2)^T$ and $b = (2, 1)^T$. We seek the point on the line through a closest to b . All points on that line have the form ta , $t \in \mathbb{R}$.

Tasks

1. Minimize $\|b - ta\|^2$ exactly, giving the unique closest point and the minimum distance.
2. Construct a matrix Q that sends any vector to its closest point on this same line. Compute Q^T and Q^2 .
3. Describe all vectors sent to zero and all vectors fixed by Q . Verify that the error $b - Qb$ is perpendicular to the line.
4. Prove $\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2$ for every x , and deduce a sharp length inequality with its equality condition.

Question 5 – Solution

Strategy. Complete the square to find the best coefficient, then express that coefficient as a linear function of the input.

Step 1: Minimize the distance. Using $a^T a = 5$, $a^T b = 4$ and $b^T b = 5$,

$$\|b - ta\|^2 = 5t^2 - 8t + 5 = 5(t - 4/5)^2 + 9/5.$$

The unique minimizer is $t = 4/5$, so

$$Qb = (4/5, 8/5)^T, \quad \text{minimum distance} = 3/\sqrt{5}.$$

Step 2: Build the projection matrix. For any input x , the minimizing coefficient is $(a^T x)/(a^T a)$. Therefore

$$Q = \frac{aa^T}{5} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad Q^T = Q, \quad Q^2 = Q.$$

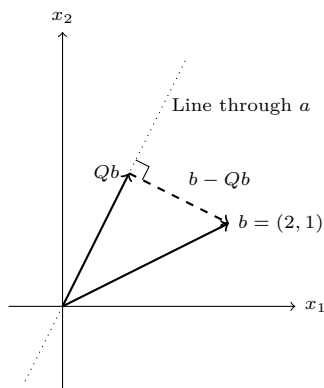
The last equality follows from $aa^T aa^T = a(a^T a)a^T = 5aa^T$.

Step 3: Describe the two perpendicular directions. The condition $Qx = 0$ is equivalent to $x_1 + 2x_2 = 0$, so the vectors sent to zero form the line through $(2, -1)^T$. The fixed vectors are exactly the multiples of a : the image lies on that line, and $Q(ta) = ta$. For the stated input, $b - Qb = (6/5, -3/5)^T$, whose dot product with a is zero.

Step 4: Prove the length identity and equality case. In general $a^T(x - Qx) = 0$, so Qx is perpendicular to $x - Qx$. Expanding the squared length of their sum gives

$$\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2, \quad \|Qx\| \leq \|x\|.$$

Equality holds exactly when $x - Qx = 0$, that is, when x lies on the line through a , including the zero vector. For b , the squared lengths are $5 = 16/5 + 9/5$. The plot uses equal axis scales and marks the right angle.



Question 6

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In three dimensions, let

$$u = (1, 1, 0)^T, \quad v = (1, 0, 1)^T, \quad w_\tau = (\tau, 1, 1)^T.$$

The plane of interest passes through $p = (1, 2, 3)^T$ and has direction vectors u, v . Use the scalar triple product $(u \times v) \cdot w$ as signed volume.

Tasks

1. Find a normal vector and a Cartesian equation for the plane. Verify that the specified point and directions satisfy the required conditions.
2. Find the area of the parallelogram spanned by u, v , and the signed and unsigned volumes spanned by u, v, w_τ .
3. Classify every τ for which the three direction vectors are linearly independent. At each dependent value, give an explicit nontrivial relation.
4. Explain how reversing the order of u, v affects the normal, plane equation, signed volume and unsigned volume. Distinguish a direction lying in the plane from a point lying on this translated plane.

Question 6 – Solution

Strategy. A cross product encodes both a perpendicular direction and an orientation. Translation affects a plane's points, not its direction space.

Step 1: Find and verify the normal. Direct calculation gives

$$n = u \times v = (1, -1, -1)^T.$$

Both dot products $n \cdot u$ and $n \cdot v$ vanish. The plane equation is $n \cdot (x - p) = 0$, or

$$x_1 - x_2 - x_3 = -4.$$

The point p satisfies it, and adding any combination of u, v to p leaves its left side unchanged.

Step 2: Compute area and volume. The parallelogram area is $\|n\| = \sqrt{3}$. The scalar triple product is

$$n \cdot w_\tau = \tau - 2, \quad \text{unsigned volume} = |\tau - 2|.$$

For example, $\tau = 0$ gives signed volume -2 and volume 2 .

Step 3: Prove the independence classification. If $\alpha u + \beta v + \gamma w_\tau = 0$, dotting with n gives $\gamma(\tau - 2) = 0$. For $\tau \neq 2$, this forces $\gamma = 0$. The second and third coordinates of $\alpha u + \beta v = 0$ then give $\alpha = \beta = 0$. Thus the vectors are independent exactly for $\tau \neq 2$. At $\tau = 2$, $w_2 = u + v$ is a nontrivial dependence, and the volume collapses to zero.

Step 4: Separate orientation from translation. Reversing the order gives $v \times u = -n$. The plane equation becomes $-x_1 + x_2 + x_3 = 4$, describing the same plane. Signed volume changes sign, while unsigned volume does not. As a direction, w_τ lies in the plane's direction space exactly when $n \cdot w_\tau = 0$, or $\tau = 2$. As a point with those coordinates, it lies on the translated plane exactly when $\tau - 2 = -4$, or $\tau = -2$. These are different questions because the plane does not pass through the origin.

Question 7

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For real t , let

$$M(t) = \begin{pmatrix} 1 & t \\ 0 & 2 \end{pmatrix}, \quad v(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}, \quad y(t) = M(t)v(t).$$

Differentiate matrices entry by entry.

Tasks

1. Derive the matrix-vector product rule from component sums. Compute y and y' in this example by two routes.
2. Find $N(t) = M(t)^{-1}$. Derive the inverse differentiation rule from $MN = I_2$ and verify it here.
3. Compute $(M^2)'$, $M'M + MM'$ and $2MM'$. Decide whether the scalar shortcut for differentiating a square is valid.
4. Compute $\det M(t)$ and examine $\|M(t)(0, 1)^T\|$. Explain why a constant nonzero determinant does not force a constant matrix or constant vector lengths.

Question 7 – Solution

Strategy. Scalar differentiation still applies to each entry, but matrix factors must retain their multiplication order.

Step 1: Derive and apply the product rule. For component i , $y_i = \sum_j M_{ij}v_j$. Differentiating the finite sum gives $y'_i = \sum_j (M'_{ij}v_j + M_{ij}v'_j)$, hence $y' = M'v + Mv'$. Direct multiplication and differentiation give

$$y = \begin{pmatrix} t^2 + te^t \\ 2e^t \end{pmatrix}, \quad y' = \begin{pmatrix} 2t + (1+t)e^t \\ 2e^t \end{pmatrix}.$$

The product-rule route gives $M'v = (e^t, 0)^T$ and $Mv' = (2t + te^t, 2e^t)^T$, confirming the same derivative.

Step 2: Differentiate the inverse with the correct order. The inverse is $N = \begin{pmatrix} 1 & -t/2 \\ 0 & 1/2 \end{pmatrix}$. Differentiating $MN = I_2$ gives $M'N + MN' = 0$; multiplying on the left by N yields

$$N' = -NM'N = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}.$$

This agrees with direct entrywise differentiation of N .

Step 3: Test the square rule explicitly. Since $M^2 = \begin{pmatrix} 1 & 3t \\ 0 & 4 \end{pmatrix}$,

$$(M^2)' = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{M'M} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{MM'}.$$

But $2MM' = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, which is different. The shortcut works when M and M' commute; they do not here.

Step 4: Interpret the constant determinant. The determinant is always 2, whereas $M(t)(0, 1)^T = (t, 2)^T$ has length $\sqrt{t^2 + 4}$. Thus signed area scaling remains fixed while shear and some vector lengths change. Determinant information alone cannot determine every entry or every geometric effect of a matrix.

Question 8

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For $n = (1, 2)^T$, define

$$H = I_2 - \frac{2nn^T}{n^T n}.$$

Consider its action on the triangle with ordered vertices $O = (0, 0)$, $a = (3, 1)$ and $b = (1, 2)$.

Tasks

1. Compute H . Prove from a perpendicular decomposition that it reflects the plane across a named line.
2. Verify $H^T H = I_2$ and $H^2 = I_2$. Deduce preservation of lengths and dot products and describe the inverse operation.
3. Find the reflected triangle, its area and orientation relative to the original. Check the midpoint and displacement of a and its image.
4. Find every vector fixed by H and every vector sent to its negative. Derive these directions directly from the reflection formula.

Question 8 – Solution

Strategy. Separate a vector into components parallel and perpendicular to the normal. The formula changes the sign of only the normal component.

Step 1: Identify the reflection axis. Here $n^T n = 5$, so

$$H = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}.$$

Write $x = x_{\perp} + x_{\parallel}$, where $x_{\parallel} = n(n^T x)/5$ and $n^T x_{\perp} = 0$. Then $Hx = x_{\perp} - x_{\parallel}$: it reflects across the line $x_1 + 2x_2 = 0$, whose direction is $(2, -1)^T$.

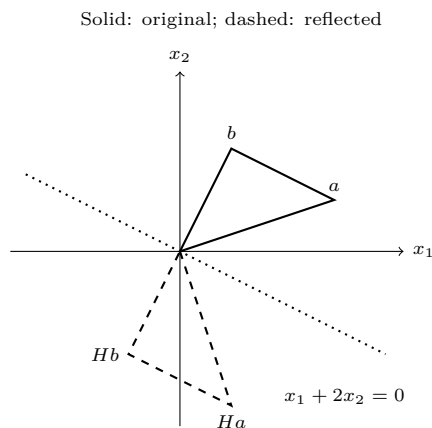
Step 2: Verify the metric and inverse identities. The matrix is symmetric, and its numerator squared is $\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$. Thus $H^2 = H^T H = I_2$. For any x, z , $(Hx)^T(Hz) = x^T H^T H z = x^T z$, preserving dot products and hence lengths. Also $H^{-1} = H$: reflecting twice restores the original vector.

Step 3: Transform and inspect the triangle. The images are

$$HO = O, \quad Ha = (1, -3)^T, \quad Hb = (-1, -2)^T.$$

The original ordered edge determinant is $3 \cdot 2 - 1 \cdot 1 = 5$, so its area is $5/2$. The reflected edge determinant is $1(-2) - (-1)(-3) = -5$. Its area remains $5/2$ but its orientation reverses, consistent with $\det H = -1$. The midpoint of a and Ha is $(2, -1)$ on the axis; the displacement $Ha - a = (-2, -4) = -2n$ is perpendicular to that axis.

Step 4: Classify fixed and reversed vectors. From the formula, $Hx = x$ exactly when $n^T x = 0$, giving the axis. For $Hx = -x$, the decomposition gives $x_{\perp} = 0$, so x is a multiple of n . The zero vector belongs to both sets. These conclusions follow from the components directly, without a characteristic-polynomial calculation. The plot uses equal axis scales.



Question 9

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

An unknown real 2×2 matrix A satisfies

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Tasks

1. Recover A uniquely, determine whether it is invertible, and verify both supplied measurements.
2. Predict $A(3, 1)^T$ without multiplying by the recovered matrix. Decide whether an additional reported image $(4, 5)^T$ is compatible.
3. In a separate experiment, replace the second input by $(2, 2)^T$ and its image by $(4, 0)^T$, keeping the first measurement. Find every matrix compatible with this experiment.
4. Classify the invertible matrices in the second experiment. Explain why one more image of an input outside the line through $(1, 1)^T$ would determine a unique matrix.

Question 9 – Solution

Strategy. Images of a basis determine a linear map, while repeated information along one input direction leaves freedom.

Step 1: Recover the columns from independent inputs. The first and second measurements are respectively the sum and difference of the columns of A . Half their sum gives column one; half their difference gives column two. Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \det A = -4 \neq 0.$$

The column sum is $(2, 0)^T$ and the difference is $(0, 4)^T$, checking both measurements. Both columns are forced, so recovery is unique.

Step 2: Predict by linearity and test a new report. Because $(3, 1)^T = 2(1, 1)^T + (1, -1)^T$, linearity predicts $A(3, 1)^T = 2(2, 0)^T + (0, 4)^T = (4, 4)^T$. Thus a report of $(4, 5)^T$ is inconsistent with the two original measurements for any linear map, not just with a particular computation of its entries.

Step 3: Describe the dependent-input experiment. Now the second input and output are exactly twice the first, so the second measurement adds no restriction. Write the first column as $(\alpha, \beta)^T$. The required column sum $(2, 0)^T$ forces the second column, giving

$$A_{\alpha, \beta} = \begin{pmatrix} \alpha & 2 - \alpha \\ \beta & -\beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

Every member gives both required outputs; every compatible matrix has this form.

Step 4: Classify inversion and the missing information. The determinant is $-\alpha\beta - (2 - \alpha)\beta = -2\beta$. Thus exactly the members with $\beta \neq 0$ are invertible. If another input w is outside the line through $(1, 1)^T$, those two inputs form a basis. Each standard coordinate vector has a unique expression in that basis, so the two measured images uniquely determine both columns. Any assigned image for this independent input defines one linear map; invertibility would additionally require independent output images.

Question 10

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = I_3 + N.$$

A discrete update is $v_{m+1} = Av_m$, starting from $v_0 = (0, 0, 1)^T$. The index m is a nonnegative integer, and each update uses the entire old vector.

Tasks

1. Compute N^2, N^3 , $\det A$ and A^{-1} . Verify the inverse without a general inverse algorithm.
2. Prove a finite formula for A^m for all nonnegative integers m , using induction or the binomial identity with justification.
3. Find v_m and describe its coordinate growth. Explain why $\det A = 1$ does not imply bounded iterates.
4. A program first updates $y \leftarrow y + z$ and then $x \leftarrow x + y$, using the already updated y . Find the matrix it actually applies and its trajectory from the same initial vector. Compare the two after two steps.

Question 10 – Solution

Strategy. The powers of the strictly upper triangular part terminate. A matrix update is simultaneous unless a different order is explicitly encoded.

Step 1: Exploit the terminating powers. Multiplication gives $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N^3 = 0$. The upper triangular matrix A has determinant 1. Moreover

$$A^{-1} = I_3 - N + N^2$$

because $(I_3 + N)(I_3 - N + N^2) = I_3 + N^3 = I_3$; the reverse product is the same.

Step 2: Derive the finite power formula. The identity matrix commutes with N , so the binomial expansion terminates:

$$A^m = I_3 + mN + \frac{m(m-1)}{2}N^2.$$

Alternatively, the formula is true at $m = 0$. Multiplication by $I_3 + N$ increases the coefficient of N to $m + 1$ and that of N^2 to $m(m-1)/2 + m = m(m+1)/2$, establishing the induction step.

Step 3: Find the exact iterates. Applying the power formula to v_0 gives

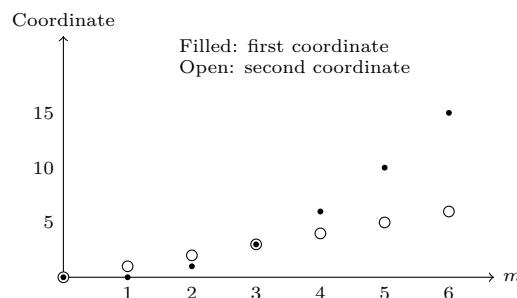
$$v_m = (m(m-1)/2, m, 1)^T.$$

The first coordinate grows quadratically, the second linearly and the third stays fixed. Determinant 1 preserves oriented volumes of three-dimensional sets, not individual lengths or boundedness under repetition. The plot marks integer-time values only; coincident coordinate values share a location.

Step 4: Diagnose the sequential-update error. The program computes $y_{\text{new}} = y + z$ and then $x_{\text{new}} = x + y + z$. It actually applies

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I_3 + N + N^2.$$

Starting from v_0 , its third coordinate remains 1, its second becomes m , and its first accumulates $1 + 2 + \dots + m = m(m+1)/2$. After two steps the correct vector is $(1, 2, 1)^T$, while the program gives $(3, 2, 1)^T$. Both matrices are invertible with determinant 1, so checking only the determinant would not expose this order-of-update mistake.



Question 1

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad u = (1, 1)^T, \quad v = (1, -1)^T,$$

and write a real starting vector as $x_0 = \alpha u + \beta v$.

Tasks

1. Find the eigenvalues, their full eigenspaces and both multiplicities. Explain why arbitrary nonzero multiples of an eigenvector are allowed but the zero vector is not.
2. Express $w = (3, 1)^T$ in the eigenvector basis and decide whether w itself is an eigenvector.
3. Find $A^m x_0$ for every nonnegative integer m , and specialize to w . Verify the result without multiplying out m matrices.
4. For every nonzero real x_0 , determine the limit of $A^m x_0 / \|A^m x_0\|$. Include the cases $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$.

Question 1 – Solution

Strategy. Resolve the starting vector into invariant directions before analyzing repeated multiplication.

Step 1: Compute the eigenspaces. The characteristic polynomial is $(\lambda - 2)^2 - 1 = (\lambda - 3)(\lambda - 1)$. Solving the two singular systems gives

$$E_3 = \text{span}\{u\}, \quad E_1 = \text{span}\{v\}.$$

Both algebraic and geometric multiplicities are 1. Scaling a nonzero eigenvector preserves its eigenvalue equation. Zero solves every homogeneous eigenvalue equation and supplies no direction, so it is excluded as an eigenvector.

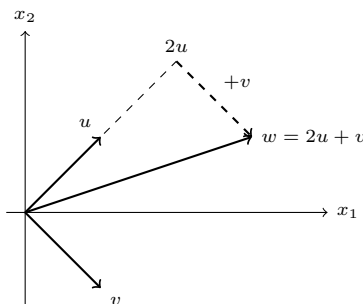
Step 2: Decompose and test the given vector. The equations $\alpha + \beta = 3$, $\alpha - \beta = 1$ give $w = 2u + v$. Both components are nonzero and their eigenvalues differ. Directly, $Aw = (7, 5)^T$ is not a scalar multiple of $(3, 1)^T$, so w is not an eigenvector.

Step 3: Compute all iterates. Linearity and $Au = 3u$, $Av = v$ give, by induction,

$$A^m x_0 = 3^m \alpha u + \beta v, \quad A^m w = (2 \cdot 3^m + 1, 2 \cdot 3^m - 1)^T.$$

The formula holds at $m = 0$, and one multiplication advances each coefficient by its eigenvalue, proving the next case.

Step 4: Classify normalized direction limits. If $\alpha \neq 0$, divide the iterate by $3^m |\alpha|$ before normalizing. The remaining v coefficient tends to zero, so the limit is $\boxed{u/\sqrt{2}}$ for $\alpha > 0$ and $\boxed{-u/\sqrt{2}}$ for $\alpha < 0$. If $\alpha = 0$, then $\beta \neq 0$ and the iterate is the unchanged vector βv , giving limit $\boxed{\text{sgn}(\beta)v/\sqrt{2}}$. The dominant eigenvalue determines the limit only when its component is present. The diagram shows the two directions and the decomposition $w = 2u + v$.



Question 2

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real a , consider

$$B_a = \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}.$$

A student claims that a repeated eigenvalue automatically provides two independent eigenvectors.

Tasks

1. Find the eigenvalue and both multiplicities for every a , treating $a = 0$ separately.
2. Classify diagonalizability over $\mathbb{R}$ and over $\mathbb{C}$. Explain whether allowing complex eigenvectors repairs any failure.
3. Find an exact formula for B_a^m for every nonnegative integer m . Derive it rather than assuming diagonalization.
4. For a fixed vector $x = (x_1, x_2)^T$, classify when $2^{-m}B_a^m x$ stays bounded as $m \rightarrow \infty$. Compare this result with what the repeated eigenvalue alone suggests.

Question 2 – Solution

Strategy. Root multiplicity and the number of independent eigenvectors answer different questions.

Step 1: Solve the eigenspace equation. The characteristic polynomial is $(\lambda - 2)^2$ for every a . The equation $(B_a - 2I)x = 0$ reduces to $ax_2 = 0$. Thus the algebraic multiplicity is always 2, while

$$a = 0 : E_2 = \mathbb{R}^2, \dim E_2 = 2; \quad a \neq 0 : E_2 = \text{span}\{(1, 0)^T\}, \dim E_2 = 1.$$

A repeated root does not itself produce a second independent vector.

Step 2: Classify diagonalization in both fields. At $a = 0$, $B_a = 2I$ is already diagonal. At $a \neq 0$, there is only one independent eigenvector, so no eigenvector basis exists. The same equation $ax_2 = 0$ holds over $\mathbb{C}$, with the same dimension count there. Hence B_a is diagonalizable exactly when $a = 0$ over either field.

Step 3: Derive the powers from a terminating product. Let $N = B_a - 2I = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, so $N^2 = 0$. The commuting binomial expansion gives

$$B_a^m = 2^m \begin{pmatrix} 1 & ma/2 \\ 0 & 1 \end{pmatrix}.$$

For $m = 0$ this is I . Multiplication by B_a adds $a/2$ to the scaled upper-right entry, confirming the formula inductively without an eigenvector basis.

Step 4: Identify the remaining growth. The scaled iterate is

$$2^{-m} B_a^m x = (x_1 + max_2/2, x_2)^T.$$

It is bounded exactly when $ax_2 = 0$, and then is constant. For every $a \neq 0$ and $x_2 \neq 0$, its first coordinate grows in magnitude linearly. Dividing out the common exponential factor 2^m does not remove this extra growth. The missing eigenvector permits behavior that the list of eigenvalues, even with multiplicity, does not describe by itself.

Question 3

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Allow complex eigenvalues and eigenvectors for the real matrix

$$C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Write $i^2 = -1$, and let $e_1 = (1, 0)^T$.

Tasks

1. Find both complex eigenvalues and their eigenspaces, and verify a representative eigenvector for each directly.
2. Prove that there is no nonzero real eigenvector, despite the existence of a complex eigenvector basis.
3. Write an eigenvector for $1 + 2i$ as $p + iq$ with p, q real, and derive the two real equations obtained from its eigenvalue equation.
4. Express C as a positive scale times a real rotation. Find $C^m e_1$ and its length for every nonnegative integer m , with the rotation angle specified unambiguously.

Question 3 – Solution

Strategy. Complex eigenvectors encode a real invariant plane; they need not describe a real line fixed in direction.

Step 1: Find and verify the complex pairs. The polynomial is $(\lambda - 1)^2 + 4$, so

$$\lambda_{\pm} = 1 \pm 2i, \quad E_{1+2i} = \text{span}_{\mathbb{C}}\{(1, -i)^T\}, \quad E_{1-2i} = \text{span}_{\mathbb{C}}\{(1, i)^T\}.$$

Indeed $C(1, -i)^T = (1 + 2i, 2 - i)^T = (1 + 2i)(1, -i)^T$. Conjugation verifies the other pair. The distinct eigenvalues give independent complex eigenvectors.

Step 2: Exclude a real eigenvector. If a real nonzero x satisfied $Cx = \lambda x$, a nonzero coordinate of x would make λ the ratio of two real numbers. Thus λ would be real, but the characteristic polynomial has no real root. No such vector exists.

Step 3: Extract the real plane relations. For $(1, -i)^T = p + iq$, take $p = (1, 0)^T$, $q = (0, -1)^T$. Equating real and imaginary parts of $C(p + iq) = (1 + 2i)(p + iq)$ gives

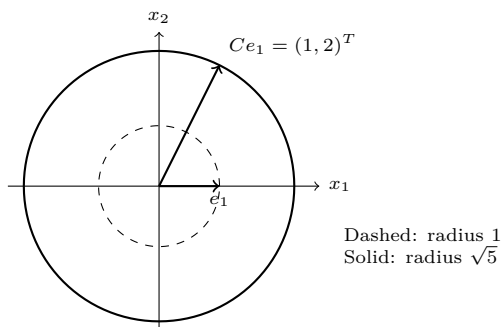
$$Cp = p - 2q, \quad Cq = 2p + q.$$

Direct multiplication verifies both. These independent real vectors span the plane even though neither is a real eigenvector.

Step 4: Recover the rotation and scale. Let $\theta \in (0, \pi/2)$ satisfy $\cos \theta = 1/\sqrt{5}$ and $\sin \theta = 2/\sqrt{5}$; equivalently $\theta = \arctan(2)$. Then

$$C = \sqrt{5} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad C^m e_1 = 5^{m/2} (\cos m\theta, \sin m\theta)^T.$$

Its length is $5^{m/2}$. The figure shows e_1 and its image $(1, 2)^T$, with unit and image circles drawn on equal scales.



Question 4

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Consider the real symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Tasks

1. Find an orthonormal eigenvector basis and an orthogonal diagonalization $S = QDQ^T$. Verify the columns directly.
2. Prove that eigenvectors of a real symmetric matrix belonging to distinct real eigenvalues are perpendicular, and connect the proof to your basis.
3. Find the sharp minimum and maximum of $x^T S x$ over real vectors with $\|x\| = 1$, giving every equality case.
4. Use the spectral decomposition to find S^{-1} and verify it. Explain why a negative eigenvalue does not prevent invertibility but does prevent positive definiteness.

Question 4 – Solution

Strategy. An orthonormal eigenvector basis turns both a quadratic form and an inverse into scalar calculations.

Step 1: Construct and check the basis. Take

$$q_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T, \quad q_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T, \quad q_3 = (0, 0, 1)^T.$$

Direct multiplication gives eigenvalues 3, 1, -1, respectively. The columns are orthonormal, so $Q = (q_1 \ q_2 \ q_3)$ has $Q^T Q = I$. Thus $SQ = QD$ with $D = \text{diag}(3, 1, -1)$, and $S = QDQ^T$.

Step 2: Prove perpendicularity. For a real symmetric matrix, if $Sa = \lambda a$ and $Sb = \mu b$, then

$$\lambda a^T b = (Sa)^T b = a^T S^T b = a^T S b = \mu a^T b.$$

When $\lambda \neq \mu$, this forces $a^T b = 0$. Normalizing the three perpendicular directions gives exactly the orthonormal basis above.

Step 3: Derive the sharp quadratic-form bounds. Write $x = c_1 q_1 + c_2 q_2 + c_3 q_3$. For a unit vector, $c_1^2 + c_2^2 + c_3^2 = 1$, and

$$x^T S x = 3c_1^2 + c_2^2 - c_3^2.$$

This is a weighted average of 3, 1, -1, so $-1 \leq x^T S x \leq 3$. The minimum occurs exactly at $x = \pm q_3$; the maximum exactly at $x = \pm q_1$. The coefficients of all other directions must vanish at an endpoint.

Step 4: Invert the nonzero eigenvalues. Since no eigenvalue is zero,

$$S^{-1} = Q \text{diag}(1/3, 1, -1) Q^T = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Multiplying by S gives I_3 . Invertibility excludes zero eigenvalues, not negative ones. But $q_3^T S q_3 = -1 < 0$, so S is not positive definite; in fact its quadratic form takes both positive and negative values.

Question 5

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

A real 2×2 matrix A is required to have

$$Au = 2u, \quad Av = -v, \quad u = (1, 1)^T, \quad v = (2, -1)^T.$$

Tasks

1. Construct the unique matrix satisfying these specifications and verify both eigenpairs.
2. Explain the effect of rescaling the two eigenvectors or swapping their order together with the corresponding eigenvalues. Contrast this with assigning the eigenvalues to the opposite directions.
3. Can the same matrix also have $(1, 0)^T$ as an eigenvector for some eigenvalue? Prove your answer.
4. Keeping the two prescribed directions, replace their eigenvalues by arbitrary real r, s . Find the full matrix formula and explain how its eigenspaces change when $r = s$.

Question 5 – Solution

Strategy. Independent eigenvectors form an input basis, so their prescribed images determine every column of the transformation.

Step 1: Build the spectral representation. The basis matrix and inverse are

$$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$A = P \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Its products with u, v are $(2, 2)^T = 2u$ and $(-2, 1)^T = -v$. Since u, v span the plane, these images determine A uniquely.

Step 2: Separate notation changes from changed assignments. Multiplying either basis column by a nonzero scalar leaves its eigenvalue relation unchanged, so the reconstructed matrix stays the same. Reordering columns and the matching diagonal entries also leaves the map unchanged. Assigning -1 to u and 2 to v instead produces $I - A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$, a different map. The correspondence between each direction and its scalar matters.

Step 3: Test a proposed third direction. Directly, $A(1, 0)^T = (0, 1)^T$, not a multiple of $(1, 0)^T$. Thus no choice of eigenvalue makes the additional request possible. Equivalently, $(1, 0)^T = (u + v)/3$ combines both distinct eigendirections.

Step 4: Recover the general two-scalar family. The same basis computation yields

$$A_{r,s} = \frac{1}{3} \begin{pmatrix} r + 2s & 2r - 2s \\ r - s & 2r + s \end{pmatrix}.$$

For $r \neq s$, the eigenspaces are exactly the two specified lines. For $r = s$, the matrix becomes rI_2 and every nonzero vector is an eigenvector; the eigenspace is the whole plane. Repeating an eigenvalue here enlarges the eigenspace, unlike a defective repeated-root example.

Question 6

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Two matrices commute when $AB = BA$.

Tasks

1. Find the full family of real matrices B commuting with A , by equating entries of the two products.
2. Find a common eigenvector basis for every matrix in this family. Explain why simultaneous diagonalization always implies commutation.
3. For $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find the eigenvalues of AB and $A + B$ using the common basis. Explain how two invertible matrices can have a singular sum.
4. Within the commuting family, classify every invertible B for which $A + B$ is singular. Identify when the sum has a one-dimensional kernel and when it is the zero matrix.

Question 6 – Solution

Strategy. Commutation forces preservation of this matrix's two distinct eigendirections, allowing scalar tests in a common basis.

Step 1: Equate the products. Subtracting the products gives

$$AB - BA = \begin{pmatrix} r - q & s - p \\ p - s & q - r \end{pmatrix}.$$

Thus $r = q$ and $s = p$, with no further restrictions. Thus the entire commuting family is $B = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$.

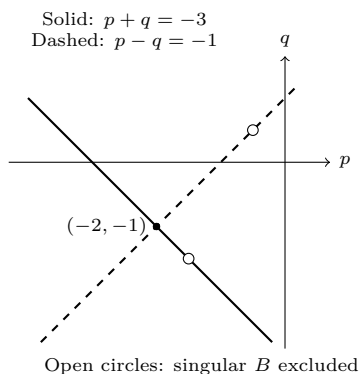
Step 2: Use the common directions. For $u = (1, 1)^T$ and $v = (1, -1)^T$, the eigenvalues of A are 3, 1, and those of B are $p + q, p - q$, respectively. These independent vectors form a common eigenvector basis, even if the two eigenvalues of B coincide. Conversely, if $A = PDP^{-1}$ and $B = PEP^{-1}$ with D, E diagonal, then $AB = PDEP^{-1} = PEDP^{-1} = BA$.

Step 3: Analyze the concrete sum and product. For $p = 1, q = 2$, the eigenvalues of B on u, v are 3, -1 . Therefore the eigenvalues of AB are 9, -1 and those of $A + B$ are 6, 0. Both A and B are invertible, but their actions on v cancel in the sum. The sum has kernel $\text{span}\{v\}$.

Step 4: Classify all cancellation cases. Invertibility of B requires $(p + q)(p - q) \neq 0$. Singularity of the sum requires $(p + q + 3)(p - q + 1) = 0$. Equivalently the full allowed set is

$$\{p + q = -3, p - q \neq 0\} \cup \{p - q = -1, p + q \neq 0\}.$$

On the first line only, the kernel is the line through u ; on the second only, it is the line through v . Their intersection is $(p, q) = (-2, -1)$, where $B = -A$ and $A + B = 0$, so the kernel is the entire plane. The excluded points are $(-3/2, -3/2)$ and $(-1/2, 1/2)$, where B itself is singular. The diagram marks these exclusions with open circles.



Question 7

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For a real parameter K , let

$$T_K = \begin{pmatrix} 1/2 & K \\ 0 & 1/4 \end{pmatrix}.$$

Both eigenvalues have magnitude less than one. We compare long-term iteration with the stronger requirement $\|T_K x\| \leq \|x\|$ for every real vector x .

Tasks

1. Find an eigenvector basis for every K and determine whether the matrix is diagonalizable.
2. Derive T_K^m for every nonnegative integer m and prove that every fixed starting vector tends to zero under iteration.
3. For the starting vector $e_2 = (0, 1)^T$, find the first-step length. Determine exactly when that first step increases length, and explain why this does not contradict the long-term limit.
4. Find the necessary and sufficient condition on K for $\|T_K x\| \leq \|x\|$ to hold for every real x . Prove it by completing a square, rather than testing only e_2 .

Question 7 – Solution

Strategy. Decaying eigencomponents need not make Euclidean length decrease at every step when the eigenvectors are not orthogonal.

Step 1: Find the distinct-eigenvalue basis. The eigenvalues are $1/2$ and $1/4$, with eigenvectors $u = (1, 0)^T$ and $v = (-4K, 1)^T$. These are independent for every K , so T_K is always diagonalizable over $\mathbb{R}$.

Step 2: Compute all powers. For $m \geq 1$, the upper-right entry in the product is

$$K \sum_{j=0}^{m-1} (1/2)^{m-1-j} (1/4)^j = 4K(2^{-m} - 4^{-m}).$$

This finite geometric sum yields

$$T_K^m = \begin{pmatrix} 2^{-m} & 4K(2^{-m} - 4^{-m}) \\ 0 & 4^{-m} \end{pmatrix}.$$

The formula is I at $m = 0$. Every entry tends to zero for each fixed K , so $T_K^m x \rightarrow 0$ for every fixed starting vector x .

Step 3: Exhibit a transient length increase. The first image of e_2 is $(K, 1/4)^T$, of length $\sqrt{K^2 + 1/16}$. It exceeds 1 exactly when $|K| > \sqrt{15}/4$. An early increase and eventual decay concern different times, so there is no contradiction. The plot uses $K = 2$ and shows the actual lengths at integer iteration counts; the initial increase is followed by decay.

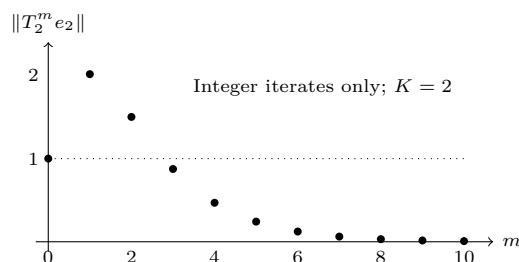
Step 4: Test all directions, not just one. For $x = (a, b)^T$, direct expansion and completion of the square give

$$\|T_K x\|^2 - \|x\|^2 = -\frac{3}{4} \left(a - \frac{2K}{3}b \right)^2 + \left(\frac{4K^2}{3} - \frac{15}{16} \right) b^2.$$

This is nonpositive for every a, b exactly when the last coefficient is nonpositive: necessity follows by choosing $a = 2Kb/3$ with $b \neq 0$. Hence the exact condition is

$$|K| \leq \frac{3\sqrt{5}}{8}.$$

This stricter threshold cannot be recovered by checking only e_2 or only the eigenvalues. At equality, nonzero vectors on $a = 2Kb/3$ retain their length in one step; outside that line they decrease.



Question 8

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real τ , consider

$$A_\tau = \begin{pmatrix} 0 & 1 \\ -\tau & 2 \end{pmatrix}.$$

Investigate diagonalization over both $\mathbb{R}$ and $\mathbb{C}$.

Tasks

1. Find the characteristic polynomial and classify its roots for $\tau < 1$, $\tau = 1$ and $\tau > 1$. Check trace and determinant against the roots.
2. Find every eigenspace and classify diagonalizability in both fields, including the repeated-root case.
3. For $\tau > 1$, write a complex eigenvector as $p + iq$ and construct a real basis in which A_τ has a real 2×2 block representing its complex pair.
4. As $\tau \rightarrow 1^-$, examine a real eigenvector basis and its inverse. Explain why the limiting matrix can fail to be diagonalizable even though every matrix before the limit is diagonalizable.

Question 8 – Solution

Strategy. Follow the eigenvectors as well as the roots. Distinct roots can merge into one direction at the transition.

Step 1: Classify the roots. The polynomial is $\lambda^2 - 2\lambda + \tau$. For $\tau < 1$, set $d = \sqrt{1 - \tau} > 0$; the roots are $1 \pm d$. At $\tau = 1$ the root is 1 twice. For $\tau > 1$, set $b = \sqrt{\tau - 1} > 0$; the roots are $1 \pm ib$. Their sum is $2 = \text{tr } A_\tau$ and product is $\tau = \det A_\tau$ in all cases.

Step 2: Find the eigenspaces and field restrictions. The first row of $A_\tau x = \lambda x$ forces $x_2 = \lambda x_1$. The second row is then exactly the characteristic equation. Thus every root has complex eigenspace spanned by $(1, \lambda)^T$; for real roots, the real eigenspace is the real span of the same vector. At $\tau = 1$ this is only one-dimensional, despite algebraic multiplicity 2. Consequently

	$\mathbb{R}$ -diagonalizable	$\mathbb{C}$ -diagonalizable
$\tau < 1$	yes	yes
$\tau = 1$	no	no
$\tau > 1$	no	yes

The real failure above 1 comes from nonreal eigenvalues, not a shortage of complex eigenvectors.

Step 3: Construct a real representation for the complex pair. For $\lambda = 1 + ib$, write $(1, 1 + ib)^T = p + iq$ with $p = (1, 1)^T$, $q = (0, b)^T$. Equating parts gives $A_\tau p = p - bq$ and $A_\tau q = bp + q$. Since $\det(p \ q) = b \neq 0$, this is a real basis, with representation

$$(p \ q)^{-1} A_\tau (p \ q) = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}.$$

It is a real block, not a real diagonalization.

Step 4: Inspect the collapsing eigenvector basis. Below 1, use $P = \begin{pmatrix} 1 & 1 \\ 1 + d & 1 - d \end{pmatrix}$. Its determinant is $-2d$, and

$$P^{-1} = \frac{1}{2d} \begin{pmatrix} d - 1 & 1 \\ 1 + d & -1 \end{pmatrix}.$$

In particular, the second column is unbounded as $d \rightarrow 0^+$. Both columns tend to $(1, 1)^T$ as $d \rightarrow 0^+$. The limiting basis is singular and cannot diagonalize the limiting matrix. Diagonalizability need not persist when distinct eigendirections coalesce.

Question 9

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = A^2.$$

For a polynomial p , define $p(A)$ by replacing each scalar power with the corresponding matrix power and each constant with that constant times I_3 .

Tasks

1. Find the eigenvalues and eigenspaces of A and B , and explain why squaring merges some eigenspaces.
2. Give a nonzero eigenvector of B that is not an eigenvector of A . Prove the general forward implication from an eigenpair of A to an eigenpair of $p(A)$, and explain why its converse can fail.
3. Find the monic polynomial of least degree satisfying $p(A) = 0$. Prove both the identity and the minimality of its degree.
4. Use that polynomial identity to express A^{-1} as a polynomial in A . Compute the inverse explicitly and verify it.

Question 9 – Solution

Strategy. A polynomial changes eigenvalues without changing each original eigenvector, but different eigenvalues can acquire the same image.

Step 1: Compare the two spectra. For A , the eigenspaces are

$$E_1 = \text{span}\{(1, 1, 0)^T\}, \quad E_{-1} = \text{span}\{(1, -1, 0)^T\}, \quad E_2 = \text{span}\{(0, 0, 1)^T\}.$$

These vectors form a basis. Since $B = \text{diag}(1, 1, 4)$, its eigenspace for 1 is the entire plane $x_3 = 0$, and its eigenspace for 4 is the third coordinate axis. Squaring sends both 1 and -1 to 1.

Step 2: Prove the forward rule and refute the converse. The vector $e_1 = (1, 0, 0)^T$ satisfies $Be_1 = e_1$ but $Ae_1 = (0, 1, 0)^T$, not a multiple of e_1 . In general, $Av = \lambda v$ implies $A^k v = \lambda^k v$ by induction; linearity then gives $p(A)v = p(\lambda)v$. The merged eigenspace permits new combinations that are eigenvectors of $p(A)$ without being eigenvectors of A .

Step 3: Find the least-degree annihilating polynomial. Any polynomial with $p(A) = 0$ must vanish at $1, -1, 2$, by applying the identity to the three nonzero eigenvectors. A nonzero polynomial therefore needs degree at least three. The monic candidate is

$$p(z) = (z - 1)(z + 1)(z - 2) = z^3 - 2z^2 - z + 2.$$

It annihilates all three basis eigenvectors, hence every vector, proving $p(A) = 0$ and minimality. It is the unique monic cubic with those roots.

Step 4: Recover the inverse from the identity. Rearrange $A^3 - 2A^2 - A + 2I = 0$ to get $A(-A^2 + 2A + I)/2 = I$. The factors commute, so the reverse product is also I . Thus

$$A^{-1} = \frac{-A^2 + 2A + I}{2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

Direct multiplication confirms the inverse without division by a matrix expression.

Question 10

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

The symmetric matrix

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

measures differences along the path 1-2-3. Let $\mathbf{1} = (1, 1, 1)^T$. A vector has zero mean when $x_1 + x_2 + x_3 = 0$.

Tasks

1. Find all eigenvalues and an orthogonal eigenvector basis. Verify each pair and show that the list is complete.
2. Express $x^T Lx$ as a sum of squares. Determine all vectors for which this expression is zero, and explain the zero eigenvalue.
3. Give a necessary and sufficient condition for $Lx = b$ to have a real solution. For $b = (2, -1, -1)^T$, find all solutions and the unique one with zero mean.
4. On the zero-mean subspace, prove the sharp bounds $\|x\|^2 \leq x^T Lx \leq 3\|x\|^2$ and give every equality case. Explain why the lower bound fails without the zero-mean restriction.

Question 10 – Solution

Strategy. Separate the constant direction, which the matrix cannot detect, from the two directions of variation.

Step 1: Verify a complete orthogonal basis. Direct multiplication gives

$$L\mathbf{1} = 0, \quad Lv = v, \quad v = (1, 0, -1)^T, \quad Lw = 3w, \quad w = (1, -2, 1)^T.$$

The three vectors are nonzero and mutually perpendicular, so they form a basis of $\mathbb{R}^3$. Thus the complete eigenvalue list is $[0, 1, 3]$, with the corresponding one-dimensional eigenspaces.

Step 2: Interpret the nonnegative quadratic form. Expanding gives

$$x^T Lx = (x_1 - x_2)^2 + (x_2 - x_3)^2 \geq 0.$$

It vanishes exactly when $x_1 = x_2 = x_3$, the constant direction. The zero eigenvalue reflects the fact that adding the same constant to all three entries changes no adjacent difference.

Step 3: Solve subject to the compatibility condition. Since $\mathbf{1}^T L = 0$, solvability requires $\mathbf{1}^T b = 0$. Conversely, every such b can be written $b = \alpha v + \beta w$, and $x = \alpha v + (\beta/3)w + c\mathbf{1}$ solves the system for every real c . For the stated b , dot products give $\alpha = 3/2$ and $\beta = 1/2$. Therefore all solutions are

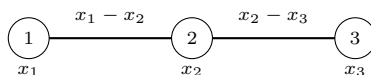
$$x = (5/3, -1/3, -4/3)^T + c\mathbf{1}, \quad c \in \mathbb{R}.$$

The displayed particular vector has zero mean; the total sum is $3c$, so zero mean uniquely selects $c = 0$.

Step 4: Prove the sharp restricted bounds. A zero-mean vector has form $x = av + bw$. Orthogonality and $\|v\|^2 = 2$, $\|w\|^2 = 6$ give

$$\|x\|^2 = 2a^2 + 6b^2, \quad x^T Lx = 2a^2 + 18b^2.$$

The lower equality holds exactly on the line through v ; the upper exactly on the line through w , with zero included in both. This proves both sharp bounds. A nonzero constant vector has positive length but zero quadratic form, so the lower bound cannot hold on all of $\mathbb{R}^3$.



Two edges of weight 1; energy is the sum of squared differences.

Question 1

Consider the scalar initial-value problem, for $t > 0$,

$$y^{(3)} + \frac{1}{t}y'' - 2y' + y = e^t, \quad y(1) = 2, \quad y'(1) = -1, \quad y''(1) = 0.$$

A first-order state must contain enough information to determine all of its first derivatives. A system is autonomous when its right-hand side has no explicit dependence on the independent variable.

Tasks

1. Use $X = (y, y', y'')^T$ to write $X' = A(t)X + g(t)$ and the complete initial state.
2. Classify the system as linear or nonlinear, homogeneous or nonhomogeneous, and autonomous or nonautonomous. State its dimension and the largest coefficient-continuity interval containing $t = 1$.
3. Use instead $Z = (y'', y, y')^T$. Find the transformed system and initial state, explaining why reordering state variables changes neither the underlying problem nor the number of initial values.
4. Prove equivalence in both directions between the scalar equation and your first state system. Explain what the coefficient singularity does, and does not, establish about continuation through $t = 0$.

Question 1 – Solution

Strategy. Store successive derivatives as independent state coordinates, then recover their required relations from the first rows.

Step 1: Build the derivative chain. Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Solving for the highest derivative gives

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -1/t \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}, \quad X(1) = (2, -1, 0)^T.$$

In particular the last row is $x_3' = -x_1 + 2x_2 - x_3/t + e^t$.

Step 2: Separate the classifications. The three-dimensional system is linear: coefficients depend on t , not on the unknown state. It is nonhomogeneous because the forcing is nonzero, and nonautonomous because $1/t$ and e^t depend explicitly on time. The largest open coefficient-continuity interval containing 1 is $(0, \infty)$.

Step 3: Reorder the coordinates consistently. Here $z_1 = x_3$, $z_2 = x_1$, $z_3 = x_2$, so

$$Z' = \begin{pmatrix} -1/t & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Z + \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad Z(1) = (0, 2, -1)^T.$$

A fixed permutation is invertible. It relabels coordinates and initial data without adding or removing information; there are still three initial values.

Step 4: Check equivalence and the domain limitation. Any scalar solution supplies a state satisfying all three rows. Conversely, for a continuously differentiable state solution, the first two rows imply $x_2 = x_1'$ and $x_3 = x_1''$; the third then makes x_1 three times differentiable and gives the original equation and initial data. The standard linear-system theorem gives a unique solution on $(0, \infty)$. It does not apply through 0, where $1/t$ is undefined. This alone does not prove that a particular solution cannot have a smooth extension; any extension would require a separately specified equation at the singular point.

Question 2

Two masses move horizontally without friction. Their displacements u, v are measured to the right from equilibrium. Mass 1 has mass 1, is attached to a fixed left wall by a spring of stiffness 2, and is coupled to mass 2 by a spring of stiffness 3. Mass 2 has mass 2 and no wall spring. A force $f(t)$ acts to the right on mass 2. Use consistent mechanical units. Initially $(u, u', v, v') = (1, 0, 0, 0)$.

Tasks

1. Derive the two second-order force equations, checking the opposite signs of the coupling forces.
2. For $X = (u, u', v, v')^T$, give the first-order matrix system and initial state. Explain why (u, v) alone is insufficient.
3. When $f = 0$, differentiate $E = \frac{1}{2}(u')^2 + (v')^2 + u^2 + \frac{3}{2}(v - u)^2$ and prove conservation. For general f , find the exact power balance.
4. Find all equilibrium states for a constant force $f(t) = F$. Explain why equilibrium requires conditions on velocities as well as displacements.

Question 2 – Solution

Strategy. Apply Newton's law to each mass and use energy to check the coupling signs independently.

Step 1: Account for the spring forces. The wall spring exerts $-2u$ on mass 1. The coupling spring exerts $3(v - u)$ on mass 1 and its opposite on mass 2. Therefore

$$\boxed{u'' = -5u + 3v, \quad 2v'' = 3u - 3v + f(t).}$$

Dividing the second equation by its mass is essential.

Step 2: Include positions and velocities. Writing $p = u'$, $q = v'$ gives

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 3/2 & 0 & -3/2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t)/2 \end{pmatrix}, \quad X(0) = (1, 0, 0, 0)^T.$$

Equal positions can have different velocities, so positions alone do not determine the first derivatives. The four coordinates supply the complete state.

Step 3: Verify the work-energy balance. Differentiate the stated energy before substituting the equations:

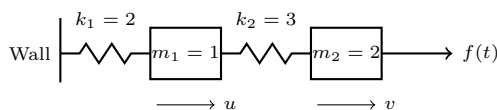
$$E' = p(-5u + 3v) + q(3u - 3v + f) + 2up + 3(v - u)(q - p).$$

Every spring term cancels, leaving $\boxed{E' = f(t)v'}$. Thus $f = 0$ conserves E ; for the stated initial data, $E(0) = 5/2$. The two kinetic terms reflect the different masses. The remaining terms are the nonnegative spring potential energies.

Step 4: Balance a constant force. An equilibrium has $p = q = 0$, $-5u + 3v = 0$, and $3u - 3v + F = 0$. Consequently $u = F/2$, $v = 5F/6$, and

$$\boxed{X_* = (F/2, 0, 5F/6, 0)^T.}$$

This is the unique equilibrium. Zero acceleration at one instant would not make a state constant if either velocity were nonzero.



Displacements from equilibrium; rightward positive.

Question 3

Two perfectly mixed tanks have constant volumes 100 L and 200 L. Fresh water enters tank 1 at 3 L/min. Tank 1 sends 5 L/min to tank 2; tank 2 returns 2 L/min to tank 1 and drains 3 L/min outside. Let x, y be the salt amounts in grams, with $x(0) = 100$, $y(0) = 0$. You may use the inward-boundary criterion: for a locally Lipschitz vector field, the nonnegative quadrant is forward invariant if its boundary derivatives point inward or are tangent.

Tasks

1. Check both volume balances and derive the first-order system for salt amounts. Identify the concentration used in each outflow term.
2. Write the amount system in matrix form. Then derive the system for concentrations $c_1 = x/100$, $c_2 = y/200$, including initial data.
3. Find the derivative of total salt and explain why the internal transfer rates cancel even though the volumes differ. Find every equilibrium.
4. Prove that nonnegative initial amounts remain nonnegative and that total salt cannot increase. Give explicit bounds for this initial state without solving the coupled system.

Question 3 – Solution

Strategy. Multiply each flow rate by the concentration in its source tank, then check the total balance.

Step 1: Balance volumes and salt separately. Tank 1 receives $3 + 2 = 5$ L/min and sends 5; tank 2 receives 5 and sends $2 + 3 = 5$. Their volumes are constant. The return concentration is $y/200$ and the forward concentration is $x/100$, giving

$$x' = -x/20 + y/100, \quad y' = x/20 - y/40.$$

Fresh water supplies no salt. Each coefficient has units min^{-1} .

Step 2: Change from amounts to concentrations. For $X = (x, y)^T$,

$$X' = \begin{pmatrix} -1/20 & 1/100 \\ 1/20 & -1/40 \end{pmatrix} X, \quad X(0) = (100, 0)^T.$$

Dividing each balance by its own tank volume gives

$$c' = \begin{pmatrix} -1/20 & 1/50 \\ 1/40 & -1/40 \end{pmatrix} c, \quad c(0) = (1, 0)^T.$$

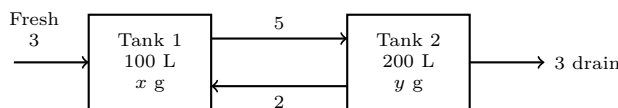
This is a constant invertible change of state; the two matrices need not match.

Step 3: Check the external balance and equilibria. Adding the amount equations gives

$$(x + y)' = -3y/200,$$

exactly the outside drain rate times its source concentration. At an equilibrium this forces $y = 0$, and then $y' = x/20 = 0$ forces $x = 0$. The only equilibrium is the salt-free state.

Step 4: Prove positivity and bounds. The field is linear and therefore locally Lipschitz. On $x = 0$, $y \geq 0$, we have $x' = y/100 \geq 0$; on $y = 0$, $x \geq 0$, we have $y' = x/20 \geq 0$. The given inward-boundary criterion therefore proves that the nonnegative quadrant is forward invariant. The constant linear system exists for all forward times, so the conclusion holds for every $t \geq 0$. Now the total balance implies $0 \leq x + y \leq 100$. Hence $0 \leq x, y \leq 100$ g, $0 \leq c_1 \leq 1$ and $0 \leq c_2 \leq 1/2$ g/L. These are guaranteed bounds, not claims that every bound is attained.



All flow rates in L/min; arrows show actual transfer directions.

Question 4

Consider the first-order system on the real time axis

$$x' = x + y, \quad y' = 2x - y + t, \quad x(0) = 1, \quad y(0) = 2.$$

An equivalent scalar equation must preserve both the dynamics and the initial information; an eliminated coordinate must be recoverable.

Tasks

1. Eliminate y to derive a second-order equation for x . Find the two corresponding initial values.
2. Prove the converse: from any solution of your scalar IVP, reconstruct y and verify both original equations and both original initial values.
3. Instead eliminate x to obtain a scalar IVP for y and a reconstruction formula for x . Verify the forcing signs.
4. A calculation uses the correct scalar equation for x but imposes $x(0) = 1$, $x'(0) = 2$. Identify exactly which original initial condition this changes. Explain why prescribing both $x'(0) = 2$ and the original $y(0) = 2$ is inconsistent.

Question 4 – Solution

Strategy. Differentiate one row, substitute the other, and retain the algebraic formula that recovers the missing coordinate.

Step 1: Eliminate y while retaining initial information. The first row gives $y = x' - x$. Differentiating it and using the second gives $x'' = x' + 2x - y + t = 3x + t$. Thus

$$x'' - 3x = t, \quad x(0) = 1, \quad x'(0) = 3.$$

The initial slope is $x(0) + y(0)$, not the initial value of y alone.

Step 2: Reconstruct and verify. Given a twice differentiable solution of this scalar IVP, define $y = x' - x$. Then $x' = x + y$ identically, and

$$y' = x'' - x' = 3x + t - x' = 2x - (x' - x) + t = 2x - y + t.$$

Also $y(0) = 3 - 1 = 2$. This proves the converse as well as the forward implication: no extra scalar solutions remain after reconstruction and data matching.

Step 3: Eliminate in the other direction. The second row gives $x = (y' + y - t)/2$. Differentiating that row yields

$$y'' = 2x' - y' + 1 = 2x + 2y - y' + 1 = 3y + 1 - t.$$

Therefore the equivalent alternative is

$$y'' - 3y = 1 - t, \quad y(0) = 2, \quad y'(0) = 0, \quad x = \frac{y' + y - t}{2}.$$

Indeed $x' = (y'' + y' - 1)/2 = (3y + y' - t)/2 = x + y$; the reconstruction also gives $x(0) = 1$ and the second row directly.

Step 4: Diagnose the mismatched data. If $x(0) = 1$ and $x'(0) = 2$, reconstruction forces $y(0) = 2 - 1 = 1$. The calculation solves a different original IVP, with initial state $(1, 1)$. Insisting simultaneously on $y(0) = 2$ would require $x'(0) = 1 + 2 = 3$, contradicting the imposed slope. Correct differential equations alone do not make two initial-value problems equivalent.

Question 5

Let

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X(0) = (0, 1)^T,$$

and introduce a time-dependent coordinate system $X = P(t)Z$, where

$$P(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The physical state X and its coordinate column Z describe the same vector using different bases at different times.

Tasks

1. Derive the differential equation for Z by differentiating $X = PZ$. Explain why the constant-basis formula $P^{-1}AP$ is insufficient.
2. Compute the coefficient matrix and initial state for Z . Classify the transformed system as linear, homogeneous, and autonomous or nonautonomous.
3. Verify the supplied physical trajectory $X(t) = (\sin t, \cos t)^T$. Obtain $Z(t)$ and check both transformed equations directly.
4. Compare $X'(0)$ with $Z'(0)$. Explain how a zero instantaneous coordinate derivative can coexist with a changing physical vector.

Question 5 – Solution

Strategy. A moving basis contributes its own derivative; keep that contribution before multiplying by the inverse.

Step 1: Differentiate the whole change of coordinates. Since $X' = P'Z + PZ' = APZ$, invertibility of P gives

$$Z' = P^{-1}(AP - P')Z.$$

Dropping P' would treat a moving basis as fixed and generally produce incorrect coordinate derivatives. Here $\det P = 1$ for all real t .

Step 2: Compute the transformed system. The matrices are

$$P^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad AP - P' = \begin{pmatrix} 0 & 0 \\ -1 & -t \end{pmatrix}.$$

Consequently

$$Z' = \begin{pmatrix} t & t^2 \\ -1 & -t \end{pmatrix} Z, \quad Z(0) = (0, 1)^T.$$

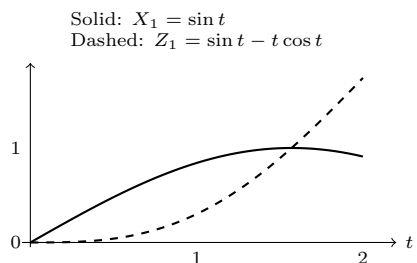
This system is linear and homogeneous, but nonautonomous. Explicit time dependence can arise entirely from a time-dependent choice of coordinates.

Step 3: Check the supplied trajectory in both descriptions. For $X = (\sin t, \cos t)^T$, $X' = (\cos t, -\sin t)^T = AX$, with the stated data. Multiplication by P^{-1} gives

$$Z = (\sin t - t \cos t, \cos t)^T.$$

Its derivative is $(t \sin t, -\sin t)^T$. The transformed right-hand side is also $(t(\sin t - t \cos t) + t^2 \cos t, -(\sin t - t \cos t) - t \cos t)^T$, which simplifies to the same vector.

Step 4: Interpret the instantaneous difference. At 0, $X' = (1, 0)^T$ but $Z' = (0, 0)^T$. The identity $X' = P'Z + PZ'$ explains this: $P'(0)Z(0) = (1, 0)^T$. The basis motion accounts for the entire instantaneous change. The figure compares the first components; the coordinate curve starts with a horizontal tangent while the physical component has slope 1.



Question 6

Consider the nonautonomous system

$$x' = tx, \quad y' = -y + \sin t, \quad x(2) = 3, \quad y(2) = -1.$$

A common way to remove explicit time is to add a clock as another state. In an autonomous system, the right-hand side depends only on the state.

Tasks

1. Write this as a two-dimensional linear system $X' = A(t)X + g(t)$ and classify homogeneity and autonomy.
2. Introduce a new independent variable τ and a clock $s(\tau)$ satisfying $s' = 1$. Give an autonomous three-dimensional system with initial data at $\tau = 0$ that reproduces the stated IVP.
3. Prove the correspondence between solutions of the two systems, specifying exactly how t and τ are related. Explain why the clock initial value is required.
4. Classify the augmented system as linear or nonlinear and find its equilibria. Explain why autonomous does not mean linear, and why adding a clock can eliminate all equilibria.

Question 6 – Solution

Strategy. Turn the old independent variable into a dependent variable whose rate is fixed and whose initial value records the time origin.

Step 1: Identify the original structure. With $X = (x, y)^T$,

$$X' = \begin{pmatrix} t & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}, \quad X(2) = (3, -1)^T.$$

It is linear, nonhomogeneous and nonautonomous. Time-dependent coefficients do not violate linearity in the dependent variables.

Step 2: Add and initialize the clock. Write $U(\tau) = (s(\tau), u(\tau), v(\tau))^T$. The autonomous system is

$$\boxed{s' = 1, \quad u' = su, \quad v' = -v + \sin s, \quad U(0) = (2, 3, -1)^T.}$$

Here every prime denotes differentiation with respect to τ . There is no explicit τ on the right-hand side.

Step 3: Prove the change-of-clock correspondence. The first row forces $s(\tau) = 2 + \tau$. Given an original solution, define $u(\tau) = x(2 + \tau)$ and $v(\tau) = y(2 + \tau)$. The chain rule verifies the last two rows and the initial data. Conversely, define $x(t) = u(t - 2)$ and $y(t) = v(t - 2)$ from an augmented solution with $s(0) = 2$; substitution recovers the original system. Intervals shift by the same amount. Choosing $s(0) = 0$ would instead sample the coefficients at $t = \tau$, changing the stated starting time.

Step 4: Distinguish autonomy, linearity and equilibrium. The augmented system is nonlinear: su multiplies two dependent variables, and $\sin s$ is a nonlinear function of a dependent variable. It cannot be classified as a homogeneous linear system merely because the original equations were linear. An equilibrium would require all three right-hand sides to vanish, but $s' = 1$ never vanishes. Thus the augmented autonomous system has no equilibrium. A clock keeps moving even if some physical state components happen to be constant.

Question 7

For a real constant κ , consider the implicit derivative equations

$$x' + y' = -x, \quad x' + \kappa y' = y.$$

A classical solution is a continuously differentiable pair satisfying both relations throughout an interval. An algebraically consistent derivative at one state is not necessarily part of a solution through that state.

Tasks

1. Write $M_\kappa X' = BX$ and determine exactly when it can be solved for a unique derivative at every state. Find the explicit system in that case.
2. At $\kappa = 1$, determine the states at which the two equations are algebraically consistent for a derivative. Describe the possible derivatives at such a state.
3. Still at $\kappa = 1$, find every classical solution on an interval of positive length. Explain why most instantaneously consistent states cannot be used as initial data.
4. For $\kappa \neq 1$ and initial state $(x, y) = (1, -1)$, compute the initial derivative and the derivative of $x + y$. Does a finite, parameter-independent initial velocity imply that a solution exists through the same state at $\kappa = 1$?

Question 7 – Solution

Strategy. Separate invertibility of the derivative matrix from the extra consistency conditions imposed when its rank drops.

Step 1: Solve when the derivative matrix is invertible. Here

$$M_\kappa = \begin{pmatrix} 1 & 1 \\ 1 & \kappa \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det M_\kappa = \kappa - 1.$$

For $\kappa \neq 1$, inversion yields

$$X' = \frac{1}{\kappa - 1} \begin{pmatrix} -\kappa & -1 \\ 1 & 1 \end{pmatrix} X.$$

This constant linear system has a unique solution for every initial state.

Step 2: Test instantaneous algebraic consistency. At $\kappa = 1$, the left sides coincide, so the right sides must satisfy $y = -x$. At a state $(a, -a)$, the derivative pairs (p, q) satisfy $p + q = -a$, an entire line of possibilities. Outside $y = -x$, no derivative can satisfy both equations even at that instant.

Step 3: Enforce the constraint along a solution. A classical solution must have $y(t) = -x(t)$ at every time. Differentiating this identity gives $x' + y' = 0$. The first equation then forces $x = 0$, and the constraint forces $y = 0$. Therefore

$$X(t) \equiv (0, 0)^T \text{ is the only classical solution at } \kappa = 1.$$

It indeed satisfies both equations. Nonzero points on the consistency line admit instantaneous derivatives but no solution interval through them. Even at the origin, most algebraic derivative choices cannot persist.

Step 4: Check the limiting initial velocity. Substituting $(1, -1)$ into the explicit system gives $x' = (-\kappa + 1)/(\kappa - 1) = -1$ and $y' = 0$ for every $\kappa \neq 1$. Thus $(x + y)' = -1$, although initially $x + y = 0$. This velocity does not keep the state on the constraint line required at $\kappa = 1$. Its finiteness does not imply a limiting solution through $(1, -1)$; Step 3 proves that no such solution exists at that parameter.

Question 8

For $t \geq 0$, consider

$$x' = 2\sqrt{|x|}, \quad y' = -y, \quad x(0) = 0, \quad y(0) = 1.$$

You may use the local uniqueness theorem: continuity of a vector field and local Lipschitz continuity in the state guarantee local uniqueness. Failure of its hypotheses alone does not prove nonuniqueness.

Tasks

1. Check continuity and test the local Lipschitz condition at states with $x = 0$. Explain exactly which uniqueness guarantee is unavailable.
2. For any waiting time $a \geq 0$, construct a solution that has $x = 0$ until time a and then becomes positive. Verify the derivative at the joining time as well as both differential equations.
3. Find y and compare the initial state and initial derivative of the waiting-time solutions. Include the solution that never leaves $x = 0$.
4. If an additional observation says $x(1) = 0$, does that restore uniqueness for later times? Exhibit two distinct solutions satisfying all the data and explain what this example says about a complete initial state.

Question 8 – Solution

Strategy. Prove nonuniqueness by constructing different differentiable solutions; do not infer it solely from a theorem's failure.

Step 1: Test the regularity hypothesis. The field $F(x, y) = (2\sqrt{|x|}, -y)^T$ is continuous everywhere. For $h > 0$, the change in its first component between (h, y) and $(0, y)$, divided by the state distance, is $2\sqrt{h}/h = 2/\sqrt{h}$. This is unbounded as $h \downarrow 0$, so no local Lipschitz constant exists near $x = 0$. The quoted uniqueness theorem cannot guarantee uniqueness there.

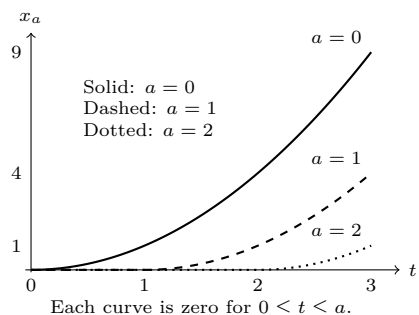
Step 2: Construct and check the waiting family. For each $a \geq 0$, define

$$x_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ (t-a)^2, & t > a. \end{cases}$$

Before a , both x'_a and $2\sqrt{|x_a|}$ are zero. After a , both are $2(t-a)$. At a , the left and right derivatives are zero, so the function is continuously differentiable and satisfies the equation there too. At $a = 0$, the same check uses the right derivative at the endpoint.

Step 3: Check the common data. The second scalar equation uniquely gives $y(t) = e^{-t}$. Every pair (x_a, e^{-t}) has initial state $(0, 1)$ and initial derivative $(0, -1)$. The pair $(0, e^{-t})$ is another solution, corresponding to waiting forever. Different finite waiting times produce different first components, despite identical initial states and slopes.

Step 4: Test whether one extra observation selects a solution. Every $a \geq 1$ gives $x_a(1) = 0$, as does the never-departing solution. For example, $a = 1$ and $a = 2$ satisfy all the data, but at $t = 3/2$ their first components are $1/4$ and 0 . Uniqueness is still absent. A complete state specifies all initial coordinates; unique evolution also requires appropriate properties of the differential equations. The figure shows three members of this family, with time increasing rightward.



Question 9

Three nonnegative dimensionless amounts undergo internal transfers: amount moves from compartment 1 to 2 at rate xy , and from 2 to 3 at rate $2y$. There are no external gains or losses. Their initial amounts are $(x, y, z) = (1, 1, 0)$. The rate laws are valid for nonnegative states.

Tasks

1. Derive the vector differential equation and classify it as autonomous and linear or nonlinear. Explain why a state-dependent coefficient does not make this a linear system.
2. Find the total-amount balance. Show that nonnegative initial states remain nonnegative and use this to bound every component for the stated data.
3. For the stated initial data, prove $x(t) > 0$ at every finite forward time. Derive a second conserved expression involving $x + y$ and $\ln x$.
4. Use the two conserved expressions to reduce the dynamics to one scalar equation for x , with formulas recovering y, z . State the restrictions that make the reduction meaningful; no explicit solution is required.

Question 9 – Solution

Strategy. Each internal transfer leaves one compartment and enters another at the same rate; this exposes conserved quantities before any solution is sought.

Step 1: Translate the transfers. The balances are

$$\boxed{x' = -xy, \quad y' = xy - 2y, \quad z' = 2y, \quad (x, y, z)(0) = (1, 1, 0).}$$

The system is autonomous, but nonlinear because of xy . Writing $x' = -yx$ does not make $-y$ a prescribed coefficient: y is another unknown state coordinate, not a known function of time.

Step 2: Prove conservation and nonnegativity. Adding the rows gives $(x + y + z)' = 0$, hence $x + y + z = 2$. Along any solution,

$$x(t) = x(0)e^{-\int_0^t y(s) ds}, \quad y(t) = y(0)e^{\int_0^t (x(s)-2) ds}, \quad z(t) = z(0) + 2 \int_0^t y(s) ds.$$

These formulas prove nonnegativity for nonnegative data on every forward existence interval. Thus $0 \leq x, y, z \leq 2$. The polynomial field is locally Lipschitz; boundedness prevents finite-time escape, so the solution continues for all $t \geq 0$.

Step 3: Obtain a second conserved quantity. With $x(0) = 1$ and $0 \leq y \leq 2$, the exponential formula gives $x(t) \geq e^{-2t} > 0$ at finite times. Therefore $\ln x$ is legitimate, and

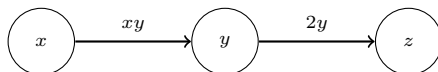
$$(x + y - 2 \ln x)' = -2y - 2(-xy)/x = 0.$$

Its initial value is 2, so $\boxed{x + y - 2 \ln x = 2}$.

Step 4: Reduce and recover the state. The two conservation laws give

$$\boxed{y = 2 - x + 2 \ln x, \quad z = -2 \ln x, \quad x' = -x(2 - x + 2 \ln x), \quad x(0) = 1.}$$

The physical branch has $x > 0$, $y \geq 0$, $z \geq 0$, in particular $x \leq 1$. Conversely, on this branch differentiation gives $y' = (x - 2)y$ and $z' = 2y$, proving recovery of the original system. Taking logarithms at $x = 0$ would be invalid; that boundary is excluded from this reduction.



Internal transfers only: $x + y + z = 2$ for the given data.

Question 10

An unknown autonomous affine system has the form $X' = AX + b$ in $\mathbb{R}^2$, where A, b are constant. Independent measurements of its velocity field F give

$$F(0, 0) = (1, -1)^T, \quad F(1, 0) = (-1, 2)^T, \quad F(0, 1) = (2, -3)^T.$$

An affine system is written as a nonhomogeneous linear differential system; its constant forcing need not vanish.

Tasks

1. Recover A and b , and prove uniqueness within the class of affine systems. Explain why the measured inputs are states rather than initial derivatives of a single trajectory.
2. Find every equilibrium and translate it to the origin. Write the differential equation in the translated state and explain what happens to the constant forcing.
3. A fourth measurement reports $F(1, 1) = (0, 1)^T$. Is it compatible with the first three under the affine assumption? Quantify the discrepancy.
4. Construct a smooth nonlinear vector field fitting all four measurements. Explain why the first three determine an affine model uniquely but do not determine an unrestricted vector field.

Question 10 – Solution

Strategy. Subtract the velocity at the origin to isolate the matrix columns, then test the model assumption with new data.

Step 1: Recover the affine coefficients. At the origin, $b = F(0, 0) = (1, -1)^T$. The differences $F(1, 0) - b = (-2, 3)^T$ and $F(0, 1) - b = (1, -2)^T$ are the first and second columns of A . Therefore

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \quad b = (1, -1)^T.$$

These columns and b are forced, proving uniqueness within the affine class. The measurements prescribe the field at three different states; they are not three independently assignable derivatives at one initial state.

Step 2: Translate the unique equilibrium. The equation $AX_* + b = 0$ reads $-2x + y + 1 = 0$ and $3x - 2y - 1 = 0$. It gives $X_* = (1, 1)^T$; uniqueness also follows from $\det A = 1$. For $Z = X - X_*$, the translation is constant, and

$$Z' = A(Z + X_*) + b = AZ, \quad \boxed{Z' = AZ.}$$

The forcing vanishes because the translation is by an equilibrium, not because every translation removes every forcing term.

Step 3: Check the fourth report. The recovered model predicts $F(1, 1) = (0, 0)^T$, while the report is $(0, 1)^T$. The reported-minus-predicted discrepancy is $(0, 1)^T$. No affine system fits all four measurements, since the first three already fix its coefficients. This conclusion tests the affine model assumption.

Step 4: Exhibit a nonlinear alternative. Define

$$\boxed{\tilde{F}(x, y) = (-2x + y + 1, 3x - 2y - 1 + xy)^T.}$$

The extra term xy vanishes at the first three states and equals 1 at $(1, 1)$, so all four measurements are met. This polynomial field is smooth and nonlinear. In it, $(1, 1)$ is not an equilibrium. More generally, fields obtained by adding $(0, cxy)^T$ all fit the first three data for any real c . Finite measurements uniquely identify this affine class, but leave unrestricted nonlinear models undetermined.

Question 1

For $X' = AX$, with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, consider

$$U(t) = e^t(1, 0)^T, \quad V(t) = e^t(t, 1)^T, \quad W(t) = e^t(1, t)^T.$$

A vector solution must satisfy both rows at every time in its interval. A fundamental matrix has solution columns that form a basis at each time.

Tasks

1. Compute the residual $F' - AF$ for each candidate $F = U, V, W$. Decide which are solutions on a nonempty open interval.
2. Use the valid candidates to form a fundamental matrix. Prove that every solution is their constant linear combination, rather than merely checking that combinations work.
3. Solve the IVP $X(0) = (2, -1)^T$ and verify both the equations and initial data.
4. For this IVP, find the first zero of its first component for $t \geq 0$. Determine the full state and derivative there, and explain why a zero component does not mean a zero solution.

Question 1 – Solution

Strategy. Check vector residuals first; then solve the triangular equations to establish completeness.

Step 1: Test all candidates. Direct differentiation gives $U' - AU = 0$ and $V' - AV = 0$. For the third candidate,

$$W' = e^t(1, t+1)^T, \quad AW = e^t(1+t, t)^T, \quad \boxed{W' - AW = e^t(-t, 1)^T}.$$

Its second residual component never vanishes, so W is not a solution on any nonempty open interval.

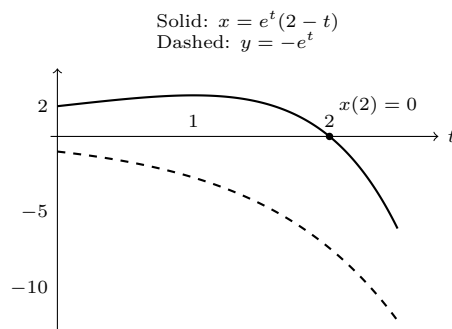
Step 2: Prove that the valid family is complete. The matrix $\Phi = (U \ V) = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has determinant $e^{2t} > 0$. More explicitly, the second equation gives $y = c_2 e^t$. Then $(e^{-t}x)' = c_2$, so

$$\boxed{X = e^t(c_1 + c_2 t, c_2)^T = c_1 U + c_2 V}.$$

This derivation reaches every solution and introduces exactly two arbitrary constants. The family is valid on the whole real axis.

Step 3: Select and verify the initial-value solution. At 0, the constants are $c_1 = 2$, $c_2 = -1$, hence $\boxed{X = e^t(2-t, -1)^T}$. Its derivative is $e^t(1-t, -1)^T$, exactly AX , and its initial state is $(2, -1)^T$ as required.

Step 4: Interpret a component zero. Since $e^t \neq 0$, the first component first vanishes at $t = 2$. There $X(2) = (0, -e^2)^T$ and $X'(2) = (-e^2, -e^2)^T$. The state is nonzero and continues moving; the first component crosses zero with negative slope. For a homogeneous linear system with continuous coefficients, only a full zero state at a time would force the identically zero solution by uniqueness. The plot displays both components.



Question 2

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}.$$

A principal fundamental matrix at 0 is a fundamental matrix Y satisfying $Y(0) = I$. For real a , also define $R_a = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $\Psi_a = \Phi R_a$.

Tasks

1. Verify $\Phi' = A\Phi$ and calculate its determinant. Explain why its columns describe a complete homogeneous solution family.
2. Construct the principal fundamental matrix $Y(t)$ and verify $Y(0) = I$. Express its entries using hyperbolic functions or exponentials.
3. For arbitrary initial data $X(0) = (p, q)^T$, find the constants in $X = \Phi c$ and the same solution in the form $X = YX(0)$.
4. Classify when Ψ_a is a fundamental matrix. At $a = 1$ and $a = -1$, determine exactly which initial states can still be represented by $X = \Psi_a d$, even though every column remains a solution.

Question 2 – Solution

Strategy. Distinguish solution columns from a complete basis, and normalize by the matrix at the initial time.

Step 1: Verify the columns and independence. Differentiating each exponential column gives $\Phi' = A\Phi$. Its determinant is -2 , never zero. Every initial vector can therefore be represented by $\Phi(0)c$; uniqueness of the linear IVP proves that the resulting constant combination supplies every solution.

Step 2: Normalize at the initial time. Since $\Phi(0)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$$Y(t) = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

Here $\cosh t = (e^t + e^{-t})/2$, $\sinh t = (e^t - e^{-t})/2$. Thus $Y(0) = I$ and $Y' = AY$ by differentiation or constant right multiplication.

Step 3: Recover arbitrary data. Solving $\Phi(0)c = (p, q)^T$ gives

$$c_1 = (p + q)/2, \quad c_2 = (p - q)/2, \quad X = (p \cosh t + q \sinh t, \quad p \sinh t + q \cosh t)^T.$$

This equals $Y(t)(p, q)^T$. The constants in a nonnormalized basis are not generally the initial state coordinates themselves.

Step 4: Detect a collapsed family. Constant combinations remain solutions, but $\det \Psi_a = -2(1 - a^2)$, so the family is fundamental exactly when $a \neq \pm 1$. At $a = 1$, both columns are $U + V$, whose value at 0 is $(2, 0)^T$; exactly the initial states $(p, 0)^T$ are representable. At $a = -1$, the columns are $U - V$ and its negative, with initial value $(0, 2)^T$; exactly $(0, q)^T$ are representable. Here U, V denote the columns of Φ . Neither singular case supplies all initial states, despite having two displayed solution columns.

Question 3

Consider the continuous time-dependent linear system

$$X' = A(t)X, \quad A(t) = \begin{pmatrix} 2t & 1 \\ 0 & -t \end{pmatrix}.$$

For two solution columns $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, write $W = u_1v_2 - u_2v_1$. Do not assume a determinant evolution formula without deriving it.

Tasks

1. Differentiate W and use the original system to derive its scalar differential equation.
2. Find $W(t)$ when the matrix of columns equals I at $t = 0$. Is the claim $W = e^{t^2}$ consistent with the system?
3. Construct the principal fundamental matrix at 0. A definite integral is an acceptable exact entry; verify the initial matrix and both column equations.
4. Find the solution with initial state $(0, 1)^T$. Explain why growth of the fundamental determinant for $t > 0$ does not imply growth of every solution component.

Question 3 – Solution

Strategy. Derive the determinant law by the product rule, and construct the columns by solving the triangular equations.

Step 1: Differentiate the determinant. Substitution gives

$$W' = (2tu_1 + u_2)v_2 + u_1(-tv_2) - (-tu_2)v_1 - u_2(2tv_1 + v_2) = tW.$$

The off-diagonal terms cancel. This is the trace law here, since $\text{tr } A = t$, obtained directly rather than assumed.

Step 2: Apply the determinant's initial value. At 0, $W(0) = 1$, and solving $W' = tW$ gives $W(t) = e^{t^2/2}$. The proposed e^{t^2} has derivative $2te^{t^2}$ and fails the required equation except at the isolated time 0; matching the initial value is insufficient.

Step 3: Construct and verify the matrix. Put $J(t) = \int_0^t e^{-3s^2/2} ds$. The second equation gives $y = c_2 e^{-t^2/2}$, and the first gives $(e^{-t^2} x)' = c_2 e^{-3t^2/2}$. Therefore

$$\Phi(t) = \begin{pmatrix} e^{t^2} & e^{t^2} J(t) \\ 0 & e^{-t^2/2} \end{pmatrix}.$$

Since $J(0) = 0$, $\Phi(0) = I$. The derivative of its upper-right entry is $2te^{t^2} J + e^{-t^2/2}$, precisely the required first-row expression. The remaining entries satisfy their rows directly. Its determinant agrees with Step 2 and never vanishes, so this is a principal fundamental matrix.

Step 4: Interpret the selected solution. The initial vector $(0, 1)^T$ selects the second column: $X = (e^{t^2} J(t), e^{-t^2/2})^T$. Its second component strictly decreases for $t > 0$, even while W grows. A determinant concerns a pair of independent columns, not the separate monotonicity of each component. The nonzero determinant guarantees a complete independent family throughout the real axis.

Question 4

For the nonautonomous system

$$x' = 2ty, \quad y' = 0,$$

define the transition matrix $T(t, s)$ by $X(t) = T(t, s)X(s)$ for arbitrary real starting time s . It must satisfy $T(s, s) = I$.

Tasks

1. Derive $T(t, s)$ from the component equations and verify its differential equation and normalization.
2. Prove $T(t, r)T(r, s) = T(t, s)$ and determine the inverse of $T(t, s)$. Explain the role of the intermediate time.
3. Let $Y(t) = T(t, 0)$. Determine whether $Y(t + s) = Y(t)Y(s)$ holds for all real t, s , and give a concrete counterexample if it fails.
4. Solve the IVP $X(-1) = (2, 3)^T$. Compare $X(1)$ with $X(-1)$ and compute $X(0)$. Does either equality imply that the solution is constant?

Question 4 – Solution

Strategy. Keep both endpoint times: a time-dependent equation need not evolve according only to elapsed time.

Step 1: Integrate from the actual starting time. The second component is $y(t) = y(s)$, and $x(t) = x(s) + y(s) \int_s^t 2u \, du$. Hence

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Its derivative with respect to t is $\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}$, which equals $A(t)T(t, s)$, and $T(s, s) = I$.

Step 2: Check composition and inversion. Multiplying two such shears adds their upper-right entries: $(t^2 - r^2) + (r^2 - s^2) = t^2 - s^2$. Thus

$$T(t, r)T(r, s) = T(t, s), \quad T(t, s)^{-1} = T(s, t).$$

The intermediate time cancels only because it is the end of the first journey and the start of the second, in the correct order.

Step 3: Test the one-parameter shortcut. The upper-right entry of $Y(t)Y(s)$ is $t^2 + s^2$, whereas that of $Y(t+s)$ is $(t+s)^2 = t^2 + s^2 + 2ts$. They agree only when $ts = 0$, not for all times. For example, $Y(1)Y(1)$ has upper-right entry 2, while $Y(2)$ has entry 4. This does not contradict the two-time composition identity.

Step 4: Apply the transition to the IVP. The solution is

$$X(t) = T(t, -1)(2, 3)^T = (3t^2 - 1, 3)^T.$$

It returns to $(2, 3)^T$ at $t = 1$, and $X'(0) = (0, 0)^T$. Nevertheless $X' = (6t, 0)^T$ is nonzero away from 0, so the solution is not constant. In a nonautonomous system, a repeated state at different times or a zero derivative at one instant does not force stationary evolution.

Question 5

Consider

$$X' = AX + f(t), \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad f(t) = (1, 2e^{-t})^T.$$

The proposed particular solution is $P(t) = (1, 2e^{-t})^T$. Distinguish combinations of homogeneous solutions from combinations of solutions of this fixed nonhomogeneous system.

Tasks

1. Verify P , and prove that the full solution family is $P + c_1(e^{-t}, 0)^T + c_2(0, e^{-2t})^T$.
2. If U, V both solve the fixed forced system, determine exactly when $\alpha U + \beta V$ also solves it. Prove the condition for real constants α, β .
3. Solve the IVP $X(0) = (0, 0)^T$ and check its initial derivative against the original right-hand side.
4. For $t \geq 0$, find the maximum of the second component of this IVP and its time. Find the limiting state and decide whether that limiting vector, held constant, is itself a solution of the forced system.

Question 5 – Solution

Strategy. Subtract one particular solution to expose the homogeneous family; keep the forcing when testing superposition.

Step 1: Verify and complete the affine family. Here $P' = (0, -2e^{-t})^T = AP + f$. For any other solution X , the difference $H = X - P$ satisfies $H' = AH$, so $H = (c_1e^{-t}, c_2e^{-2t})^T$. Conversely each such sum solves the forced equation. Thus the proposed family is complete.

Step 2: Test a weighted combination. For $Z = \alpha U + \beta V$,

$$Z' - AZ = (\alpha + \beta)f(t).$$

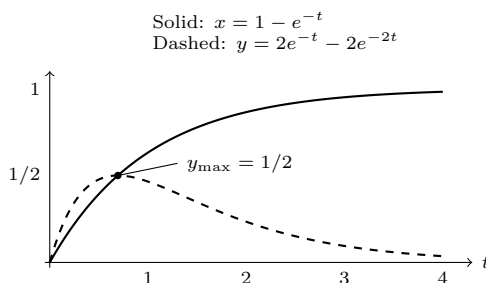
Because the first component of f is always 1, equality to f holds exactly when $\boxed{\alpha + \beta = 1}$. In particular an average is a solution, while a sum generally is not. The difference of two forced solutions solves the homogeneous equation.

Step 3: Impose the zero initial state. The constants are $c_1 = -1$, $c_2 = -2$, yielding

$$\boxed{X(t) = (1 - e^{-t}, 2e^{-t} - 2e^{-2t})^T}.$$

At 0 this is zero, but $X'(0) = (1, 2)^T = f(0)$. A zero initial state therefore does not imply a zero solution in this nonhomogeneous problem.

Step 4: Locate the peak and interpret the limit. For $y = 2e^{-t}(1 - e^{-t})$, $y' = 2e^{-t}(2e^{-t} - 1)$. It is positive before $\ln 2$ and negative after, so the unique maximum on $t \geq 0$ is $\boxed{y(\ln 2) = 1/2}$. The state tends to $(1, 0)^T$. However the constant vector $(1, 0)^T$ has zero derivative while $A(1, 0)^T + f(t) = (0, 2e^{-t})^T \neq 0$ at finite times. A limiting state need not itself be a constant solution of a time-dependent forced equation. The plot marks the actual second-component peak.



Question 6

For $X' = AX$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, consider

$$U(t) = (\cos t, -\sin t)^T, \quad V(t) = (\sin t, \cos t)^T.$$

You may use uniqueness of linear initial-value problems with continuous coefficients. The independent variable is real time.

Tasks

1. Verify both solutions and state their initial vectors.
2. Find all real times when their first components agree. At those times compare their second components and decide whether the full states coincide.
3. Prove that two solutions of the same continuous linear system agreeing in their full state at one time must agree throughout their common interval. Apply this to the zero solution of a homogeneous system.
4. Find a constant time shift relating V to U . Explain why identical states at different times are compatible with your uniqueness statement.

Question 6 – Solution

Strategy. Keep equality of one coordinate, equality of full states at the same time, and a time-shift relation separate.

Step 1: Verify the two vector equations. Differentiation gives $U' = (-\sin t, -\cos t)^T = AU$ and $V' = (\cos t, -\sin t)^T = AV$. Their initial vectors are $U(0) = (1, 0)^T$ and $V(0) = (0, 1)^T$.

Step 2: Find the coordinate coincidences. The equation $\cos t = \sin t$ holds exactly at

$$t = \pi/4 + k\pi, \quad k \in \mathbb{Z}.$$

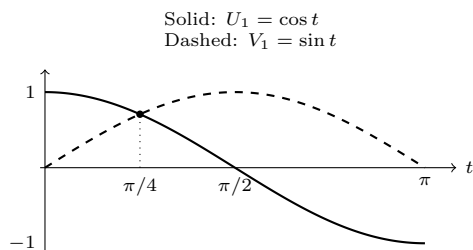
At such a time the common first component is $(-1)^k/\sqrt{2}$. The second components are its negative for U and its positive for V , so they differ. More generally,

$$\|U(t) - V(t)\|^2 = (\cos t - \sin t)^2 + (-\sin t - \cos t)^2 = 2.$$

The full states never agree at the same time. The figure displays first components only, and marks one coordinate coincidence.

Step 3: State what uniqueness actually implies. Suppose two solutions share the full vector $X(t_0) = x_0$. They solve the same IVP at t_0 , so uniqueness makes them identical on their common interval. Equivalently their difference solves the homogeneous equation with zero initial data and must vanish throughout that interval. Taking one solution to be zero proves that a homogeneous solution reaching the full zero state cannot later depart from it.

Step 4: Check the shifted identity. Trigonometric identities give $V(t) = U(t - \pi/2)$. This compares states at different time arguments; it does not assert $U(t) = V(t)$. An autonomous system permits translating a solution in time, and the shifted function normally has different initial data at time 0. Thus the shift creates no conflict with same-time IVP uniqueness.



Question 7

For a real parameter a , solve the nonlinear IVP

$$x' = x^2, \quad y' = -y, \quad x(0) = a, \quad y(0) = 1.$$

A maximal interval means the largest connected open time interval containing 0 on which the classical solution exists. The vector field is smooth.

Tasks

1. Derive the solution for $a \neq 0$, and handle $a = 0$ without dividing by the zero solution.
2. Find the maximal interval for each sign of a . Explain why smoothness of the field guarantees local uniqueness but not necessarily existence for all real times.
3. At $a = 1$, the rational formula is finite at $t = 2$. Does that value belong to a continuation of the stated IVP through $t = 1$? Justify your answer.
4. Compare $a = 0$ with $a = \varepsilon > 0$. Prove a small-data bound on any fixed $[0, T]$ when $\varepsilon T \leq 1/2$, and evaluate the difference at $t = 1/\varepsilon - 1$ for $0 < \varepsilon < 1$. Explain why the two conclusions are consistent.

Question 7 – Solution

Strategy. Derive explicit solutions, then distinguish the algebraic formula's domain from the connected interval of an initial-value solution.

Step 1: Solve and verify, retaining the zero case. For $a \neq 0$, separation gives $1/x = 1/a - t$, hence

$$X_a(t) = \left(\frac{a}{1-at}, e^{-t} \right)^T.$$

The first derivative is $a^2/(1-at)^2 = x^2$, and the data match at 0. For $a = 0$, the solution is $X_0 = (0, e^{-t})^T$; local uniqueness of the smooth field excludes other departures from $x = 0$.

Step 2: Find the maximal connected intervals. The first component blows up at $1/a$ when $a \neq 0$. Thus

$$a > 0 : (-\infty, 1/a); \quad a = 0 : \mathbb{R}; \quad a < 0 : (1/a, \infty).$$

The exponential component introduces no further restriction. A finite-valued continuous extension across the pole is impossible. Smoothness is a local regularity condition and does not prevent unbounded growth in finite time.

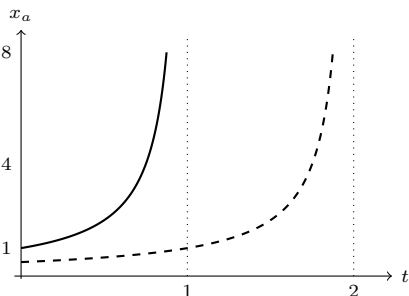
Step 3: Reject a disconnected continuation. At $a = 1$, the formula gives $x(2) = -1$, but the IVP through 0 exists only for $t < 1$. Its first component tends to $+\infty$ as $t \uparrow 1$. The branch for $t > 1$ solves the differential equation separately; it cannot be joined into a classical solution through the intervening pole.

Step 4: Compare finite-time and long-time sensitivity. For $0 \leq t \leq T$ with $\varepsilon T \leq 1/2$,

$$\|X_\varepsilon(t) - X_0(t)\| = \frac{\varepsilon}{1 - \varepsilon t} \leq 2\varepsilon.$$

But at $t = 1/\varepsilon - 1$, still before the pole, the difference equals 1. That comparison time tends to infinity as $\varepsilon \downarrow 0$. There is no contradiction: closeness on each fixed finite interval does not give a uniform bound over arbitrarily long, parameter-dependent times. The plot shows only the initial-value branches before their poles.

Solid: $a = 1$; dashed: $a = 1/2$
Dotted verticals: poles; $a = 0$ stays on the time axis.



Positive branches shown up to $x = 8$; no continuation across a pole.

Question 8

Let $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e_1 = (1, 0)^T$. Consider the periodically forced system

$$X' = AX + R(t)e_1.$$

You may use $R' = AR$, $R^{-1} = R(-t)$ and $R(t+2\pi) = R(t)$. A 2π -periodic solution satisfies $X(t+2\pi) = X(t)$ for all real t .

Tasks

1. Verify that $X_c(t) = R(t)(c + te_1)$ is a solution for every constant vector c . Determine its initial state.
2. Prove that this family contains every solution, by differentiating $R(-t)X(t)$.
3. Compute the change in state over one forcing period and decide whether any choice of initial data produces a 2π -periodic solution.
4. Replace e_1 in the forcing by an arbitrary constant vector $b = (\alpha, \beta)^T$. Find the full solution family and classify exactly when a 2π -periodic solution exists.

Question 8 – Solution

Strategy. Factor out the supplied rotating matrix and test periodicity on the full solution, not just on its coefficients and forcing.

Step 1: Verify the proposed family directly. The product rule gives

$$X'_c = AR(t)(c + te_1) + R(t)e_1 = AX_c + R(t)e_1.$$

Also $X_c(0) = c$. Thus every initial state has a member of this family.

Step 2: Prove completeness independently. Set $Z = R(-t)X$. Differentiating the inverse identity gives $(R^{-1})' = -R^{-1}A$. Consequently

$$Z' = -R^{-1}AX + R^{-1}(AX + Re_1) = e_1.$$

It follows that $Z = c + te_1$, so $\boxed{X = R(t)(c + te_1)}$ is the entire family, valid on the real axis.

Step 3: Examine one full forcing period. Periodicity of R gives

$$\boxed{X(t + 2\pi) - X(t) = 2\pi R(t)e_1.}$$

This difference has length 2π and never vanishes, independently of c . In particular $X(2\pi) = X(0) + (2\pi, 0)^T$. No initial state gives a 2π -periodic solution. A periodic forcing can produce a response with an accumulating change rather than a periodic response.

Step 4: Classify the general constant-amplitude forcing. The same calculation gives

$$\boxed{X = R(t)(c + tb), \quad X(t + 2\pi) - X(t) = 2\pi R(t)b.}$$

Since $R(t)$ is invertible, a periodic solution exists exactly when $b = 0$, that is, $\alpha = \beta = 0$. In this case every member $X = R(t)c$ is 2π -periodic, including the zero solution. If $b \neq 0$, no choice of c can cancel the per-period difference. The classification concerns this specified forcing family; it does not claim that every nonzero periodic forcing precludes a periodic response.

Question 9

For

$$x' = y, \quad y' = -x,$$

only the measurements $x(0) = 1$ and $x(T) = b$ are given, where $T > 0$. The second initial component is unknown. You may verify solutions using sine and cosine identities; no phase-plane analysis is required.

Tasks

1. Find every solution consistent with $x(0) = 1$, using one arbitrary constant, and identify what that constant means.
2. Classify existence and uniqueness for all $T > 0$ and $b \in \mathbb{R}$, including the exceptional times $T = k\pi$.
3. Apply the classification to $(T, b) = (\pi, -1)$ and $(\pi/2, 2)$. Give the full solution set in each case.
4. When $\sin T \neq 0$, suppose the reported value of b has an error δ . Find the exact resulting error in the recovered second initial component. Explain what happens as $T \rightarrow \pi$ and why this is not failure of uniqueness for a fully specified IVP.

Question 9 – Solution

Strategy. Treat the unknown initial velocity as a parameter and ask whether the observations actually determine it.

Step 1: Parametrize all compatible initial states. Since $x'' = -x$ and $x(0) = 1$, the complete family is

$$x = \cos t + c \sin t, \quad y = -\sin t + c \cos t.$$

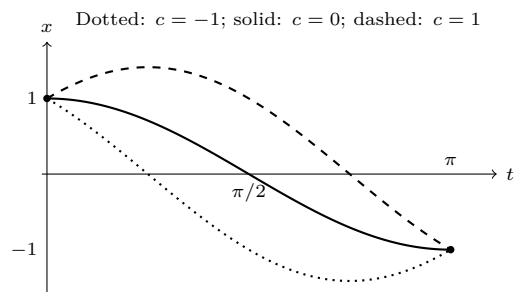
Direct differentiation verifies both rows. The arbitrary constant is $c = y(0) = x'(0)$, the missing initial component.

Step 2: Test the observation equation. The second measurement requires $\cos T + c \sin T = b$. If $\sin T \neq 0$, there is exactly one solution, with $c = (b - \cos T)/\sin T$. If $T = k\pi$ for a positive integer k , the measurement is independent of c : there are infinitely many solutions when $b = (-1)^k$, and none when $b \neq (-1)^k$. These cases exhaust all positive times.

Step 3: Evaluate the two specified experiments. At $(\pi, -1)$ every real c is allowed, giving the full family in Step 1. The figure shows three of these indistinguishable endpoint measurements. At $(\pi/2, 2)$, $c = 2$, so the unique solution is

$$X = (\cos t + 2 \sin t, -\sin t + 2 \cos t)^T.$$

Step 4: Quantify sensitivity and interpret it. Replacing b by $b + \delta$ changes c by exactly $\Delta c = \delta / \sin T$. Thus the absolute amplification factor is $1/|\sin T|$, which is unbounded as $T \rightarrow \pi$ through nonexceptional times. A fixed nonzero measurement error can then create a large recovered initial-velocity error. At $T = \pi$, that component is unobservable from these two measurements. Each fully specified initial vector still has a unique solution; the difficulty lies in recovering it from partial observations.



All share $x(0) = 1$ and $x(\pi) = -1$; initial velocities differ.

Question 10

An unknown continuous homogeneous linear system has the two specified solution columns

$$\Phi(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.$$

We ask whether these columns determine a unique system $X' = A(t)X$, rather than assuming that a displayed pair automatically comes from a chosen matrix.

Tasks

1. Find $A(t)$ and verify $\Phi' = A\Phi$. Prove uniqueness of the recovered coefficient matrix at every real time.
2. Write the complete solution family and solve the IVP $X(1) = (2, 3)^T$. Verify the recovered initial state.
3. Could both columns instead solve an uncoupled system $x' = a(t)x$, $y' = b(t)y$ on any nonempty open interval? Give a direct contradiction using their first components.
4. Compute the determinant and trace relevant to this fundamental matrix. Does its constant determinant imply that every solution has constant Euclidean length? Test the second column and explain the conclusion.

Question 10 – Solution

Strategy. An invertible matrix of solution columns determines its coefficient matrix through its derivative, while its determinant alone does not determine lengths.

Step 1: Recover and verify the matrix. The determinant of Φ is 1, so

$$\Phi^{-1} = \begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}, \quad \boxed{A = \Phi' \Phi^{-1} = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}.}$$

Multiplication gives $A\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix} = \Phi'$. Any coefficient matrix satisfying the column equations must equal $\Phi' \Phi^{-1}$, proving uniqueness at every real time.

Step 2: Select the constants for the IVP. The full solution is $\boxed{X = \Phi(t)c}$ with constant $c \in \mathbb{R}^2$. Indeed $(\Phi^{-1}X)' = 0$ for any solution, proving completeness. At time 1,

$$c = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} (2, 3)^T = (1, 1)^T.$$

Thus $\boxed{X = (1+t, 1+t+t^2)^T}$, which equals $(2, 3)^T$ at 1. Its derivative follows either directly or from the verified column identity.

Step 3: Rule out an uncoupled alternative. The first component of the first column is the constant 1. Its equation would force $a(t) = 0$ throughout the interval. The first component of the second column is t , whose derivative is 1; with $a = 0$ its proposed equation would instead give derivative 0. This is impossible on any nonempty interval. The coupling is essential.

Step 4: Separate independence from length preservation. Here $\det \Phi = 1$ and $\text{tr } A = -t + t = 0$, consistent with the constant determinant. But the second solution column has squared length

$$\boxed{t^2 + (1+t^2)^2 = 1 + 3t^2 + t^4,}$$

which is not constant. A nonzero determinant ensures independence of the columns; even a constant determinant does not force individual solution lengths to stay fixed. The explicit column supplies a counterexample.

Question 1

Consider $x' = y$, $y' = -4x$, with initial state $(x, y) = (1, 0)$ at $t = 0$. A phase trajectory is the curve traced by $(x(t), y(t))$, with its direction as time increases. Nullclines are sets where one coordinate derivative is zero.

Tasks

1. Find the equilibrium and both nullclines. Determine the directions of the vector field on each coordinate axis away from the origin.
2. Find a conserved quadratic expression and the phase curve through the initial state. Explain why the curve must be drawn with equal coordinate scales.
3. Find a parametrization solving the IVP, its least positive period, and the direction in which the phase curve is traversed.
4. Locate every horizontal and vertical tangent on this phase curve. Explain why writing it globally as a single-valued function $y(x)$ would lose information.

Question 1 – Solution

Strategy. Use a conserved quantity to identify the geometric curve, then use the original equations to recover direction and timing.

Step 1: Locate zero derivatives. The x -nullcline is $y = 0$, and the y -nullcline is $x = 0$. Their only common point, $(0, 0)$, is the equilibrium. At $(x, 0)$ the field is $(0, -4x)$: downward for $x > 0$, upward for $x < 0$. At $(0, y)$ it is $(y, 0)$: rightward for $y > 0$, leftward for $y < 0$.

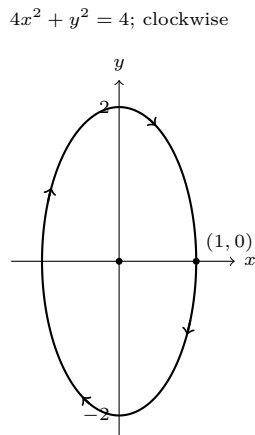
Step 2: Determine the conserved curve. Differentiating gives $(4x^2 + y^2)' = 8xy - 8xy = 0$. The initial state therefore lies on

$$4x^2 + y^2 = 4,$$

an ellipse with horizontal semiaxis 1 and vertical semiaxis 2. Equal coordinate scales preserve this actual geometric aspect ratio.

Step 3: Restore orientation and time. Direct substitution verifies $x = \cos(2t)$, $y = -2\sin(2t)$ and the initial data. The least positive period is π . At the initial point $(1, 0)$ the velocity is $(0, -4)$, so the ellipse is traversed clockwise. The conserved equation alone does not specify this orientation or period.

Step 4: Identify all tangent directions. Horizontal tangents occur where $y' = 0$, namely $(0, 2)$ and $(0, -2)$, where $x' \neq 0$. Vertical tangents occur where $x' = 0$, namely $(1, 0)$ and $(-1, 0)$, where $y' \neq 0$. For $-1 < x < 1$, the same x has two possible values $y = \pm 2\sqrt{1 - x^2}$ on this orbit. A single branch cannot represent the full closed trajectory. The origin is a separate equilibrium, not part of this ellipse. Arrows in the figure indicate increasing time.



Question 2

For $x' = x$, $y' = -y$, a student divides the equations to obtain $dy/dx = -y/x$ and concludes that every trajectory is a hyperbola $xy = C$. Investigate both the usefulness and the limitations of this calculation.

Tasks

1. Derive the conserved quantity without dividing by x or y . Find all equilibria.
2. Classify the trajectories lying on the coordinate axes, including their time directions. Explain what division can omit.
3. For $xy \neq 0$, identify the separate trajectory branches and their directions in all four quadrants. Explain why an entire two-branch hyperbola is not one trajectory.
4. Solve the IVP $(x(0), y(0)) = (2, 1/2)$ and describe both time limits. Determine all initial states whose solutions tend to the origin as $t \rightarrow \infty$.

Question 2 – Solution

Strategy. Keep zero-coordinate cases before separating variables, and use explicit signs to orient each connected branch.

Step 1: Find a division-free invariant. The product rule gives $(xy)' = x'y + xy' = xy - xy = 0$. Thus $xy = C$ along every solution, including axis solutions. The only equilibrium is $(0, 0)$.

Step 2: Recover the axis trajectories. If $y = 0$, then $x = x_0 e^t$; each nonzero horizontal half-axis is a trajectory directed away from the origin. If $x = 0$, then $y = y_0 e^{-t}$; each nonzero vertical half-axis is directed toward the origin. The origin itself is a constant trajectory. Division by x loses the vertical axis; subsequent division by y can also lose the horizontal axis.

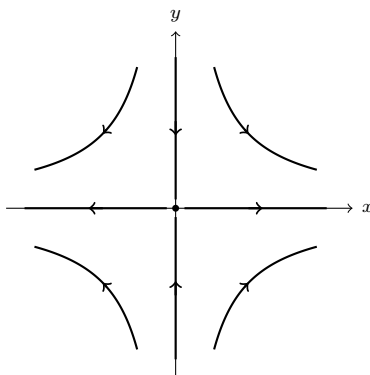
Step 3: Separate the hyperbola branches. For nonzero coordinates, $x = x_0 e^t$ and $y = y_0 e^{-t}$ preserve both signs. In quadrants I, II, III, IV the directions are respectively right/down, left/down, left/up, right/up. Each fixed nonzero C has two disconnected branches; no continuous solution switches between them. The equation $xy = 0$ likewise combines several distinct trajectories rather than describing one orbit through the origin.

Step 4: Apply the initial data and characterize attraction. For the stated IVP,

$$x = 2e^t, \quad y = \frac{1}{2}e^{-t}, \quad xy = 1.$$

As $t \rightarrow \infty$, $x \rightarrow \infty$, $y \rightarrow 0$; as $t \rightarrow -\infty$, $x \rightarrow 0$, $y \rightarrow \infty$. Neither limit is the origin. For general initial data, convergence to the origin in forward time occurs exactly when $x_0 = 0$, including the zero state. The figure distinguishes the axis directions from representative hyperbola branches.

$xy = \pm 1$ and the four axis half-orbits



Question 3

Consider the autonomous nonlinear system

$$x' = y - x^2, \quad y' = 1 - x.$$

A nullcline describes a zero coordinate velocity. It need not be a trajectory. No explicit solution of the full system is required.

Tasks

1. Find the equilibrium and both nullclines.
2. Use inequalities relative to the nullclines to determine the signs of x' , y' in every region. Give the actual velocity at $(0, 0)$, $(0, 1)$ and $(1, 0)$.
3. Determine tangent directions along each nullcline away from the equilibrium. Prove that neither whole nullcline is an invariant trajectory.
4. For a solution starting at $(0, 0)$, calculate $x'(0)$, $y'(0)$, $x''(0)$, $y''(0)$. Explain how it can have a vertical tangent while immediately leaving the x -nullcline.

Question 3 – Solution

Strategy. Read component signs from the two defining expressions, then distinguish instantaneous tangency from remaining on a curve.

Step 1: Locate the nullclines and their intersection. The x -nullcline is $y = x^2$, and the y -nullcline is $x = 1$. Their sole intersection is the equilibrium $(1, 1)$.

Step 2: Determine the regional directions. Above the parabola $x' > 0$; below it $x' < 0$. To the left of $x = 1$, $y' > 0$; to its right, $y' < 0$. Combining these independent signs gives right/up, left/up, right/down or left/down wherever the corresponding inequalities hold. The requested velocities are $F(0, 0) = (0, 1)$, $F(0, 1) = (1, 1)$ and $F(1, 0) = (-1, 0)$.

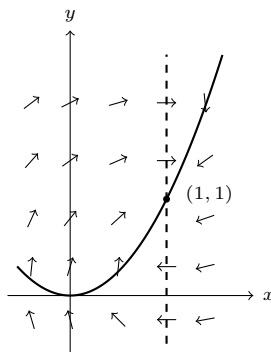
Step 3: Check nullcline tangency correctly. On $y = x^2$, away from $(1, 1)$, the nonzero velocity is vertical. The parabola has finite-slope tangent $(1, 2x)$, so this velocity is not tangent to that parabola. On $x = 1$, away from the equilibrium, the velocity is horizontal and not tangent to the vertical line. Thus neither complete nullcline is invariant. Each is crossed by typical solutions; the equilibrium is the exception where both velocities vanish.

Step 4: Examine departure from the origin. Differentiating the equations gives $x'' = y' - 2xx'$ and $y'' = -x'$. At the initial state,

$$x' = 0, \quad y' = 1, \quad x'' = 1, \quad y'' = 0.$$

Hence $x(t) = t^2/2 + o(t^2)$ and $y(t) = t + o(t^2)$ as $t \rightarrow 0$. For small positive t , $y - x^2 = t + o(t) > 0$, so the solution leaves the nullcline and acquires positive horizontal velocity. A vertical tangent at one instant does not mean that x stays constant. The figure shows nullclines and local velocity arrows, not guessed solution curves.

Solid: $y = x^2$
Dashed: $x = 1$; small arrows: velocities



Question 4

Compare the autonomous vector fields

$$F(x, y) = (-y, x), \quad G(x, y) = (1 + x^2 + y^2)(-y, x), \quad H(x, y) = (y, -x).$$

All three have the origin as an equilibrium. A geometric orbit and the time needed to traverse it are different pieces of information.

Tasks

1. Prove that every nonzero trajectory of each field lies on a circle centered at the origin.
2. On a circle of radius $R > 0$, compute the angular velocity for each field and determine its time direction.
3. Find the least positive period on that circle for each field. Give explicit solutions starting at $(2, 0)$ for F and G .
4. Explain precisely which information is shared by the three phase pictures and which requires arrows or time parametrization. Does a positive scalar multiple of a field reverse its directions?

Question 4 – Solution

Strategy. Compute radial and angular rates separately; multiplication of a field can change timing without changing its curves.

Step 1: Verify the common invariant circles. For F , $(x^2 + y^2)' = 2x(-y) + 2yx = 0$. Multiplication by $1 + x^2 + y^2$ preserves this cancellation for G , and changing the sign preserves it for H . A nonzero initial radius stays fixed and each field is nonzero on that circle.

Step 2: Compute orientation from angular velocity. For $R^2 = x^2 + y^2 > 0$,

$$\theta' = \frac{xy' - yx'}{R^2}.$$

Substitution gives $\theta'_F = 1$, $\theta'_G = 1 + R^2$, $\theta'_H = -1$. Thus F, G traverse circles counterclockwise, while H traverses them clockwise.

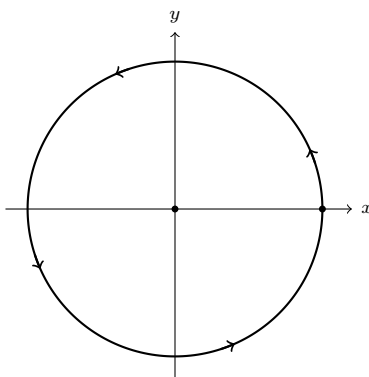
Step 3: Calculate periods and the given trajectories. A full revolution requires angular change of magnitude 2π , so

$$T_F = T_H = 2\pi, \quad T_G = \frac{2\pi}{1 + R^2}.$$

At radius 2, the requested solutions are $X_F = (2 \cos t, 2 \sin t)$ and $X_G = (2 \cos 5t, 2 \sin 5t)$. They start at the same state and cover the same circle at different rates. The zero equilibrium is not assigned a least positive revolution period.

Step 4: Distinguish geometry, direction and speed. All three fields have the same nonzero orbit sets and the same equilibrium. Arrows separate H from F, G ; the time parametrization separates F from G . A strictly positive multiplier preserves velocity direction wherever the field is nonzero, so it cannot reverse an orbit. A negative multiplier does reverse it. The plotted circle is shared by F, G ; its arrows show their common orientation, and the caption records their different periods at radius 2.

Same radius-2 orbit for F, G : counterclockwise



$T_F = 2\pi$, $T_G = 2\pi/5$; H reverses the arrows.

Question 5

The time-dependent system $x' = 1$, $y' = 2t$ has two proposed solutions

$$U(t) = (t, t^2), \quad V(t) = (t + 1, t^2), \quad t \in \mathbb{R}.$$

Investigate whether their projected curves in the (x, y) plane can cross without contradicting uniqueness of the original initial-value problem.

Tasks

1. Verify the candidates, derive the full solution family, and find the equations of their two projected curves.
2. Find the intersection of those curves, the time at which each solution visits it, and the velocity of each at its visit.
3. Decide whether $U(t)$ and $V(t)$ ever agree at the same time. Explain why the usual no-crossing conclusion for a locally Lipschitz autonomous plane field does not apply to these projections.
4. Add time as a state to obtain an autonomous three-dimensional system. Show that the two lifted trajectories do not intersect, including above their projected intersection.

Question 5 – Solution

Strategy. Record the time attached to each projected point; the vector field here is not determined by the plane coordinates alone.

Step 1: Verify and eliminate time. Both derivatives are $(1, 2t)$, so both candidates solve the system. Integration gives the full family $(x, y) = (t + a, t^2 + b)$. The candidate plane curves are respectively $y = x^2$ and $y = (x - 1)^2$, each directed toward increasing x .

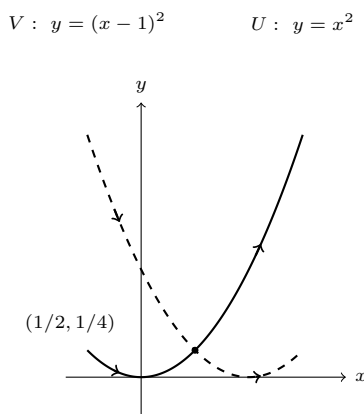
Step 2: Locate the projected crossing and its times. Solving $x^2 = (x - 1)^2$ gives $(x, y) = (1/2, 1/4)$. The first solution reaches it at $t = 1/2$, with velocity $(1, 1)$; the second reaches it at $t = -1/2$, with velocity $(1, -1)$. The different tangent directions reflect different evaluation times.

Step 3: Check what uniqueness prohibits. At any common time, $V(t) - U(t) = (1, 0)$, so the full states never agree. The IVP is unique, but the projected point (x, y) alone does not fix $y' = 2t$. There is no single autonomous plane velocity at the projected crossing. Thus the autonomous no-crossing rule is inapplicable. Geometric intersection at different times is not an intersection of these solutions at the same specified time and state.

Step 4: Restore the clock in the state. With new independent variable τ , use

$$s' = 1, \quad x' = 1, \quad y' = 2s.$$

The lifted candidates are $(s, x, y) = (t, t, t^2)$ and $(t, t + 1, t^2)$. An intersection of lifted curves would first require equal s , hence equal t , and would then contradict the fixed horizontal difference 1. Above the plane crossing their clock coordinates are $1/2$ and $-1/2$, so the lifted states are distinct. The figure deliberately labels only projected plane curves and shows their separate rightward directions.



Projected curves; intersection occurs at different times.

Question 6

For $r^2 = x^2 + y^2$, consider

$$x' = x(1 - r^2) - y, \quad y' = y(1 - r^2) + x.$$

For $r > 0$, use polar coordinates with a continuously followed angle along each trajectory. A periodic orbit is a nonconstant closed trajectory.

Tasks

1. Find every equilibrium. Derive differential equations for r and the angular coordinate.
2. Identify the invariant unit circle, its orientation and least positive period. Determine whether nearby trajectories move toward it from each side.
3. For any initial radius $r_0 > 0$, solve the equation for $u = r^2$ and prove convergence to the unit circle in forward time. State separately what happens at $r_0 = 0$.
4. Does a nonzero solution converge to one particular point on the unit circle? Prove your answer and distinguish approaching an orbit from approaching an equilibrium.

Question 6 – Solution

Strategy. Separate the radial attraction from the angular motion; approaching a circle does not require stopping on it.

Step 1: Find the radial and angular equations. For a nonzero state, the equilibrium coefficient matrix has determinant $(1 - r^2)^2 + 1 > 0$, so it cannot annihilate that state. The only equilibrium is the origin. Direct differentiation gives

$$r' = r(1 - r^2), \quad \theta' = 1 \quad (r > 0).$$

The formulas follow from $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.

Step 2: Identify the periodic orbit and radial directions. At $r = 1$, the radius is constant and the angle increases at unit rate. This circle is a counterclockwise periodic orbit of least period 2π . For $0 < r < 1$, $r' > 0$; for $r > 1$, $r' < 0$. Both sides move radially toward the circle rather than away from it.

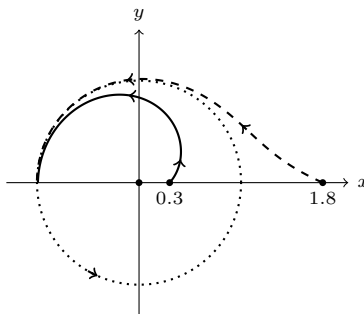
Step 3: Prove the forward radial limit. The squared radius satisfies $u' = 2u(1 - u)$, with solution

$$u(t) = \frac{1}{1 + (r_0^{-2} - 1)e^{-2t}} \quad (t \geq 0).$$

The denominator is positive for every $t \geq 0$, and direct differentiation and evaluation at zero verify the formula. It tends to 1, hence $r \rightarrow 1$. With $\theta = \theta_0 + t$, this also supplies a forward-global solution. At $r_0 = 0$ the solution remains at the equilibrium; the reciprocal formula is not used there.

Step 4: Rule out convergence to a single point. Along times $t = 2\pi n$ the states tend to $(\cos \theta_0, \sin \theta_0)$. Along $t = 2\pi n + \pi$ they tend to its negative. These are distinct, so no nonzero solution has a single limiting point. Its distance to the unit circle, $|r - 1|$, tends to zero while it continues rotating. The figure shows the unit orbit and sample inward/outward spirals; all arrows represent increasing time.

Solid: $r_0 = 0.3$; dashed: $r_0 = 1.8$
Dotted: invariant unit orbit



Question 7

Consider the damped nonlinear system

$$x' = y, \quad y' = -x - x^3 - cy, \quad c > 0,$$

with initial state $(1, 0)$. Define $E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$. A level curve of E need not itself be a solution curve.

Tasks

1. Find the equilibrium and derive the exact change of E along a solution.
2. Use the initial energy to prove bounds on both coordinates for every forward time. Explain why the smooth system then has a global forward solution.
3. At the initial state, $E' = 0$. Does this mean that the solution is stationary or remains on its original energy level? Use the original equations to decide.
4. Prove that no nonconstant periodic orbit exists, by integrating the energy balance over a hypothetical period. Explain what the enclosing energy curve represents in the phase picture.

Question 7 – Solution

Strategy. A decreasing energy supplies a containing region and rules out a closed return without requiring an explicit trajectory.

Step 1: Compute the equilibrium and dissipation. Equilibrium requires $y = 0$ and $x + x^3 = 0$, hence only $(0, 0)$. Along solutions,

$$E' = y(-x - x^3 - cy) + (x + x^3)y = \boxed{-cy^2 \leq 0}.$$

Thus energy decreases except at instants when $y = 0$.

Step 2: Bound the forward solution. Initially $E = 3/4$. Since all energy terms are nonnegative, $y^2/2 \leq 3/4$ gives $|y| \leq \sqrt{3/2}$. Also $x^4 + 2x^2 \leq 3$, so $(x^2 + 1)^2 \leq 4$ and $|x| \leq 1$. Therefore

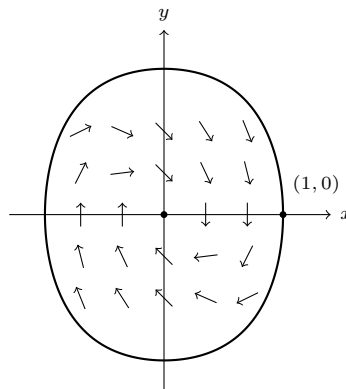
$$\boxed{|x(t)| \leq 1, \quad |y(t)| \leq \sqrt{3/2} \quad (t \geq 0).}$$

The polynomial field is locally Lipschitz. These bounds keep the state in a compact region, preventing finite-time escape and allowing continuation for all forward times.

Step 3: Interpret zero instantaneous energy loss. At $(1, 0)$, the velocity is $(0, -2)$, not zero. Moreover $y(t) = -2t + o(t)$ for small positive t , so y becomes nonzero and $E' = -cy^2 < 0$. The solution enters lower energy levels. A horizontal zero of E' at one instant does not make energy constant on an interval or make its level curve a trajectory.

Step 4: Exclude nonconstant periodic motion. If a solution had period $T > 0$, integrating gives $0 = E(T) - E(0) = -c \int_0^T y(t)^2 dt$. Continuity and $c > 0$ force $y \equiv 0$, then $x' = 0$ and $x + x^3 = 0$, so the solution is the zero equilibrium. Thus no nonconstant periodic orbit exists. The drawn $E = 3/4$ curve bounds a forward-invariant region; it is not a closed orbit. Interior arrows show velocities for the illustrative choice $c = 1$.

$E = 3/4$ is a containing boundary, not an orbit
Interior velocity arrows use $c = 1$



Question 8

Start with $x' = -y$, $y' = x$, and introduce the nonlinear change of coordinates $u = x$, $v = y + x^2$. The original unit-circle trajectory is $(x, y) = (\cos t, \sin t)$. Both coordinate planes should be drawn with equal scales when interpreting their geometry.

Tasks

1. Find the inverse coordinate change and derive the autonomous system for (u, v) .
2. Find the image of the unit circle as an implicit equation and a time parametrization. Verify the parametrization in the transformed differential equations.
3. Determine the equilibrium, the least positive period of the transformed unit orbit, and its velocity at the image of $(1, 0)$.
4. Can this invertible coordinate change turn the unit circle into a self-intersecting trajectory? Prove your conclusion and explain which geometric features can change even though the time parametrization is preserved.

Question 8 – Solution

Strategy. Transform both the curve and its velocity by the chain rule; invertibility preserves distinct states even when shapes change.

Step 1: Derive the inverse and new field. The inverse is $x = u$, $y = v - u^2$, valid on the whole plane. Therefore

$$u' = u^2 - v, \quad v' = u + 2u(u^2 - v).$$

The term $2xx'$ in $v' = y' + 2xx'$ is essential. This is a nonlinear autonomous system, despite the linear original field.

Step 2: Transform and verify the orbit. Substituting into $x^2 + y^2 = 1$ gives

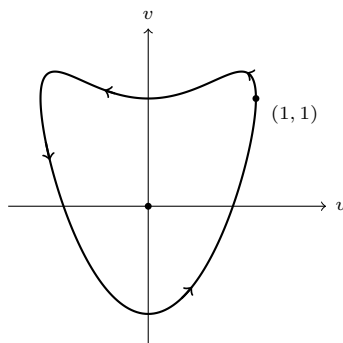
$$u^2 + (v - u^2)^2 = 1, \quad (u, v) = (\cos t, \sin t + \cos^2 t).$$

Here $u' = -\sin t = u^2 - v$, and $v' = \cos t - 2\cos t \sin t = u + 2u(u^2 - v)$. Thus this is the actual transformed trajectory, not merely a guessed level curve.

Step 3: Check equilibrium, period and orientation locally. From $u' = 0$ we have $v = u^2$; then $v' = u$ forces $u = v = 0$. The only equilibrium is the image of the original origin. The transformed orbit has period 2π . A smaller period would, through the inverse map, give a smaller period of $(\cos t, \sin t)$, which is impossible. The image of $(1, 0)$ is $(1, 1)$, with velocity $(0, 1)$: upward.

Step 4: Exclude a self-intersection. If two transformed states agree, the inverse gives equal original states. On the unit circle this occurs only for times differing by an integer multiple of 2π . No new self-intersection can appear within one traversal. The circle becomes a distorted closed curve, so Euclidean lengths, angles and curvature can change. The same time variable and invertible state map preserve the period and prevent distinct states from being identified. The figure plots the transformed orbit in equal u, v scales.

$$u^2 + (v - u^2)^2 = 1; \text{ equal coordinate scales}$$



Question 9

For $x' = 1 - x^2$, $y' = -y$, focus on the strip $-1 < x < 1$ and its boundaries. You may use $(\tanh t)' = 1 - \tanh^2 t$ and $\operatorname{artanh} x = \frac{1}{2} \ln((1+x)/(1-x))$ on this strip.

Tasks

1. Find the equilibria and verify the trajectory $(x, y) = (\tanh t, 0)$, including its direction and its limits as $t \rightarrow \pm\infty$.
2. Explain why the closed segment joining the two equilibria is not a single trajectory, even though it is the closure of the connecting orbit.
3. For initial state $(0, b)$, find a parametrization and eliminate time on the strip. Describe the forward and backward limits, treating $b = 0$ separately.
4. Determine trajectories on $x = -1$ and $x = 1$. Can a nonconstant one reach its equilibrium in finite time? Distinguish approaching an endpoint from attaining it.

Question 9 – Solution

Strategy. Keep equilibria separate from nearby nonconstant orbits, and retain the open domain when eliminating time.

Step 1: Identify the connecting trajectory. The equilibria are $(-1, 0), (1, 0)$. The function $(\tanh t, 0)$ solves both equations, has $x' > 0$, and tends to $(-1, 0)$ backward in time and $(1, 0)$ forward in time. Its orbit is the open horizontal segment $-1 < x < 1$, directed rightward.

Step 2: Separate the orbit from its closure. For every finite t , $-1 < \tanh t < 1$, so neither endpoint is reached. Each endpoint is its own constant trajectory. Uniqueness for the smooth field also prevents a solution from reaching an equilibrium and then leaving it. The closed segment is the union of three trajectories, not one.

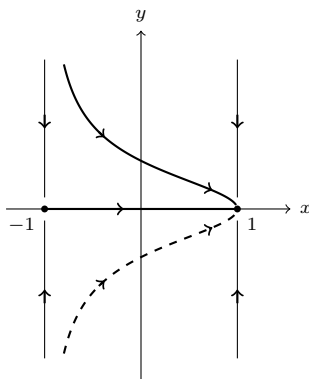
Step 3: Recover the whole one-parameter family. For initial $(0, b)$,

$$x = \tanh t, \quad y = be^{-t}, \quad y = b\sqrt{\frac{1-x}{1+x}}, \quad -1 < x < 1.$$

The inverse hyperbolic tangent formula proves the eliminated equation. Every member tends to $(1, 0)$ as $t \rightarrow \infty$. If $b = 0$, the backward limit is $(-1, 0)$. If $b \neq 0$, then $x \rightarrow -1$ while $y \rightarrow +\infty$ for $b > 0$ or $-\infty$ for $b < 0$; there is no finite backward limiting state.

Step 4: Check the invariant boundary lines. On either $x = \pm 1$, the first coordinate stays fixed and $y = y_0 e^{-(t-t_0)}$. For $y_0 \neq 0$, each vertical half-line is an orbit directed toward $(\pm 1, 0)$, reached only as $t \rightarrow \infty$. At $y_0 = 0$ the state is already the equilibrium. The figure marks equilibria as separate filled points; connecting arrows and nearby trajectories approach them without asserting finite-time arrival.

$b = 1/2$ (solid), $b = -1/2$ (dashed), $b = 0$
Filled equilibria are not part of the connecting orbit.



Question 10

On the positive quadrant $x > 0$, $y > 0$, consider

$$x' = x(1 - y), \quad y' = y(x - 1), \quad (x(0), y(0)) = (2, 1).$$

You may use the elementary geometric fact that a compact regular level of a smooth strictly convex function in the plane, enclosing its unique minimum, is a simple closed curve.

Tasks

1. Find the positive equilibrium and nullclines. Determine the velocity and direction at the stated initial point.
2. Prove that $H = x - \ln x + y - \ln y$ is conserved, and find its level for the initial data.
3. Use convexity, boundary behavior and the vector field to show that this nonconstant orbit is closed and traversed in a finite period. No period formula is required.
4. Find exact equations for the minimum and maximum of each population on this orbit, identify where those extrema occur, and explain why their maxima are not simultaneous.

Question 10 – Solution

Strategy. A proper conserved function confines the positive orbit, and nonzero tangential velocity turns its level curve into a periodic trajectory.

Step 1: Read the positive nullclines. In the positive quadrant, $x' = 0$ exactly at $y = 1$, and $y' = 0$ exactly at $x = 1$. The only positive equilibrium is $(1, 1)$. At $(2, 1)$ the velocity is $(0, 1)$, so the orbit initially moves upward. The positive axes are not included in the stated domain.

Step 2: Derive the conserved level. Differentiating yields

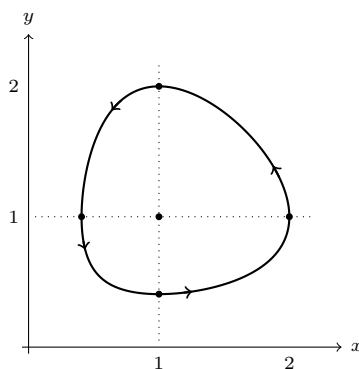
$$H' = (1 - 1/x)x(1 - y) + (1 - 1/y)y(x - 1) = 0.$$

For the stated data, $\boxed{H = 3 - \ln 2}$. The Hessian is diagonal with entries $1/x^2, 1/y^2 > 0$, so H is strictly convex with unique minimum $H(1, 1) = 2$.

Step 3: Justify a closed periodic orbit. The function $z - \ln z$ tends to infinity as $z \downarrow 0$ or $z \rightarrow \infty$. Thus this level is compact within the positive quadrant. Its gradient vanishes only at $(1, 1)$, below the stated level; it is regular. The given convex-level fact makes it a simple closed curve. The vector field is tangent by conservation and never zero on this curve. Its continuous speed therefore has a positive minimum, so traversal of the finite-length smooth closed curve takes finite time. Uniqueness then gives a periodic orbit. It is counterclockwise, as its velocity at the rightmost point $(2, 1)$ is upward.

Step 4: Locate and compare the extrema. Extrema of x occur at $y = 1$, hence $x - \ln x = 2 - \ln 2$. This has roots 2 and a unique $a \in (0, 1)$, since $z - \ln z$ decreases on $(0, 1)$ and increases on $(1, \infty)$. By symmetry, $\boxed{x_{\min} = y_{\min} = a, \quad x_{\max} = y_{\max} = 2}$. The four extreme points are $(a, 1), (2, 1), (1, a), (1, 2)$. In particular the two maxima occur at different states, so they are not simultaneous. The plotted level uses the exact conserved equation.

$$x - \ln x + y - \ln y = 3 - \ln 2$$



Counterclockwise; dotted nullclines $x = 1, y = 1$.

Question 1

Let $X = (x, y)^T$ solve

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

For geometric interpretation use modal coordinates $u = (x + y)/2$ and $v = (x - y)/2$. A stable node means that all nearby states tend to the origin forward in time, with two real negative eigenvalues.

Tasks

1. Find both eigenpairs and solve the IVP. Explain why the two modes give every solution.
2. Eliminate time for this IVP in the (u, v) coordinates, retaining the domain. Translate the result to the original phase plane.
3. Determine the time direction and the limiting tangent line at the origin. Find $\lim_{t \rightarrow \infty} y(t)/x(t)$.
4. For arbitrary $X(0) = (p, q)^T$, identify precisely the initial states that approach along the faster eigendirection instead. Classify the origin.

Question 1 – Solution

Strategy. Decompose into independent eigendirections, then distinguish the dominant slow mode from initial data that cancel it.

Step 1: Resolve the modes. The characteristic polynomial is $(\lambda+1)(\lambda+3)$. Eigenvectors are $v_1 = (1, 1)^T$ for -1 and $v_2 = (1, -1)^T$ for -3 . They form a basis, so modal coordinates satisfy $u' = -u$, $v' = -3v$. Initially $u = v = 1$, giving

$$X = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This decoupling also proves completeness for arbitrary initial data.

Step 2: Retain the orbit domain. Here $u = e^{-t} > 0$ and $v = e^{-3t} = u^3$ for all real t . Thus the orbit is

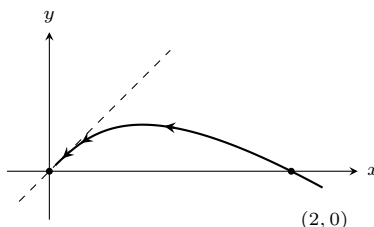
$$\frac{x-y}{2} = \left(\frac{x+y}{2} \right)^3, \quad x+y > 0.$$

The origin is a limiting point, not a point attained at finite time.

Step 3: Read the approach direction. As time increases, u decreases toward zero. Moreover $y/x = (1 - e^{-2t})/(1 + e^{-2t}) \rightarrow 1$. Both the state direction and the velocity direction approach the line $y = x$; indeed $dy/dx = (1 - 3e^{-2t})/(1 + 3e^{-2t}) \rightarrow 1$. At $(2, 0)$ the velocity is $(-4, 2)$, consistent with the arrows.

Step 4: Keep the exceptional line. For initial (p, q) , the slow and fast coefficients are $(p+q)/2$ and $(p-q)/2$. Every solution tends to zero. A nonzero solution approaches along the fast line $y = -x$ exactly when $p+q = 0$; otherwise its limiting line is $y = x$. The zero state remains fixed and has no approach direction. Both eigenvalues are negative, so the origin is a stable node.

Stable node: slow line $y = x$ (dashed)



Question 2

Consider

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X.$$

Use the Euclidean distance to the line $y = -x$. In the perturbed initial data below assume $0 < \varepsilon < 1/\sqrt{2}$.

Tasks

1. Find the eigenpairs and general solution. Identify every initial state producing a bounded solution for $t \geq 0$.
2. Explain the saddle geometry and the forward directions on its two eigenlines. Can a nonconstant trajectory cross either eigenline?
3. For $X(0) = (1 + \varepsilon, -1 + \varepsilon)^T$, find when the magnitudes of its stable and unstable modal coefficients first become equal.
4. Find the first $t \geq 0$ at which this solution has distance 1 from $y = -x$. Explain why small initial error can eventually cause a large departure.

Question 2 – Solution

Strategy. Track stable and unstable coefficients separately; perpendicular distance measures departure from the stable line.

Step 1: Separate decay from growth. Eigenpairs are $3, (1, 1)^T$ and $-1, (1, -1)^T$. For data (p, q) ,

$$X = \frac{p+q}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{p-q}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These independent vectors give the full family. A forward solution is bounded exactly when $p+q=0$, and then it tends to zero.

Step 2: Interpret the invariant lines. The unstable line $y=x$ points away from the origin; the stable line $y=-x$ points toward it. The opposite signs make the origin a saddle. Each modal coefficient preserves its sign unless identically zero. Thus off-line trajectories cannot cross either eigenline at finite time. Each equilibrium or nonzero half-line is a separate trajectory.

Step 3: Compute modal balance. The perturbed solution is $X = \varepsilon e^{3t}(1, 1)^T + e^{-t}(1, -1)^T$. Equality of modal magnitudes requires $\varepsilon e^{3t} = e^{-t}$, hence

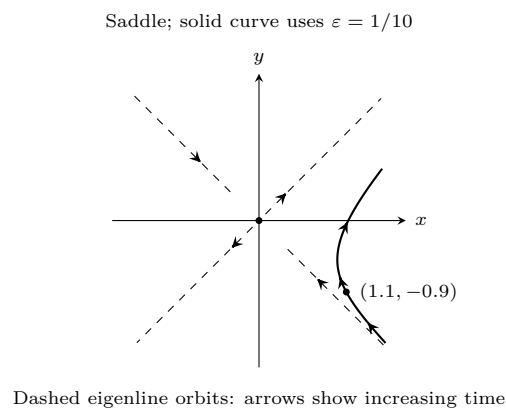
$$t_{\text{balance}} = \frac{1}{4} \ln(1/\varepsilon).$$

The eigenvectors have equal length, so the same time balances their Euclidean contributions. It is nonnegative under the stated assumption.

Step 4: Measure the delayed departure. Distance to $y=-x$ is $|x+y|/\sqrt{2} = \sqrt{2}\varepsilon e^{3t}$, strictly increasing from a value below one. Its first value one occurs at

$$t_{\text{exit}} = \frac{1}{3} \ln\left(\frac{1}{\sqrt{2}\varepsilon}\right).$$

The delay diverges as $\varepsilon \downarrow 0$, but every positive error has a growing unstable component. Only exact cancellation stays on the stable line. The figure illustrates $\varepsilon = 1/10$.



Question 3

For

$$X' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

a student claims that positive eigenvalues force both coordinates to increase for all real time. Investigate the entire trajectory, including $t < 0$.

Tasks

1. Find the eigenpairs and the solution, and classify the equilibrium.
2. Eliminate time to find the exact phase curve and its domain. Determine its time direction.
3. Find every stationary point of $x(t)$ and classify it. Determine when $x(t)$ is negative and when it vanishes.
4. Identify the limiting eigenline as $t \rightarrow -\infty$ and the limiting direction as $t \rightarrow \infty$. Explain what is wrong with the student's claim.

Question 3 – Solution

Strategy. Growing modal amplitudes can combine with opposite signs; compute coordinate derivatives before making a monotonicity claim.

Step 1: Solve in an eigenbasis. Eigenpairs are $2, (1, 0)^T$ and $1, (-1, 1)^T$. The initial vector is their sum, so

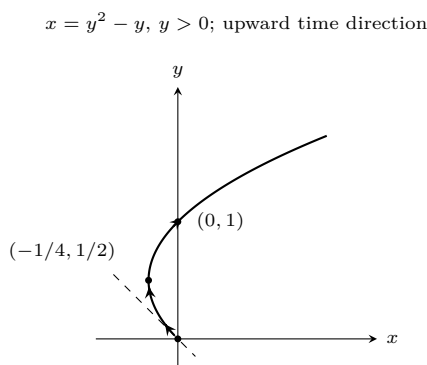
$$x = e^{2t} - e^t, \quad y = e^t.$$

Both eigenvalues are positive, making the origin an unstable node (a source). Every solution tends to zero backward in time, not forward.

Step 2: Recover the geometric branch. Since $y = e^t > 0$, the phase curve is $x = y^2 - y, y > 0$. It is traversed toward increasing y . The limiting origin is excluded; the finite-time point with $x = 0$ is $(0, 1)$. The curve's leftmost point has a vertical tangent, not an equilibrium.

Step 3: Locate coordinate reversal. We have $x' = e^t(2e^t - 1)$, zero only at $t = -\ln 2$. It is negative before this time and positive after it. Therefore the unique global minimum is $x = -1/4$, at $y = 1/2$. For $t < 0$, $x = e^t(e^t - 1) < 0$; it vanishes only at $t = 0$ and is positive for $t > 0$. The limit $x \rightarrow 0$ as $t \rightarrow -\infty$ is not another zero.

Step 4: Distinguish modal and coordinate behavior. Backward, $X/e^t \rightarrow (-1, 1)^T$, so the approach line is $y = -x$. Forward, $X/e^{2t} \rightarrow (1, 0)^T$, giving the positive horizontal direction. Positive eigenvalues describe growth of modes with fixed coefficients; they do not force each original coordinate to increase. Here the two contributions to x have opposite signs, producing a genuine reversal. The faster eigenmode eventually dominates because its coefficient is nonzero.



Question 4

Let

$$X' = \begin{pmatrix} -1 & 6 \\ 0 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Write $\|X\| = \sqrt{x^2 + y^2}$. Investigate whether negative eigenvalues alone force this ordinary distance from the origin to decrease at every time.

Tasks

1. Find both eigenpairs and solve the IVP. Show that all initial states decay as $t \rightarrow \infty$.
2. Find the maximum of $x(t)$ on $t \geq 0$. Evaluate $\|X\|$ there and compare it with $\|X(0)\|$.
3. Compute $(\|X\|^2)'$ for a general state. Give a state at which it is positive, and determine its sign at the stated initial point.
4. Construct a positive quadratic quantity using $u = x + 6y$, $v = y$ that strictly decreases along every nonzero solution. Explain how this reconciles decay with transient growth.

Question 4 – Solution

Strategy. Euclidean distance mixes nonorthogonal modes; a quadratic quantity in modal coordinates separates their decay.

Step 1: Decouple the eigenmodes. The eigenpairs are $-1, (1, 0)^T$ and $-2, (-6, 1)^T$. Since $(0, 1)^T = 6(1, 0)^T + (-6, 1)^T$,

$$x = 6(e^{-t} - e^{-2t}), \quad y = e^{-2t}.$$

Every initial vector has this two-mode form with suitable coefficients, so all solutions tend to zero.

Step 2: Exhibit transient amplification. The derivative $x' = 6(-e^{-t} + 2e^{-2t})$ changes from positive to negative at $t = \ln 2$. Thus $x_{\max} = 3/2$, while $y = 1/4$ there. Consequently

$$\|X(\ln 2)\| = \sqrt{37}/4 > 1 = \|X(0)\|.$$

This is the maximum of x , not a claim about the time of maximum norm.

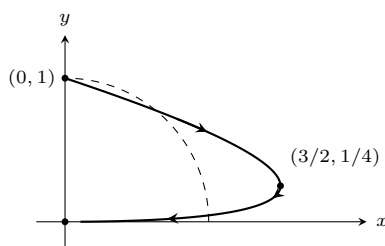
Step 3: Test instantaneous norm growth. Direct differentiation gives

$$(\|X\|^2)' = -2x^2 + 12xy - 4y^2.$$

At $(1, 1)$ it equals $6 > 0$, so the norm can increase. At the given initial state $(0, 1)$ it equals -4 : this trajectory first moves closer, but later exceeds its initial distance. Negative eigenvalues do not make this quadratic expression negative at every state.

Step 4: Use a norm adapted to the modes. The invertible coordinates satisfy $u' = -u$, $v' = -2v$. Thus $E = (x + 6y)^2 + y^2 = u^2 + v^2$ is positive except at zero and $E' = -2u^2 - 4v^2 < 0$ for every nonzero state. The Euclidean circles and the level ellipses of E measure different sizes. Decrease of the latter is compatible with temporary Euclidean growth, and the explicit modes still establish eventual decay.

Solid: IVP; dashed: unit-circle arc



Question 5

For a real parameter $-1 < a < 1$, consider

$$X' = A_a X, \quad A_a = \frac{1}{2} \begin{pmatrix} a-2 & a+2 \\ a+2 & a-2 \end{pmatrix}.$$

Treat the zero-eigenvalue case explicitly; do not assume that every equilibrium is isolated.

Tasks

1. Find the eigenvalues, an eigenbasis independent of a , and the solution with arbitrary initial state $(p, q)^T$.
2. Classify the origin for $a < 0$ and $a > 0$, and identify the initial states that converge to it in each case.
3. At $a = 0$, find every equilibrium and the limiting state of every solution. Is the origin attracting all nearby states?
4. For initial state $(2, 0)^T$, compare $\lim_{a \uparrow 0} \lim_{t \rightarrow \infty} X_a(t)$ with $\lim_{t \rightarrow \infty} \lim_{a \uparrow 0} X_a(t)$. Explain the difference.

Question 5 – Solution

Strategy. Keep the parameter-dependent mode visible; its decay disappears at the zero eigenvalue.

Step 1: Use fixed eigendirections. The vectors $(1, 1)^T$ and $(1, -1)^T$ have eigenvalues a and -2 . They remain independent, and their eigenvalues are distinct throughout the stated parameter interval. Setting $m = (p + q)/2$, $d = (p - q)/2$ gives

$$X_a(t) = me^{at} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + de^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This formula satisfies the system and the initial data for every allowed a .

Step 2: Classify the nonzero-parameter cases. For $a < 0$, both modes decay, so the origin is a stable node and all initial states converge to it. For $a > 0$, the origin is a saddle; convergence occurs exactly when $m = 0$, that is, on $y = -x$. The $y = x$ mode then grows. There are no other equilibria in these two cases because $\det A_a = -2a \neq 0$.

Step 3: Resolve the equilibrium line. At $a = 0$, every point on $y = x$ is an equilibrium and $X_0(t) \rightarrow (m, m)^T$. This is the orthogonal projection of the initial state onto that line. The origin does not attract all nearby states: nearby nonzero points of the equilibrium line remain fixed. Nevertheless it is stable in the distance sense, since the orthogonal constant and decaying modes never increase the Euclidean norm.

Step 4: Compare the orders of limiting. For the specified data, $m = d = 1$. At each fixed $a < 0$ the time limit is zero; hence the first iterated limit is $\boxed{(0, 0)^T}$. At each fixed finite t , taking $a \uparrow 0$ instead gives $(1, 1)^T + e^{-2t}(1, -1)^T$, whose time limit is $\boxed{(1, 1)^T}$. The decay e^{at} becomes arbitrarily slow as a approaches zero from below. It is not uniform over such parameters on an unbounded time interval, so these two operations need not commute.

Question 6

An unknown constant real 2×2 matrix A is known to satisfy

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These are exact eigenvector observations, not merely measured slopes.

Tasks

1. Reconstruct A and justify its uniqueness.
2. Solve $X' = AX$ with $X(0) = (3, 3)^T$. Verify the initial state and derivative directly.
3. Find the entire set of initial states that decay forward in time. Determine the limiting direction for the stated IVP.
4. Would knowing only $\text{tr } A = 1$ and $\det A = -2$ determine that decaying line? Give a concrete comparison matrix and explain the effect of rescaling either supplied eigenvector.

Question 6 – Solution

Strategy. An eigenbasis fixes the linear map, while eigenvalues alone do not fix its directions.

Step 1: Reconstruct the operator. Let $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \text{diag}(-1, 2)$. Since $\det P = -3 \neq 0$,

$$A = PDP^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

A linear map is uniquely determined by its values on a basis, so there is no other matrix satisfying both given observations.

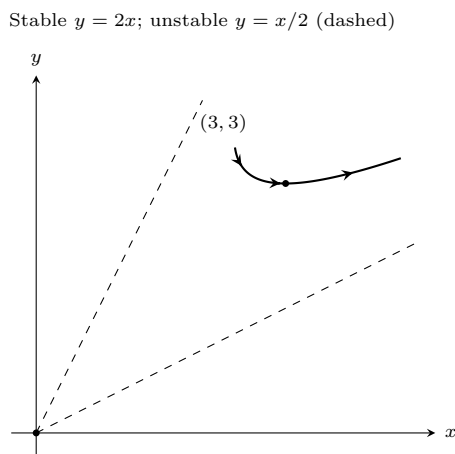
Step 2: Solve and check the IVP. The initial vector is $(1, 2)^T + (2, 1)^T$, giving

$$X = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

At zero this is $(3, 3)^T$, and its derivative is $(-1, -2)^T + (4, 2)^T = (3, 0)^T = A(3, 3)^T$. More generally each mode differentiates to its eigenvalue times itself, verifying the ODE for every time.

Step 3: Separate stable data from generic growth. For initial (p, q) , the unstable coefficient is $(2p - q)/3$. Decay therefore occurs exactly on $q = 2p$, the line spanned by $(1, 2)^T$. For the stated IVP the unstable coefficient is one, so $X/e^{2t} \rightarrow (2, 1)^T$ and $X/\|X\| \rightarrow (2, 1)^T/\sqrt{5}$. The drawn eigenlines are invariant; the off-line curve cannot cross them.

Step 4: Distinguish rates from directions. The matrix $B = \text{diag}(-1, 2)$ has the same trace and determinant, hence the same eigenvalues, but its decaying line is the horizontal axis. Trace and determinant alone do not determine the stable line. Multiplying either supplied eigenvector by any nonzero scalar changes its coordinate coefficient but leaves PDP^{-1} unchanged: the diagonal rescaling commutes with D . Zero rescaling is not allowed.



Question 7

Consider the three-dimensional homogeneous system

$$X' = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A solution is called bounded forward (backward) if its Euclidean norm is bounded on $[0, \infty)$ (respectively $(-\infty, 0]$).

Tasks

1. Find an eigenbasis and use it to write the complete solution for arbitrary (p, q, r) .
2. Find exactly the initial states giving bounded forward solutions. Within that set, distinguish the slow and fast nonzero decay directions.
3. Find exactly the initial states giving bounded backward solutions, and describe their forward behavior.
4. Determine every solution bounded on the entire real line. Explain why cancellation between distinct growing modes cannot produce additional cases.

Question 7 – Solution

Strategy. Express the initial state in a basis of three real modes, then read boundedness from individual modal coordinates.

Step 1: Resolve all three components. The triangular matrix has distinct eigenvalues $-2, -1, 1$. Corresponding vectors are $v_1 = (1, 0, 0)^T$, $v_2 = (1, 1, 0)^T$, $v_3 = (1, 1, 1)^T$, forming an invertible triangular basis. Solving for coefficients yields

$$X = (p - q)e^{-2t}v_1 + (q - r)e^{-t}v_2 + re^tv_3.$$

Each term satisfies the ODE; their sum at zero is $(p, q, r)^T$. Equivalently the independent coordinates $x - y, y - z, z$ solve scalar equations with rates $-2, -1, 1$, proving completeness.

Step 2: Find the stable plane and its exception. Forward boundedness requires and is implied by $r = 0$; this is the plane $z = 0$ of initial states. If $q \neq 0$, then $X/e^{-t} \rightarrow qv_2$, so the slow limiting line is $\text{span}(v_2)$. If $q = 0$ and $p \neq 0$, only the faster mode $pe^{-2t}v_1$ remains. The zero state is separate. All states in this plane tend to zero.

Step 3: Find the backward-bounded line. As $t \rightarrow -\infty$, the negative-rate exponentials grow. Their coefficients must vanish: $p - q = 0$ and $q - r = 0$. Thus the exact set is the line $\boxed{p = q = r}$. On it $X = re^tv_3$, tending to zero backward and, for $r \neq 0$, growing without bound forward. At $r = 0$ it is the equilibrium.

Step 4: Intersect the two conditions. A solution bounded for all real time lies both in $r = 0$ and in $p = q = r$, so $\boxed{X \equiv 0}$ is the only one. This is not merely an inference from the norm of a sum. The inverse basis map is fixed and linear; bounded X forces all its modal coordinates $x - y, y - z, z$ to be bounded. Each is one scalar exponential, so a forbidden growing coefficient cannot be hidden by cancellation with another mode.

Question 8

Let

$$X' = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} X, \quad x(0) = 1, \quad y(T) = b, \quad T > 0.$$

Only the initial value $q = y(0)$ is unknown. These conditions concern different times; they are not a complete initial state at one time.

Tasks

1. Find the real eigenpairs and write the solution in terms of q .
2. Derive the equation that q must satisfy. Identify every positive time at which its coefficient vanishes.
3. At the exceptional time, classify all values of b by whether there are zero, one, or infinitely many solutions. Explain why uniqueness for an IVP is not contradicted.
4. For other times, give q explicitly and its change under a measurement error $b \mapsto b + \delta$. Describe what happens near the exceptional time.

Question 8 – Solution

Strategy. Two real exponential modes can cancel in an endpoint observation even when the full state remains uniquely determined by its initial data.

Step 1: Find the two modes. The characteristic polynomial is $\lambda^2 - 1$. Eigenvectors are $(1, 1)^T$ for 1 and $(1, 2)^T$ for -1 . The data $(1, q)^T$ give coefficients $2 - q$ and $q - 1$, so

$$X = (2 - q)e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (q - 1)e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Step 2: Locate the singular observation time. The endpoint condition becomes

$$b = 2e^T - 2e^{-T} + q(2e^{-T} - e^T).$$

The coefficient of q vanishes exactly when $e^{2T} = 2$, hence $T_* = \frac{1}{2} \ln 2$, the unique positive exceptional time.

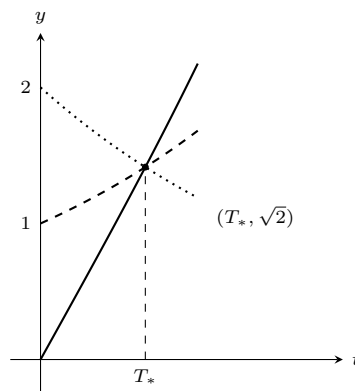
Step 3: Classify compatibility at that time. At T_* , the equation reduces to $b = \sqrt{2}$. If $b = \sqrt{2}$, every real q gives a solution; otherwise none does. There is never exactly one at this time. Distinct q specify distinct initial states. In fact $x(T_*) = (3 - q)/\sqrt{2}$, so the full states remain different even though their second components agree. Each individual IVP still has its unique solution. The figure shows three such second components.

Step 4: Quantify sensitivity away from the exception. For $T \neq T_*$ there is exactly one solution, with

$$q = \frac{b - 2e^T + 2e^{-T}}{2e^{-T} - e^T}, \quad \Delta q = \frac{\delta}{2e^{-T} - e^T}.$$

The denominator tends to zero as $T \rightarrow T_*$, so the absolute amplification factor diverges. This is a poorly conditioned recovery of missing initial data, not loss of uniqueness for a fully specified IVP. For a fixed nonzero δ , the recovered initial error can become arbitrarily large.

Solid $q = 0$; dashed $q = 1$; dotted $q = 2$



Question 9

Let

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}, \quad P = A + 2I, \quad Q = -(A + I).$$

Here a projection means a matrix R with $R^2 = R$; it need not be an orthogonal projection. Develop a solution formula without choosing normalized eigenvectors.

Tasks

1. Find the eigenvalues and verify $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$.
2. Identify the range and kernel of each projection. Are these projections orthogonal in the Euclidean plane?
3. Prove that $F(t) = e^{-t}P + e^{-2t}Q$ satisfies $F' = AF$ and $F(0) = I$. Use it to solve the IVP $X(0) = (1, 0)^T$.
4. Prove $F(t+s) = F(t)F(s)$ and find its inverse. Explain why these identities hold for every pair of real times here.

Question 9 – Solution

Strategy. Polynomial projections isolate the two real eigenspaces and make the evolution law a scalar calculation on each one.

Step 1: Verify complementary projections. The characteristic polynomial is $(\lambda+1)(\lambda+2)$, and direct multiplication gives $A^2 + 3A + 2I = 0$. Here $P = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$. Thus $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$. For example $PQ = -(A + 2I)(A + I) = 0$; the other identities follow by multiplication or from $Q = I - P$.

Step 2: Identify the oblique directions. The range of P is $\text{span}(2, 3)^T$, the -1 eigenspace; its kernel is $\text{span}(1, 1)^T$, the -2 eigenspace. For Q the range and kernel are interchanged. These two directions have nonzero dot product 5, so neither projection is orthogonal. The matrices are also visibly nonsymmetric.

Step 3: Construct and verify evolution. The same polynomial identity gives $AP = -P$, $AQ = -2Q$. Consequently $F' = AF$ and $F(0) = P + Q = I$. For arbitrary initial vector X_0 , $F(t)X_0$ solves the IVP; constant linear-system uniqueness gives the complete solution. In particular

$$X = e^{-t} \begin{pmatrix} -2 \\ -3 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Its initial state is $(1, 0)^T$ and its initial derivative $(-4, -3)^T = A(1, 0)^T$.

Step 4: Multiply by modes. All mixed products vanish, so

$$F(t)F(s) = e^{-(t+s)}P + e^{-2(t+s)}Q = F(t+s).$$

Taking $s = -t$ yields $F(t)^{-1} = F(-t) = e^tP + e^{2t}Q$. No finite real time makes either scalar exponential vanish. This is the normalized evolution of one constant matrix; the fixed complementary projections justify the multiplication rule for every $s, t \in \mathbb{R}$. No orthogonality or eigenvector normalization was used.

Question 10

An unknown real matrix has eigenvectors $(1, 1)^T$ and $(1, -1)^T$ with distinct negative eigenvalues λ and μ , respectively. For the solution with $X(0) = (2, 0)^T$, define $u = (x + y)/2$, $v = (x - y)/2$. The observed entire phase curve is $v = u^2$, $u > 0$, with direction toward the origin. A clock measurement gives $u(\ln 2) = 1/2$.

Tasks

1. Determine what the phase curve alone implies about λ and μ . Explain why its shape and arrows do not determine their absolute values.
2. Use the clock measurement to recover both eigenvalues and the matrix.
3. Write and verify the full IVP solution, including the observed curve, its domain, and the clock measurement.
4. Find the limiting tangent at the origin and the maximum of $y(t)$ on $t \geq 0$. Explain how uniformly speeding up this system would affect the curve, maximum value, and time of that maximum.

Question 10 – Solution

Strategy. Geometry determines a ratio of modal rates; one elapsed-time observation fixes their common scale.

Step 1: Extract a rate ratio. Initially $u = v = 1$, so $u = e^{\lambda t}$ and $v = e^{\mu t}$. The identity $v = u^2$ for every time implies $\mu = 2\lambda$. Any $\lambda < 0$ with this ratio gives the same oriented orbit as t ranges over the real line. Thus the curve and its arrows alone leave one positive speed factor undetermined.

Step 2: Calibrate the clock. The measurement yields $e^{\lambda \ln 2} = 1/2$, so $\lambda = -1$ and $\mu = -2$. Using the specified eigenbasis reconstructs

$$A = \frac{1}{2} \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.$$

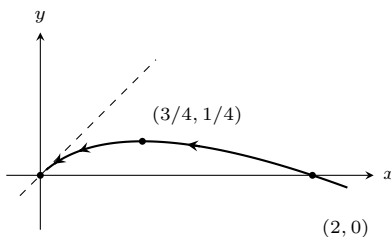
Step 3: Verify the reconstructed trajectory. The solution is

$$x = e^{-t} + e^{-2t}, \quad y = e^{-t} - e^{-2t}.$$

Differentiating and multiplying by the displayed matrix agree component by component, and the initial state is $(2, 0)^T$. Moreover $u = e^{-t} > 0$, $v = e^{-2t} = u^2$, and $u(\ln 2) = 1/2$. Every $u > 0$ occurs at a finite real time; the limiting origin is excluded from the orbit.

Step 4: Separate geometric and temporal conclusions. As $t \rightarrow \infty$, $dy/dx = (1 - 2e^{-t})/(1 + 2e^{-t}) \rightarrow 1$, giving the tangent line $y = x$. On $t \geq 0$, $y' = e^{-t}(2e^{-t} - 1)$ changes from positive to negative at $t = \ln 2$. Hence $y_{\max} = 1/4$, attained at $x = 3/4$. Replacing A by kA , $k > 0$, changes the solution to $X(kt)$: the oriented curve and maximum value remain the same, but the maximum occurs at $(\ln 2)/k$. Except for $k = 1$, it fails the given clock measurement.

Recovered orbit; dashed tangent $y = x$



Question 1

Let

$$X' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use a complex eigenvector to construct real solutions. Angles in the original (x, y) plane are measured counterclockwise from the positive x -axis.

Tasks

1. Find the eigenvalues and an eigenvector for the eigenvalue with positive imaginary part. Separate its exponential solution into real and imaginary parts.
2. Prove that the resulting two real solutions are independent and solve the IVP.
3. Find the radius, an unwrapped angle, and the first positive time at which this trajectory reaches the positive x -axis.
4. Find the radius multiplier after one full revolution. Is the nonzero solution periodic? Explain why the conjugate eigenpair does not supply two additional independent real solutions.

Question 1 – Solution

Strategy. Take real and imaginary parts with their signs intact, then use polar coordinates to separate decay from rotation.

Step 1: Extract real solutions. The eigenvalues are $-1 \pm 2i$. For $\lambda = -1 + 2i$, choose $v = (1, -i)^T$, which satisfies $Av = \lambda v$. Writing $e^{\lambda t}v = U + iV$ gives

$$U = e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}, \quad V = e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}.$$

Because A is real, both parts solve the original real system.

Step 2: Resolve the initial state. The determinant of the columns (U, V) is $-e^{-2t} \neq 0$. At zero they are $(1, 0)^T$ and $(0, -1)^T$, so $X = U - 2V$:

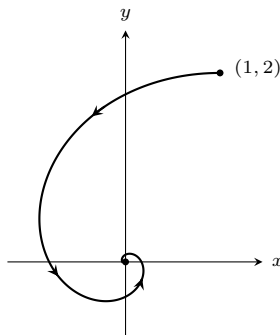
$$X = e^{-t} \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \sin 2t + 2 \cos 2t \end{pmatrix}.$$

The nonzero determinant and linear-system uniqueness give the complete real family as arbitrary real combinations of U, V .

Step 3: Locate the first positive-axis crossing. Let $\theta_0 = \arctan 2$, which lies in $(0, \pi/2)$. Here $r = \sqrt{5}e^{-t}$ and a continuous angle is $\theta = \theta_0 + 2t$. Rotation is counterclockwise. The first positive time on the positive x -axis requires $\theta = 2\pi$, hence $t_* = (2\pi - \arctan 2)/2$. The earlier negative-axis crossing is not the requested event.

Step 4: Separate revolution from repetition. One revolution takes π , reducing the radius by $e^{-\pi}$. Since the radius strictly decreases, no nonzero solution is periodic; the origin is a stable spiral. Conjugating the exponential solution gives $U - iV$, whose parts are $U, -V$, already in the same real span. Thus a conjugate pair supplies two real degrees of freedom, not four. The figure shows a finite forward segment of the stated IVP.

Counterclockwise decay; $r = \sqrt{5}e^{-t}$



Question 2

Consider

$$x' = 4y, \quad y' = -x, \quad (x(0), y(0)) = (2, 0).$$

A student sees eigenvalues $\pm 2i$ and claims that the trajectory must be a circle rotating with constant Euclidean angular speed 2.

Tasks

1. Find the eigenvalues and solve the IVP in real form.
2. Find a conserved positive quadratic expression and determine the complete phase curve. Locate its intercepts.
3. Find the least positive period and the direction of traversal. Identify coordinates in which the motion is a uniform circular rotation.
4. Compute the original Euclidean angular velocity $\theta' = (xy' - yx')/(x^2 + y^2)$. Find its range on the orbit and evaluate the student's claim.

Question 2 – Solution

Strategy. A linear coordinate change can turn a circle into an ellipse while preserving elapsed time and period.

Step 1: Solve the oscillatory IVP. The characteristic polynomial is $\lambda^2 + 4$, giving $\pm 2i$. From $x'' = -4x$ and $x'(0) = 0$,

$$x = 2 \cos 2t, \quad y = -\sin 2t.$$

Differentiation verifies both first-order equations and the initial data.

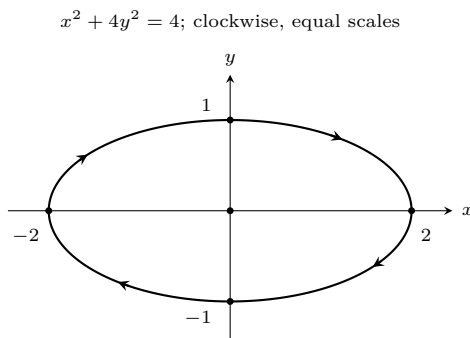
Step 2: Recover the invariant ellipse. The derivative of $H = x^2 + 4y^2$ is $8xy - 8xy = 0$. Thus the orbit is $x^2 + 4y^2 = 4$, with intercepts $(\pm 2, 0)$ and $(0, \pm 1)$. The parametrization covers the whole ellipse. Its unequal semiaxes must remain unequal in an equal-scale drawing; it is not a circle in the original Euclidean coordinates.

Step 3: Determine period and orientation. The least positive period is π . At $(2, 0)$ the velocity is $(0, -2)$, so traversal is clockwise. The coordinates $u = x$, $v = 2y$ satisfy $u' = 2v$, $v' = -2u$ and $u^2 + v^2 = 4$. In that transformed plane the angular velocity is constantly -2 . The invertible transformation preserves the least period.

Step 4: Compute the actual angular speed. In the original plane,

$$\theta' = -\frac{x^2 + 4y^2}{x^2 + y^2} = -\frac{4}{x^2 + y^2}.$$

On the ellipse, $1 \leq x^2 + y^2 \leq 4$, so $-4 \leq \theta' \leq -1$. It equals -1 at the horizontal intercepts and -4 at the vertical ones. Thus imaginary eigenvalues determine a temporal frequency, but do not force a Euclidean circle or constant Euclidean angular speed. The average signed angular velocity over a complete traversal is nevertheless -2 .



Question 3

Let

$$X' = \begin{pmatrix} -1 & 8 \\ -1/2 & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Use $\|X\| = \sqrt{x^2 + y^2}$. A stable spiral means that all nearby states approach the origin with continuing rotation, not necessarily that their ordinary distance decreases at every instant.

Tasks

1. Find the eigenvalues and solve the IVP. Determine its rotation direction.
2. Evaluate the norm at $t = \pi/4$ and compare it with its initial value. Compute $(\|X\|^2)'$ at a general state.
3. Find a positive quadratic expression $H = x^2 + cy^2$ with $H' = -2H$. Use it to establish decay of every solution.
4. For the given IVP and $0 < \rho < 1$, give an explicit time after which the state is guaranteed to remain in $\|X\| \leq \rho$. Explain why this guarantee need not be the first entry time.

Question 3 – Solution

Strategy. Use coordinates that balance the oscillatory coupling, then compare the adapted quadratic size with Euclidean distance.

Step 1: Solve the damped rotation. The characteristic polynomial is $(\lambda + 1)^2 + 4$, so the rates are $-1 \pm 2i$. Direct substitution verifies

$$x = 4e^{-t} \sin 2t, \quad y = e^{-t} \cos 2t.$$

For any nonzero state, $xy' - yx' = -x^2/2 - 8y^2 < 0$; rotation is clockwise. The transformed coordinates $(x, 4y)$ rotate uniformly with decay.

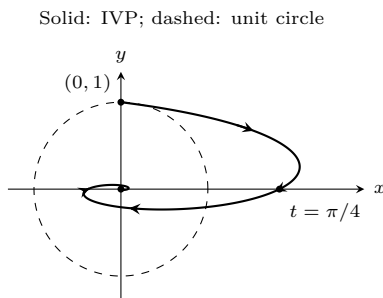
Step 2: Exhibit temporary growth. Initially the norm is one, whereas $\|X(\pi/4)\| = 4e^{-\pi/4} > 1$. More generally,

$$(\|X\|^2)' = -2x^2 + 15xy - 2y^2,$$

which equals $11 > 0$ at $(1, 1)$. Its value at the given initial state is -2 . Hence even this trajectory can first shrink and subsequently exceed its initial distance. The negative real parts guarantee eventual decay, not a sign for this derivative at every state.

Step 3: Find a decreasing quadratic size. Differentiating $x^2 + cy^2$ cancels the mixed term when $c = 16$. Then $H = x^2 + 16y^2$, $H' = -2H$, so $H(t) = H(0)e^{-2t}$. Since $\|X\|^2 \leq H$, every initial state tends to zero. Together with the strict rotation sign this identifies a stable spiral, even though the drawn orbit can move outside the unit circle.

Step 4: Give a permanent-entry bound. For the specified data, $H(0) = 16$ and $\|X(t)\| \leq 4e^{-t}$. Thus every $t \geq \ln(4/\rho)$ satisfies $\|X(t)\| \leq \rho$. This bound controls all later times. It need not be the first entry: H overestimates the ordinary squared norm except where $y = 0$. Transient contraction and expansion can occur before the guaranteed time; no claim of a sharp first-entry time is made.



Question 4

For $a \in \mathbb{R}$, consider

$$X' = \begin{pmatrix} a & -3 \\ 3 & a \end{pmatrix} X.$$

For $r > 0$, define the positive-axis return map $P_a(r)$ as the radius the solution starting at $(r, 0)$ has on its first later return to the positive x -axis. A return to this ray need not be a return to the same point.

Tasks

1. Find the eigenvalues and the solution from an arbitrary nonzero initial state in polar form.
2. Classify the origin for $a < 0$, $a = 0$, and $a > 0$. Determine exactly when a nonzero solution is periodic.
3. Find the first return time, the map P_a , and all its fixed radii in $r > 0$. Explain their relation to periodic trajectories.
4. An experiment finds $P_a(r) = r/2$ for one nonzero radius. Recover a , and find the radius after n complete turns for every integer $n \geq 0$.

Question 4 – Solution

Strategy. The real part sets radial growth and the imaginary part sets turning time; a ray return becomes periodic only when the radius is restored.

Step 1: Read radial and angular evolution. The eigenvalues are $a \pm 3i$. For initial radius $r_0 > 0$ and angle θ_0 , the solution has

$$r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 + 3t.$$

Equivalently $X(t) = e^{at} R(3t) X(0)$, where $R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$. The only equilibrium is zero because the determinant is $a^2 + 9 > 0$.

Step 2: Classify all parameter cases. For $a < 0$ the origin is a stable spiral; for $a > 0$ it is an unstable spiral. At $a = 0$ it is a center: every nonzero state follows a circle with least period $2\pi/3$. For $a \neq 0$, the radius is strictly monotone, so no nonzero solution is periodic. Zero remains constant and has no least positive period.

Step 3: Compute the return map. The first return to the positive ray takes $T = 2\pi/3$, independent of r and a . Consequently

$$P_a(r) = e^{2\pi a/3} r.$$

For $r > 0$, the fixed-point equation holds for every radius if $a = 0$ and for none if $a \neq 0$. These fixed radii are exactly the nonzero periodic trajectories. Returning to an angle alone is insufficient.

Step 4: Recover damping from one revolution. The measurement gives $e^{2\pi a/3} = 1/2$, hence

$$a = -\frac{3 \ln 2}{2\pi}, \quad r_n = r_0 2^{-n}.$$

The coefficient a is unique because the real exponential is one-to-one. The same multiplier applies to every radius by linearity; a single nonzero measurement therefore determines the damping rate in this specified family.

Question 5

An unknown real matrix A has eigenvalues $-1 \pm 3i$. Every solution of $X' = AX$ satisfies

$$\frac{d}{dt}(x^2 + 4y^2) = -2(x^2 + 4y^2).$$

At the state $(2, 0)$, the velocity has a negative second component. Use all three observations to reconstruct the system.

Tasks

1. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and extract the restrictions imposed by the quadratic identity at arbitrary states.
2. Use the eigenvalues to find every matrix consistent with that identity before applying the direction observation.
3. Select the correct matrix and solve the IVP starting at $(2, 0)$. Verify its quadratic size and rotation direction.
4. Find the time and contraction factor for one complete turn. Explain precisely which ambiguity would remain if the direction observation were omitted.

Question 5 – Solution

Strategy. An identity valid for all initial states determines quadratic coefficients; the remaining sign is geometric orientation.

Step 1: Compare polynomial coefficients. Differentiation gives $2ax^2 + 2(b+4c)xy + 8dy^2 = -2x^2 - 8y^2$. Every point can be an initial state, so coefficients must agree:

$$\boxed{a = d = -1, \quad b = -4c.}$$

Step 2: Use the spectral determinant. The eigenvalues require trace -2 and determinant 10 . The trace is already correct, while $\det A = 1 - bc = 1 + 4c^2 = 10$ gives $c = \pm 3/2$. Thus precisely two matrices remain:

$$\begin{pmatrix} -1 & 6 \\ -3/2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -6 \\ 3/2 & -1 \end{pmatrix}.$$

Both have the required quadratic decay and the required eigenvalues.

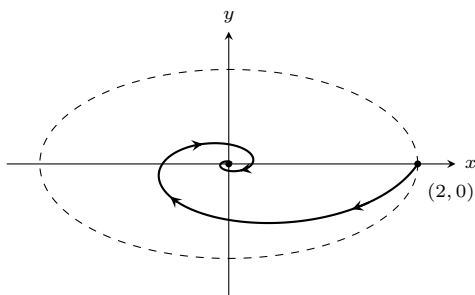
Step 3: Use orientation and solve. At $(2, 0)$, $y' = 2c < 0$, so select $c = -3/2$, $b = 6$. The IVP is

$$\boxed{x = 2e^{-t} \cos 3t, \quad y = -e^{-t} \sin 3t.}$$

It satisfies the selected equations, initial data and $x^2 + 4y^2 = 4e^{-2t}$. Moreover $xy' - yx' = -3x^2/2 - 6y^2 < 0$ off zero, confirming clockwise rotation. The shrinking ellipses are size contours, not individual orbits.

Step 4: Separate timing, contraction and sign. The coordinates $(x, 2y)$ make one clockwise revolution in $2\pi/3$. An invertible positive diagonal rescaling preserves a full turn, so this is also the time to return to the initial positive ray. The whole state then equals $e^{-2\pi/3}X(0)$; this is not a periodic return. Without the direction observation, the second matrix would remain equally possible, with counterclockwise rotation and the same contraction factor. The unordered conjugate eigenvalues do not determine orientation.

Clockwise spiral; dashed initial-size contour



Question 6

Let

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 0 \end{pmatrix}, \quad B = A - I.$$

Find the normalized evolution matrix for $X' = AX$ without selecting or normalizing complex eigenvectors. A normalized evolution matrix F satisfies $F' = AF$ and $F(0) = I$.

Tasks

1. Find the eigenvalues and prove a simple identity for B^2 .
2. Construct a real formula for $F(t)$ using that identity, and verify both defining conditions.
3. Solve the IVP $X(0) = (1, 1)^T$. Compute $F(\pi)$ and explain why a complete oscillation does not produce a periodic nonzero solution.
4. Prove the multiplication law $F(t + s) = F(t)F(s)$, find $F(t)^{-1}$, and compute its determinant. Does the determinant alone determine how every vector length changes?

Question 6 – Solution

Strategy. Remove the real part of the spectrum; the remaining matrix acts algebraically like multiplication by an imaginary number.

Step 1: Isolate the oscillatory matrix. The characteristic polynomial is $\lambda^2 - 2\lambda + 5$, with roots $1 \pm 2i$. Here $B = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$ and direct multiplication gives $B^2 = -4I$.

Step 2: Construct a real evolution formula. Set

$$F(t) = e^t \left(I \cos 2t + \frac{1}{2} B \sin 2t \right).$$

Its value at zero is I . Differentiating the bracket gives $-2I \sin 2t + B \cos 2t$, which equals B times that bracket because $B^2 = -4I$. The product rule therefore yields $F' = (I + B)F = AF$. Uniqueness verifies that this formula evolves every initial state.

Step 3: Solve and inspect one oscillation. Since $B(1, 1)^T = (-4, 0)^T$,

$$X = e^t \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix}.$$

The initial state is $(1, 1)^T$ and initial derivative $(-3, 1)^T = A(1, 1)^T$. Also $F(\pi) = e^\pi I$: a full trigonometric cycle multiplies the state by e^π . More generally the periodic bracket and its inverse are bounded, so every nonzero solution grows without bound forward in time. Thus there is no nonzero periodic solution.

Step 4: Verify the evolution algebra. Multiplying two brackets and using $B^2 = -4I$ gives the sine and cosine addition formulas, hence $F(t + s) = F(t)F(s)$. Therefore $F(t)^{-1} = F(-t)$. For scalars c, s , $\det(cI + sB/2) = c^2 + s^2$, so $\det F(t) = e^{2t}$. This controls area, not every vector length: at the state $(1, 1)$ the squared-length derivative is $2(1, 1) \cdot (-3, 1) = -4 < 0$, despite increasing area. Shape distortion and eventual growth can coexist.

Question 7

A damped oscillator is written as a state system

$$x' = y, \quad y' = -5x - 2y, \quad (x(0), y(0)) = (1, 0).$$

Here x is displacement and y is velocity. A turning point means a time when the displacement has a local maximum or minimum.

Tasks

1. Find the complex eigenvalues and the real IVP solution.
2. Find the first positive zero of the displacement and the velocity there, retaining its sign.
3. Find the first positive turning time and classify that turning point. Determine the ratio of the magnitudes of consecutive displacement extrema.
4. Compute the derivative of the mechanical energy $E = (y^2 + 5x^2)/2$. Explain why $E'(0) = 0$ does not make the motion constant or periodic.

Question 7 – Solution

Strategy. The state components have different phases: displacement zeros and velocity zeros mark different events on the spiral.

Step 1: Solve the state IVP. The characteristic polynomial is $\lambda^2 + 2\lambda + 5$, giving $-1 \pm 2i$. The initial conditions yield

$$x = e^{-t}(\cos 2t + \tfrac{1}{2} \sin 2t), \quad y = -\tfrac{5}{2}e^{-t} \sin 2t.$$

Differentiation gives $x' = y$ and $y' = -5x - 2y$, including $y'(0) = -5$.

Step 2: Locate the first displacement crossing. Write $\alpha = \arctan 2 \in (0, \pi/2)$. The first positive root of $\cos 2t + \frac{1}{2} \sin 2t = 0$ occurs at

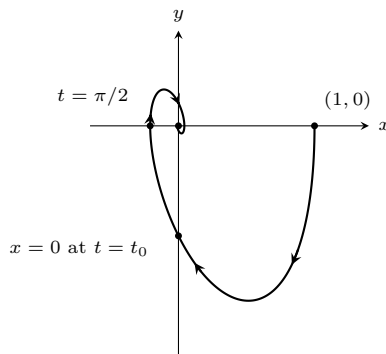
$$t_0 = (\pi - \alpha)/2, \quad y(t_0) = -\sqrt{5}e^{-t_0}.$$

Indeed $\sin(\pi - \alpha) = 2/\sqrt{5}$ and $\cos(\pi - \alpha) = -1/\sqrt{5}$. The negative velocity shows a crossing from positive to negative displacement, not a turning point.

Step 3: Separate the turning events. Velocity vanishes exactly at $t = n\pi/2$, for integers n . The first positive turning time is $\pi/2$, with displacement $-e^{-\pi/2}$. There $x'' = y' = 5e^{-\pi/2} > 0$, so it is a minimum. Successive extrema have $x(n\pi/2) = (-1)^n e^{-n\pi/2}$; their absolute values have ratio $e^{-\pi/2}$ for consecutive forward extrema. Successive positive maxima are one full oscillation apart and have ratio $e^{-\pi}$.

Step 4: Interpret dissipation at a zero derivative. The energy derivative is $E' = yy' + 5xx' = -2y^2 \leq 0$. At zero, $y = 0$ so $E' = 0$, but the acceleration is -5 and the state immediately changes. On any nontrivial time interval this solution has $y \neq 0$ somewhere, so its energy strictly drops over that interval. A nonconstant periodic solution would instead require zero total energy loss over a period, forcing $y \equiv 0$ and then $x \equiv 0$, a contradiction. The phase arrows distinguish the crossing and turning points.

Displacement–velocity phase curve



Question 8

For a nonzero real frequency ω , consider

$$X' = \begin{pmatrix} -1 & -\omega \\ \omega & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An instrument records only selected times. At every integer $n \geq 0$ it reports $X(n) = e^{-n}(1, 0)^T$. Assume the measurements are exact.

Tasks

1. Solve the IVP and determine every nonzero frequency consistent with all the integer-time measurements.
2. Can these measurements determine rotation direction or the number of revolutions per unit time? Explain why the observed points do not imply a straight-line continuous trajectory.
3. An additional measurement is $X(1/4) = e^{-1/4}(0, 1)^T$. Find every frequency still allowed, including negative frequencies.
4. If one also knows $0 < \omega < 4\pi$, recover the frequency uniquely. Exhibit two different frequencies that fit both sets of measurements when that extra bound is absent.

Question 8 – Solution

Strategy. Sampling observes angles only modulo complete revolutions; additional information is needed to recover the continuous frequency.

Step 1: Solve and impose integer samples. The eigenvalues are $-1 \pm i\omega$, and

$$X(t) = e^{-t} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}.$$

The measurement at $n = 1$ requires $\omega = 2\pi k$, $k \in \mathbb{Z}$. Every such value satisfies all integer samples; the nonzero-frequency assumption excludes $k = 0$.

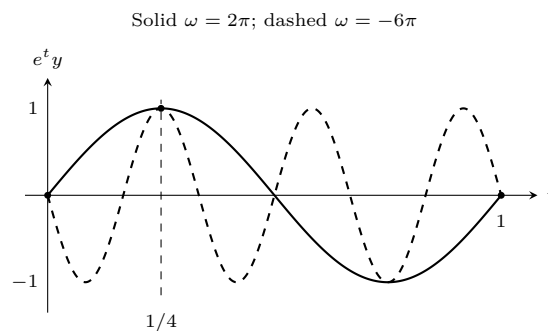
Step 2: Identify what the samples conceal. The sign of k gives counterclockwise or clockwise rotation, and $|k|$ gives revolutions per unit time. Neither is fixed by the integer samples. For every permitted nonzero frequency the continuous angle changes and the radius decays, tracing a spiral. The sampled points all lie on one ray because whole turns occur between them, not because that ray is an invariant trajectory.

Step 3: Use the quarter-time observation. The additional angle condition is $\omega/4 = \pi/2 + 2\pi m$. Thus

$$\omega = 2\pi(1 + 4m), \quad m \in \mathbb{Z}.$$

These values already satisfy the integer constraints. Negative possibilities remain, for example $m = -1$ gives $\omega = -6\pi$. No zero frequency occurs in this family.

Step 4: Use the external frequency bound. Under $0 < \omega < 4\pi$, only $m = 0$ remains, giving $\omega = 2\pi$. Without that bound, 2π and -6π both fit all stated observations, despite opposite directions and different speeds. The figure plots $e^t y(t)$ for these two solutions: they agree at the recorded times but differ between them. The vertical variable is the rescaled component, not the radius or a phase-plane coordinate.



Question 9

Consider

$$X' = \begin{pmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use Euclidean boundedness on $[0, \infty)$ or $(-\infty, 0]$ as appropriate. For a nonzero state its direction means the normalized vector $X/\|X\|$.

Tasks

1. Find the eigenvalues and the complete real solution for arbitrary initial data.
2. Classify the initial states giving bounded solutions forward in time, backward in time, or for all real time.
3. When $r \neq 0$, determine the limiting direction as $t \rightarrow \infty$ and justify why the rotating components do not affect it.
4. When $r = 0$ but $(p, q) \neq (0, 0)$, determine the state limit and decide whether its direction has a limit. Give a sequence-based justification.

Question 9 – Solution

Strategy. Decouple the rotating plane from the real exponential axis, and distinguish convergence of a state from convergence of its direction.

Step 1: Separate the blocks. The eigenvalues are $-1 \pm 2i$ and 1. The complete solution is

$$\begin{aligned} x &= e^{-t}(p \cos 2t - q \sin 2t), \\ y &= e^{-t}(p \sin 2t + q \cos 2t), \\ z &= r e^t. \end{aligned}$$

The planar rotation is invertible at every time, so the arbitrary three initial components are represented without loss.

Step 2: Classify boundedness in both time directions. Exactly $r = 0$ gives forward boundedness. For backward boundedness, $x^2 + y^2 = (p^2 + q^2)e^{-2t}$ forces $p = q = 0$; the remaining z component then tends to zero backward. The intersection of this axis with the forward-bounded plane is zero, so $X \equiv 0$ is the only solution bounded on the entire real line. Rotation cannot cancel the planar radius.

Step 3: Find the direction with an unstable component. If $r \neq 0$, the ratio of planar radius to $|z|$ is $\sqrt{p^2 + q^2} e^{-2t} / |r| \rightarrow 0$. Consequently $X / \|X\| \rightarrow (0, 0, \operatorname{sgn} r)^T$. The state itself grows without bound; its direction can still converge because the real growing mode dominates both decaying components.

Step 4: Test direction on the stable plane. If $r = 0$ and $s = \sqrt{p^2 + q^2} > 0$, then $X \rightarrow 0$, but its normalized planar state is $R(2t)(p, q)^T / s$. At $t_n = n\pi$ it equals $(p, q)^T / s$; at $t_n = n\pi + \pi/2$ it equals its negative. Both sequences tend to infinity and the two unit vectors are distinct, so no directional limit exists. At $n\pi + \pi/4$ the direction is perpendicular to its value at $n\pi$, so even the unoriented line of approach has no limit. A convergent state need not behave like a nonzero real dominant eigenmode.

Question 10

For $\gamma > 0$, consider the four-dimensional system

$$x'_1 = -x_2, \quad x'_2 = x_1, \quad x'_3 = -\gamma x_4, \quad x'_4 = \gamma x_3.$$

A state is periodic if there is $T > 0$ with $X(t+T) = X(t)$ for all real t . A constant solution has no least positive period.

Tasks

1. Find the eigenvalues and solve the IVP $X(0) = (1, 0, 1, 0)^T$. Describe its projections onto the two coordinate planes.
2. For $\gamma = \sqrt{2}$, prove that this full state is not periodic, even though both planar projections are periodic.
3. Still with $\gamma = \sqrt{2}$, classify exactly which arbitrary initial states give periodic solutions, and give their least periods when nonconstant.
4. Replace γ by $3/2$. Find the least period for the stated IVP and explain why purely imaginary eigenvalues alone do not guarantee a common period in higher dimensions.

Question 10 – Solution

Strategy. A full state repeats only when every active rotating block repeats at the same time.

Step 1: Solve each oscillatory block. The eigenvalues are $\pm i$ and $\pm i\gamma$. For the stated initial data,

$$X(t) = (\cos t, \sin t, \cos \gamma t, \sin \gamma t)^T.$$

The two projections are unit circles with respective least periods 2π and $2\pi/\gamma$. Their closure as plane curves alone does not settle repetition of the combined state.

Step 2: Exclude a common period for irrational frequency ratio. At $\gamma = \sqrt{2}$, any period must satisfy $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$ for positive integers m, n . This would imply $\sqrt{2} = n/m$, impossible. Hence the four-dimensional state is not periodic. Its norm nevertheless remains $\sqrt{2}$, so the obstruction is incompatible timing, not growth or damping.

Step 3: Classify all initial states in the irrational case. Each initially zero two-component block remains zero. A nonzero first block repeats exactly at integer multiples of 2π ; a nonzero second block repeats exactly at integer multiples of $2\pi/\sqrt{2}$. Thus precisely the states with at least one block zero give periodic solutions. If only the first block is nonzero, the least period is 2π ; if only the second is nonzero, it is $\sqrt{2}\pi$. If both vanish, the solution is constant. If both are nonzero, the same irrationality argument rules out every positive period, regardless of their initial phases.

Step 4: Recover a common period for rational frequency ratio. With $\gamma = 3/2$ and both blocks active, $T = 2\pi m$ must also satisfy $3m/2 \in \mathbb{Z}$. The least positive choice is $m = 2$, giving $T = 4\pi$. In these independent rotation systems all states are bounded, but common periodicity requires commensurate active frequencies. Two planar centers therefore need not combine into a periodic full state; purely imaginary eigenvalues alone do not provide the missing frequency ratio.

Question 1

Compare the systems $X' = AX$ and $X' = BX$, where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Both have the characteristic polynomial $(\lambda + 2)^2$.

Tasks

1. Find the eigenspace of each matrix and explain why the repeated characteristic root does not determine the number of independent eigenvectors.
2. Write the complete real solution of each system for initial state $(p, q)^T$. Verify any term containing t .
3. For the B system starting at $(0, 1)$, eliminate time with the correct domain. Find the maximum of $x(t)$ on $t \geq 0$.
4. Compare the two systems' phase trajectories and forward limits. Does every repeated eigenvalue require a polynomial factor in every solution?

Question 1 – Solution

Strategy. The eigenspace dimension decides whether a repeated root already supplies a full basis or needs another solution.

Step 1: Distinguish the eigenspaces. For A , every nonzero vector is a -2 eigenvector, so the eigenspace has dimension two. For B , $(B + 2I)(x, y)^T = (y, 0)^T$, so its eigenspace is $\text{span}(1, 0)^T$, of dimension one. Algebraic multiplicity is two in both cases; the geometric multiplicities differ.

Step 2: Recover both complete families. For initial (p, q) ,

$$X_A = e^{-2t} \begin{pmatrix} p \\ q \end{pmatrix}, \quad X_B = e^{-2t} \begin{pmatrix} p + qt \\ q \end{pmatrix}.$$

For B , first solve $y' = -2y$, then $(e^{2t}x)' = q$. This derives every solution, and direct differentiation verifies the qt term. In particular $te^{-2t}(1, 0)^T$ alone is not the missing solution; the accompanying $e^{-2t}(0, 1)^T$ is essential.

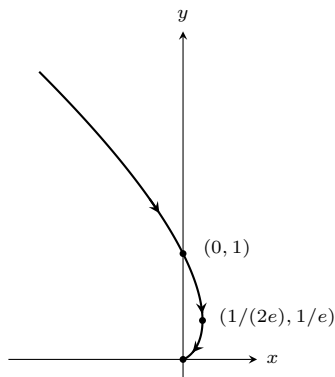
Step 3: Analyze the selected orbit. For B with $(p, q) = (0, 1)$, $x = te^{-2t}$ and $y = e^{-2t} > 0$. Eliminating t gives

$$x = -\frac{1}{2}y \ln y, \quad y > 0.$$

Forward time decreases y . On $t \geq 0$, $x' = e^{-2t}(1 - 2t)$ changes sign at $t = 1/2$, giving $x_{\max} = 1/(2e)$ at $y = e^{-1}$.

Step 4: Interpret multiplicity and geometry. Every nonzero A trajectory is a straight ray toward zero. For B , only the eigenline $y = 0$ gives straight rays; otherwise $x/y = p/q + t$ changes, giving curved trajectories. All solutions of both systems decay. A repeated root needs no polynomial factor when there is a full eigenbasis; even in the defective system, eigenline initial data have $q = 0$ and no t term. The figure shows the selected B orbit, not a generic ray.

$x = -\frac{1}{2}y \ln y$; arrows show increasing time



Question 2

Let

$$X' = NX, \quad N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Both eigenvalues are zero. A matrix is nilpotent if some positive power of it is zero; investigate the effect on the actual motion.

Tasks

1. Compute N^2 , find the entire equilibrium set, and determine its dimension.
2. Find the normalized evolution matrix and the solution from arbitrary $(p, q)^T$. Verify the evolution law directly.
3. For initial data outside the equilibrium line, find the phase trajectory and its time direction. Can it reach that line at a finite or infinite forward time?
4. Classify all forward-bounded solutions. Decide whether the origin is stable in the sense that sufficiently small initial states remain small for all $t \geq 0$.

Question 2 – Solution

Strategy. A zero eigenvalue can preserve an equilibrium line while a nilpotent part creates unbounded drift parallel to it.

Step 1: Identify the nilpotent part and equilibria. Direct multiplication gives $N^2 = 0$ but $N \neq 0$. The equation $NX = 0$ requires $x = y$, so the equilibria form the one-dimensional line $L = \{(s, s) : s \in \mathbb{R}\}$. The double zero eigenvalue has only one independent eigendirection.

Step 2: Construct the complete flow. The matrix $F(t) = I + tN$ has $F(0) = I$ and $F' = N = NF$, since $N^2 = 0$. Thus

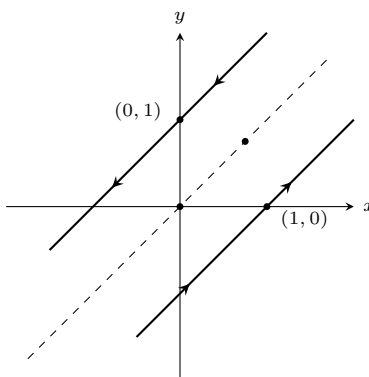
$$X(t) = \begin{pmatrix} p \\ q \end{pmatrix} + t(p - q) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Also $(I + tN)(I + sN) = I + (t + s)N$, verifying the evolution law and $F(t)^{-1} = F(-t)$. All real initial states and all real times are allowed.

Step 3: Find the drifting straight lines. Put $d = p - q$. If $d \neq 0$, then $x - y = d$ for every time and the velocity is the constant vector $d(1, 1)^T$. Thus the full orbit is the entire line $x - y = d$, traversed up/right for $d > 0$ and down/left for $d < 0$. Its distance to L stays $|d|/\sqrt{2} > 0$, so it neither reaches nor approaches the equilibrium line. If $d = 0$, the orbit is one fixed point, not the entire line L .

Step 4: Test boundedness and stability. Exactly $p = q$ gives a forward-bounded solution; all other solutions drift without bound. The origin is unstable: initial $(\varepsilon, 0)$ can be arbitrarily small, yet evolves to $(\varepsilon(1 + t), \varepsilon t)$, eventually leaving any fixed neighborhood. Zero real parts alone do not ensure boundedness or stability when a nonzero nilpotent part is present. The dashed line in the figure consists of separate equilibria.

Solid: drifting orbits; dashed: equilibrium line



Question 3

Consider

$$X' = AX, \quad A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For a repeated eigenvalue λ , a length-two chain means vectors v, w with $(A - \lambda I)v = 0$ and $(A - \lambda I)w = v \neq 0$.

Tasks

1. Find the eigenvalue, its eigenspace, and a chain with $v = (1, 1)^T$.
2. Derive and verify two independent real solutions using this chain. Explain why $te^{\lambda t}v$ alone fails.
3. Solve the IVP and verify the initial state and derivative.
4. Find every generalized vector w compatible with the fixed v . Explain how changing w changes the coefficients without changing the IVP solution.

Question 3 – Solution

Strategy. The generalized vector cancels the extra derivative introduced by the polynomial factor.

Step 1: Build a chain explicitly. The characteristic polynomial is $(\lambda+2)^2$. Let $N = A+2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; then $N^2 = 0$. Its kernel is $\text{span}(1, 1)^T$. With $v = (1, 1)^T$, choose $w = (1, 0)^T$, since $Nw = v$. The two vectors are independent.

Step 2: Verify the missing solution. Set $U = e^{-2t}v$ and $V = e^{-2t}(w + tv)$. Using $Nv = 0$, $Nw = v$, differentiation gives $U' = AU$ and $V' = AV$. Their determinant is $e^{-4t} \det(v, w) = -e^{-4t} \neq 0$, so they form a complete real basis. For $Z = te^{-2t}v$, however, $Z' - AZ = e^{-2t}v \neq 0$: the polynomial term by itself is not a solution.

Step 3: Determine the IVP coefficients. Since $(2, 1)^T = v + w$, the required solution is $U + V$:

$$X = e^{-2t} \begin{pmatrix} 2+t \\ 1+t \end{pmatrix}.$$

At zero it is $(2, 1)^T$, and its derivative is $(-3, -1)^T$, agreeing with $A(2, 1)^T$. The mode calculation verifies the equation for every time, not just at the initial point.

Step 4: Account for chain freedom. The equation $Nw = v$ means $w_1 - w_2 = 1$, so every choice is $w_c = (1 + c, c)^T = w + cv$, $c \in \mathbb{R}$. The new second solution is $V_c = V + cU$. Hence the same IVP becomes $X = (1 - c)U + V_c$. In general a combination $aU + bV$ becomes $(a - bc)U + bV_c$. The basis changes, but its span and every physical solution remain the same. Scaling or shifting a chain must be accompanied by the corresponding coefficient change.

Question 4

For $\varepsilon > 0$, let

$$X'_\varepsilon = \begin{pmatrix} -1 & 1 \\ 0 & -1 - \varepsilon \end{pmatrix} X_\varepsilon, \quad X_\varepsilon(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Study the limit as the two distinct eigenvalues merge. Let X_0 denote the solution of the same IVP with $\varepsilon = 0$.

Tasks

1. Solve the IVP for $\varepsilon > 0$ by using the distinct real modes.
2. Find the limit at each fixed time as $\varepsilon \downarrow 0$ and verify the resulting repeated-root solution.
3. For $0 \leq t \leq T$, prove bounds on $|x_\varepsilon - x_0|$ and $|y_\varepsilon - y_0|$ that tend uniformly to zero as $\varepsilon \downarrow 0$. You may use $0 \leq 1 - e^{-u} \leq u$ for $u \geq 0$.
4. At $t = 1/\varepsilon$, compare x_ε/x_0 with 1. Explain how coefficients that diverge and a relative error that persists can coexist with the fixed-time limit.

Question 4 – Solution

Strategy. Keep the difference of nearby exponentials together; an integral representation exposes the limit without subtracting divergent pieces.

Step 1: Solve before the roots merge. Eigenvectors are $(1, 0)^T$ for -1 and $(1, -\varepsilon)^T$ for $-1 - \varepsilon$. Decomposition of $(0, 1)^T$ gives coefficients $1/\varepsilon, -1/\varepsilon$, hence

$$x_\varepsilon = e^{-t} \frac{1 - e^{-\varepsilon t}}{\varepsilon}, \quad y_\varepsilon = e^{-(1+\varepsilon)t}.$$

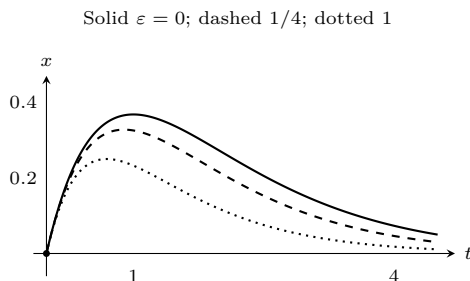
Step 2: Pass to the repeated-root solution. Since $(1 - e^{-\varepsilon t})/\varepsilon = \int_0^t e^{-\varepsilon s} ds$, its limit at each fixed real t is t . Thus $X_0 = e^{-t}(t, 1)^T$. Direct differentiation gives $x'_0 = -x_0 + y_0$, $y'_0 = -y_0$ and the stated initial data. The polynomial factor is the limit of the combined modes.

Step 3: Prove uniform finite-interval control. For $0 \leq t \leq T$, the supplied inequality yields

$$0 \leq x_0 - x_\varepsilon = e^{-t} \int_0^t (1 - e^{-\varepsilon s}) ds \leq \frac{1}{2} \varepsilon t^2 e^{-t} \leq \frac{1}{2} \varepsilon T^2.$$

Similarly $0 \leq y_0 - y_\varepsilon = e^{-t}(1 - e^{-\varepsilon t}) \leq \varepsilon t e^{-t} \leq \varepsilon T$. Both bounds tend uniformly to zero on every fixed finite interval.

Step 4: Separate relative and absolute effects. At $t = 1/\varepsilon$, $x_\varepsilon/x_0 = 1 - e^{-1}$, so the relative discrepancy is e^{-1} , independent of ε . This time moves to infinity, and $x_0 = \varepsilon^{-1} e^{-1/\varepsilon} \rightarrow 0$, so the absolute discrepancy still vanishes there. The separate modal coefficients diverge, but their exponential contributions cancel to a finite limit. Neither divergent coefficients nor this moving-time relative error contradicts the proved finite-interval convergence.



Question 5

Let

$$X' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Examine the whole trajectory for $-\infty < t < \infty$, including its approach to the origin backward in time.

Tasks

1. Find the repeated eigenvalue and eigenspace, and solve the IVP.
2. Eliminate time and retain the full orbit domain. Find the unique minimum of $x(t)$ and all its finite zeros.
3. Find the limiting normalized directions as $t \rightarrow -\infty$ and $t \rightarrow \infty$. Does approaching an eigenline imply that the trajectory lies on it?
4. Determine whether this orbit ever reaches the origin or repeats a state. Explain why its coordinate reversal is not an oscillation.

Question 5 – Solution

Strategy. The exponential controls overall growth, while the polynomial factor can reverse one coordinate and dominate the limiting direction.

Step 1: Recover the defective solution. The double eigenvalue is 1, with eigenspace $y = 0$. Solving $y' = y$ and then $(e^{-t}x)' = 3$ gives

$$x = 3te^t, \quad y = e^t.$$

The factor 3 is part of the coupling and cannot be omitted from the generalized mode. Both components satisfy the stated initial data.

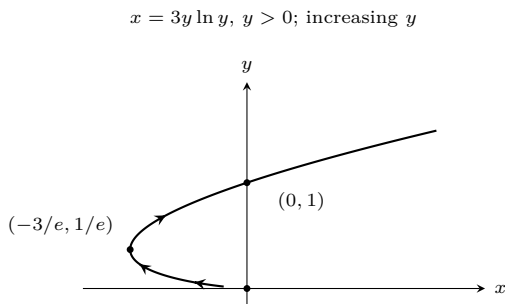
Step 2: Find the curve and minimum. Since $y > 0$, the orbit is $x = 3y \ln y, \quad y > 0$, traversed toward increasing y . Its derivative $x' = 3e^t(t+1)$ vanishes only at $t = -1$, changing from negative to positive. The global minimum is $x = -3/e$ at $y = 1/e$. The only finite zero of x is $t = 0$; its backward limit zero is not another attained zero.

Step 3: Find the two limiting directions. The normalized state is

$$\frac{X}{\|X\|} = \frac{(3t, 1)^T}{\sqrt{9t^2 + 1}}.$$

It tends to $(-1, 0)^T$ backward and $(1, 0)^T$ forward. Although the limiting line is the eigenline $y = 0$, every finite-time state has $y > 0$ and lies off that line. The origin is an unstable defective node; this IVP tends to it only as $t \rightarrow -\infty$.

Step 4: Distinguish reversal from repetition. The origin is never reached at finite time because $e^t > 0$. Also $y = e^t$ is strictly increasing, so two different times cannot give the same state. The single minimum of x comes from the factor te^t , not recurring oscillations. There is no periodic orbit or repeated cycle: a coordinate can reverse even with a positive repeated real eigenvalue. The equal-scale figure marks the minimum and the initial state.



Question 6

Let

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad X' = AX, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use the Euclidean vector norm. For $N = A + I$, the operator bounds $\|NX\| \leq \|X\|$ and $\|N^2X\| \leq \|X\|$ may be used.

Tasks

1. Compute the powers of N needed to find the normalized evolution and the complete solution.
2. For initial $(0, 0, 1)^T$, find the maximum of each of the first two components on $t \geq 0$ and its time.
3. For arbitrary nonzero initial data, identify when the largest polynomial factor in the solution is quadratic, linear, or constant. Do all solutions decay?
4. Can one constant C satisfy $\|X(t)\| \leq Ce^{-t}\|X(0)\|$ for all $t \geq 0$ and all initial data? Prove instead a bound with decay $e^{-t/2}$ and an explicit constant.

Question 6 – Solution

Strategy. A longer nilpotent chain truncates the exponential after a quadratic term; its degree matters for sharp decay estimates.

Step 1: Truncate the nilpotent exponential. Here N^2 has only entry $(1, 3)$ equal to one; $N^3 = 0$. Direct differentiation verifies $F(t) = e^{-t}(I + tN + t^2N^2/2)$ and $F(0) = I$. Consequently

$$X = e^{-t} \begin{pmatrix} p + qt + rt^2/2 \\ q + rt \\ r \end{pmatrix}.$$

This gives all initial states and shows why a length-three chain needs $t^2/2$, rather than another unrelated eigenvector.

Step 2: Locate the component peaks. For $(p, q, r) = (0, 0, 1)$, the first components are $t^2e^{-t}/2$ and te^{-t} . Their derivatives are $e^{-t}t(2-t)/2$ and $e^{-t}(1-t)$. Thus their maxima are $2e^{-2}$ at $t = 2$ and e^{-1} at $t = 1$, respectively. These peaks are not simultaneous.

Step 3: Classify polynomial degrees. If $r \neq 0$, the leading vector factor is $(r/2)t^2(1, 0, 0)^T$. If $r = 0$ but $q \neq 0$, it is $qt(1, 0, 0)^T$. If $q = r = 0$ and $p \neq 0$, it is the constant $p(1, 0, 0)^T$. Every polynomial factor times e^{-t} tends to zero, so all solutions decay; algebraic multiplicity three does not force degree two for every state.

Step 4: Prove an honest exponential estimate. For initial $(0, 0, 1)$, $e^t\|X(t)\| \geq t^2/2$, ruling out the proposed constant C . The given operator bounds instead imply $\|X(t)\| \leq e^{-t}(1 + t + t^2/2)\|X(0)\|$. Using $e^{-t/2} \leq 1$, $te^{-t/2} \leq 2/e$ and $(t^2/2)e^{-t/2} \leq 8/e^2$ for $t \geq 0$ gives

$$\|X(t)\| \leq (1 + 2/e + 8/e^2)e^{-t/2}\|X(0)\|.$$

A slightly slower exponential absorbs the polynomial transient.

Question 7

An unknown constant real 2×2 matrix A has characteristic polynomial $(\lambda + 1)^2$. Its normalized evolution F is measured exactly at one time:

$$F(1) = e^{-1} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

You may use the Cayley–Hamilton identity $(A + I)^2 = 0$.

Tasks

1. Recover A from the measured matrix and prove uniqueness under the stated spectral assumption.
2. Find $F(t)$ for every real time and verify the evolution and inverse laws.
3. Solve the IVP $X(0) = (1, 0)^T$. Does its first component cross zero for $t \geq 0$?
4. Would measuring only $\det F(1) = e^{-2}$ recover the same matrix? Give a different admissible matrix and explain what information the full observation supplies.

Question 7 – Solution

Strategy. Under the repeated-root assumption the nilpotent exponential is linear in time, making the full one-step observation invertible algebraically.

Step 1: Recover the nilpotent part. Let $N = A + I$. Since $N^2 = 0$, direct verification gives $F(t) = e^{-t}(I + tN)$. Therefore

$$N = eF(1) - I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.$$

This N indeed squares to zero and is nonzero. Every matrix with the stated characteristic polynomial has the same formula, so the full measurement determines it uniquely within the specified class.

Step 2: Verify all-time evolution. The normalized evolution is

$$F(t) = e^{-t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.$$

Its derivative equals AF and $F(0) = I$. Since $N^2 = 0$, multiplying $(I + tN)(I + sN)$ gives $I + (t + s)N$. Thus $F(t + s) = F(t)F(s)$ and $F(t)^{-1} = F(-t)$ for all real times.

Step 3: Inspect the recovered IVP. Selecting the first column gives $X = e^{-t}(1 + t, -t)^T$. At zero the state is $(1, 0)^T$ and the derivative is $(0, -1)^T = A(1, 0)^T$. For $t \geq 0$, the first component stays strictly positive because $1 + t > 0$, though it tends to zero. Its only finite zero on the entire real line is $t = -1$, outside the requested forward interval.

Step 4: Separate area data from the full map. The different matrix $\tilde{A} = -I$ has the same repeated eigenvalue and $\det \tilde{F}(1) = e^{-2}$, but $\tilde{F}(1) = e^{-1}I$. Indeed $\det(I + tN) = 1$, so the nilpotent shear is invisible to this area multiplier. The full observed matrix supplies directional and coupling information, which the determinant alone discards. Uniqueness above uses both the spectral assumption and the complete matrix observation.

Question 8

For $k \geq 0$, consider

$$X' = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix} X.$$

Investigate Euclidean monotonicity and construct an alternative quadratic size when the standard squared norm $x^2 + y^2$ is unsuitable.

Tasks

1. Find the complete solution and classify the eigenspace dimension as k varies. Prove forward decay for all initial data.
2. Classify all k for which $(x^2 + y^2)'$ is negative at every nonzero state, nonpositive with nonzero equality states, or positive at some state.
3. At the boundary parameter between those cases, decide whether the norm strictly decreases between any two distinct times along every nonzero solution. Address the instantaneous equality line.
4. Choose a positive coefficient $c = c(k)$ so that $H = x^2 + cy^2$ has strictly negative derivative at every nonzero state for every $k \geq 0$. Give an explicit choice and proof.

Question 8 – Solution

Strategy. The nilpotent coupling does not change the eigenvalues, but it can change which quadratic size decreases at every state.

Step 1: Solve for every coupling strength. The repeated eigenvalue is -1 . If $k = 0$, the eigenspace is the whole plane; if $k > 0$, it is $y = 0$. For initial (p, q) ,

$$X = e^{-t}(p + kqt, q)^T.$$

The component equations directly verify completeness. Constant and linear factors times e^{-t} both vanish forward, so every initial state decays.

Step 2: Locate the exact norm threshold. The squared-norm derivative is

$$(x^2 + y^2)' = -2x^2 + 2kxy - 2y^2.$$

For $0 \leq k < 2$, $2|xy| \leq x^2 + y^2$ makes this strictly negative off zero. At $k = 2$ it equals $-2(x - y)^2$, with equality on $x = y$. For $k > 2$, its value at $(1, 1)$ is $2k - 4 > 0$. Thus the threshold is $k = 2$, despite identical eigenvalues throughout the family.

Step 3: Distinguish instantaneous equality from an interval. At $k = 2$, a trajectory cannot stay on $x = y$ over an interval unless it is zero. On that line the velocity is $(x, -x)$, which is tangent to the line only when $x = 0$. Integrating $-2(x - y)^2$ over any interval therefore gives strictly negative change for every nonzero solution. The norm strictly decreases between distinct times even if its derivative vanishes at an isolated time.

Step 4: Construct a uniformly valid quadratic choice. Choose $c = 1 + k^2/2 > 0$. Completing the square yields

$$H' = -2(x - ky/2)^2 - (2 + k^2/2)y^2 < 0 \quad (X \neq 0).$$

The quadratic $H = x^2 + (1 + k^2/2)y^2$ is positive definite for every k , including zero. Its decrease is compatible with temporary Euclidean growth when $k > 2$; the two quantities measure different families of ellipses. The spectral decay is unchanged by this choice of measurement.

Question 9

The critically damped equation

$$x'' + 2x' + x = 0, \quad x(0) = 1, \quad x'(0) = -a, \quad a \geq 0$$

has a repeated negative characteristic root. A student claims that critical damping prevents the displacement from crossing its equilibrium for every choice of initial velocity.

Tasks

1. Write the equation as a state system with $y = x'$, find its eigenspace, and solve the stated IVP.
2. Classify exactly the values of a for which $x(t)$ crosses zero at a positive time. Give that time when it exists.
3. For the crossing case, find the subsequent minimum displacement and its time. For the other cases, describe forward monotonicity.
4. If the first displacement crossing is observed at a prescribed $T > 0$, recover the initial velocity. Explain why a single crossing does not imply oscillatory motion.

Question 9 – Solution

Strategy. The linear polynomial multiplying the repeated-root exponential determines whether the initial velocity produces an overshoot.

Step 1: Find the defective state solution. The state matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, with double eigenvalue -1 and eigenspace $\text{span}(1, -1)^T$. The initial data give

$$x = e^{-t}[1 + (1-a)t], \quad y = e^{-t}[-a - (1-a)t].$$

These formulas verify $x' = y$, $y' = -x - 2y$, and $(x(0), y(0)) = (1, -a)$.

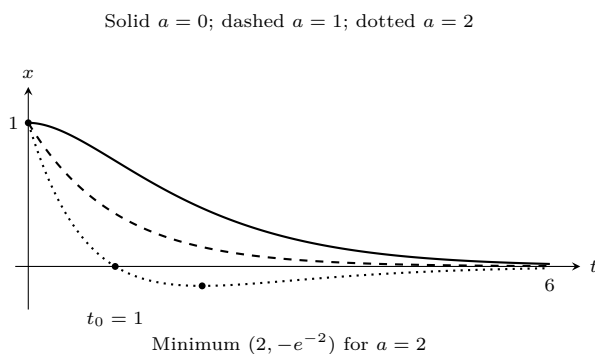
Step 2: Classify positive-time crossings. Because $e^{-t} > 0$, a positive-time zero occurs exactly when $a > 1$. It is unique and equals $t_0 = 1/(a-1)$. The polynomial has negative slope, so this is a crossing from positive to negative displacement. For $0 \leq a \leq 1$, the displacement remains positive at every finite forward time. Thus critical damping alone does not prevent overshoot.

Step 3: Locate the minimum or prove monotonicity. If $a > 1$, $y = e^{-t}[(a-1)t - a]$ changes from negative to positive at

$$t_{\min} = \frac{a}{a-1} = t_0 + 1, \quad x_{\min} = -(a-1)e^{-a/(a-1)}.$$

The displacement then increases toward zero from below. For $0 \leq a \leq 1$, $y < 0$ for all $t > 0$, so the positive displacement strictly decreases to zero; when $a = 0$, only the initial derivative vanishes.

Step 4: Infer the velocity from the crossing. Solving $T = 1/(a-1)$ gives $x'(0) = -a = -(1 + 1/T)$. The solution has only one zero because its exponential factor never vanishes and its polynomial factor is linear. After the single minimum it approaches zero from below without further crossings. This is an initial-data-dependent overshoot, not a repeating oscillation. The time plot compares $a = 0, 1, 2$; only the $a = 2$ curve crosses in those examples.



Question 10

For $\varepsilon \geq 0$, consider

$$A_\varepsilon = \begin{pmatrix} -1 & \varepsilon & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad X' = A_\varepsilon X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A matrix is diagonalizable if it has a basis of eigenvectors. For nonzero solutions, compare their normalized directions $X/\|X\|$.

Tasks

1. Find the eigenvalues and eigenspaces. For which ε is the matrix diagonalizable?
2. Find the complete solution for arbitrary initial data, including the polynomial term when present.
3. Identify all forward-bounded initial states. Within that set, classify the limiting forward directions for every nonzero initial state.
4. For initial $(0, 1, 0)^T$, compare the iterated limits of $X_\varepsilon/\|X_\varepsilon\|$ as $t \rightarrow \infty$ and $\varepsilon \downarrow 0$, in both orders. Explain what changes even though the eigenvalues do not.

Question 10 – Solution

Strategy. A repeated stable eigenvalue can have a two-dimensional eigenspace or a defective chain; the limiting direction detects that difference.

Step 1: Determine the eigenspaces. The eigenvalues are -1 (multiplicity two) and 2 . For -1 , $z = 0$ and $\varepsilon y = 0$. Thus its eigenspace is the whole plane $z = 0$ when $\varepsilon = 0$, but only the x -axis when $\varepsilon > 0$. The 2 eigenspace is always $\text{span}(1, 0, 1)^T$. There is a full eigenbasis exactly when $\boxed{\varepsilon = 0}$; this diagonalizable three-dimensional matrix is not a scalar matrix.

Step 2: Solve the coupled components. First $z = re^{2t}$ and $y = qe^{-t}$. Then $(e^t x)' = \varepsilon q + 3re^{3t}$, giving

$$X = e^{-t} \begin{pmatrix} p - r + \varepsilon q t \\ q \\ 0 \end{pmatrix} + re^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

This satisfies every component and recovers $(p, q, r)^T$ at zero.

Step 3: Classify the stable-plane directions. Forward boundedness holds exactly when $r = 0$. On that plane all states decay. For $\varepsilon = 0$, the nonzero direction stays constantly $(p, q, 0)^T / \sqrt{p^2 + q^2}$. For $\varepsilon > 0$ and $q \neq 0$, the normalized direction tends to $(\text{sgn } q, 0, 0)^T$. If $q = 0$, $p \neq 0$, it is constantly $(\text{sgn } p, 0, 0)^T$. The zero solution has no normalized direction.

Step 4: Compare the two orders of limits. For the specified initial state, the normalized vector is $(\varepsilon t, 1, 0)^T / \sqrt{\varepsilon^2 t^2 + 1}$. Taking $t \rightarrow \infty$ first with $\varepsilon > 0$, then $\varepsilon \downarrow 0$, gives $\boxed{(1, 0, 0)^T}$. Reversing the operations gives $\boxed{(0, 1, 0)^T}$. At every fixed finite time the solution depends continuously on ε , but long-time direction is not uniform near zero. The eigenvalues stay fixed while the eigenspace dimension changes.

Question 1

Consider the constant-input system

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Separate the equilibrium response from the transient response.

Tasks

1. Find the equilibrium and transform the equation into a homogeneous system by translating the state.
2. Use the homogeneous eigenmodes to find the complete family, then solve the stated IVP.
3. Determine whether either component overshoots its equilibrium value for $t \geq 0$. Give the limiting state and justify the signs of the derivatives.
4. Explain why the nonhomogeneous solution family is not a vector space. Which linear combinations of two solutions necessarily solve the same forced equation?

Question 1 – Solution

Strategy. A constant particular solution shifts the equilibrium; the translated state evolves through the familiar homogeneous modes.

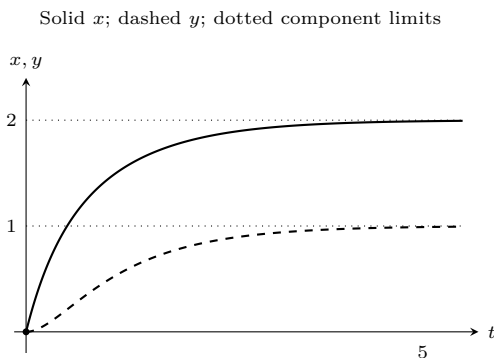
Step 1: Translate to the forced equilibrium. Solving $-2x + y + 3 = 0$, $x - 2y = 0$ gives $X_* = (2, 1)^T$. With $Z = X - X_*$, the equation becomes $Z' = AZ$. The origin is not an equilibrium of the forced system.

Step 2: Resolve the transient coefficients. The eigenpairs of A are $-1, (1, 1)^T$ and $-3, (1, -1)^T$. Thus every solution is $X_* + c_1 e^{-t}(1, 1)^T + c_2 e^{-3t}(1, -1)^T$. The initial state requires $c_1 = -3/2$, $c_2 = -1/2$, giving

$$x = 2 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

Step 3: Check monotonicity and the limit. Here $x' = \frac{3}{2}(e^{-t} + e^{-3t}) > 0$ for $t \geq 0$, while $y' = \frac{3}{2}(e^{-t} - e^{-3t}) > 0$ for $t > 0$ and $y'(0) = 0$. Both start at zero and tend to 2 and 1, respectively. Their monotone approach excludes overshoot; the initial velocity is $(3, 0)^T$, as the original equation requires. The two horizontal dotted lines in the figure are component limits, not additional time-dependent solutions.

Step 4: Identify the affine superposition rule. If U, V solve the same forced equation, then $(\alpha U + \beta V)' - A(\alpha U + \beta V) = (\alpha + \beta)(3, 0)^T$. It is a solution exactly when $\alpha + \beta = 1$. In particular the zero function is not a solution, and the sum of two solutions generally fails. Differences of solutions are homogeneous; adding any homogeneous solution to one particular solution gives the complete affine family derived above.



Question 2

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both homogeneous eigenvalues are negative, but the forcing has an unbounded component. Use a polynomial particular solution.

Tasks

1. Find a particular solution of the form $X_p = ut + v$, with constant vectors u, v . Explain why a constant trial cannot suffice.
2. Add the complete homogeneous family and impose the initial data.
3. Verify the result in the original equations and determine the limits of $x(t) - t$ and $y(t)$.
4. Decide which components are bounded and whether two solutions with different initial data approach one another. Explain why homogeneous stability does not force convergence to a constant here.

Question 2 – Solution

Strategy. Match polynomial coefficients before imposing initial data; stability controls differences of responses, not the size of an unbounded input.

Step 1: Match the polynomial coefficients. Substituting $X_p = ut + v$ gives $u = A(ut + v) + (t, 1)^T$. The coefficient of t requires $Au = -(1, 0)^T$, hence $u = (1, 0)^T$. The constant equation $u = Av + (0, 1)^T$ gives $v = (0, 1)^T$. Thus $X_p = (t, 1)^T$. A constant trial cannot cancel the forcing's nonzero coefficient of t .

Step 2: Complete the family and solve the IVP. The homogeneous solution is $e^{-t}(c_1 + c_2t, c_2)^T$. Initial zero gives $c_1 = 0$, $c_2 = -1$, so

$$x = t(1 - e^{-t}), \quad y = 1 - e^{-t}.$$

Step 3: Check the original equation and asymptotics. We have $x' = 1 - e^{-t} + te^{-t}$ and $y' = e^{-t}$. The right sides $-x + y + t$ and $-y + 1$ give those same expressions. The initial values vanish. Moreover $x - t = -te^{-t} \rightarrow 0$ and $y \rightarrow 1$. The first component tracks a growing ramp; its tracking error decays, but its value does not tend to a finite constant.

Step 4: Separate stability from bounded response. On $t \geq 0$, y is bounded between zero and one, while $x \rightarrow \infty$. For any two solutions, their difference has the form $e^{-t}(d_1 + d_2t, d_2)^T \rightarrow 0$. Thus initial-data effects disappear even though the common forced response grows. Negative eigenvalues of A guarantee decay of the homogeneous transient; they do not turn an unbounded polynomial forcing into a bounded steady state. A complete answer therefore describes both the particular response and the transient.

Question 3

For real constants a, b , consider

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X + e^t \begin{pmatrix} a \\ b \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A student says that because the forcing exponent equals an eigenvalue, a factor t must occur in a particular solution for every nonzero (a, b) .

Tasks

1. Determine exactly when a trial $X_p = e^t v$ with constant vector v works, and describe all such v .
2. Construct a particular solution valid for every (a, b) and then solve the zero-state IVP.
3. Explain how the component of the forcing in the resonant eigendirection decides whether a polynomial factor is unavoidable.
4. For $a = 0, b \neq 0$, determine the forward growth of the solution. Is growth by itself evidence of resonance? Address the student's claim.

Question 3 – Solution

Strategy. Resonance is a compatibility problem for the forcing direction, not merely a comparison of two scalar exponents.

Step 1: Test the simple exponential trial. Substitution gives $(I - A)v = (a, b)^T$, or $0 = a$, $3v_2 = b$. Thus the trial works exactly when $a = 0$, with $v = (c, b/3)^T$, $c \in \mathbb{R}$. The free first component is a homogeneous contribution, not a new forced effect.

Step 2: Construct the general forced response. A particular solution for all parameters is $X_p = (ate^t, (b/3)e^t)^T$. Adding $(c_1e^t, c_2e^{-2t})^T$ and imposing zero initial data gives

$$x = ate^t, \quad y = \frac{b}{3}(e^t - e^{-2t}).$$

Differentiation gives $x' - x = ae^t$ and $y' + 2y = be^t$, verifying both forcing components and the initial values.

Step 3: Identify the resonant component. The resonant eigendirection is $(1, 0)^T$. Its forced scalar equation is $(e^{-t}x)' = a$, so a nonzero a necessarily creates ate^t . When $a = 0$, there is no forcing in that direction and no polynomial factor is needed. No choice of homogeneous constants can cancel a nonzero ate^t for all time. The condition is directional, not just spectral.

Step 4: Separate forced growth from resonance. For $a = 0, b \neq 0$, the zero-state solution has $x = 0$ and $y/e^t \rightarrow b/3$. It grows because its forcing grows, although the forced second component has homogeneous rate -2 and is not resonant. The first eigenvalue 1 is present in the matrix but receives no input. Thus growth alone does not prove resonance, and the student's claim fails for every nonzero forcing vector of the form $(0, b)^T$.

Question 4

Consider the time-dependent system

$$X' = A(t)X + f(t), \quad A(t) = \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use variation of parameters and check the correct two-time transition matrix.

Tasks

1. Find a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ and its inverse.
2. Derive the variation-of-parameters equation by writing $X = \Phi C$, then compute the IVP solution.
3. Find $T(t, s) = \Phi(t)\Phi(s)^{-1}$ and express the same solution as an integral of $T(t, s)f(s)$. Evaluate it.
4. A proposed shortcut replaces $T(t, s)$ by $\Phi(t - s)$. Compute the resulting candidate and its residual in the original system. Explain why the shortcut fails.

Question 4 – Solution

Strategy. For time-dependent coefficients, propagation depends on both the departure time and the arrival time.

Step 1: Normalize the homogeneous matrix. Solving $y' = 0$, $x' = 2ty$ gives

$$\Phi(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}, \quad \Phi(t)^{-1} = \begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix}.$$

Its determinant is one, and direct differentiation verifies $\Phi' = A(t)\Phi$ and $\Phi(0) = I$.

Step 2: Derive and integrate the varying coefficients. Substituting $X = \Phi C$ cancels $\Phi' C = A\Phi C$, leaving $C' = \Phi^{-1}f = (t - t^2, 1)^T$. Zero initial data gives $C = (t^2/2 - t^3/3, t)^T$. Therefore

$$X = (t^2/2 + 2t^3/3, t)^T.$$

Indeed $x' = t + 2t^2 = 2ty + t$ and $y' = 1$.

Step 3: Use the genuine two-time transition. Matrix multiplication yields

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Thus $X(t) = \int_0^t T(t, s)f(s) ds = \int_0^t (s + t^2 - s^2, 1)^T ds$, reproducing the same formula. This expression also follows directly from $\Phi(t) \int_0^t \Phi(s)^{-1}f(s) ds$.

Step 4: Expose the incorrect shortcut. Using $\Phi(t-s)$ instead gives $\tilde{X} = (t^2/2 + t^3/3, t)^T$. Its residual is

$$\tilde{X}' - A(t)\tilde{X} - f(t) = (-t^2, 0)^T,$$

which is not identically zero. The two matrices differ because $t^2 - s^2 \neq (t-s)^2$ in general. A fundamental matrix normalized at zero cannot automatically be shifted in time when the coefficient matrix itself changes with time.

Question 5

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The forcing matches a repeated homogeneous exponent and enters through the second component of a length-two chain.

Tasks

1. Use $Z = e^t X$ to derive a simpler forced system and solve the IVP.
2. Test the trial $X_p = te^{-t}v$ with constant vector v . Explain why it cannot supply a particular solution here.
3. Find the forward maximum of each component and its time. Do the maxima occur together?
4. Determine the long-time behavior and compare the polynomial degree of this forced response with the homogeneous family. Does resonance necessarily imply an unbounded response?

Question 5 – Solution

Strategy. Removing the common exponential reveals successive integrations along the nilpotent chain.

Step 1: Integrate along the chain. With $N = A + I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the transformed system is $Z' = NZ + (0, 1)^T$, $Z(0) = 0$. Thus $z_2' = 1$, $z_1' = z_2$, giving $z_2 = t$, $z_1 = t^2/2$ and

$$x = \frac{1}{2}t^2e^{-t}, \quad y = te^{-t}.$$

Substitution verifies the original equation and zero initial state.

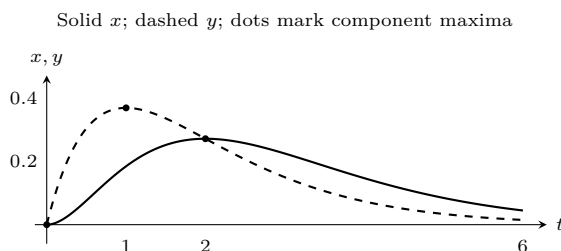
Step 2: Reject an insufficient polynomial trial. The trial would give $Z = tv$, so its equation requires $v = tNv + (0, 1)^T$ for every time. Matching coefficients forces $Nv = 0$ and $v = (0, 1)^T$, but $N(0, 1)^T = (1, 0)^T \neq 0$. These conditions contradict each other. A factor t multiplying a constant vector is insufficient; the chain requires a quadratic term in the first component.

Step 3: Locate the separate response peaks. For $t \geq 0$, $x' = e^{-t}t(2-t)/2$ and $y' = e^{-t}(1-t)$. Their sign changes show

$$x_{\max} = 2e^{-2} \text{ at } t = 2, \quad y_{\max} = e^{-1} \text{ at } t = 1.$$

The maxima occur at different states and different times. The figure marks these component peaks on the two time graphs.

Step 4: Interpret resonance without growth to infinity. Both components tend to zero, since every fixed polynomial times e^{-t} vanishes. The homogeneous family is $e^{-t}(c_1 + c_2t, c_2)^T$, whose largest degree is one; the forcing raises the first component's degree to two. This is a resonant polynomial response, even though the negative exponential ultimately wins. Resonance describes an enhanced polynomial factor, not a universal guarantee of an unbounded solution. Adding any homogeneous solution still leaves forward decay.



Question 6

For $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, consider

$$X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Investigate the response to a forcing that rotates at the homogeneous oscillation frequency.

Tasks

1. Use $X = R(t)Z$ to derive and solve a constant-forcing equation for Z . Give the full family.
2. Find the unique 2π -periodic solution and prove its uniqueness.
3. For $X(0) = 0$, find the state, its distance from the periodic solution at the same time, and its distance from the unit circle.
4. Does the zero-state response become exactly periodic at a finite time or tend to a single point? Explain how its state curve should be interpreted in this time-dependent forced problem.

Question 6 – Solution

Strategy. A rotating coordinate frame removes the oscillation and leaves scalar damping toward a constant vector.

Step 1: Pass to the rotating frame. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $R' = JR$ and $A = -I + J$. Substitution cancels the rotational terms, giving $Z' = -Z + (1, 0)^T$. Hence

$$X(t) = R(t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \left(c - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right],$$

where $c = X(0)$ is an arbitrary real vector. This gives every solution.

Step 2: Identify and prove the periodic response. Taking $c = (1, 0)^T$ gives $P(t) = (\cos t, \sin t)^T$. If another solution were 2π -periodic, its difference from P would be $e^{-t}R(t)d$ and would satisfy $d = e^{-2\pi}d$ at times zero and 2π . Thus $d = 0$, proving uniqueness. Damping prevents a secular term despite the matched rotation frequency.

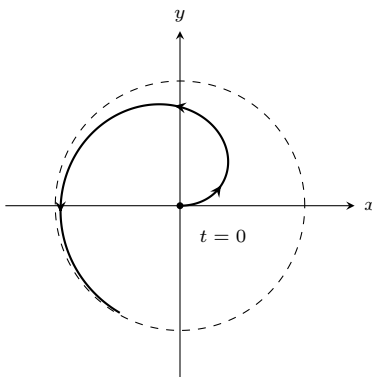
Step 3: Quantify zero-state convergence. For $c = 0$,

$$X(t) = (1 - e^{-t})(\cos t, \sin t)^T \quad (t \geq 0).$$

Since rotations preserve length, $\|X(t) - P(t)\| = e^{-t}$. Its radius is $1 - e^{-t}$, so its distance to the unit circle is also e^{-t} . Its initial velocity is $(1, 0)^T$, agreeing with the input at zero.

Step 4: Distinguish asymptotic response from exact return. The radius is strictly increasing at every finite forward time, so the solution neither repeats a state nor becomes exactly periodic later. Along $t = 2\pi n$ and $t = 2\pi n + \pi$, its states tend to $(1, 0)^T$ and $(-1, 0)^T$, respectively; there is no single point limit. The drawn curve is a nonautonomous state path approaching a periodic response, not an autonomous phase portrait or a claim of a finite-time merger with the circle. Only a finite initial segment is drawn.

Solid: initial state path; dashed: periodic-response circle



Question 7

On $t \geq 0$, consider the bounded input in

$$x' = x + e^{-t}, \quad y' = -y + 1, \quad (x(0), y(0)) = (p, q).$$

Investigate which initial states suppress the unstable homogeneous mode.

Tasks

1. Find the complete solution and classify every initial state yielding a bounded forward response.
2. Find the limiting state for those bounded solutions. Is that limiting vector, held constant, itself a solution of the original forced equation?
3. Express the bounded first component using an integral from the current time to infinity. Explain how this condition selects its initial value.
4. Perturb the selected initial first component by δ , with $0 < |\delta| < 1$. Find when its difference from the bounded response first reaches magnitude one. Does bounded forcing ensure bounded responses here?

Question 7 – Solution

Strategy. Boundedness can require an exact cancellation of an unstable mode rather than follow from bounded forcing alone.

Step 1: Solve and isolate the growing coefficient. Integrating factors give

$$x = (p + \frac{1}{2})e^t - \frac{1}{2}e^{-t}, \quad y = 1 + (q - 1)e^{-t}.$$

The second component is bounded for every q . The first is bounded exactly when $p = -1/2$, so the full bounded family has that fixed first initial value and arbitrary q .

Step 2: Check the limit against the original equation. For bounded data, $X(t) \rightarrow (0, 1)^T$. However the constant function $X = (0, 1)^T$ has zero derivative, whereas the original right side is $(e^{-t}, 0)^T \neq 0$ at finite times. It is the limiting equilibrium of the limiting equation, not an exact constant solution of the stated time-dependent equation.

Step 3: Recover the cancellation from a future integral. Solving the first equation with no growing term gives

$$x(t) = - \int_t^\infty e^{t-s} e^{-s} ds = -\frac{1}{2}e^{-t}.$$

At zero it selects $p = -1/2$. This is a boundedness condition imposed over the whole future, used to choose an initial value. It does not replace the ordinary forward initial-value evolution once that value is fixed.

Step 4: Measure sensitivity to the unstable mode. Changing p by δ adds δe^t to the first component. Its absolute difference starts below one and grows strictly, first reaching one at $t = \ln(1/|\delta|)$. The input remains bounded throughout, but every such perturbed response is unbounded. The unstable homogeneous coefficient, not the input size alone, determines the failure. Exact cancellation selects a special family; it does not make the underlying system stable.

Question 8

For $t \geq 0$, let

$$x' = -x + y, \quad y' = -y + u(t), \quad x(0) = y(0) = 0,$$

where $u(t) = 1$ for $0 \leq t < 1$ and $u(t) = 0$ for $t > 1$. Take the state to be continuous at the switch; use one-sided derivatives there. The value assigned to $u(1)$ does not change the continuous solution.

Tasks

1. Solve on $0 \leq t \leq 1$, then continue for $t \geq 1$ using $\tau = t - 1$ and the state at the switch.
2. Find the one-sided state derivatives at $t = 1$. Which component derivative is continuous, and which jumps?
3. Find the global forward maximum of x and its time. Explain why the maximum occurs after the input has switched off.
4. Find the long-time limit and the integrals $\int_0^\infty x(t) dt$ and $\int_0^\infty y(t) dt$ by integrating the differential equations.

Question 8 – Solution

Strategy. Match the state at the switch, then track the stored second-component response as it continues to feed the first.

Step 1: Solve and match the two intervals. For $0 \leq t \leq 1$, integration gives $x = 1 - (1+t)e^{-t}$, $y = 1 - e^{-t}$. Set $a = 1 - 2/e$, $b = 1 - 1/e$, the switch values. For $\tau = t - 1 \geq 0$, the homogeneous continuation is

$$x = e^{-\tau}(a + b\tau), \quad y = be^{-\tau}.$$

At $\tau = 0$ these match the first formulas, so both components are continuous.

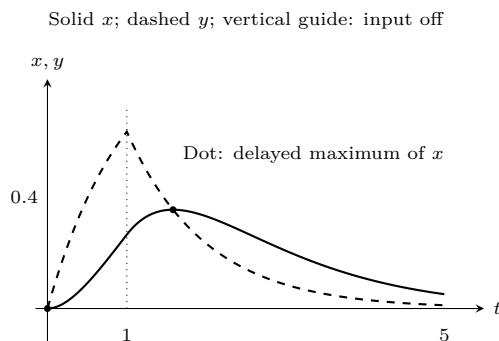
Step 2: Compare one-sided derivatives. Before the switch, $(x', y') = (te^{-t}, e^{-t})$. Thus $X'(1^-) = (1/e, 1/e)^T$. Immediately after it, $X'(1^+) = (-a+b, -b)^T = (1/e, -1+1/e)^T$. The derivative x' is continuous; y' has jump -1 . A finite step input changes a derivative, not the state itself.

Step 3: Find the delayed first-component peak. Before switching, $x' > 0$ for $t > 0$. Afterward, $x' = e^{-\tau}(b - a - b\tau)$, with $b - a = 1/e$. It changes sign once, at $\tau = 1/(e - 1)$, hence

$$t_{\max} = 1 + \frac{1}{e - 1}, \quad x_{\max} = (1 - 1/e)e^{-1/(e-1)}.$$

The still-positive y initially exceeds x after the input ends, so $x' = -x + y$ remains positive until the two components become equal.

Step 4: Integrate the total response. Both components tend to zero exponentially times a linear factor, so their integrals converge. Integrating $y' = -y + u$ over $[0, \infty)$ and using its zero endpoints gives $\int y = \int u = 1$. Integrating $x' = -x + y$ similarly gives $\int_0^\infty x(t) dt = \int_0^\infty y(t) dt = 1$. Equal total areas do not mean equal graphs or simultaneous peaks. The figure marks the switch and the delayed maximum of x .



Question 9

Consider the input-driven system

$$x' = y, \quad y' = u(t), \quad x(0) = y(0) = 0, \quad 0 \leq t \leq 1.$$

Choose an input so that $(x(1), y(1)) = (1, 0)$. First restrict attention to inputs of the form $u(t) = a + bt$.

Tasks

1. Express the endpoint conditions as two integral conditions on a general continuous input u .
2. Find the unique affine input $a + bt$ satisfying both conditions. Explain why a constant input cannot work.
3. Compute the entire state path for that input. Determine whether displacement is monotone and find the maximum velocity.
4. Is the input unique among all continuous functions? Test additions of $c(6t^2 - 6t + 1)$, with arbitrary real c , and distinguish endpoint agreement from identical intermediate motion.

Question 9 – Solution

Strategy. The endpoint conditions constrain two weighted integrals of the input, not all of its time-dependent values.

Step 1: Derive the two input moments. Integrating gives $y(t) = \int_0^t u(s) ds$. A second integration yields $x(t) = \int_0^t (t-s)u(s) ds$, verified by differentiation and the zero initial data. The endpoint requirements are

$$\int_0^1 u(s) ds = 0, \quad \int_0^1 (1-s)u(s) ds = 1.$$

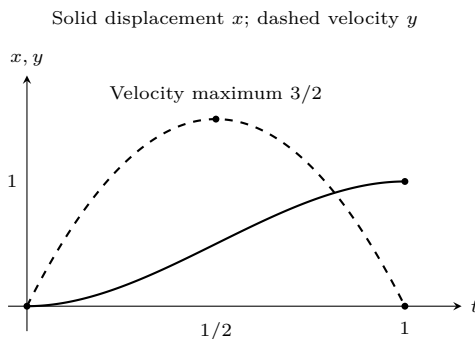
Step 2: Solve within the affine family. For $u = a + bt$, these conditions become $a + b/2 = 0$ and $a/2 + b/6 = 1$. Their unique solution is $a = 6, b = -12$, so $u = 6 - 12t$. A constant input would need $a = 0$ from the final velocity condition, then could not create displacement one.

Step 3: Recover and interpret the state path. Integration gives

$$y = 6t(1-t), \quad x = 3t^2 - 2t^3.$$

The state starts at zero and ends at $(1, 0)$. Since $y > 0$ for $0 < t < 1$, displacement strictly increases through the interior. Velocity has its unique maximum $3/2$ at $t = 1/2$, where the input changes sign. The negative input in the second half slows the positive velocity to zero; it does not reverse the displacement.

Step 4: Expose the remaining freedom. Let $g(t) = 6t^2 - 6t + 1$. Direct integration gives $\int_0^1 g = 0$ and $\int_0^1 sg(s) ds = 0$, hence also $\int_0^1 (1-s)g(s) ds = 0$. Therefore every $u = 6 - 12t + cg(t)$ reaches the same endpoint. For $c \neq 0$, g is not identically zero, so the velocity derivative and intermediate motion differ. Uniqueness holds within the affine trial family, not among all continuous inputs. No monotonicity claim is made for the added-input family.



Question 10

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Consider

$$X' = JX + e^{-t^2} R(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \geq 0.$$

You may use $\int_0^\infty e^{-s^2} ds = \sqrt{\pi}/2$. Exact definite-integral answers are expected; no elementary antiderivative is assumed.

Tasks

1. Use variation of parameters to find the solution for arbitrary $X(0) = c$. Specialize to $c = 0$.
2. For $c = 0$, find the limiting norm. Does the state converge to a point or become periodic? Explain why forcing that tends to zero need not produce a state that tends to zero.
3. Find exactly the initial state that makes $X(t) \rightarrow 0$. Express that solution with an integral from t to infinity.
4. Prove a quantitative decay bound for this selected solution using $\int_t^\infty e^{-s^2} ds \leq e^{-t^2}/(2t)$ for $t > 0$, deriving the inequality as part of the solution.

Question 10 – Solution

Strategy. A rotating frame converts the forcing into accumulated scalar amplitude; cancelling its total integral removes the persistent free rotation.

Step 1: Integrate in rotating coordinates. Since $R' = JR$ and $R^{-1} = R(-t)$, writing $X = R(t)Z$ gives $Z' = e^{-t^2}(1, 0)^T$. Define $G(t) = \int_0^t e^{-s^2} ds$. Then

$$X(t) = R(t) \begin{bmatrix} c + G(t) \\ 0 \end{bmatrix}.$$

For $c = 0$ the state is $G(t)(\cos t, \sin t)^T$. Differentiation verifies the equation, including the non-elementary amplitude derivative.

Step 2: Interpret the zero-state asymptotics. Let $L = \sqrt{\pi}/2$. For $c = 0$, the norm is $G(t) \uparrow L$, strictly increasing at every finite time, so the solution is not periodic. At times $2\pi n$ and $2\pi n + \pi$ it tends to $(L, 0)^T$ and $(-L, 0)^T$, so no point limit exists. The input decays, but the undamped homogeneous rotation retains the accumulated amplitude; there is no dissipative mechanism to erase it.

Step 3: Select the unique cancelling initial state. Because rotations preserve length, $\|X(t)\| \rightarrow \|c + L(1, 0)^T\|$. Thus $X(t) \rightarrow 0$ exactly when $c = (-L, 0)^T$. For that state,

$$X(t) = -R(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \int_t^\infty e^{-s^2} ds.$$

Any other initial state leaves a nonzero limiting amplitude.

Step 4: Bound the remaining tail. For $s \geq t > 0$, $1 \leq s/t$. Therefore

$$0 \leq \int_t^\infty e^{-s^2} ds \leq \frac{1}{t} \int_t^\infty s e^{-s^2} ds = \frac{e^{-t^2}}{2t}.$$

The selected solution consequently obeys $\|X(t)\| \leq e^{-t^2}/(2t) \quad (t > 0)$. This is a tail estimate, not a formula at $t = 0$; the initial norm there is L . The definite integral and the bound avoid inventing an elementary antiderivative or replacing the exact response by numerical samples.

Question 1

For $t \geq 0$, consider the coupled initial-value problem

$$x' = -2x + y, \quad y' = 2x - 3y, \quad x(0) = 1, \quad y(0) = 0.$$

Write $F(s) = \mathcal{L}\{x\}(s)$ and $G(s) = \mathcal{L}\{y\}(s)$, using one-sided Laplace transforms. Recover both components by solving the transformed algebraic system.

Tasks

1. Derive the two transformed equations, retaining the initial terms. Solve for F and G and state a common half-plane of convergence.
2. Invert both transforms and verify the initial values and original differential equations.
3. Find the transform and time response of $w = x - y$. Explain why one pole of the full system is absent from this observation.
4. Find the exact time and value of the largest $y(t)$ on $t \geq 0$. Explain how y can initially increase while both system modes decay.

Question 1 – Solution

Strategy. Keep the initial data on the right side of the transformed system, then distinguish component responses from observed combinations.

Step 1: Solve the algebraic system. The derivative rule gives $(s+2)F - G = 1$ and $-2F + (s+3)G = 0$. Their determinant is $(s+1)(s+4)$, so

$$F = \frac{s+3}{(s+1)(s+4)}, \quad G = \frac{2}{(s+1)(s+4)}.$$

Both transforms converge for $\operatorname{Re} s > -1$. Dropping the initial term 1 would incorrectly give the zero solution.

Step 2: Invert and check. Partial fractions give

$$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}, \quad y = \frac{2}{3}(e^{-t} - e^{-4t}).$$

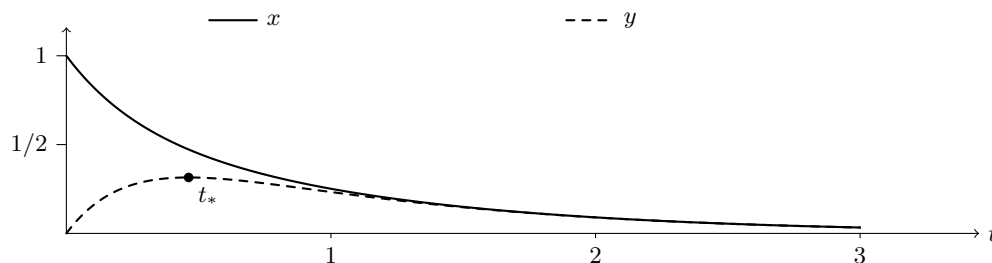
At zero these are $(1, 0)$. Differentiation gives $x' = -\frac{2}{3}e^{-t} - \frac{4}{3}e^{-4t}$ and $y' = -\frac{2}{3}e^{-t} + \frac{8}{3}e^{-4t}$, which equal $-2x + y$ and $2x - 3y$.

Step 3: Explain the cancelled pole. Subtracting the transforms gives $F - G = 1/(s+4)$, hence $w = e^{-4t}$. The slow mode is a multiple of $(1, 1)^T$; subtracting its components annihilates it. The state still contains this mode. Cancellation in an observation does not remove a mode from the system.

Step 4: Locate the component peak. Since $y' = \frac{2}{3}e^{-t}(4e^{-3t} - 1)$, it changes sign once, from positive to negative, at $t_* = \log(4)/3$. Thus

$$\max_{t \geq 0} y(t) = \frac{1}{2} 4^{-1/3}, \quad t_* = \log(4)/3.$$

Here $y'(0) = 2$ because the first component feeds the second. Decay of each mode does not imply monotonic decay of every component: their signed combination initially grows. The curve tends back to zero.



Question 2

Let $Z = (x, y, z)^T$ satisfy $Z' = AZ$ for $t \geq 0$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad Z(0) = (a, b, c)^T.$$

The characteristic polynomial has a triple root. Investigate how the initial vector determines the poles actually present in each transformed component.

Tasks

1. Compute $(sI - A)^{-1}$ directly, and verify its product with $sI - A$ is I . State where the inverse matrix exists.
2. Use the resolvent to find all three Laplace transforms and invert them for arbitrary real a, b, c .
3. For the initial state $(0, 0, 1)^T$, verify the original equations and explain the different polynomial degrees in the three components.
4. Classify the pole order of $\mathcal{L}\{x\}$ at $s = -2$ for every (a, b, c) , including the case of no pole. Does a triple characteristic root force a triple pole in every component?

Question 2 – Solution

Strategy. Invert the triangular matrix before substituting data; its powers of $(s+2)^{-1}$ display how a chain transmits the initial state.

Step 1: Compute the resolvent. Put $q = s + 2$. For $q \neq 0$,

$$(sI - A)^{-1} = \begin{pmatrix} q^{-1} & q^{-2} & q^{-3} \\ 0 & q^{-1} & q^{-2} \\ 0 & 0 & q^{-1} \end{pmatrix}.$$

Multiplication by the upper triangular matrix with diagonal q and superdiagonal -1 gives diagonal entries 1; each off-diagonal entry cancels. The inverse exists exactly for $s \neq -2$. This algebraic domain is larger than the convergence half-plane $\operatorname{Re} s > -2$ for the nonzero responses.

Step 2: Multiply by the initial vector. The transforms are $a/q + b/q^2 + c/q^3$, $b/q + c/q^2$, and c/q . Since $\mathcal{L}\{t^n e^{-2t}\} = n!/(s+2)^{n+1}$,

$$x = e^{-2t}(a + bt + ct^2/2), \quad y = e^{-2t}(b + ct), \quad z = ce^{-2t}.$$

The factorial is essential: the inverse of q^{-3} is $t^2 e^{-2t}/2$.

Step 3: Check transmission along the chain. For the specified initial state,

$$Z = (t^2 e^{-2t}/2, t e^{-2t}, e^{-2t})^T.$$

Directly, $z' = -2z$, $y' = -2y + z$ and $x' = -2x + y$, with the required initial state. After multiplication by e^{2t} , the equations become $(e^{2t}z)' = 0$, $(e^{2t}y)' = e^{2t}z$ and $(e^{2t}x)' = e^{2t}y$. Two successive integrations produce the quadratic factor.

Step 4: Classify actual pole orders. The numerator of the first transform over q^3 is $aq^2 + bq + c$. Its order at $q = 0$ gives the exhaustive classification:

initial-data condition	pole order of $\mathcal{L}\{x\}$
$c \neq 0$	3
$c = 0, b \neq 0$	2
$c = b = 0, a \neq 0$	1
$a = b = c = 0$	no pole

Thus the characteristic multiplicity supplies a possible order, not an order every response must attain. Even for $c \neq 0$, the third component has only a simple pole, whereas the first has a triple pole.

Question 3

A delayed input drives the zero-state system

$$x' = -x + H(t - 2), \quad y' = x - 2y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

Here $H(t - a)$ is zero for $t < a$ and one for $t > a$; its value at the switch does not affect the continuous state. Interpret derivatives there one-sidedly.

Tasks

1. Find the transforms of both components and identify the factor that encodes the input delay.
2. Invert them to obtain a piecewise solution. Explain why multiplying an undelayed response by e^{-2t} does not implement a delay.
3. Determine continuity of x, y, x', y' at $t = 2$, and find the one-sided values of y'' there.
4. Prove both components increase after the switch and approach their limits without overshoot. Sketch their time responses, marking the switch and limiting levels.

Question 3 – Solution

Strategy. First invert the undelayed rational factors, then translate the entire response using the second shifting theorem.

Step 1: Transform both equations. With $F = \mathcal{L}\{x\}$ and $G = \mathcal{L}\{y\}$, $(s+1)F = e^{-2s}/s$ and $(s+2)G = F$. Hence

$$F = \frac{e^{-2s}}{s(s+1)}, \quad G = \frac{e^{-2s}}{s(s+1)(s+2)}, \quad \operatorname{Re} s > 0.$$

It is e^{-2s} , rather than an exponential in time, that encodes the delay.

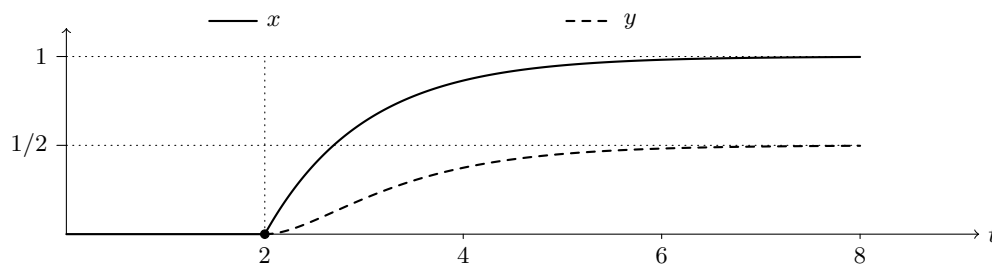
Step 2: Shift the inverse transforms. The undelayed inverses are $1 - e^{-t}$ and $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2}(1 - e^{-t})^2$. Writing $\tau = t - 2$, the solution is zero for $t < 2$, and for $t \geq 2$,

$$x = 1 - e^{-\tau}, \quad y = \frac{1}{2}(1 - e^{-\tau})^2.$$

These formulas give zero at the joining point. Multiplication by e^{-2t} would change decay rates while generally leaving a nonzero response before $t = 2$; it is a different transform operation.

Step 3: Check the switch regularity. Both states are continuous. Before the switch their derivatives vanish; after it $x' = e^{-\tau}$ and $y' = e^{-\tau} - e^{-2\tau}$. Thus $x'(2^-) = 0$, $x'(2^+) = 1$, whereas $y'(2^-) = y'(2^+) = 0$. Afterward $y'' = -e^{-\tau} + 2e^{-2\tau}$, so $y''(2^-) = 0$, $y''(2^+) = 1$. In particular y is continuously differentiable at the switch but is not twice differentiable there.

Step 4: Establish shape and limits. For $\tau > 0$, $x' > 0$ and $y' = e^{-\tau}(1 - e^{-\tau}) > 0$. Also $0 < x < 1$ and $0 < y < 1/2$, with limits 1 and $1/2$. These inequalities prove the absence of overshoot for all later times. The dotted horizontal guides in the sketch are limits; the solid and dashed curves are the components. Both remain zero throughout the waiting interval.



Question 4

For $a > 0$, a single impulse acts on an undamped system:

$$x' = y + \alpha\delta(t - a), \quad y' = -x + \beta\delta(t - a), \quad (x(0), y(0)) = (1, 0).$$

Here $\delta(t - a)$ has unit mass, α, β are real, and states are piecewise smooth with jumps determined by integrating the equations. Use right-hand values at the impulse when writing the response.

Tasks

1. Transform the system and find $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ for general a, α, β .
2. Invert the transforms and verify both jump conditions directly. Explain why this impulse may create jumps in the states themselves.
3. Find the unique pair (α, β) that leaves the state identically zero for all $t > a$.
4. If only the second equation can receive an impulse ($\alpha = 0$), find all admissible stopping times and amplitudes. Identify the earliest time and sketch its two component responses, showing one-sided values.

Question 4 – Solution

Strategy. The shifted impulse produces a shifted homogeneous response; stopping requires cancellation of the complete pre-impulse state.

Step 1: Solve in the transform domain. The equations are $sF - G = 1 + \alpha e^{-as}$ and $F + sG = \beta e^{-as}$. Thus, for $\text{Re } s > 0$,

$$F = \frac{s}{s^2 + 1} + e^{-as} \frac{\alpha s + \beta}{s^2 + 1}, \quad G = \frac{-1}{s^2 + 1} + e^{-as} \frac{-\alpha + \beta s}{s^2 + 1}.$$

This half-plane suffices for all parameter choices, even when cancellation makes the actual response have compact support.

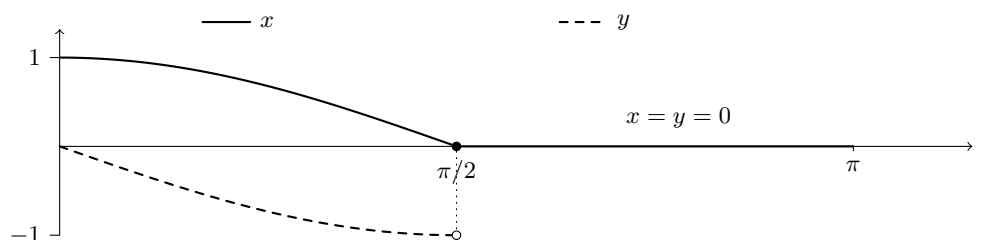
Step 2: Invert and check jumps. For $t < a$, $(x, y) = (\cos t, -\sin t)$. For $t \geq a$, with $\tau = t - a$,

$$x = \cos t + \alpha \cos \tau + \beta \sin \tau, \quad y = -\sin t - \alpha \sin \tau + \beta \cos \tau.$$

Hence $[x]_a = \alpha$, $[y]_a = \beta$, where $[f]_a = f(a^+) - f(a^-)$. Integrating the original equations across a gives these same jumps: the ordinary bounded terms contribute zero in the shrinking interval, while the impulses contribute their masses. Away from a the ordinary equations hold.

Step 3: Cancel the full state. The state just before a is $(\cos a, -\sin a)$. The only kick making the right-hand state zero is $(\alpha, \beta) = (-\cos a, \sin a)$. The ensuing homogeneous IVP then stays zero by uniqueness. Conversely, an identically zero future requires that same right-hand state, so the kick is unique.

Step 4: Restrict the available impulse. With $\alpha = 0$, cancellation requires $\cos a = 0$. Thus $a = \pi/2 + k\pi$, $\beta = (-1)^k$, $k = 0, 1, 2, \dots$. The earliest is $a = \pi/2$, $\beta = 1$. There x is continuous at zero, while y jumps from -1 to 0 . Both vanish afterward. The vertical dotted segment marks a jump, not values traversed continuously by the solution.



Question 5

Consider, for $t \geq 0$,

$$Z' = AZ + B, \quad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Z(0) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

A student cancels a factor $s - 2$ in the zero-initial-state transform and concludes that every solution of this forced system is bounded.

Tasks

1. Compute $(sI - A)^{-1}B$ and find the response when $p = q = 0$ using Laplace transforms.
2. Explain the cancellation using the direction of B , and identify the system mode that the constant input does not excite from zero.
3. Retain the initial-state term in the transformed system and determine exactly which (p, q) produce bounded solutions on $[0, \infty)$.
4. Perturb the zero initial state to $(\varepsilon, 2\varepsilon)^T$, with $\varepsilon \neq 0$. Find the exact difference from the zero-state response and assess the student's conclusion.

Question 5 – Solution

Strategy. Separate the input contribution from the initial-state contribution before cancelling rational factors.

Step 1: Calculate the zero-state response. The determinant is $s^2 - s - 2 = (s - 2)(s + 1)$, and

$$(sI - A)^{-1}B = \frac{1}{(s - 2)(s + 1)} \begin{pmatrix} s - 1 & 1 \\ 2 & s \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The constant input contributes an additional $1/s$. Inverting yields $\boxed{Z_0(t) = (1 - e^{-t})(1, -1)^T}$, which is bounded. Its transform converges for $\operatorname{Re} s > 0$; the cancelled expression extends through $s = 2$, where the full matrix resolvent itself is undefined.

Step 2: Identify why the pole disappears. We have $AB = -B$, so the input lies entirely in the stable eigendirection. The other eigenpair is $2, (1, 2)^T$. From zero, this input never excites that direction. The pole cancellation describes the chosen input and data; it does not change either eigenvalue of A .

Step 3: Include all initial conditions. Write $(p, q)^T = c_+(1, 2)^T + c_-(1, -1)^T$, where $c_+ = (p + q)/3$ and $c_- = (2p - q)/3$. The transform contains

$$\frac{c_+}{s - 2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{c_-}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{s(s + 1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$Z = c_+ e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + [1 + (c_- - 1)e^{-t}] \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The growing vector cannot be cancelled by a bounded term, so boundedness holds exactly when $\boxed{p + q = 0}$. Then the limit is $(1, -1)^T$. A common convergence half-plane for the general family is $\operatorname{Re} s > 2$.

Step 4: Test sensitivity to the omitted term. For the specified perturbation, $c_+ = \varepsilon$ and $c_- = 0$. The exact difference is $\boxed{Z - Z_0 = \varepsilon e^{2t}(1, 2)^T}$, whose Euclidean norm is $\sqrt{5}|\varepsilon|e^{2t} \rightarrow \infty$. Thus arbitrarily small initial perturbations in that direction destroy boundedness. The student's inference confuses a bounded zero-state response with boundedness for all initial states.

Question 6

Two inputs with the same frequency drive the same system from rest:

$$x' = -y + \cos t, \quad y' = x + \sigma \sin t, \quad x(0) = y(0) = 0, \quad \sigma \in \{1, -1\}.$$

Use Laplace transforms to investigate why a frequency match alone does not settle whether this vector input generates growing oscillations.

Tasks

1. Derive both transformed components for general σ , showing the denominator before any cancellation.
2. For $\sigma = 1$, invert the transforms and verify the original equations. Determine the Euclidean norm for $t \geq 0$.
3. For $\sigma = -1$, simplify before inversion, solve the IVP and prove boundedness.
4. Compare the pole orders and the input directions in the two cases. Is every sinusoidal vector forcing at the natural frequency resonant in the sense of an unbounded zero-state response?

Question 6 – Solution

Strategy. Solve the coupled algebra first; vector numerators can either retain or cancel the repeated oscillator poles.

Step 1: Derive the rational responses. The transformed equations are $sF + G = s/(s^2 + 1)$ and $-F + sG = \sigma/(s^2 + 1)$. Solving gives, on the sufficient half-plane $\operatorname{Re} s > 0$,

$$F = \frac{s^2 - \sigma}{(s^2 + 1)^2}, \quad G = \frac{(1 + \sigma)s}{(s^2 + 1)^2}.$$

The numerator, not the unsimplified denominator alone, decides actual pole order.

Step 2: Invert the co-rotating input. For $\sigma = 1$, differentiating the standard transforms with respect to s gives $\mathcal{L}\{t \cos t\} = (s^2 - 1)/(s^2 + 1)^2$ and $\mathcal{L}\{t \sin t\} = 2s/(s^2 + 1)^2$. Hence

$$x = t \cos t, \quad y = t \sin t, \quad \|Z(t)\| = t.$$

Indeed $x' = \cos t - t \sin t = -y + \cos t$ and $y' = \sin t + t \cos t = x + \sin t$, with zero initial state. The response is unbounded even though each forcing component is bounded.

Step 3: Invert the opposite rotation. For $\sigma = -1$, cancellation gives $F = 1/(s^2 + 1)$ and $G = 0$. Consequently $x = \sin t, y = 0$, with norm $|\sin t| \leq 1$. The first equation reads $\cos t = 0 + \cos t$; the second reads $0 = \sin t - \sin t$. Both initial conditions hold. No polynomial factor survives in this response.

Step 4: Interpret the distinction. For $\sigma = 1$, both transforms have double poles at $s = \pm i$; for $\sigma = -1$, the first has simple poles and the second is zero. The free system rotates counterclockwise. The input $(\cos t, \sin t)$ rotates with it, whereas $(\cos t, -\sin t)$ rotates oppositely. More explicitly, write $Z = R(t)V$, where R is the counterclockwise rotation matrix. Then $V' = (1, 0)^T$ in the first case and $V' = (\cos 2t, -\sin 2t)^T$ in the second. Their integrals respectively grow linearly and remain bounded. A frequency match alone therefore does not force an unbounded response; the vector direction and resulting pole cancellations matter.

Question 7

A scalar input drives a two-stage system:

$$x' = -x + u(t), \quad y' = x - 3y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

Assume u is continuous and of exponential order. In the inverse task, assume a prescribed target y and its first two derivatives are continuous and of exponential order. All claims about input bounds apply for every $t \geq 0$.

Tasks

1. Derive both transfer expressions in the Laplace domain and invert their kernels to express x and y as convolutions with u .
2. If $0 \leq u \leq 1$, prove a sharp constant upper bound on $y(t)$. Explain in what sense the bound is sharp.
3. Recover x and u from a prescribed y . State the initial compatibility conditions and justify uniqueness of the recovered input.
4. Apply the inverse formula to $y(t) = t^2 e^{-t}$. Find x, u , decide whether $0 \leq u \leq 1$ is possible, and give a second obstruction using the target's maximum.

Question 7 – Solution

Strategy. Positive inverse kernels give all-time bounds; elimination then turns the same system into an exact input-recovery formula.

Step 1: Invert the transfer kernels. If $U = \mathcal{L}\{u\}$, the zero initial values give $F = U/(s+1)$ and $G = U/[(s+1)(s+3)]$. Partial fractions yield

$$x(t) = \int_0^t e^{-(t-r)} u(r) dr, \quad y(t) = \frac{1}{2} \int_0^t (e^{-(t-r)} - e^{-3(t-r)}) u(r) dr.$$

These identities hold on any common right half-plane of convergence and, after inversion, for all $t \geq 0$.

Step 2: Use positivity to obtain a sharp bound. The kernel $k(v) = (e^{-v} - e^{-3v})/2$ is nonnegative for $v \geq 0$. For $0 \leq u \leq 1$,

$$0 \leq y(t) \leq \int_0^t k(v) dv = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} < \frac{1}{3} \quad (t < \infty).$$

The strict upper inequality follows from the positive remaining tail of k ; at $t = 0$ it also holds. With $u \equiv 1$, $y(t) \rightarrow 1/3$. Thus $1/3$ is the least uniform constant upper bound, although it is not attained at finite time from zero.

Step 3: Recover the input and its compatibility conditions. The second equation forces $x = y' + 3y$. Substitution into the first gives

$$u = y'' + 4y' + 3y, \quad y(0) = 0, \quad y'(0) = 0.$$

The conditions are necessary for both initial states to vanish. Conversely, under the stated regularity, these formulas satisfy both equations and the zero initial values whenever the conditions hold. They prove uniqueness: any input producing that y must give exactly this x and u .

Step 4: Test the proposed target. For $y = t^2 e^{-t}$, the compatibility conditions hold, and direct calculation gives $x = (2t + 2t^2)e^{-t}$, $u = (2 + 4t)e^{-t}$. The unique input is nonnegative but already has $u(0) = 2 > 1$; its maximum is $4e^{-1/2} > 1$ at $t = 1/2$. Independently, the target peaks at $t = 2$ with value $4e^{-2} > 1/3$, contradicting the kernel bound. It is therefore achievable by a continuous input, but not by an input confined to $[0, 1]$.

Question 8

For $\varepsilon \geq 0$, let

$$x' = -\varepsilon x - y + 1, \quad y' = x - \varepsilon y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

A student evaluates $\lim_{s \rightarrow 0^+} s\mathcal{L}\{x\}$ and $\lim_{s \rightarrow 0^+} s\mathcal{L}\{y\}$ and declares these the limiting state for every ε . For rational transforms, the final-value theorem requires all poles of the simplified $s\mathcal{L}\{f\}$ to have negative real part.

Tasks

1. Find both transformed components for general ε and calculate the student's two proposed limits.
2. Solve the case $\varepsilon = 0$. Check the theorem's hypothesis and determine whether either component has a limit as $t \rightarrow \infty$.
3. For $\varepsilon > 0$, invert the transforms and establish the true limiting state. Explain why the theorem now applies.
4. Show that $y_\varepsilon \rightarrow y_0$ at each fixed time as $\varepsilon \rightarrow 0^+$ but not uniformly on $[0, \infty)$. Use times $2\pi n$ to obtain a lower bound on the uniform error. Sketch y_0 and $y_{1/2}$.

Question 8 – Solution

Strategy. Check pole locations before applying the final-value theorem, and distinguish fixed-time convergence from an infinite-time assertion.

Step 1: Solve the transformed system. With $D = (s + \varepsilon)^2 + 1$, the equations give

$$F = \frac{s + \varepsilon}{sD}, \quad G = \frac{1}{sD}.$$

A sufficient convergence half-plane is $\operatorname{Re} s > 0$. The proposed final values are $\varepsilon/(1 + \varepsilon^2)$ and $1/(1 + \varepsilon^2)$.

Step 2: Examine the undamped case. At $\varepsilon = 0$, inversion yields $\boxed{x = \sin t, y = 1 - \cos t}$. Neither has a limit: the values along successive maxima and minima differ. Both sF and sG have poles at $\pm i$, so the stated theorem's hypothesis fails. The finite algebraic values 0 and 1 do not prove convergence in time.

Step 3: Resolve the damped response. For $\varepsilon > 0$, decompose, for example,

$$G = \frac{1}{1 + \varepsilon^2} \left(\frac{1}{s} - \frac{s + \varepsilon}{D} - \frac{\varepsilon}{D} \right).$$

Inverting this and the analogous decomposition of F gives

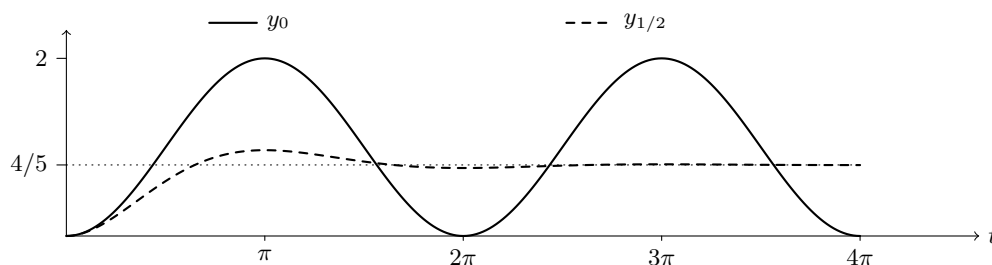
$$x_\varepsilon = \frac{\varepsilon - e^{-\varepsilon t}(\varepsilon \cos t - \sin t)}{1 + \varepsilon^2}, \quad y_\varepsilon = \frac{1 - e^{-\varepsilon t}(\cos t + \varepsilon \sin t)}{1 + \varepsilon^2}.$$

Both start at zero. The exponentially decaying terms leave the limit $\boxed{(\varepsilon, 1)^T / (1 + \varepsilon^2)}$. Now the poles of sF and sG are $-\varepsilon \pm i$, strictly in the left half-plane, as required. The limit also solves the equilibrium equations.

Step 4: Test uniformity on the whole half-line. At fixed t , the formula gives $y_\varepsilon(t) \rightarrow 1 - \cos t = y_0(t)$. But $y_0(2\pi n) = 0$, whereas $y_\varepsilon(2\pi n) = (1 - e^{-2\pi n \varepsilon})/(1 + \varepsilon^2)$. Consequently

$$\sup_{t \geq 0} |y_\varepsilon(t) - y_0(t)| \geq \frac{1}{1 + \varepsilon^2}.$$

This lower bound tends to 1, not 0. Fixed-time convergence therefore cannot be used to pass a limiting-state conclusion to the undamped system. The dotted guide is the damped limit $4/5$ for $\varepsilon = 1/2$.



Question 9

A rectangular input drives an initially resting oscillator:

$$x' = y, \quad y' = -4x + u_T(t), \quad x(0) = y(0) = 0, \quad u_T(t) = H(t) - H(t - T), \quad T > 0.$$

Use the causal step convention for $t \geq 0$. States are continuous at T , and equations at the switch are understood one-sidedly.

Tasks

1. Find both Laplace transforms and express the response as an undelayed step response minus its delayed copy.
2. Calculate $(x(T), y(T))$ and determine exactly which positive pulse durations leave the state at rest forever after the pulse ends.
3. For arbitrary T , calculate the constant value of $y^2 + 4x^2$ after switch-off. Relate complete cancellation to zeros of the input transform at the oscillator poles.
4. Compare $T = \pi$ and $T = \pi/2$: give their post-pulse responses and sketch $x(t)$ on $0 \leq t \leq 2\pi$. Explain why a nonzero, nonnegative pulse can leave no residual oscillation.

Question 9 – Solution

Strategy. Shift a single step response, then check both state components at switch-off; displacement alone is not a stopping condition.

Step 1: Transform and invert by shifting. Since $sF = G$ and $sG = -4F + (1 - e^{-Ts})/s$,

$$F = \frac{1 - e^{-Ts}}{s(s^2 + 4)}, \quad G = \frac{1 - e^{-Ts}}{s^2 + 4}, \quad \text{Re } s > 0.$$

Let $h(t) = (1 - \cos 2t)/4$ and $j(t) = \sin 2t/2$. Then

$$x = h(t) - H(t - T)h(t - T), \quad y = j(t) - H(t - T)j(t - T).$$

Both delayed terms begin at zero, proving state continuity.

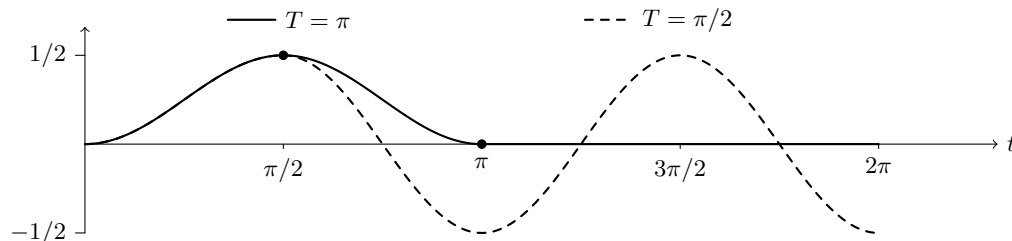
Step 2: Enforce complete rest at the endpoint. At T , the state is $((1 - \cos 2T)/4, \sin 2T/2)^T$. Both components vanish exactly when $\cos 2T = 1$, giving $T = k\pi$, $k = 1, 2, \dots$; the earliest is π . Zero endpoint data then yield the identically zero unforced continuation. Conversely, rest for all later times requires these endpoint values by continuity.

Step 3: Compute the residual and its spectral cancellation. After the pulse, differentiation gives $(y^2 + 4x^2)' = 2y(-4x) + 8xy = 0$. Evaluation at T yields

$$y^2 + 4x^2 = \frac{1}{4} \sin^2 2T + \frac{1}{4} (1 - \cos 2T)^2 = \sin^2 T.$$

The finite-pulse transform $U_T = (1 - e^{-Ts})/s$ has zeros at $s = \pm 2i$ exactly when $e^{2iT} = 1$. These are precisely the stopping durations; they cancel the oscillator poles. At $s = 0$ its singularity is removable, with value T , so a nonzero total input area does not prevent this cancellation.

Step 4: Compare two durations. For $T = \pi$, $x = y = 0$ after π . For $T = \pi/2$, writing $\tau = t - \pi/2 \geq 0$, the endpoint state is $(1/2, 0)$ and $x = \frac{1}{2} \cos 2\tau$, $y = -\sin 2\tau$ afterward. During either pulse, $x = h(t)$ and $y = j(t)$; the plotted curves coincide up to $\pi/2$. A full-period pulse lets the forced trajectory return to rest before switch-off; accumulated state contributions have different oscillatory phases and can cancel even when the input never becomes negative. Equal input sign is not equal state phase.



Question 10

A known system has an unknown continuous input u and unknown initial state:

$$x' = -x + y, \quad y' = -2x - 4y + u(t), \quad t \geq 0.$$

The input and states are of exponential order. A proposed pair of measured one-sided transforms is

$$F(s) = \frac{s+3}{(s+1)(s+2)}, \quad G(s) = \frac{1}{s+2}.$$

Assess compatibility before trying to infer the input.

Tasks

1. Recover both initial values from the transforms using the initial-value rule, then invert F and G .
2. Check the first transformed equation, including its initial term, and verify the corresponding time-domain equation.
3. Recover $\mathcal{L}\{u\}$ and $u(t)$ from the second equation. Prove whether knowledge of $x(t)$ alone would determine both $y(t)$ and $u(t)$ under the stated assumptions.
4. Suppose F is unchanged but the proposed second transform is instead $\tilde{G}(s) = 1/(s+3)$. Can any continuous input in the second equation repair the discrepancy? Quantify the violated transformed identity.

Question 10 – Solution

Strategy. Initial terms and the equation without the unknown input provide independent consistency checks before input reconstruction.

Step 1: Recover the initial state and signals. The initial-value rule gives $x(0) = \lim_{s \rightarrow \infty} sF(s) = 1$ and $y(0) = \lim_{s \rightarrow \infty} sG(s) = 1$. Partial fractions give

$$F = \frac{2}{s+1} - \frac{1}{s+2}, \quad \boxed{x = 2e^{-t} - e^{-2t}, \quad y = e^{-2t}.$$

Both recovered signals indeed start at one; their transforms share the convergence half-plane $\operatorname{Re} s > -1$.

Step 2: Check the input-free constraint. The first transformed equation is $(s+1)F - G = x(0) = 1$. Using the measured expressions, $(s+1)F - G = (s+3)/(s+2) - 1/(s+2) = 1$. In time, $x' = -2e^{-t} + 2e^{-2t} = -x + y$. Thus the two measurements pass a constraint that adjusting the second-equation input cannot alter.

Step 3: Reconstruct and establish uniqueness. The second equation gives $U = sG - y(0) + 2F + 4G$. Therefore

$$U = \frac{s+4}{s+2} - 1 + \frac{2(s+3)}{(s+1)(s+2)} = \frac{4}{s+1}, \quad \boxed{u = 4e^{-t}.$$

Directly, $y' = -2e^{-2t} = -2x - 4y + 4e^{-t}$. Even x alone determines $y = x' + x$ and then $\boxed{u = x'' + 5x' + 6x}$. The continuous input makes these derivatives well defined through the system. Thus two inputs giving the same exact x must give the same y and the same u ; the initial state is recovered as well. This is an exact-data uniqueness claim, not a claim about noise sensitivity.

Step 4: Reject an incompatible second signal. The replacement still has initial value one, but

$$(s+1)F - \tilde{G} - 1 = \frac{1}{s+2} - \frac{1}{s+3} = \boxed{\frac{1}{(s+2)(s+3)} \neq 0}.$$

Equivalently, $x' + x - \tilde{y} = e^{-2t} - e^{-3t}$ is nonzero for $t > 0$. No choice of input confined to the second equation can repair this failure of the first equation. Matching initial values is necessary but does not guarantee compatibility of the complete measured signals.

Question 1

Two perfectly mixed tanks contain 100 L and 200 L of solution. Two pumps transfer 10 L/min in each direction; there is no external inflow, outflow, or reaction. Initially the first tank contains 10 kg of salt and the second contains none. Let $x(t), y(t)$ be their salt amounts in kg, with t in minutes.

Tasks

1. Derive the amount equations from salt fluxes, explaining why the volumes stay constant and why the two transfer coefficients differ.
2. Find a conserved quantity and the equilibrium amounts. Does equilibrium mean equal amounts or equal concentrations?
3. Solve the initial-value problem and verify nonnegativity and conservation for all $t \geq 0$.
4. Find the earliest time at which the concentration difference is at most 0.005 kg/L, and show it stays below this tolerance thereafter. Sketch the two concentrations and their common limiting level.

Question 1 – Solution

Strategy. Use volume flow times concentration for each salt flux, then use conservation to reduce the coupled balances.

Step 1: Construct the two balances. Each tank gains and loses 10 L/min, so its volume is fixed. Its outgoing salt rate is 10 times its own concentration. Thus

$$x' = -\frac{1}{10}x + \frac{1}{20}y, \quad y' = \frac{1}{10}x - \frac{1}{20}y.$$

The coefficients have units min^{-1} : equal volume flows remove different fractions of the unequal tank volumes per minute.

Step 2: Use the conserved total. Adding gives $(x + y)' = 0$, so $x + y = 10$ kg. At equilibrium $x/100 = y/200$. Consequently $(x_*, y_*) = (10/3, 20/3)$ kg and the common concentration is $1/30$ kg/L. Equal amounts would leave unequal concentrations and hence a nonzero net salt transfer.

Step 3: Solve and check the physical range. Substituting $y = 10 - x$ gives $x' = 1/2 - 3x/20$. The initial data yield

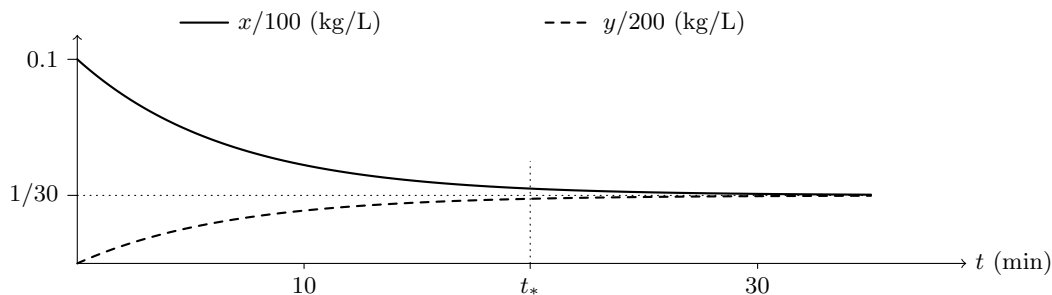
$$x = \frac{10}{3} + \frac{20}{3}e^{-3t/20}, \quad y = \frac{20}{3}(1 - e^{-3t/20}).$$

Their sum is 10, both are nonnegative, and their initial values are $(10, 0)$. Differentiation gives $x' = -e^{-3t/20}$ and $y' = e^{-3t/20}$ kg/min; substitution into the original flux balances gives the same rates.

Step 4: Compute a lasting tolerance time. The concentration difference is

$$\frac{x}{100} - \frac{y}{200} = \frac{1}{10}e^{-3t/20} > 0.$$

It decreases strictly, so its first value 0.005 occurs at $t = \frac{20}{3} \log 20 \text{ min} \approx 19.97 \text{ min}$. The inequality holds for this time and every later time. The first concentration approaches the dotted limit from above; the second approaches it from below.



Question 2

An idealized drug model has a central compartment and a peripheral compartment. Let x, y be drug amounts in mg, with time in hours. Each hour, transfer from central to peripheral is 1 times the current central amount; reverse transfer is 1 times the peripheral amount. Elimination occurs only from the central compartment, at rate 1 times its amount per hour. Immediately after a bolus, $(x(0), y(0)) = (D, 0)$, where $D > 0$ is the numerical dose in mg; no further drug enters.

Tasks

1. Derive the two amount equations and the rate of change of total drug. State which modeling assumptions justify linear transfer laws.
2. Explain why nonnegative initial amounts remain nonnegative. Add an eliminated-drug state $e(t)$ and find the conserved total, with $e(0) = 0$.
3. Solve the two-compartment IVP exactly in terms of D . Verify the initial data and determine the limiting retained and eliminated amounts.
4. Find $\int_0^\infty x(t) dt$ and $\int_0^\infty y(t) dt$ directly from the balance equations. Explain their units and why adding them does not count an eliminated mass twice.

Question 2 – Solution

Strategy. Distinguish retained mass from cumulative elimination; integrated compartment balances recover exposure without integrating long formulas.

Step 1: Write transfer and loss rates. Well-mixed compartments and constant, unsaturated first-order rate constants give $x' = -2x + y$, $y' = x - y$. The numerical coefficients have units h^{-1} . Adding gives $(x + y)' = -x$: internal transfers cancel, but central elimination removes drug. Amounts, rather than concentrations, are the states; no unspecified compartment volume is needed for these laws.

Step 2: Establish the physical state constraints. On $x = 0, y \geq 0$, $x' = y \geq 0$; on $y = 0, x \geq 0$, $y' = x \geq 0$. These inward boundary directions and uniqueness preserve the nonnegative quadrant. With $e' = x$, $e(0) = 0$, $x + y + e = D$. Thus retained amounts lie between 0 and D , and e is nondecreasing. The third state records removal, not return flow.

Step 3: Resolve the two decay modes. The roots are $\lambda_{\pm} = (-3 \pm \sqrt{5})/2$, both negative. Solving for the constants gives, with $r = \sqrt{5}$,

$$x = De^{-3t/2}(\cosh(rt/2) - r^{-1} \sinh(rt/2)), \quad y = \frac{2D}{r}e^{-3t/2} \sinh(rt/2).$$

These give $x(0) = D, y(0) = 0, x'(0) = -2D, y'(0) = D$. Differentiating the formulas gives $x' = -2x + y$ and $y' = x - y$ for all t . Since $r < 3$, both modal exponentials decay: $x, y \rightarrow 0$ and $e \rightarrow D$.

Step 4: Integrate the balances. Decay makes both integrals finite. Integrating $(x + y)' = -x$ gives $\int_0^\infty x \, dt = D$ numerically, and integrating $y' = x - y$ then gives $\int_0^\infty y \, dt = D$. Restoring the rate units,

$$\int_0^\infty x(t) \, dt = \int_0^\infty y(t) \, dt = D \text{ mg} \cdot \text{h}.$$

These are amount–time exposures in $\text{mg} \cdot \text{h}$, not eliminated masses. Their sum is $2D \text{ mg} \cdot \text{h}$ of total residence exposure, while the eliminated mass is only $D \text{ mg}$. A molecule can contribute residence time in both compartments.

Question 3

Two uniform bodies each have heat capacity $1 \text{ kJ}/^\circ\text{C}$. Their mutual thermal conductance is $1 \text{ kJ}/(\text{min}\cdot^\circ\text{C})$, and each has that same conductance to a room maintained at 20°C . Heat flow equals conductance times temperature difference; there are no internal heat sources. Initially $T_1 = 80^\circ\text{C}$ and $T_2 = 20^\circ\text{C}$. Time is in minutes.

Tasks

1. Derive the two energy balances. Introduce excess temperatures $\theta_i = T_i - 20$ and write the homogeneous system.
2. Find the total excess thermal energy and its decay law. Explain which heat transfers cancel in this sum.
3. Solve for both temperatures. Find the exact maximum temperature of the initially cooler body and when it occurs.
4. Explain why one body can warm while total excess energy decreases. Prove neither body leaves the interval $[20, 80]^\circ\text{C}$, and sketch the temperatures.

Question 3 – Solution

Strategy. Write heat-capacity balances before shifting the reference temperature; sum and difference variables isolate two decay rates.

Step 1: Balance heat for each body. For body 1, capacity times T_1' equals $(T_2 - T_1) + (20 - T_1)$ in kJ/min; the analogous balance holds for body 2. Hence

$$\theta_1' = -2\theta_1 + \theta_2, \quad \theta_2' = \theta_1 - 2\theta_2, \quad (\theta_1(0), \theta_2(0)) = (60, 0).$$

The numerical decay coefficients have units min^{-1} .

Step 2: Track total excess energy. With each capacity equal to 1 kJ/°C, excess energy is $E = \theta_1 + \theta_2$ numerically in kJ. Interbody exchange cancels, leaving $E' = -E$ and $E(t) = 60e^{-t}$ kJ. Only heat delivered to the room decreases this total.

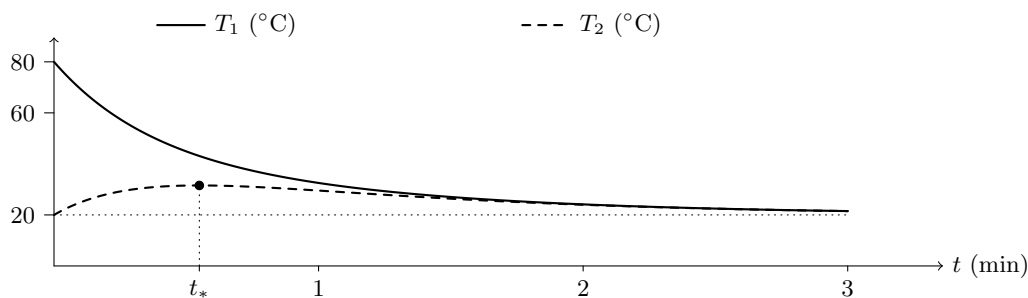
Step 3: Solve the modes and locate the warm-up peak. The difference obeys $(\theta_1 - \theta_2)' = -3(\theta_1 - \theta_2)$. Together with the sum this gives

$$T_1 = 20 + 30(e^{-t} + e^{-3t}), \quad T_2 = 20 + 30(e^{-t} - e^{-3t}).$$

Here $T_2' = 30(-e^{-t} + 3e^{-3t})$, which changes sign from positive to negative at $t_* = (\log 3)/2$. Therefore

$$T_{2,\max} = (20 + 20/\sqrt{3})^\circ\text{C}, \quad t_* = (\log 3)/2 \text{ min.}$$

Step 4: Reconcile local warming with total cooling. Initially the hotter body supplies heat to body 2 faster than body 2 loses heat to the room. Nevertheless $E' < 0$ throughout. The formulas give $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 60e^{-t} \leq 60$, so each temperature stays in $[20, 80]^\circ\text{C}$. Both tend to room temperature; the second has a genuine temporary maximum, not a higher final equilibrium.



Question 4

Two 1 kg masses slide without friction on a horizontal line. Each is attached to its outer fixed wall by a spring of stiffness 1 N/m, and a third spring of stiffness 1 N/m joins the masses. Let q_1, q_2 be displacements in metres from equilibrium, positive to the right. Initially $q_1 = a$, $q_2 = 0$ and both velocities vanish, where $a > 0$. Time is in seconds; Hooke's law applies.

Tasks

1. Derive the forces, paying attention to the change in length of the middle spring. Write a first-order system with four states and its initial vector.
2. Use the sum and difference of the displacements to solve for q_1, q_2 and their velocities.
3. Construct the total mechanical energy from the three springs and two masses. Show it is constant and find its initial value.
4. Determine whether the full initial state ever returns at a positive time. Justify your answer exactly, and sketch q_1, q_2 for $a = 0.1$ m on $0 \leq t \leq 10$ s.

Question 4 – Solution

Strategy. The coupling spring exerts opposite forces on the masses; symmetric and antisymmetric motions then become independent oscillators.

Step 1: Derive the complete state model. The middle spring's extension is $q_2 - q_1$, giving force $q_2 - q_1$ on mass 1 and its negative on mass 2. Thus $q_1'' = -2q_1 + q_2$ and $q_2'' = q_1 - 2q_2$, with numerical acceleration coefficients in s^{-2} . For $Z = (q_1, v_1, q_2, v_2)^T$,

$$Z' = (v_1, -2q_1 + q_2, v_2, q_1 - 2q_2)^T, \quad Z(0) = (a, 0, 0, 0)^T.$$

Step 2: Solve the two normal modes. Set $s = q_1 + q_2$, $d = q_1 - q_2$. Then $s'' = -s$, $d'' = -3d$, with $s(0) = d(0) = a$ and zero initial derivatives. Therefore

$$q_1 = \frac{a}{2}(\cos t + \cos \sqrt{3}t), \quad q_2 = \frac{a}{2}(\cos t - \cos \sqrt{3}t).$$

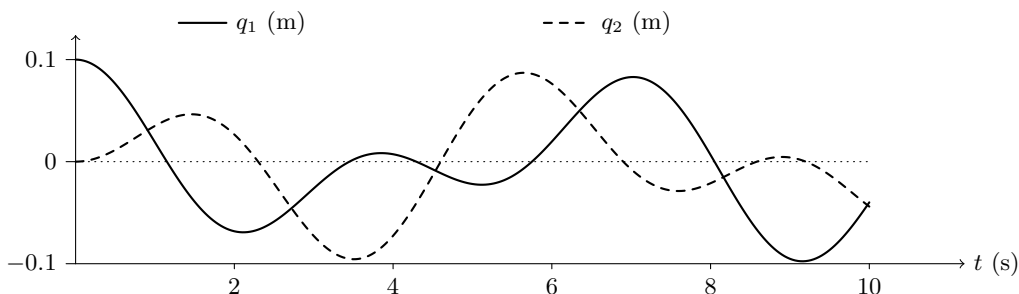
The velocities are $v_1 = -a(\sin t + \sqrt{3} \sin \sqrt{3}t)/2$ and $v_2 = -a(\sin t - \sqrt{3} \sin \sqrt{3}t)/2$. They satisfy all four initial data.

Step 3: Check the energy balance. In joules, with the stated numerical masses and stiffnesses,

$$E = \frac{1}{2}(v_1^2 + v_2^2) + \frac{1}{2}(q_1^2 + q_2^2 + (q_2 - q_1)^2).$$

Its derivative is $v_1(q_1'' + 2q_1 - q_2) + v_2(q_2'' + 2q_2 - q_1) = 0$. At the initial state $E = a^2 \text{ J}$ when a is the numerical initial displacement in metres. Energy includes the middle spring as well as the wall springs.

Step 4: Test an exact return. A return of the full state requires $s = d = a$ and $s' = d' = 0$. Thus $t = 2\pi m$ and $\sqrt{3}t = 2\pi n$ for integers m, n . For $t > 0$ this would imply $\sqrt{3} = n/m$, impossible. No exact positive-time return occurs. A near return visible in a plot cannot prove periodicity.



Question 5

Two magnetically coupled loops have inductances $L_1 = L_2 = 2$ H, mutual inductance $M = 1$ H, and resistances $R_1 = R_2 = 1$ Ω . Reference current directions and coil orientations are chosen so the mutual derivative terms have positive signs in both voltage-drop balances. At $t = 0$ a constant 3 V source is connected to loop 1; loop 2 has no source. Initially both currents are zero. Time is in seconds; the ideal linear circuit model applies.

Tasks

1. Use Kirchhoff's voltage law to derive the coupled derivative equations, then solve them for i'_1, i'_2 . Explain why the inductance matrix is invertible.
2. Solve the initial-value problem using current sum and difference, and find the limiting currents.
3. Determine the sign of $i_2(t)$ for $t > 0$ and its most negative value and time. Explain what a negative current means here, and sketch both currents.
4. With magnetic energy $W = (2i_1^2 + 2i_1i_2 + 2i_2^2)/2$, derive the power balance. Explain what changes when both voltage sources are zero.

Question 5 – Solution

Strategy. Invert the inductance matrix, not just its diagonal; the sum and difference currents see different inductances.

Step 1: Form the coupled voltage balances. Voltage drops give $2i_1' + i_2' = 3 - i_1$ and $i_1' + 2i_2' = -i_2$. The inductance determinant is $4 - 1 = 3 \text{ H}^2 > 0$, so

$$i_1' = 2 - \frac{2}{3}i_1 + \frac{1}{3}i_2, \quad i_2' = -1 + \frac{1}{3}i_1 - \frac{2}{3}i_2.$$

Each right side has units A/s. Ignoring the mutual terms would predict $i_2 \equiv 0$ and would fail the second voltage balance during the transient.

Step 2: Solve the sum and difference. Let $s = i_1 + i_2$, $d = i_1 - i_2$. Adding and subtracting the voltage equations gives $3s' = 3 - s$ and $d' = 3 - d$, with $s(0) = d(0) = 0$. Thus $s = 3(1 - e^{-t/3})$, $d = 3(1 - e^{-t})$, and

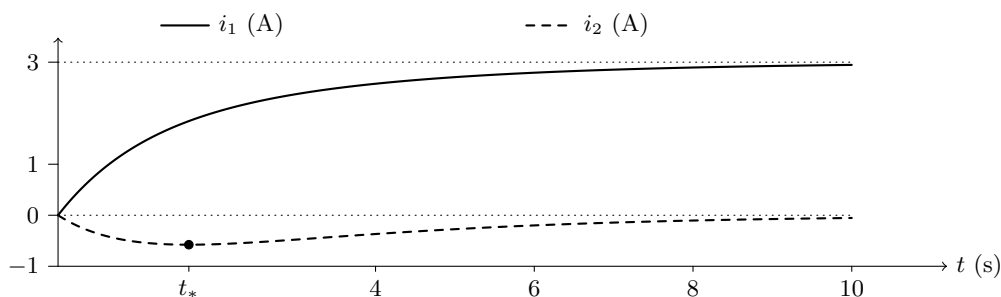
$$i_1 = 3 - \frac{3}{2}(e^{-t/3} + e^{-t}), \quad i_2 = \frac{3}{2}(e^{-t} - e^{-t/3}).$$

They tend to (3, 0) A, the resistive steady state. Their initial slopes are (2, -1) A/s, agreeing with the inverted model.

Step 3: Locate the induced-current extremum. For $t > 0$, $e^{-t} < e^{-t/3}$, so $i_2 < 0$. Its derivative vanishes when $e^{-t/3}/3 = e^{-t}$, at $t_* = (3/2) \log 3$ s. The derivative changes from negative to positive, giving

$i_{2,\min} = -1/\sqrt{3}$ A. The sign means actual current flows opposite its chosen reference direction; it is not a negative stored energy or a violation of the circuit model.

Step 4: Verify the power identity. Writing $L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $W' = i^T L i' = i^T (V - i)$ gives $W' = 3i_1 - i_1^2 - i_2^2$ in watts. With $V = 0$ instead, $W' = -i_1^2 - i_2^2 \leq 0$. Since L has positive eigenvalues 1, 3, $W > 0$ for nonzero currents. The driven circuit can gain magnetic energy from its source; the source-free circuit can only dissipate it.



Question 6

In an idealized closed-population epidemic model, s, i, r are susceptible, infectious and removed fractions. There are no births, deaths or reinfections. Homogeneous mixing gives new infections at rate $2si$ per day and removal at rate i per day. Initially $(s, i, r) = (0.99, 0.01, 0)$, and t is in days. Assume the stated rates remain constant throughout the modeled outbreak.

Tasks

1. Derive the three balance equations and explain conservation and nonnegativity. Reduce to two independent states.
2. Eliminate time to derive a relation between i and s . Find the susceptible fraction and infectious fraction at the unique infectious peak; justify that this peak is reached.
3. Show $i(t) \rightarrow 0$ and $s(t) \rightarrow s_\infty > 0$. Derive an equation uniquely selecting s_∞ in $(0, 1/2)$ and estimate it to three decimals.
4. Sketch the physical path in the (s, i) plane with equal scales, its initial point, peak and limiting endpoint. Explain why the peak condition does not mean infections have ended.

Question 6 – Solution

Strategy. Conservation confines the dynamics; eliminating time gives an exact curve and identifies both the peak and final-size constraint.

Step 1: Balance the three populations. The equations are $\boxed{s' = -2si, i' = 2si - i, r' = i}$, with $s + i + r = 1$. Boundary rates cannot create negative fractions. More explicitly, $s = 0.99e^{-2\int_0^t i}$ and $i = 0.01e^{\int_0^t (2s-1)}$ are positive at finite times; $r' = i \geq 0$. Use $r = 1 - s - i$ to reduce the system.

Step 2: Eliminate time and locate the peak. Since $s, i > 0$, $di/ds = -1 + 1/(2s)$. Integrating through the initial point gives

$$\boxed{i = 1 - s + \frac{1}{2} \log(s/0.99)}.$$

Here $i' = i(2s - 1)$ changes sign at $s = 1/2$. This value must be reached: if $s \geq 1/2$ forever, then $i \geq 0.01$ and $s' \leq -0.02s$, a contradiction. Strict decrease of s permits only one crossing. Hence

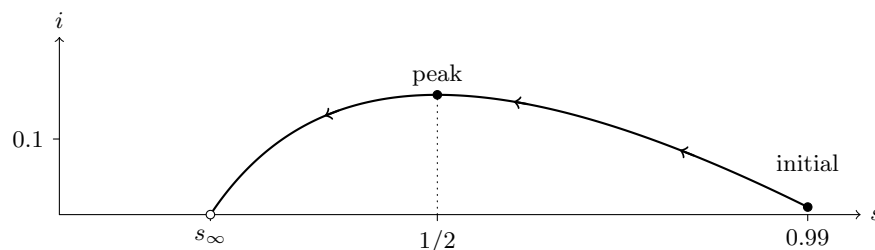
$$\boxed{i_{\max} = \frac{1}{2} + \frac{1}{2} \log(0.5/0.99) \approx 0.15845}.$$

Step 3: Identify the physical final size. Because $\int_0^\infty i = r_\infty \leq 1$, $s \geq 0.99e^{-2} > 0$. Thus decreasing s has a positive limit. Also i is integrable and has a bounded derivative on the invariant simplex, so $i \rightarrow 0$: nonvanishing peaks would have widths bounded below and contradict integrability. The peak crossing implies $s_\infty < 1/2$. Substitution gives

$$\boxed{1 - s_\infty + \frac{1}{2} \log(s_\infty/0.99) = 0, \quad s_\infty \approx 0.200}.$$

On $(0, 1/2)$ the left side is strictly increasing, tends to $-\infty$ at zero and is positive at $1/2$, so this root is unique.

Step 4: Interpret the physical trajectory. Time moves toward smaller s , up to the peak and then down toward zero infectious fraction. The limiting endpoint is open because $i(t) > 0$ at finite times. At the peak, incidence and removal balance; both are positive. The peak marks zero net infectious growth, not the end of infection.



Question 7

A continuous two-stage population model uses juvenile and adult abundances J, A , measured in thousands, with time in years. Juveniles mature at per-capita rate 1 yr^{-1} and die at rate 1 yr^{-1} . Adults die at rate 1 yr^{-1} and produce juveniles at per-capita rate $b \text{ yr}^{-1}$, where $b \geq 0$. Assume no crowding and strictly positive initial J_0, A_0 .

Tasks

1. Derive the stage equations and explain why maturation must appear with opposite signs in the two balances.
2. Find the eigenvalues and classify long-time extinction, a critical threshold, and unbounded growth as b varies. Justify that the growing mode is actually present for the stated initial data.
3. At the critical value, find a conserved weighted total, the exact solution and its limiting stage abundances.
4. For every $b \geq 0$, find the limiting ratio J/A and interpret it. Explain why this linear model cannot determine a realistic carrying capacity.

Question 7 – Solution

Strategy. Separate transfers between stages from births and deaths, then use the dominant mode and the exceptional zero eigenvalue.

Step 1: Construct the stage balances. Maturation removes juveniles and adds adults, whereas juvenile death removes individuals without adding to the adult stage. Therefore $J' = bA - 2J$, $A' = J - A$. All coefficients have units yr^{-1} . Nonnegative off-diagonal rates preserve the nonnegative quadrant; the strictly positive initial populations remain positive.

Step 2: Determine the growth threshold. The characteristic equation is $\lambda^2 + 3\lambda + 2 - b = 0$, so $\lambda_{\pm} = (-3 \pm \sqrt{1 + 4b})/2$. Put $d = \sqrt{1 + 4b}$. The adult solution has dominant coefficient

$$A(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}, \quad c_+ = \frac{J_0 + (1 + d)A_0/2}{d} > 0.$$

Thus both populations tend to zero for $0 \leq b < 2$, approach a nonzero critical equilibrium for $b = 2$, and grow without bound for $b > 2$. For $b > 2$, the dominant juvenile coefficient $(\lambda_+ + 1)c_+$ is also positive.

Step 3: Solve the critical case exactly. When $b = 2$, $K = J + 2A = J_0 + 2A_0$ is conserved. Since $A' = K - 3A$,

$$A = K/3 + (A_0 - K/3)e^{-3t}, \quad J = K/3 - 2(A_0 - K/3)e^{-3t}.$$

Both tend to $K/3$ thousand. The critical equilibria form the line $J = A$; there is no single equilibrium amount independent of initial data.

Step 4: Interpret the asymptotic stage ratio. Since $J = A' + A$ and $c_+ > 0$, division by the dominant adult mode gives

$$\lim_{t \rightarrow \infty} \frac{J}{A} = \lambda_+ + 1 = \frac{\sqrt{1 + 4b} - 1}{2}.$$

For $b = 0$, this limit is zero: juveniles decay at rate 2, adults at rate 1. The ratio describes relative stage composition even when the entire population tends to zero; it does not imply constant abundance. For $b > 2$ there is no crowding term to halt growth. A carrying capacity requires additional density-dependent assumptions absent from this model.

Question 8

An ideal temperature controller is modeled in dimensionless variables. The room temperature is 0, the target is 1, and the scaled heater output is $P(t)$. The plant obeys $T' = -T + P$. A controller with memory sets

$$P(t) = k \int_0^t [1 - T(\tau)] d\tau, \quad T(0) = 0, \quad k > 0.$$

There is no heater saturation, delay or heat-source noise in this ideal model.

Tasks

1. Convert the plant and controller into a two-state autonomous IVP. Find its equilibrium and explain why P is a state rather than a fixed parameter.
2. Shift to the equilibrium and show it attracts every initial state for every $k > 0$. Classify the eigenvalues at the critical gain.
3. For $k = 1/4$, solve the specified IVP. Prove both T and P stay in $[0, 1)$ at finite times and increase toward 1. Sketch them.
4. Replace the memory controller by the proportional rule $P = k(1 - T)$, with the same $k > 0$. Find its limiting temperature and explain the difference in steady error and the practical scope of the ideal memory model.

Question 8 – Solution

Strategy. The accumulated error supplies an additional state; its equilibrium condition eliminates the temperature offset of proportional feedback.

Step 1: Introduce the controller state. Differentiation gives

$$T' = -T + P, \quad P' = k(1 - T), \quad (T(0), P(0)) = (0, 0).$$

At equilibrium, $T = 1$ and $P = 1$. Heater output depends on accumulated past error, so two states with the same temperature can have different future motion if their current P values differ.

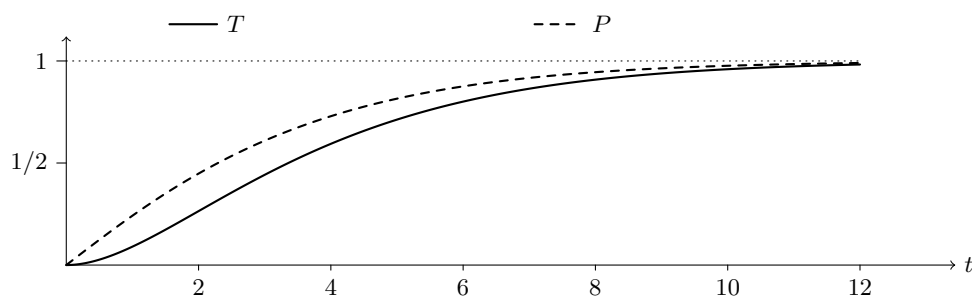
Step 2: Check stability and the critical gain. For $x = T - 1$, $u = P - 1$, the matrix is $\begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix}$. Its roots are $(-1 \pm \sqrt{1 - 4k})/2$. For $0 < k < 1/4$ both are real and negative; at $k = 1/4$ there is a repeated root $-1/2$; for $k > 1/4$ they are complex with real part $-1/2$. Every homogeneous mode decays, including the polynomial times exponential at the repeated root. Thus the equilibrium attracts every initial state within this ideal linear model.

Step 3: Solve the critical-gain response. Eliminating P gives $T'' + T' + T/4 = 1/4$, with $T(0) = T'(0) = 0$. The repeated-root solution and $P = T' + T$ yield

$$T = 1 - (1 + t/2)e^{-t/2}, \quad P = 1 - (1 + t/4)e^{-t/2}.$$

The initial values vanish and $T' = te^{-t/2}/4 \geq 0$, $P' = (1/4 + t/8)e^{-t/2} > 0$. The latter equals $(1 - T)/4$, checking the controller equation. The positive subtracted terms and monotonicity imply $0 \leq T, P < 1$ at finite t , and both tend to 1. Here the heater output stays nonnegative without needing a saturation rule.

Step 4: Compare steady errors. Proportional feedback instead gives $T' = k - (1 + k)T$, so $T \rightarrow k/(1 + k)$ and the steady error is $1/(1 + k) > 0$. The memory controller can retain $P = 1$ even when the current error is zero; a proportional controller would then set $P = 0$. Real actuator limits and delays can change stability or feasibility, so the ideal conclusions apply under the stated assumptions, not to every physical controller.



Question 9

In a closed, well-mixed vessel of constant volume, the reversible reaction $A \rightleftharpoons 2B$ has forward reaction rate $k_f a$ and reverse reaction rate $k_r b^2$ in $\text{mol}/(\text{L}\cdot\text{min})$, where a, b are concentrations in mol/L . Here $k_f = 1 \text{ min}^{-1}$ and $k_r = 1 \text{ L}/(\text{mol}\cdot\text{min})$. Initially $a = 1$, $b = 0 \text{ mol}/\text{L}$. There are no other reactions. Use numerical concentration values in these units.

Tasks

1. Derive the concentration equations from the stoichiometric coefficients. Identify a conserved weighted concentration and explain why $a + b$ is not conserved.
2. Reduce to a scalar equation for b and identify its physically allowed interval. Show this interval is invariant.
3. Find the unique physical equilibrium and prove the stated initial response approaches it monotonically in b .
4. Solve explicitly for $b(t)$ by separation, then recover $a(t)$. Explain why the second algebraic root of the equilibrium equation is inadmissible and why the physical equilibrium is not reached at finite time.

Question 9 – Solution

Strategy. Each forward reaction consumes one A and creates two B; the resulting weighted conservation law reduces the nonlinear system.

Step 1: Respect the reaction stoichiometry. A net forward reaction rate $a - b^2$ gives

$$a' = -a + b^2, \quad b' = 2a - 2b^2$$

in the chosen numerical units. Hence $(2a + b)' = 0$ and $2a + b = 2$ mol/L. The sum $a + b$ has derivative $a - b^2$ and is not conserved: the reaction changes the number of molecules, while preserving the weighted building-block total.

Step 2: Reduce within the physical domain. With $a = 1 - b/2$, the scalar equation is $b' = 2 - b - 2b^2$, with $0 \leq b \leq 2$ required by $a, b \geq 0$. At $b = 0$ the derivative is $2 > 0$, and at $b = 2$ it is $-8 < 0$. Uniqueness and inward boundary directions make the interval invariant. The concentrations cannot be extended to negative values merely because an algebraic equation has another root.

Step 3: Identify the attracting physical equilibrium. Let $d = \sqrt{17}$ and $r_{\pm} = (-1 \pm d)/4$. Then $b' = -2(b - r_+)(b - r_-)$, with $r_- < 0 < r_+ < 2$. The only physical equilibrium is

$$b_* = r_+, \quad a_* = 1 - r_+/2.$$

For $0 \leq b < r_+$, $b' > 0$. Uniqueness prevents crossing the equilibrium, so the solution from zero increases and stays below r_+ . Its limit must be a zero of the scalar right side; otherwise its positive derivative would remain bounded away from zero. Thus it tends to r_+ .

Step 4: Separate and impose the initial concentration. Since $2(r_+ - r_-) = d$, integration gives

$$\log \left| \frac{b - r_+}{b - r_-} \right| = -dt + \log \left| \frac{r_+}{r_-} \right|.$$

The ratio is negative on the physical branch starting at zero. Solving gives

$$b(t) = \frac{r_+(1 - e^{-dt})}{1 - (r_+/r_-)e^{-dt}}, \quad a(t) = 1 - \frac{b(t)}{2}.$$

The denominator exceeds 1 at finite times because $r_+/r_- < 0$. The formulas give $(a, b) = (1, 0)$ initially and the stated equilibrium as $t \rightarrow \infty$. For every finite t , $b < r_+$, so arrival is asymptotic.

Question 10

A conserved tracer moves among three well-mixed compartments. Transfers occur only along $1 \rightleftharpoons 2 \rightleftharpoons 3$, with constant first-order rates, no external input and no loss. Let x_j be fractions of the total tracer, and use time in hours. Three experiments start with all tracer in compartment j ; their measured initial derivative vectors are

$$v_1 = (-2, 2, 0)^T, \quad v_2 = (1, -4, 3)^T, \quad v_3 = (0, 2, -2)^T.$$

Treat these measurements as exact within the stated model class.

Tasks

1. Recover all four transfer rates and the matrix in $X' = AX$. Explain why the experiment vectors give columns rather than rows.
2. Verify conservation and nonnegativity, and explain why the unit simplex is the physical state space.
3. Find the unique equilibrium with total fraction 1. Check balance of each pair of opposing transfer fluxes.
4. Solve the first experiment exactly using the three modes. Determine whether equilibrium is reached at finite time, and state what the identification result does and does not establish about alternative model classes.

Question 10 – Solution

Strategy. Initial states equal to coordinate vectors reveal matrix columns; conservation and nonnegative transfer rates then provide physical checks.

Step 1: Identify the directed rates. Since $X'(0) = Ae_j$, the j th experimental vector is column j . Thus

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 3 & -2 \end{pmatrix}.$$

The rates are $k_{1 \rightarrow 2} = 2$, $k_{2 \rightarrow 1} = 1$, $k_{2 \rightarrow 3} = 3$, $k_{3 \rightarrow 2} = 2 \text{ h}^{-1}$. Each diagonal entry is minus the total outgoing rate from its compartment. Transposing the measurements would incorrectly replace these column balances with row balances.

Step 2: Check the physical state space. Every column sums to zero, so $(x_1 + x_2 + x_3)' = 0$. At a boundary $x_j = 0$ with the other fractions nonnegative, the corresponding derivative is a sum of nonnegative incoming fluxes. The simplex $\boxed{x_j \geq 0, x_1 + x_2 + x_3 = 1}$ is therefore invariant. Conservation bounds every fraction by 1.

Step 3: Balance the equilibrium fluxes. The end-compartment equations require $x_2 = 2x_1$ and $2x_3 = 3x_2$. Normalization then gives $\boxed{X_* = (1, 2, 3)^T/6}$. Indeed the opposing fluxes on the first edge are $2(1/6) = 1(2/6)$, and on the second are $3(2/6) = 2(3/6)$. These two relations and the total also prove uniqueness of the normalized equilibrium.

Step 4: Resolve the first experiment and its interpretation. The eigenpairs are $0, (1, 2, 3)^T$, $-2, (1, 0, -1)^T$ and $-6, (1, -4, 3)^T$. Resolving $(1, 0, 0)^T$ gives

$$X = X_* + \frac{3}{4}e^{-2t}(1, 0, -1)^T + \frac{1}{12}e^{-6t}(1, -4, 3)^T.$$

Equivalently,

$$\boxed{x_1 = \frac{1}{6} + \frac{3}{4}e^{-2t} + \frac{1}{12}e^{-6t}, \quad x_2 = \frac{1}{3}(1 - e^{-6t}), \quad x_3 = \frac{1}{2} - \frac{3}{4}e^{-2t} + \frac{1}{4}e^{-6t}.$$

The coefficients reproduce the initial state, and each mode satisfies the matrix equation. Since $x_2 < 1/3$ at finite times, equilibrium is approached but never reached then. The experiments uniquely identify this constant linear transfer matrix; they do not rule out nonlinear or time-dependent models sharing the same three measured initial slopes.