

Question 1

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = e^{2t}(1+t)$ for $t \geq 0$. A student obtains the expression $1/(s-2) + 1/(s-2)^2$ and claims that it gives a transform value at every real $s \neq 2$.

Tasks

1. Evaluate the truncated defining integral from zero to R , treating $s = 2$ separately.
2. Determine exactly which real s give convergence, and find $F(s)$ on that set.
3. Evaluate the student's expression at $s = 1$ and explain why it cannot be the Laplace integral there.
4. Prove directly from the integral, without differentiating the final formula, that F is positive and strictly decreasing throughout its convergence interval.

Question 1 – Solution

Strategy. Keep the upper endpoint finite until the parameter-dependent boundary terms have been examined.

Step 1: Evaluate the finite integral. Write $a = s - 2$. Integration by parts gives, for $a \neq 0$,

$$I_R = \int_0^R (1+t)e^{-at} dt = \frac{1}{a} + \frac{1}{a^2} - e^{-aR} \left[\frac{1+R}{a} + \frac{1}{a^2} \right].$$

At $a = 0$, direct integration gives $I_R = R + R^2/2$. The formula with denominators cannot be used at that parameter.

Step 2: Determine the convergence set. For $a > 0$, both e^{-aR} and Re^{-aR} tend to zero. Therefore

$$F(s) = \frac{1}{s-2} + \frac{1}{(s-2)^2}, \quad s > 2.$$

For $a = 0$, the displayed polynomial diverges. For $a < 0$, the integrand $(1+t)e^{|a|t}$ is at least one, so $I_R \geq R \rightarrow \infty$. Thus $s > 2$ is the entire real convergence set, and convergence there is absolute.

Step 3: Reject an algebraic continuation as an integral value. At $s = 1$, the rational expression equals $-1 + 1 = 0$. But the actual integral is

$$I_R = \int_0^R (1+t)e^t dt = Re^R \rightarrow \infty.$$

The discarded boundary term grows instead of vanishing. A finite value of the algebraic expression outside the convergence interval does not assign a value to the defining improper integral.

Step 4: Prove order properties from the kernel. For $s > 2$, the integrand is positive, so $F(s) > 0$. If $s_2 > s_1 > 2$, then for every $t > 0$,

$$0 < e^{-s_2 t} f(t) < e^{-s_1 t} f(t).$$

Both integrals converge, and their difference is strictly positive on any fixed interval of positive length inside $(0, \infty)$. Hence $F(s_2) < F(s_1)$. This argument shows why positivity and monotonicity hold, independently of the rational simplification.

Question 2

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Consider the finite-duration triangular signal

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 < t \leq 2, \\ 0, & t > 2. \end{cases}$$

Tasks

1. Compute $F(s)$ directly by integrating over the two nonzero pieces, first for $s \neq 0$.
2. Determine its full real convergence set and compute $F(0)$ directly. Is the apparent singularity at zero a true divergence?
3. Prove $F(-s) = e^{2s} F(s)$ by a change of variable, using the symmetry of the triangle.
4. If the value at $t = 1$ alone is changed to 100, does the transform change? Explain what this example says about recovering individual point values from an integral transform.

Question 2 – Solution

Strategy. Finite support guarantees convergence for every real parameter. Apparent poles in a simplified expression must be checked against the original integral.

Step 1: Integrate both pieces. For $s \neq 0$, integration by parts gives

$$\int_0^1 te^{-st} dt = \frac{1 - (1+s)e^{-s}}{s^2}, \quad \int_1^2 (2-t)e^{-st} dt = \frac{(s-1)e^{-s} + e^{-2s}}{s^2}.$$

Adding yields

$$F(s) = \frac{(1 - e^{-s})^2}{s^2} \quad (s \neq 0).$$

Step 2: Restore the missing parameter value. For any fixed real s , the integrand is bounded and piecewise continuous on $[0, 2]$, and vanishes afterward. Thus the real convergence set is all of $\mathbb{R}$. At zero, the integral is the triangle's area:

$$F(0) = 1.$$

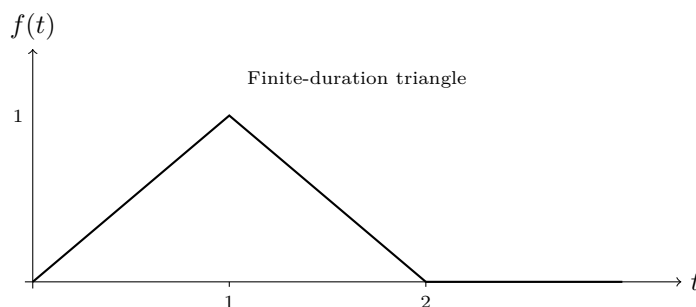
Also $(1 - e^{-s})/s \rightarrow 1$ as $s \rightarrow 0$, so the simplified formula has a removable singularity, not divergent integral behavior.

Step 3: Use reflection symmetry. On $[0, 2]$, $f(2-u) = f(u)$. Substituting $u = 2-t$ gives

$$F(-s) = \int_0^2 e^{st} f(t) dt = e^{2s} \int_0^2 e^{-su} f(2-u) du = e^{2s} F(s).$$

This holds at zero as well. Equivalently, $e^s F(s)$ is an even function of s , expressing symmetry about the triangle's midpoint.

Step 4: Distinguish a point from an interval. Changing one finite point value does not change any of these Riemann integrals, so the transform is unchanged. The modified function is different at one point but agrees elsewhere. Therefore the transform alone cannot recover arbitrary isolated point assignments; this observation does not contradict uniqueness results that impose continuity.



Question 3

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin 3t$ for $t \geq 0$. Its truncated transform is $F_R(s) = \int_0^R e^{-st} \sin 3t dt$.

Tasks

1. Evaluate $F_R(s)$ for real s , retaining its upper-endpoint contribution.
2. Find the exact real convergence interval for $F(s)$ and determine whether convergence there is absolute.
3. At $s = 0$, exhibit two sequences of upper endpoints giving different values of $F_R(0)$.
4. Compare $\lim_{s \rightarrow 0^+} F(s)$ with the existence of $F(0)$. Why does a finite parameter limit not justify interchanging the two limits $R \rightarrow \infty$ and $s \rightarrow 0^+$?

Question 3 – Solution

Strategy. Damping by e^{-st} can create convergence that disappears at the boundary parameter.

Step 1: Keep the endpoint term. Two integrations by parts, or differentiation of the resulting antiderivative, give

$$F_R(s) = \frac{3 - e^{-sR}[s \sin 3R + 3 \cos 3R]}{s^2 + 9}.$$

There is no real zero of the denominator, but that alone says nothing about the limit as $R \rightarrow \infty$.

Step 2: Classify convergence. If $s > 0$, the endpoint term vanishes and

$$F(s) = \frac{3}{s^2 + 9}, \quad s > 0.$$

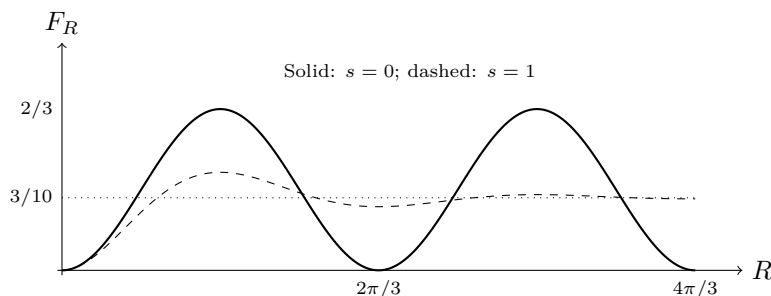
Absolute convergence follows from $|e^{-st} \sin 3t| \leq e^{-st}$. If $s < 0$, take $R_n = 2n\pi/3$. Then $F_{R_n}(s) = 3(1 - e^{|s|R_n})/(s^2 + 9) \rightarrow -\infty$, so the improper integral cannot converge.

Step 3: Test the boundary directly. At $s = 0$, $F_R(0) = (1 - \cos 3R)/3$. For

$$R_n = 2n\pi/3, \quad F_{R_n}(0) = 0; \quad Q_n = (2n+1)\pi/3, \quad F_{Q_n}(0) = 2/3.$$

The two sequences tend to infinity but have different integral values. Thus $F(0)$ does not exist, and the full real convergence interval is $(0, \infty)$.

Step 4: Separate the two limiting procedures. The damped transforms satisfy $\lim_{s \rightarrow 0^+} F(s) = 1/3$. In contrast, setting $s = 0$ first produces an oscillating truncated integral with no limit. For fixed R , the limit in s exists, but convergence in R is not uniform as damping disappears. The value $1/3$ is a limit of damped integrals, not the value of the undamped improper integral. The graph compares accumulated integrals, not the original sine signal.



Question 4

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For $t > 0$, let $f(t) = t^{-1/2}e^{-t}$; assign any finite value to $f(0)$. For the defining integral, treat zero as an improper endpoint as well as infinity. You may use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2$.

Tasks

1. Analyze convergence near zero and near infinity separately, and find the exact real convergence set.
2. Evaluate $F(s)$ on that set by a substitution.
3. Repeat the endpoint analysis for $h(t) = t^{-1}e^{-t}$, $t > 0$. Does exponential decay guarantee a transform for this function?
4. Explain why the first example does not contradict a theorem giving piecewise continuity on each finite interval and exponential order as sufficient conditions for existence.

Question 4 – Solution

Strategy. An improper transform can fail at either endpoint. Rapid decay at infinity cannot repair a nonintegrable singularity at zero.

Step 1: Separate the endpoints. Write $a = s + 1$. Near zero, e^{-at} is bounded above and below by positive constants, and $\int_0^1 t^{-1/2} dt = 2$, so this endpoint is integrable for every fixed real s . At infinity, $a > 0$ gives exponential decay. If $a = 0$, the integral of $t^{-1/2}$ diverges; if $a < 0$, the integrand eventually exceeds one. Hence convergence occurs exactly for $s > -1$, and is absolute.

Step 2: Evaluate the convergent integral. For $a > 0$, use $u = \sqrt{at}$, so $t = u^2/a$ and $dt = 2u du/a$. Then

$$F(s) = \frac{2}{\sqrt{a}} \int_0^\infty e^{-u^2} du = \boxed{\frac{\sqrt{\pi}}{\sqrt{s+1}}, \quad s > -1.}$$

Both the substitution and the square root require the already established condition $a > 0$. An assigned value at the single point zero does not change the improper integral.

Step 3: Test the stronger singularity. For h , the integrand is $t^{-1}e^{-(s+1)t}$. On a sufficiently small interval $(0, \delta)$, its exponential factor is at least $1/2$. Thus

$$\int_0^\delta t^{-1} e^{-(s+1)t} dt \geq \frac{1}{2} \int_0^\delta \frac{dt}{t} = \infty.$$

This holds for every real s . Consequently h has no transform under this definition, even for parameters that give fast decay at infinity.

Step 4: Interpret the sufficient theorem. The first function is unbounded as $t \rightarrow 0^+$, so it does not satisfy the usual finite-limit requirement for piecewise continuity on $[0, A]$. Yet its endpoint singularity is integrable, and the direct calculation establishes existence for $s > -1$. A sufficient theorem guarantees existence when its hypotheses hold; failure of a hypothesis does not prove nonexistence. Here direct endpoint analysis gives the sharper conclusion.

Question 5

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin t/t$ for $t > 0$, with $f(0) = 1$. In this problem examine parameters $s \geq 0$. You may derive the positive-parameter transform using $\sin t/t = \int_0^1 \cos(ut) du$.

Tasks

1. For $s > 0$, justify interchanging the two integrations and evaluate $F(s)$.
2. For $s \geq 0$ and $R > 0$, prove the tail bound $|\int_R^\infty e^{-st} \sin t/t dt| \leq 2/R$, including existence of the tail at $s = 0$.
3. Use that uniform tail bound and finite-interval convergence as $s \rightarrow 0^+$ to determine $F(0)$.
4. Determine whether the integral at $s = 0$ converges absolutely, and contrast it with the positive-parameter integrals.

Question 5 – Solution

Strategy. A uniform tail estimate can justify passage to the boundary parameter; absolute and conditional convergence must still be distinguished.

Step 1: Evaluate for positive damping. For $s > 0$, the double integral of the absolute integrand is at most $\int_0^\infty e^{-st} dt = 1/s$, so interchange is justified. Integration by parts gives $\int_0^\infty e^{-st} \cos(ut) dt = s/(s^2 + u^2)$. Therefore

$$F(s) = \int_0^1 \frac{s}{s^2 + u^2} du = \boxed{\arctan(1/s), \quad s > 0.}$$

Step 2: Control the tail uniformly. Set $w(t) = e^{-st}/t$ for $t \geq R$. For every $s \geq 0$, w decreases to zero and $w' \leq 0$. Integration by parts on $[R, L]$ gives

$$\int_R^L w(t) \sin t dt = [-w(t) \cos t]_R^L + \int_R^L w'(t) \cos t dt.$$

The integral of $|w'|$ is $w(R) - w(L)$, so the right side has magnitude at most $2w(R) \leq 2/R$. Applying the same bound with any larger lower endpoint proves the Cauchy criterion as $L \rightarrow \infty$, including at $s = 0$. The bound persists for the improper tail.

Step 3: Pass to the boundary parameter. On each fixed $[0, R]$, $e^{-st}f(t) \rightarrow f(t)$ uniformly as $s \rightarrow 0^+$, since f is continuous and bounded there. The tails at both s and zero have magnitude at most $2/R$. Thus

$$|F(s) - F(0)| \leq \left| \int_0^R (e^{-st} - 1)f(t) dt \right| + 4/R.$$

First choose large R , then let $s \rightarrow 0^+$. This proves continuity at the boundary and hence $\boxed{F(0) = \pi/2}$.

Step 4: Test absolute convergence. On $[n\pi + \pi/6, n\pi + 5\pi/6]$, $|\sin t| \geq 1/2$ and $t \leq (n+1)\pi$. Each such interval contributes at least $1/[3(n+1)]$ to $\int |\sin t|/t dt$. The harmonic series diverges, so convergence at zero is conditional. For $s > 0$, $|\sin t/t| \leq 1$ gives absolute convergence by comparison with e^{-st} . The boundary value here exists as an ordinary improper integral, not merely as a limit of damped values.

Question 6

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Compare $f_+(t) = e^{t^2}$ and $f_-(t) = e^{-t^2}$ for $t \geq 0$, and write their transforms as F_+ and F_- .

Tasks

1. Determine the real convergence set of each transform directly from the defining integral.
2. Complete the square to express $F_-(s)$ through a Gaussian tail for every real s . No special-function notation is required.
3. For a truncation at T with $T + s/2 > 0$, prove an explicit upper bound for the omitted tail of $F_-(s)$.
4. At $s = -2$, find the smallest integer $T > 1$ for which your bound certifies an error at most 10^{-4} . Explain why negative s does not destroy convergence for f_- .

Question 6 – Solution

Strategy. Compare the quadratic growth or decay with the linear term supplied by the Laplace weight, and use the completed square to bound numerical truncation.

Step 1: Classify both integrals. For any real s , $t^2 - st \geq 0$ when $t \geq \max\{s, 0\}$, so $e^{t^2-st} \geq 1$ on a half-line. Hence F_+ diverges for every real s . For F_- ,

$$-t^2 - st = -(t + s/2)^2 + s^2/4.$$

A shifted Gaussian has an integrable tail and no finite-endpoint singularity, so F_- converges absolutely for every real s .

Step 2: Express the exact transform. Substitution $u = t + s/2$ gives

$$F_-(s) = e^{s^2/4} \int_{s/2}^{\infty} e^{-u^2} du, \quad s \in \mathbb{R}.$$

The lower endpoint is allowed to be negative; the finite portion of the integral then remains harmless. This definite integral is an exact answer.

Step 3: Bound the omitted tail. Let $a = T + s/2 > 0$. Since $1 \leq u/a$ for $u \geq a$,

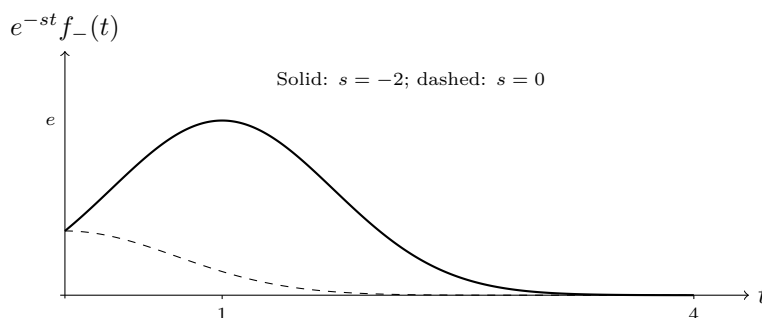
$$\int_a^{\infty} e^{-u^2} du \leq \frac{1}{a} \int_a^{\infty} u e^{-u^2} du = \frac{e^{-a^2}}{2a}.$$

Consequently the positive truncation error satisfies

$$\int_T^{\infty} e^{-t^2-st} dt \leq \frac{e^{-T^2-sT}}{2T+s}.$$

The condition $2T + s > 0$ is part of the bound and cannot be dropped.

Step 4: Certify a cutoff. For $s = -2$, the bound is $e^{-T^2+2T}/(2T-2)$, decreasing for $T > 1$. At $T = 3$ it is $e^{-3}/4 \approx 0.01245$, while at $T = 4$ it is $e^{-8}/6 \approx 5.59 \times 10^{-5}$. Thus the smallest integer cutoff certified by this bound is $\boxed{T = 4}$. Negative s makes the weight grow exponentially, but the quadratic decay still dominates. The graph shows the weighted integrands, whose areas give the transforms.



Question 7

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For each integer $n \geq 1$, let $w_n = e^{-2n^2}$. Define a nonnegative signal by

$$f(t) = \begin{cases} e^{n^2}, & n \leq t \leq n + w_n \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

A function is of exponential order if $|f(t)| \leq M e^{at}$ for all sufficiently large t , for some constants $M > 0$ and real a .

Tasks

1. Explain why f is piecewise continuous on every finite interval but is not of exponential order.
2. Prove that its Laplace integral nevertheless converges absolutely for every real s .
3. Express $F(s)$ as a convergent series, treating $s = 0$ separately.
4. Explain how the spike widths resolve the apparent conflict between arbitrarily large heights and convergence, and why the usual exponential-order theorem is not contradicted.

Question 7 – Solution

Strategy. The integral depends on weighted areas, not just pointwise heights. Quantify the widths before invoking a growth criterion.

Step 1: Examine local regularity and height. Since $w_n < 1$, the spike intervals are disjoint. Any bounded time interval meets only finitely many spikes, each with finite height and finite one-sided limits. Thus f is piecewise continuous locally. If an exponential-order bound held, evaluating at $t = n$ for large integers would give $e^{n^2} \leq Me^{an}$, or $n^2 - an \leq \ln M$. The left side tends to infinity, a contradiction.

Step 2: Estimate each weighted area. Fix real s . On the n th spike, $t \leq n + 1$ and $e^{-st} \leq e^{|s|(n+1)}$. Its contribution is therefore at most

$$e^{n^2} w_n e^{|s|(n+1)} = e^{-n^2 + |s|(n+1)}.$$

For sufficiently large n , $|s|(n+1) \leq n^2/2$, so these terms are at most $e^{-n^2/2} \leq e^{-n/2}$. The comparison series converges. Since the integrand is nonnegative, the bound proves absolute convergence for every real s .

Step 3: Sum the exact contributions. Split finite integrals into their finitely many spike intervals, then pass to the limit using the convergent positive series. For $s \neq 0$,

$$F(s) = \sum_{n=1}^{\infty} e^{n^2 - sn} \frac{1 - e^{-sw_n}}{s}.$$

The quotient is positive even for negative s . At $s = 0$, each contribution is simply height times width, giving

$$F(0) = \sum_{n=1}^{\infty} e^{-n^2}.$$

No closed form for this convergent series is required.

Step 4: Interpret the theorem correctly. The heights grow faster than every exponential, but the widths shrink fast enough that their areas are e^{-n^2} . Any fixed exponential weight still leaves a summable sequence of areas. The common existence theorem provides sufficient conditions; exponential order is not necessary for an individual function to have a Laplace transform. This example establishes that distinction by direct estimates, rather than by trying to apply a theorem whose hypothesis fails.

Question 8

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

An unknown signal belongs to the family $f(t) = 0$ for $0 \leq t < a$ and $f(t) = A$ for $t \geq a$, where $A > 0$ and $a \geq 0$. Exact measurements are

$$F(1) = 2/e, \quad F(2) = e^{-2}.$$

Tasks

1. Derive the transform and its real convergence set directly, then recover A and a from the measurements.
2. For arbitrary positive measurements $M_1 = F(1)$ and $M_2 = F(2)$, give a necessary and sufficient consistency condition for this family and explicit recovery formulas.
3. For $h(t) = e^{-t}(t^2 - 5t/3 + 1/3)$, compute its transform at $s > 0$ directly by integration by parts and show that it vanishes at both measured parameters.
4. Use h to explain precisely why the two measurements identify the signal within the stated family but do not identify an arbitrary real-valued signal.

Question 8 – Solution

Strategy. Distinguish uniqueness of a few model parameters from uniqueness among all possible functions.

Step 1: Integrate and calibrate. For $s > 0$,

$$F(s) = A \int_a^\infty e^{-st} dt = \frac{Ae^{-as}}{s}.$$

For $s \leq 0$, the positive tail integral diverges. Dividing the two measured equations gives $2F(2)/F(1) = e^{-a} = e^{-1}$, so $\boxed{a = 1, A = 2}$. These values reproduce both measurements.

Step 2: Characterize all consistent data. For positive M_1, M_2 , the family requires $2M_2/M_1 = e^{-a} \in (0, 1]$. Thus consistency is exactly $\boxed{2M_2 \leq M_1}$. In that case the unique parameters are

$$\boxed{a = \ln \frac{M_1}{2M_2}, \quad A = \frac{M_1^2}{2M_2}.$$

They satisfy $a \geq 0, A > 0$ and both equations, proving sufficiency as well as necessity. Equality corresponds to onset at zero.

Step 3: Construct an invisible perturbation. Set $b = s + 1 > 0$. Integration by parts gives the three integrals of $1, t, t^2$ against e^{-bt} as $1/b, 1/b^2, 2/b^3$. Therefore

$$H(s) = \frac{2}{(s+1)^3} - \frac{5}{3(s+1)^2} + \frac{1}{3(s+1)} = \boxed{\frac{(s-1)(s-2)}{3(s+1)^3}}.$$

The exponential boundary terms vanish, justifying those integrations. In particular, $H(1) = H(2) = 0$, although h is not the zero function.

Step 4: State the identification limit. For every real λ , the signal $f + \lambda h$ has a convergent transform for $s > 0$. Splitting its defining integral gives $F(s) + \lambda H(s)$, so both recorded measurements are unchanged. Distinct λ give distinct real-valued signals; for nonzero λ they are not members of the original two-parameter family. The recovery formulas establish uniqueness only under that family assumption, not among arbitrary signals.

Question 9

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let f have period two, with $f(t) = 1$ for $0 \leq t < 1$ and $f(t) = 0$ for $1 \leq t < 2$, repeated for all $t \geq 0$. Use the definition rather than a precomputed periodic-transform formula.

Tasks

1. Split the integral into complete periods, sum the resulting series, and determine the exact real convergence set.
2. Simplify the transform on its convergence interval.
3. Compute $\lim_{s \rightarrow 0^+} sF(s)$ and $\lim_{s \rightarrow \infty} sF(s)$. Relate these to the period average and to the signal just after zero.
4. If the defining integral is truncated after N complete periods, find its exact relative error for $s > 0$. At $s = 0.2$, find the smallest integer N making that error at most 10^{-3} .

Question 9 – Solution

Strategy. A time translation by one period multiplies the weighted area by a fixed factor, producing a geometric series.

Step 1: Build the series and its domain. For $s > 0$, the n th nonzero interval contributes

$$\int_{2n}^{2n+1} e^{-st} dt = e^{-2ns} \frac{1 - e^{-s}}{s}.$$

The ratio e^{-2s} lies in $(0, 1)$, so the sum converges. At $s = 0$, each interval contributes one and the sum diverges. For $s < 0$, $e^{-st} \geq 1$ on each such interval, so it also diverges. The real convergence set is exactly $(0, \infty)$, with absolute convergence since $f \geq 0$.

Step 2: Simplify the geometric sum. Summing gives

$$F(s) = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \boxed{\frac{1}{s(1 + e^{-s})}, \quad s > 0.}$$

The integral over only one period is the first term of this sum, not the transform of the endlessly repeated signal.

Step 3: Interpret the two normalized limits. The formula gives

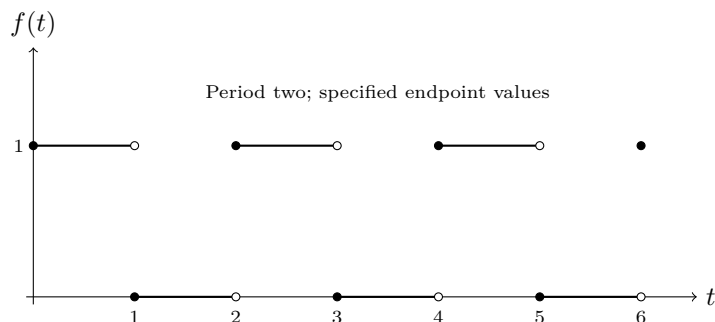
$$\boxed{\lim_{s \rightarrow 0^+} sF(s) = \frac{1}{2}, \quad \lim_{s \rightarrow \infty} sF(s) = 1.}$$

The first value is the period average $(1/2) \int_0^2 f(t) dt = 1/2$. The second agrees with $f(0^+) = 1$: large s weights short times most strongly. These normalized limits do not assert that $F(0)$ exists; it does not.

Step 4: Quantify finite-window error. Writing $F_N = \int_0^{2N} e^{-st} f(t) dt$, the finite geometric sum gives

$$F_N = F(s)(1 - e^{-2Ns}), \quad \boxed{\frac{F(s) - F_N}{F(s)} = e^{-2Ns}.}$$

At $s = 0.2$, the requirement is $e^{-0.4N} \leq 10^{-3}$, so $N \geq \ln(1000)/0.4 \approx 17.27$. The least integer is $\boxed{N = 18}$, corresponding to a cutoff time of 36. Endpoint assignments do not affect the integrals; the graph uses the values specified in the question.



Question 10

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

A continuous signal on $[0, \infty)$ satisfies $0 \leq f(t) \leq 1$, and its observed part is $f(t) = t$ on $[0, 1]$. Its future after one is unknown. Consider $s > 0$.

Tasks

1. Derive explicit lower and upper bounds for $F(s)$ using only this information.
2. Decide which bound is attained within the continuous signal class. Construct admissible signals approaching any unattained bound.
3. State the exact resulting range of possible values of $F(1)$, justifying that every value in that range is possible.
4. Find the width of the uncertainty interval for general $s > 0$, prove it decreases with s , and find the smallest positive integer s making the width at most 0.01.

Question 10 – Solution

Strategy. Positivity of the Laplace weight gives bounds; continuity at the last observed point decides whether an extremum is actually attained.

Step 1: Bound the unknown tail. The observed contribution is

$$L(s) = \int_0^1 t e^{-st} dt = \frac{1 - (1+s)e^{-s}}{s^2}.$$

The remaining integral lies between zero and $\int_1^\infty e^{-st} dt = e^{-s}/s$. Therefore

$$L(s) \leq F(s) \leq U(s), \quad \boxed{U(s) = \frac{1 - e^{-s}}{s^2}}.$$

All integrals converge absolutely because $0 \leq f \leq 1$ and $s > 0$.

Step 2: Resolve attainment. The upper bound is attained by $f(t) = \min\{t, 1\}$. The lower bound is not attained: continuity and $f(1) = 1$ force a strictly positive contribution on some interval immediately after one. For $\varepsilon > 0$, instead continue linearly from $(1, 1)$ to $(1 + \varepsilon, 0)$ and then stay zero. This continuous f_ε satisfies

$$0 < F_\varepsilon(s) - L(s) \leq \varepsilon e^{-s} \rightarrow 0.$$

Thus $L(s)$ is the infimum, while $U(s)$ is a maximum.

Step 3: Give the full range at one. Substitution gives the interval

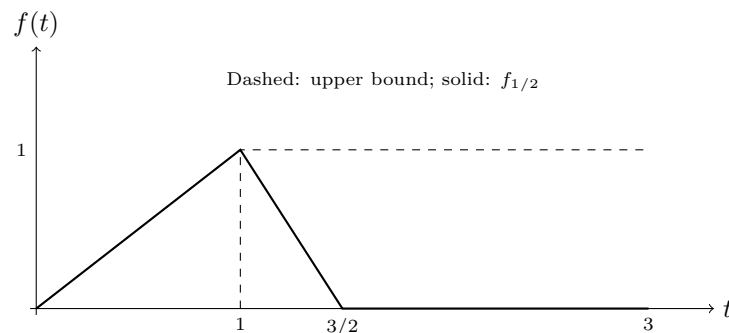
$$\boxed{1 - 2/e < F(1) \leq 1 - 1/e}.$$

For any target strictly above the lower endpoint, choose ε small enough that $F_\varepsilon(1)$ is below it. A convex combination of f_ε and the upper-bound signal remains continuous, in $[0, 1]$, and agrees with the observed ramp. Splitting the defining integral shows its transform is the same convex combination. Choosing the weight appropriately reaches the target, proving no intermediate values are missing.

Step 4: Measure the uncertainty. The interval width is $W(s) = U(s) - L(s) = e^{-s}/s$, with

$$W'(s) = -e^{-s}(s+1)/s^2 < 0.$$

Since $W(3) = e^{-3}/3 \approx 0.01660$ and $W(4) = e^{-4}/4 \approx 0.00458$, the least positive integer meeting the tolerance is $\boxed{s = 4}$. Increasing s reduces sensitivity to the unknown late-time portion; it does not supply missing pointwise information about that future.



Question 1

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = \sin^2(3t)$. A student proposes

$$\mathcal{L}\{\sin^2(3t)\}(s) = [\mathcal{L}\{\sin 3t\}(s)]^2.$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$ and $\mathcal{L}\{\sin bt\} = b/(s^2 + b^2)$ for $s > 0$.

Tasks

1. Find the correct transform using a trigonometric identity and linearity.
2. Determine the full real convergence set and compare the correct result with the student's proposed expression.
3. Use $|\sin x| \leq |x|$ to prove a direct upper bound for the transform when $s > 0$.
4. Use $u = st$ in the defining integral to derive $\lim_{s \rightarrow \infty} s^3 F(s)$ independently of the rational formula. Explain how this limit diagnoses the proposed product rule.

Question 1 – Solution

Strategy. Linearity applies to sums and scalar multiples, not to pointwise products. A large-parameter check can expose the distinction.

Step 1: Rewrite the function before transforming. Since $\sin^2(3t) = (1 - \cos 6t)/2$, linearity gives

$$F(s) = \frac{1}{2s} - \frac{s}{2(s^2 + 36)} = \boxed{\frac{18}{s(s^2 + 36)}}, \quad s > 0.$$

Step 2: Check convergence and the proposed rule. For $s > 0$, boundedness of f gives absolute convergence. For $s \leq 0$, the weight is at least one and $\sin^2(3t)$ has positive integral on each period, so the nonnegative improper integral diverges. Thus the real convergence set is exactly $(0, \infty)$. The proposed expression is

$$\frac{9}{(s^2 + 9)^2},$$

which is not the correct transform. For example, at $s = 3$ the correct value is $2/15$, while the proposed value is $1/36$. Squaring a time function is not transformed by squaring its transform.

Step 3: Obtain a bound from the integral. The inequality $0 \leq \sin^2(3t) \leq 9t^2$ yields

$$0 < F(s) \leq 9 \int_0^\infty t^2 e^{-st} dt = \boxed{18/s^3}.$$

The last integral is $2/s^3$, by two integrations by parts. Its boundary terms vanish for $s > 0$.

Step 4: Recover the leading scale independently. With $u = st$,

$$s^3 F(s) = \int_0^\infty e^{-u} s^2 \sin^2(3u/s) du.$$

The integrand tends to $9u^2 e^{-u}$ and is bounded by that integrable function. Dominated convergence therefore gives $\boxed{s^3 F(s) \rightarrow 18}$. This reflects the local behavior $\sin^2(3t) \sim 9t^2$. In contrast, multiplying the proposed expression by s^3 gives a limit of zero. The proposed rule even predicts the wrong leading power of $1/s$.

Question 2

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Find the transform of $f(t) = t^2 e^{-2t} \cos 3t$ by combining exponential shifting with differentiation in the transform parameter. You may use $C(p) = \mathcal{L}\{\cos 3t\}(p) = p/(p^2 + 9)$ for $p > 0$.

Tasks

1. Justify the identity $\mathcal{L}\{t^2 g(t)\}(s) = G''(s)$ in this example, including the convergence condition needed for differentiation.
2. Compute and simplify the transform explicitly.
3. Find all its zeros and determine its sign within the convergence interval. Explain why a zero transform value need not mean that the original function is zero.
4. The simplified expression also has value zero at $s = -2$. Does that value represent an ordinary Laplace integral? Check the boundary directly and determine whether any smaller real parameter can work.

Question 2 – Solution

Strategy. Account for the exponential shift first, then differentiate the correct parameter expression and retain its convergence domain.

Step 1: Justify parameter differentiation. The factor e^{-2t} changes the effective parameter to $p = s + 2$. For any $p_0 > 0$, a sufficiently small neighborhood has $p \geq p_0/2$. Differentiating the kernel twice produces $t^2 e^{-pt} \cos 3t$, dominated by $t^2 e^{-p_0 t/2}$, an integrable function. The first derivative is justified similarly. Thus, for $s > -2$, the desired transform is $C''(s + 2)$.

Step 2: Differentiate and simplify. Direct calculation gives

$$C'(p) = \frac{9 - p^2}{(p^2 + 9)^2}, \quad C''(p) = \frac{2p(p^2 - 27)}{(p^2 + 9)^3}.$$

Therefore

$$F(s) = \frac{2(s + 2)[(s + 2)^2 - 27]}{[(s + 2)^2 + 9]^3}, \quad s > -2.$$

The exponential times a polynomial gives absolute convergence on this interval.

Step 3: Interpret its zero and sign. For $p > 0$, the denominator and $2p$ are positive. Hence the only zero in the convergence interval is $s = 3\sqrt{3} - 2$. The transform is negative below that value and positive above it. At the zero, weighted positive and negative portions of $t^2 e^{-2t} \cos 3t$ cancel; the function itself is certainly not identically zero.

Step 4: Test the boundary rather than the rational expression. At $s = -2$, the integral becomes $\int_0^R t^2 \cos 3t \, dt$. Integration by parts gives

$$\frac{R^2 \sin 3R}{3} + \frac{2R \cos 3R}{9} - \frac{2 \sin 3R}{27}.$$

At $R_n = 2n\pi/3$, this equals $2R_n/9 \rightarrow \infty$. For $p = s + 2 < 0$, consider $[R_n, R_n + \pi/9]$, where $\cos 3t \geq 1/2$. The integral on that interval is at least $(\pi/18)R_n^2 e^{-pR_n}$ and does not tend to zero, violating the Cauchy criterion. Thus no real $s \leq -2$ works; the algebraic boundary value is not a transform value.

Question 3

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \sin 2t$, with $G(s) = 2/[(s+1)^2 + 4]$ for $s > -1$. Define the compressed signals $f(t) = g(3t)$ and $h(t) = 3g(3t)$.

Tasks

1. Derive the time-scaling identity for $g(at)$, $a > 0$, by substitution in the defining integral. Track the convergence parameter.
2. Find the transforms of f and h , and their full real convergence intervals. Explain why replacing $G(s)$ by $G(3s)$ gives the wrong scaling.
3. Compute the signed total integrals of g, f, h . Which compression preserves the total integral?
4. Use parameter differentiation to derive the first-moment identity. Compute $\int_0^\infty t f(t) dt$ and $\int_0^\infty t h(t) dt$. Explain their different scaling factors.

Question 3 – Solution

Strategy. Time compression changes both the exponential weight and the integration measure; amplitude compensation affects area separately.

Step 1: Derive the scaling rule. For $a > 0$, substituting $u = at$ gives

$$\int_0^\infty e^{-st} g(at) dt = \frac{1}{a} \int_0^\infty e^{-(s/a)u} g(u) du = \boxed{\frac{1}{a} G(s/a)}.$$

The effective parameter is s/a , so $s/a > -1$ here. Both the reciprocal parameter scaling and the factor $1/a$ are essential.

Step 2: Apply the rule to both signals. For $a = 3$,

$$\boxed{F(s) = \frac{6}{(s+3)^2 + 36}, \quad H(s) = \frac{18}{(s+3)^2 + 36}, \quad s > -3.}$$

These are the transforms of $e^{-3t} \sin 6t$ and three times that signal. At the boundary the undamped sine integral fails to converge; below it the exponentially growing sine fails the Cauchy criterion on positive half-wave subintervals. Thus $s > -3$ is exact. $G(3s)$ would stretch the parameter in the opposite direction and omit the measure factor.

Step 3: Compare signed areas. Since zero lies in the convergence intervals, substitution of $s = 0$ gives

$$\int_0^\infty g = \frac{2}{5}, \quad \int_0^\infty f = \frac{2}{15}, \quad \int_0^\infty h = \frac{2}{5}.$$

Compression alone reduces signed area by three; multiplying the amplitude by three restores it. These are signed integrals, not integrals of absolute value.

Step 4: Track first moments. Exponential decay justifies differentiation near $s = 0$, so $G'(s) = -\int_0^\infty t e^{-st} g(t) dt$. As $G'(0) = -4/25$, the original first moment is $4/25$. Differentiating $F(s) = G(s/3)/3$ and $H(s) = G(s/3)$ yields

$$\boxed{\int_0^\infty t f(t) dt = \frac{4}{225}, \quad \int_0^\infty t h(t) dt = \frac{4}{75}.$$

For f , one factor $1/3$ comes from dt and another from $t = u/3$. The amplitude factor in h cancels only one of them.

Question 4

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \cos t$ and $y(t) = \int_0^t g(u) du$. The transform of g is $G(s) = (s+1)/[(s+1)^2 + 1]$ for $s > -1$.

Tasks

1. Derive the transform rule for y by integration by parts, showing where the zero initial value and the boundary at infinity enter.
2. Find an elementary expression for $y(t)$, its transform $Y(s)$, and the exact real convergence interval of Y .
3. Set $z(t) = y(t) - 1/2$. Find its transform and convergence interval, including its value at $s = 0$. Explain why cancellation can enlarge the interval.
4. Find the first maximum of y and its height. Does integration of this exponentially decaying input produce an exponentially decaying or monotone cumulative response?

Question 4 – Solution

Strategy. A cumulative integral may retain a nonzero constant even when its input decays. Its transform domain must be checked anew.

Step 1: Derive the cumulative rule. Here $y(0) = 0$, $y' = g$, and y is bounded because g is absolutely integrable. For $s > 0$, integration by parts gives

$$G(s) = [e^{-st}y(t)]_0^\infty + s \int_0^\infty e^{-st}y(t) dt = sY(s).$$

The initial term is zero and the upper term vanishes because y is bounded. Thus $Y = G/s$ on this half-line.

Step 2: Determine the response and its domain. An antiderivative gives

$$y(t) = \frac{1 + e^{-t}(\sin t - \cos t)}{2}, \quad Y(s) = \frac{s+1}{s[(s+1)^2 + 1]}, \quad s > 0.$$

The formula satisfies $y(0) = 0$ and $y' = g$. Since $y(t) \rightarrow 1/2$, it is eventually bounded below by a positive constant. Its transform therefore diverges for every $s \leq 0$. The cumulative response has a smaller convergence interval than the input.

Step 3: Remove the persistent constant. Now $z = e^{-t}(\sin t - \cos t)/2$. Transforming these two terms directly gives

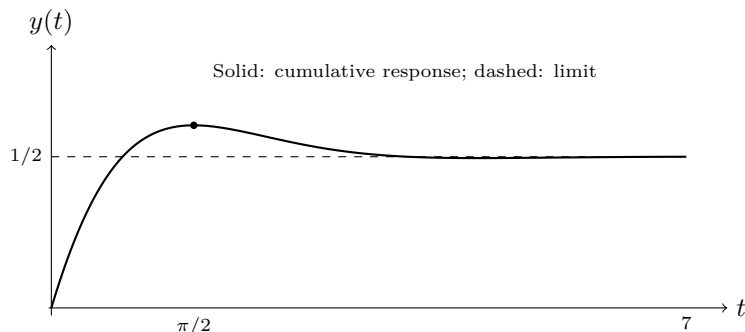
$$Z(s) = -\frac{s}{2[(s+1)^2 + 1]}, \quad s > -1.$$

It converges absolutely there; at and below -1 the nondecaying or growing trigonometric tail prevents ordinary convergence. In particular, $Z(0) = 0$: its signed areas cancel. The identity $Z = Y - 1/(2s)$ is initially valid only for $s > 0$. The wider domain follows from the actual centered function, not by subtracting divergent integrals at $s = 0$.

Step 4: Locate the overshoot. Since $y' = e^{-t} \cos t$, the first positive turning time is $t = \pi/2$, with derivative changing from positive to negative. Its height is

$$y(\pi/2) = \frac{1}{2}(1 + e^{-\pi/2}) > \frac{1}{2}.$$

The response overshoots and oscillates about its limiting constant, so it is neither monotone nor decaying to zero. Its centered part does decay exponentially.



Question 5

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = e^{-t}(1+t)$. A student claims that the second derivative has transform $s^2 F(s)$, without any initial terms.

Tasks

1. Derive the first- and second-derivative transform formulas using integration by parts on a finite interval before taking the limit.
2. Compute $F(s)$, $f'(t)$ and $f''(t)$, and obtain their transforms explicitly.
3. Identify the omitted term in the student's claim and verify the corrected result independently from the expression for f'' .
4. Evaluate the transforms of f' and f'' at $s = 0$ and check them using endpoint differences. Explain how a nonzero second derivative can have zero total signed integral.

Question 5 – Solution

Strategy. Differentiation in time creates boundary terms. Keep them until the initial data and exponential decay have been used.

Step 1: Retain the boundaries. Integration by parts on $[0, R]$ gives

$$\int_0^R e^{-st} f'(t) dt = e^{-sR} f(R) - f(0) + s \int_0^R e^{-st} f(t) dt.$$

Repeating for f'' and then taking $R \rightarrow \infty$ yields

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0),$$

provided the upper boundary terms vanish. Here they do for every $s > -1$, since f, f' are polynomials times e^{-t} .

Step 2: Compute the functions and transforms. Using elementary exponential integrals,

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} = \frac{s+2}{(s+1)^2}.$$

Direct differentiation gives $f' = -te^{-t}$ and $f'' = (t-1)e^{-t}$, with $f(0) = 1$ and $f'(0) = 0$. Hence

$$\boxed{\mathcal{L}\{f'\} = -\frac{1}{(s+1)^2}, \quad \mathcal{L}\{f''\} = -\frac{s}{(s+1)^2}, \quad s > -1.}$$

These intervals are exact: at or below -1 , the resulting polynomial or growing exponential tails do not have convergent improper integrals.

Step 3: Repair and check the claim. The omitted initial contribution is $-sf(0) - f'(0) = -s$. Thus the corrected expression is $s^2F - s = -s/(s+1)^2$. Independently, transforming $(t-1)e^{-t}$ gives

$$\frac{1}{(s+1)^2} - \frac{1}{s+1} = -\frac{s}{(s+1)^2}.$$

This agreement verifies both the boundary terms and the algebra.

Step 4: Check zero-parameter integrals. At zero, the first-derivative transform equals -1 , consistent with $\int_0^\infty f' = f(\infty) - f(0) = 0 - 1$. The second-derivative transform is zero, consistent with $\int_0^\infty f'' = f'(\infty) - f'(0) = 0$. The function f'' is negative before $t = 1$ and positive afterward; its two signed contributions cancel. Zero integral is not the same as a zero function.

Question 6

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

For $t > 0$, define

$$h(t) = \frac{e^{-t} - e^{-4t}}{t},$$

and extend it continuously to $t = 0$. This example concerns division by time; do not assume a transform rule for dividing an arbitrary function by t .

Tasks

1. Find $h(0)$ and express $h(t)$ as an integral of e^{-ut} over a finite interval in u .
2. Use that representation to derive the transform $H(s)$ and justify the order of integration.
3. Find the exact real convergence interval. Explain why transforming e^{-t}/t and e^{-4t}/t separately is invalid.
4. Find $H'(s)$ and $\lim_{s \rightarrow \infty} sH(s)$. Relate them to positivity of h and its value at zero.

Question 6 – Solution

Strategy. Cancellation removes the origin singularity. Preserve that cancellation by integrating over an exponential parameter.

Step 1: Remove the apparent singularity. Expansion at zero, or l'Hopital's rule, gives $\boxed{h(0) = 3}$. Direct integration in u gives

$$h(t) = \int_1^4 e^{-ut} du,$$

including at $t = 0$ by continuity. In particular, h is positive and bounded above by three.

Step 2: Derive the logarithmic transform. For $s > -1$, $s + u > 0$ throughout $[1, 4]$. The absolute double integral is finite, as seen by integrating first in t :

$$H(s) = \int_1^4 \int_0^\infty e^{-(s+u)t} dt du = \int_1^4 \frac{du}{s+u} = \boxed{\ln \frac{s+4}{s+1}}, \quad s > -1.$$

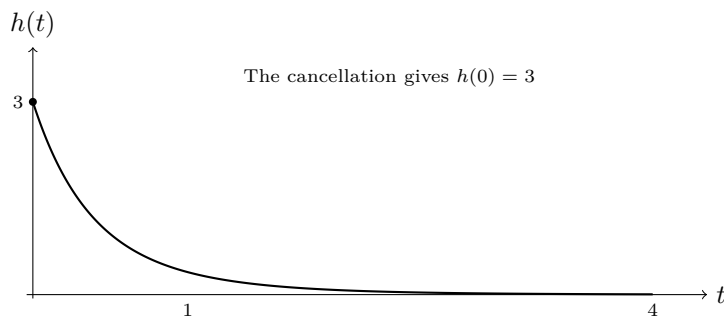
The finite value justifies interchange; both logarithm arguments are positive on this domain.

Step 3: Check the domain and invalid separation. At $s = -1$, the integrand is $(1 - e^{-3t})/t$, whose tail is at least $1/(2t)$ for sufficiently large t . It diverges. For smaller s , a larger positive exponential weight makes divergence persist. Thus $s > -1$ is the full real convergence interval. Each separate term e^{-t}/t or e^{-4t}/t has a nonintegrable origin singularity for every s . Their divergent transforms cannot be subtracted; it is the difference of the functions that is transformable.

Step 4: Interpret derivative and initial scale. Differentiating the logarithm gives

$$\boxed{H'(s) = \frac{1}{s+4} - \frac{1}{s+1} = -\frac{3}{(s+1)(s+4)} < 0.}$$

This agrees with $H'(s) = -\int_0^\infty te^{-st}h(t) dt$, since $h > 0$. Also $H(s) = \ln[1 + 3/(s+1)]$, so $\ln(1+x)/x \rightarrow 1$ implies $\boxed{sH(s) \rightarrow 3 = h(0)}$. The graph displays the continuous signal; its apparent division-by-zero point has been correctly filled in.



Question 7

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

An unknown signal belongs to the family $f(t) = e^{-at} \cos bt$, with $a \geq 0$ and $b > 0$. Its transform satisfies the exact observations

$$\lim_{s \rightarrow \infty} s[sF(s) - 1] = -2, \quad \lim_{s \rightarrow \infty} s^3[F(s) - 1/s + 2/s^2] = -5.$$

Tasks

1. Derive the transform in terms of a, b and expand it through the term in s^{-3} .
2. Recover a, b from the observations and state the resulting transform with its convergence interval.
3. Compute $f(0)$, $f'(0)$ and $f''(0)$ directly. Explain how the expansion coefficients encode these initial quantities.
4. For $q(t) = t^3 e^{-t}$, compute its transform and show that $f + \lambda q$ has the same two observed limits for every real λ . Explain the limit of the parameter recovery claim.

Question 7 – Solution

Strategy. Large transform parameters emphasize early time. Within a specified family, the first few coefficients can determine parameters, but they do not determine an arbitrary signal.

Step 1: Transform and expand. Exponential shifting in the cosine pair gives

$$F(s) = \frac{s+a}{(s+a)^2 + b^2}, \quad s > -a.$$

Expanding this rational expression for large positive s yields

$$F(s) = \frac{1}{s} - \frac{a}{s^2} + \frac{a^2 - b^2}{s^3} + O(s^{-4}).$$

For example, factor s^2 from the denominator and expand its reciprocal through order s^{-2} before multiplying by $1/s + a/s^2$.

Step 2: Recover the parameters. The first limit equals $-a$, so $a = 2$. With this value, the second equals $a^2 - b^2 = -5$, giving $b^2 = 9$. Since $b > 0$,

$$a = 2, \quad b = 3, \quad F(s) = \frac{s+2}{(s+2)^2 + 9}, \quad s > -2.$$

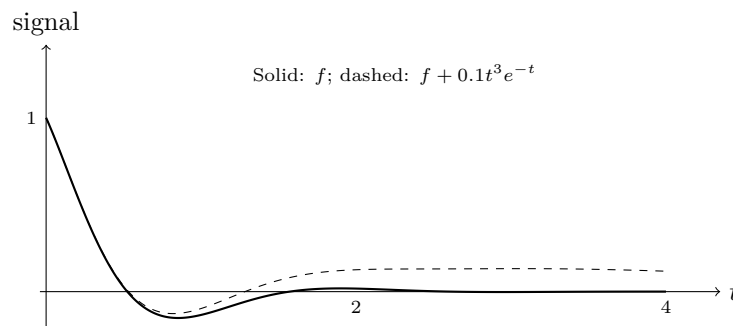
The boundary has an undamped cosine integral that does not converge; smaller parameters produce growing oscillations. Thus the interval is exact.

Step 3: Read the initial information. Direct differentiation gives

$$f(0) = 1, \quad f'(0) = -a = -2, \quad f''(0) = a^2 - b^2 = -5.$$

These are precisely the coefficients of $1/s$, $1/s^2$ and $1/s^3$ in the expansion. Repeated integration by parts explains the same pattern when the needed derivative bounds hold.

Step 4: Show what the measurements cannot identify. Three integrations by parts give $\mathcal{L}\{q\}(s) = 6/(s+1)^4$ for $s > -1$. Thus $f + \lambda q$ has transform $F(s) + 6\lambda/(s+1)^4$ on their common domain. The added term is $O(s^{-4})$, so it changes neither observed limit. Also $q(0) = q'(0) = q''(0) = 0$. Distinct λ therefore give different signals with the same recorded information. The parameters are uniquely recovered only within the stated exponential-cosine family.



Question 8

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Suppose f is continuous, bounded and nonnegative on $[0, \infty)$. Consider the candidate expression

$$R(s) = \frac{s}{(s+1)^2}, \quad s > 0.$$

Tasks

1. Derive the sign requirements on $F(s)$, $F'(s)$ and $F''(s)$ imposed by these hypotheses. State when $F(s) > 0$.
2. Although $R(s) > 0$ for every $s > 0$, prove that it cannot be the transform of such a nonnegative f .
3. Construct a continuous bounded sign-changing function g whose transform is R , using basic exponential pairs. Verify its sign change and full real convergence interval.
4. Compute $\int_0^\infty g(t) dt$ and explain why positivity of a transform throughout $s > 0$ does not imply positivity of its original function.

Question 8 – Solution

Strategy. Positivity of the transform is only one consequence of a nonnegative signal; parameter derivatives supply stronger necessary tests.

Step 1: Derive the necessary signs. If $0 \leq f \leq M$, then the kernels obtained by differentiating once or twice are dominated locally in $s > 0$ by constant multiples of $te^{-s_0 t/2}$ and $t^2 e^{-s_0 t/2}$. Differentiation under the integral is therefore justified, and

$$F(s) \geq 0, \quad F'(s) = - \int_0^\infty te^{-st} f(t) dt \leq 0,$$

$$F''(s) = \int_0^\infty t^2 e^{-st} f(t) dt \geq 0.$$

If f is not identically zero, continuity gives a positive interval contribution and hence $F(s) > 0$ for every $s > 0$.

Step 2: Test the positive candidate. Although $R(s) > 0$,

$$R'(s) = \frac{1-s}{(s+1)^3} > 0 \quad (0 < s < 1).$$

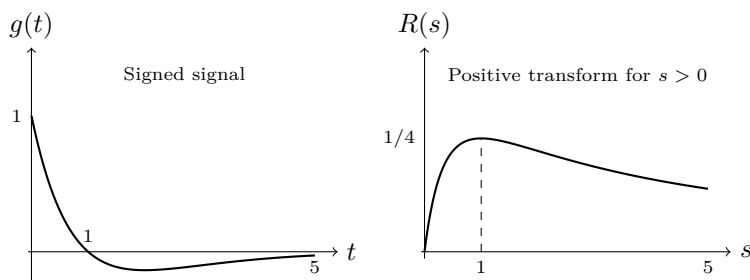
This contradicts the required nonpositive derivative. Thus no continuous bounded nonnegative signal can have this transform.

Step 3: Find a valid signed signal. Rewrite $R = 1/(s+1) - 1/(s+1)^2$. The basic exponential integrals give

$$g(t) = e^{-t}(1-t), \quad \mathcal{L}\{g\}(s) = R(s), \quad s > -1.$$

The function is positive for $0 \leq t < 1$, zero at one, and negative for $t > 1$. It is continuous and bounded. At or below $s = -1$, its eventually negative polynomial or growing tail has no finite improper integral, so the domain is exact.

Step 4: Explain the weighted cancellation. At $s = 0$, the integral exists and equals $R(0) = 0$, also verified by $\int e^{-t} = 1$ and $\int te^{-t} = 1$. For positive s , the extra weight suppresses the late negative portion more strongly, producing a positive transform. Consequently $F > 0$ alone is insufficient to infer $f \geq 0$. The two plots show a sign-changing signal and its everywhere-positive transform on $s > 0$.



Question 9

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

A nonnegative density on $t \geq 0$ has the form $f(t) = cte^{-at}$ with $a > 0$, $c > 0$ and $\int_0^\infty f(t) dt = 1$. Define its mean by $\mu = \int_0^\infty tf(t) dt$ and variance by $\sigma^2 = \int_0^\infty t^2 f(t) dt - \mu^2$. An exact measurement gives $\mu = 4$.

Tasks

1. Determine the relation between a, c from normalization and compute the transform.
2. Derive the first two moment identities using parameter differentiation. Recover a, c and the variance from the measured mean.
3. Could this same family have mean 4 and variance 4? Give an exact consistency relation for arbitrary positive mean.
4. Find the location of the density maximum for the recovered parameters and compare it with the mean. Explain why those locations differ.

Question 9 – Solution

Strategy. Normalization fixes one parameter, while derivatives of the transform at zero recover moments. A density's peak is not its mean.

Step 1: Normalize and transform. Integration by parts gives $\int_0^\infty te^{-at} dt = 1/a^2$. Thus $c = a^2$, and exponential shifting yields

$$F(s) = \frac{a^2}{(s+a)^2}, \quad s > -a.$$

The nonnegative polynomial-exponential tail diverges at or below $-a$. In particular, $F(0) = 1$, as normalization requires.

Step 2: Extract moments and parameters. Since zero lies strictly inside the convergence interval, exponential decay justifies differentiation near zero:

$$-F'(0) = \int_0^\infty tf(t) dt, \quad F''(0) = \int_0^\infty t^2 f(t) dt.$$

The rational expression gives $F'(s) = -2a^2/(s+a)^3$ and $F''(s) = 6a^2/(s+a)^4$. Hence

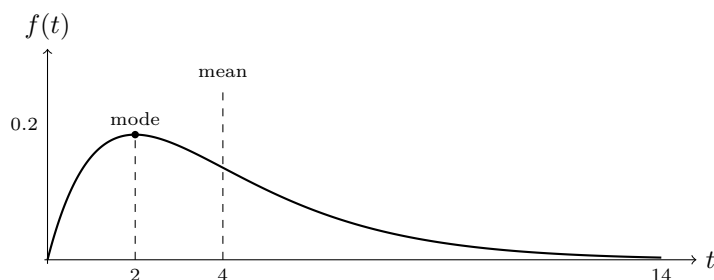
$$\mu = \frac{2}{a}, \quad \sigma^2 = \frac{6}{a^2} - \frac{4}{a^2} = \frac{2}{a^2}.$$

From $\mu = 4$, we obtain

$$a = \frac{1}{2}, \quad c = \frac{1}{4}, \quad \sigma^2 = 8, \quad F(s) = \frac{1}{(2s+1)^2}.$$

Step 3: Test consistency rather than refitting. Eliminating a gives $\sigma^2 = \mu^2/2$. Thus mean four forces variance eight, and the pair $(4, 4)$ is incompatible with this family. Changing c cannot repair it because normalization already fixes $c = a^2$.

Step 4: Locate the peak. For general a , $f'(t) = a^2e^{-at}(1-at)$, positive before $t = 1/a$ and negative afterward. The unique density maximum is therefore at $t = 1/a = 2$, while the mean is four. The long right tail contributes to the mean; the point of largest density measures a different feature. The graph marks both the mode and the mean for $f(t) = te^{-t/2}/4$.



Question 10

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let f be twice continuously differentiable, with f, f', f'' of exponential order. Work at sufficiently large real s so that all transforms below converge and all exponential boundary terms vanish. You may use

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{tf\} = -F', \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0).$$

Tasks

1. Compute $\mathcal{L}\{(tf)'\}$ and $\mathcal{L}\{tf'\}$ separately. Explain their difference using the product rule in time.
2. Verify both results explicitly for $f(t) = e^{-t}$.
3. Derive $\mathcal{L}\{t^2 f''\}$, carefully differentiating every s -dependent factor.
4. A student keeps only $s^2 F''$ in the last result. Identify the missing terms and verify the correct formula for $f = e^{-t}$ by transforming $t^2 e^{-t}$ directly.

Question 10 – Solution

Strategy. Multiplication by time and differentiation in time correspond to operations in s that must be applied in the correct order.

Step 1: Compare the two orders. Since $(tf)(0) = 0$, applying the time-derivative rule after time multiplication gives

$$\boxed{\mathcal{L}\{(tf)'\} = -sF'(s)}.$$

In the opposite order, parameter differentiation applies to the entire transform of f' :

$$\boxed{\mathcal{L}\{tf'\} = -\frac{d}{ds}[sF(s) - f(0)] = -F(s) - sF'(s)}.$$

Their difference is $F(s)$, exactly as required by $(tf)' - tf' = f$. The initial value $f(0)$ is constant in s and differentiates to zero.

Step 2: Check a concrete example. For $f = e^{-t}$, $F = 1/(s+1)$ and $F' = -1/(s+1)^2$. The formulas predict

$$\mathcal{L}\{(1-t)e^{-t}\} = \frac{s}{(s+1)^2}, \quad \mathcal{L}\{-te^{-t}\} = -\frac{1}{(s+1)^2}.$$

Direct use of the exponential and te^{-t} integrals verifies both, for $s > -1$. Their difference is $1/(s+1) = F$.

Step 3: Differentiate the complete expression twice. Multiplication by t^2 corresponds to two parameter derivatives with positive sign. Thus

$$\mathcal{L}\{t^2 f''\} = \frac{d^2}{ds^2}[s^2 F - sf(0) - f'(0)] = \boxed{2F + 4sF' + s^2 F''}.$$

The first derivative of $s^2 F$ is $2sF + s^2 F'$; differentiating again produces both product-rule contributions. The initial terms are at most linear in s , so their second derivatives vanish.

Step 4: Diagnose and verify. The proposed $s^2 F''$ omits $2F + 4sF'$. For $f = e^{-t}$, the correct result simplifies to

$$\frac{2}{s+1} - \frac{4s}{(s+1)^2} + \frac{2s^2}{(s+1)^3} = \boxed{\frac{2}{(s+1)^3}}.$$

Since $f'' = e^{-t}$, direct integration of $t^2 e^{-(s+1)t}$ gives this same value for $s > -1$. Differentiating only F while treating its prefactors as constants is the source of the error.

Question 1

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$F(s) = \frac{s+5}{(s+1)(s+3)}.$$

The basic pair $\mathcal{L}\{e^{-at}\} = 1/(s+a)$ holds for $s > -a$.

Tasks

1. Find the inverse by partial fractions and verify the decomposition.
2. Determine the full real convergence interval of the recovered function.
3. Prove that the inverse is positive for all $t \geq 0$. Does its negative partial-fraction coefficient contradict positivity?
4. Find its initial value, its unique maximum and its limit at infinity. Explain why a sum of decaying modes need not decrease from the start.

Question 1 – Solution

Strategy. Resolve the two poles, then factor the resulting time expression before deciding its sign or shape.

Step 1: Resolve and invert. Writing $F = A/(s+1) + B/(s+3)$ gives $A+B=1$ and $3A+B=5$. Thus $A=2$, $B=-1$, and

$$f(t) = 2e^{-t} - e^{-3t}.$$

Transforming these two terms gives $2/(s+1) - 1/(s+3)$, whose numerator is $2(s+3) - (s+1) = s+5$. This verifies the entire rational expression, not just its pole locations.

Step 2: Retain the actual domain. For $s > -1$, both exponential integrals converge absolutely. Since $f = e^{-t}(2 - e^{-2t}) \geq e^{-t}$ on $t \geq 0$, its weighted integral diverges for $s \leq -1$. Hence the exact real domain is $s > -1$; other algebraic values of F are not ordinary transform values.

Step 3: Determine the sign. The factor $2 - e^{-2t}$ lies in $[1, 2)$, so $f(t) > 0$ everywhere. The negative coefficient belongs to one term in a decomposition; it does not determine the sign of the whole function. Here the faster-decaying negative term never dominates the positive term.

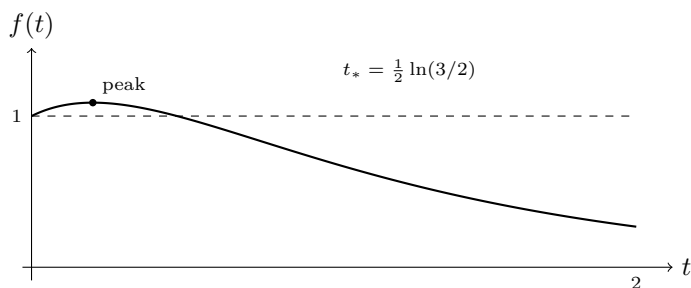
Step 4: Locate the peak. We have $f(0) = 1$, $f(t) \rightarrow 0$, and

$$f'(t) = e^{-t}(-2 + 3e^{-2t}).$$

The bracket decreases through zero once. Therefore f increases up to, then decreases after,

$$t_* = \frac{1}{2} \ln(3/2), \quad f(t_*) = \frac{4}{3} \sqrt{2/3}.$$

In particular $f'(0) = 1 > 0$: decay of each mode does not force monotonic decay of their signed combination. The graph marks the small but real overshoot.



Question 2

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{s^2 + 3s + 4}{(s + 1)^3}.$$

You may use $\mathcal{L}\{t^n e^{-t}\} = n!/(s + 1)^{n+1}$ for nonnegative integers n and $s > -1$.

Tasks

1. Find the complete partial-fraction expansion at the repeated pole.
2. Invert every term, explaining the factorial factors. Verify the transform of your answer.
3. A student includes only a term $C/(s + 1)^3$. Show why this cannot represent F , and identify the missing time terms.
4. Compute $f(0)$, $f'(0)$, $f''(0)$ and the exact convergence interval. Check the three initial quantities against the first three large- s coefficients of F .

Question 2 – Solution

Strategy. A pole of order three allows all three reciprocal powers, and each power has its own factorial normalization.

Step 1: Expand around the pole. Set $p = s + 1$. Then $s^2 + 3s + 4 = p^2 + p + 2$, so

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Equivalently, multiplying by $(s+1)^3$ gives the polynomial identity $s^2 + 3s + 4 = (s+1)^2 + (s+1) + 2$.

Step 2: Apply the correctly normalized pairs. The inverse of $(s+1)^{-3}$ is $t^2 e^{-t}/2!$. Thus

$$\boxed{f(t) = e^{-t}(1 + t + t^2)}.$$

Its three forward transforms are $1/(s+1)$, $1/(s+1)^2$ and $2/(s+1)^3$, confirming the result. Forgetting $2!$ would double the quadratic term incorrectly.

Step 3: Explain the incomplete ansatz. An expression $C/(s+1)^3$ has constant numerator after multiplication by the denominator. It cannot equal the nonconstant polynomial $s^2 + 3s + 4$ for every s . The proposed inverse $Ct^2 e^{-t}/2$ lacks both e^{-t} and te^{-t} ; no choice of C can repair that omission.

Step 4: Check initial data and convergence. Differentiation gives

$$f' = e^{-t}(t - t^2), \quad f'' = e^{-t}(1 - 3t + t^2),$$

so $\boxed{f(0) = 1, f'(0) = 0, f''(0) = 1}$. Independently, expansion at large positive s gives

$$F(s) = \frac{1}{s} + \frac{0}{s^2} + \frac{1}{s^3} + O(s^{-4}).$$

These are the initial-value coefficients obtained by repeated integration by parts; this polynomial-exponential function satisfies all needed derivative bounds. Its defining integral converges absolutely for $s > -1$. At $s = -1$ the integrand is the positive polynomial $1 + t + t^2$, and for smaller s it grows still faster. The exact domain is $\boxed{s > -1}$.

Question 3

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Recover the inverse of

$$F(s) = \frac{3s + 1}{s^2 + 4s + 13}.$$

Use the pairs for $e^{-at} \cos bt$ and $e^{-at} \sin bt$, whose numerators are $s + a$ and b , respectively, over $(s + a)^2 + b^2$.

Tasks

1. Complete the square and rewrite the numerator in terms of the shifted parameter.
2. Find and forward-check the inverse. Diagnose the candidate $e^{-2t}(3 \cos 3t + \frac{1}{3} \sin 3t)$.
3. Express the inverse as $Re^{-2t} \cos(3t + \phi)$ with $R > 0$ and $0 < \phi < \pi/2$. Find its first positive zero.
4. Give its exact real convergence interval and explain the roles of the decay envelope and oscillation frequency.

Question 3 – Solution

Strategy. Shift the numerator as well as the denominator; then distinguish phase, amplitude and exponential decay.

Step 1: Match the shifted pairs. The denominator is $(s+2)^2 + 9$, while $3s+1 = 3(s+2) - 5$. Therefore

$$F(s) = 3 \frac{s+2}{(s+2)^2 + 9} - \frac{5}{3} \frac{3}{(s+2)^2 + 9}.$$

Step 2: Invert and diagnose the sign error. The matching time function is

$$f(t) = e^{-2t} \left[3 \cos 3t - \frac{5}{3} \sin 3t \right].$$

Its forward transform reproduces $3(s+2) - 5 = 3s+1$. The proposed candidate instead has numerator $3(s+2) + 1 = 3s+7$. Completing the square without adjusting the numerator loses a constant contribution.

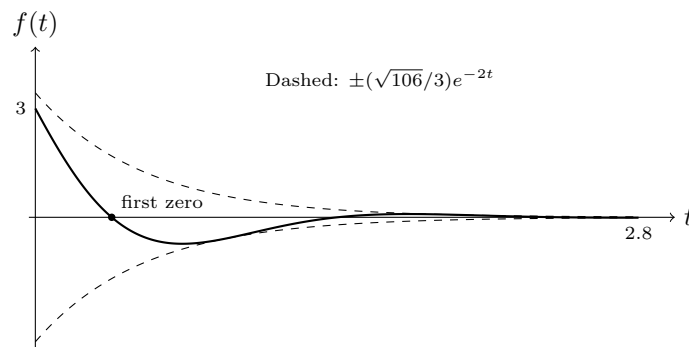
Step 3: Recover phase and the first zero. In $R \cos(3t + \phi) = R \cos \phi \cos 3t - R \sin \phi \sin 3t$, compare coefficients:

$$R = \frac{\sqrt{106}}{3}, \quad \phi = \arctan(5/9).$$

Both sine and cosine of ϕ are positive, so the specified quadrant removes any phase ambiguity. Starting at $f(0) = 3 > 0$, the first zero occurs when $3t + \phi = \pi/2$:

$$t_0 = \frac{\pi/2 - \arctan(5/9)}{3} = \frac{1}{3} \arctan(9/5).$$

Step 4: Separate envelope from oscillation. The bound $|f(t)| \leq R e^{-2t}$ gives absolute convergence for $s > -2$. At $s = -2$, the weighted signal is a nonzero sinusoid, whose primitive oscillates without a limit. For $s < -2$, positive fixed-length subintervals centered at successive cosine peaks have integrals growing exponentially, contradicting the Cauchy criterion. Thus $s > -2$ is exact. The decay rate is two and the angular frequency is three; neither gives the phase by itself. The graph shows both signed envelopes and the first zero.



Question 4

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{5}{(s+1)(s^2+4)}.$$

The required basic pairs are e^{-t} , $\cos 2t$ and $\sin 2t$.

Tasks

1. Choose a complete real partial-fraction form and solve for its coefficients.
2. Find the inverse and verify its forward transform by recombining numerators.
3. Compute the first three initial quantities $f(0), f'(0), f''(0)$. Relate their cancellations to the leading large- s behavior of F .
4. Determine the exact real convergence interval. Does the inverse approach a limit as $t \rightarrow \infty$? Justify your answer using explicit sequences.

Question 4 – Solution

Strategy. An irreducible quadratic requires a linear numerator. The transient and persistent parts then have different long-time roles.

Step 1: Resolve the real factors. Use

$$F = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Clearing denominators gives $5 = A(s^2+4) + (Bs+C)(s+1)$, so $A+B=0$, $B+C=0$, $4A+C=5$. Hence $A=1$, $B=-1$, $C=1$.

Step 2: Invert each numerator term. Since the sine pair has numerator two,

$$f(t) = e^{-t} - \cos 2t + \frac{1}{2} \sin 2t.$$

The forward numerator is $(s^2+4) - s(s+1) + (s+1) = 5$, giving the required transform. A constant numerator over the quadratic alone would have omitted the cosine contribution.

Step 3: Check the onset cancellations. Direct differentiation gives

$$f' = -e^{-t} + 2 \sin 2t + \cos 2t, \quad f'' = e^{-t} + 4 \cos 2t - 2 \sin 2t.$$

Therefore $f(0) = 0$, $f'(0) = 0$, $f''(0) = 5$. Also $F(s) = 5/s^3 + O(s^{-4})$: the $1/s$ and $1/s^2$ terms vanish. Repeated integration by parts connects precisely these coefficients to the displayed initial values. In time, the first nonzero Taylor term is $5t^2/2$.

Step 4: Identify the persistent oscillation. The exponential term decays, but $-\cos 2t + \frac{1}{2} \sin 2t$ does not. In fact,

$$f(n\pi) \longrightarrow -1, \quad f(n\pi + \pi/2) \longrightarrow 1.$$

Thus f has no limit at infinity, although it is bounded. Boundedness gives absolute convergence for $s > 0$. At $s = 0$, the primitive contains the nonconstant periodic term $-\sin 2R/2 + (1 - \cos 2R)/4$, so it has no limit. For $s < 0$, the persistent sinusoid has positive fixed-length intervals with exponentially growing weighted integrals; the decaying time term cannot cancel it on late such intervals. Consequently the exact domain is $s > 0$.

Question 5

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider the repeated quadratic factor

$$F(s) = \frac{1}{(s^2 + 4)^2}.$$

You may use $\mathcal{L}\{\sin 2t\} = 2/(s^2 + 4)$, $\mathcal{L}\{\cos 2t\} = s/(s^2 + 4)$ and $\mathcal{L}\{tg(t)\} = -G'(s)$ for $s > 0$.

Tasks

1. Derive the transform of $t \cos 2t$ by differentiating the cosine transform.
2. Find constants A, B such that the inverse is $A \sin 2t + Bt \cos 2t$. Verify the resulting rational numerator.
3. Show that the inverse is unbounded despite having no positive exponential factor. Give an explicit sequence witnessing growth.
4. Find its leading term near $t = 0$ and its exact real convergence interval. Compare the local term with the large- s scale $F(s) \sim s^{-4}$.

Question 5 – Solution

Strategy. A repeated quadratic can be handled with parameter differentiation and linear combinations, without assuming a product rule for inverses.

Step 1: Produce the repeated denominator. Differentiating the cosine pair gives

$$\mathcal{L}\{t \cos 2t\} = -\frac{d}{ds} \frac{s}{s^2 + 4} = \frac{s^2 - 4}{(s^2 + 4)^2}.$$

Polynomial factors times the damping kernel are integrable for $s > 0$, justifying the differentiation.

Step 2: Cancel the unwanted numerator. The transform of $A \sin 2t + Bt \cos 2t$ has numerator $2A(s^2 + 4) + B(s^2 - 4)$. Requiring it to equal one gives $2A + B = 0$, $8A - 4B = 1$, hence

$$f(t) = \frac{\sin 2t - 2t \cos 2t}{16}.$$

Substitution gives $(s^2 + 4)/8 - (s^2 - 4)/8 = 1$, verifying the numerator exactly. Inverting a product by multiplying inverses would not give this result.

Step 3: Exhibit growth, not just an envelope. The bound $|f(t)| \leq (1 + 2t)/16$ is only an upper bound. Actual unboundedness follows from

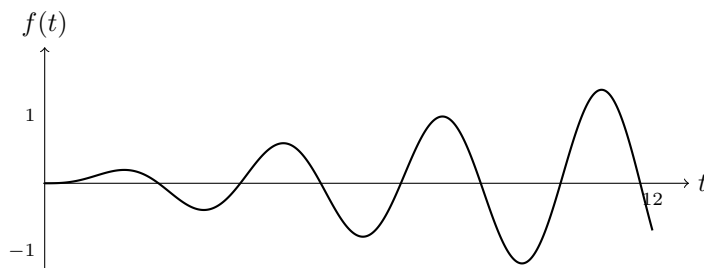
$$f(n\pi) = -n\pi/8 \longrightarrow -\infty.$$

The factor t produced by the repeated poles causes increasing oscillation size. The graph displays several oscillations at their actual scale.

Step 4: Check onset and convergence. Taylor expansion gives $\sin 2t - 2t \cos 2t = (8/3)t^3 + O(t^5)$, so $f(t) = t^3/6 + O(t^5)$. Since $\mathcal{L}\{t^3/6\} = s^{-4}$, the initial scale agrees with $F(s) \sim s^{-4}$. Absolute convergence holds for $s > 0$ by the linear envelope. At $s = 0$, direct integration gives

$$\int_0^R f(t) dt = \frac{1 - \cos 2R - R \sin 2R}{16},$$

which has no limit. For $s < 0$, intervals near successive positive peaks of $-\cos 2t$ have eventually positive integrand of order $te^{|s|t}$; their integrals fail the Cauchy criterion. Thus $s > 0$ is exact.



Question 6

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

For $a > 0$, let

$$F_a(s) = \frac{1}{(s+1)(s+a)}.$$

The two poles coincide when $a = 1$.

Tasks

1. Find the inverse for $a \neq 1$ and separately for $a = 1$. Explain why substitution into the unsimplified partial fractions fails at $a = 1$.
2. Prove the inverse is positive for every $t > 0$, and give its exact real convergence interval for all $a > 0$.
3. Derive the representation $f_a(t) = te^{-t} \int_0^1 e^{-(a-1)tv} dv$ directly from your formula, including the case $a = 1$.
4. Use the representation to prove uniform convergence $f_a \rightarrow f_1$ on the whole half-line as $a \rightarrow 1$. Obtain an explicit bound proportional to $|a - 1|$ for $1/2 \leq a \leq 3/2$.

Question 6 – Solution

Strategy. Combine the two large opposite coefficients before taking the coincident-pole limit. An integral representation controls the difference uniformly.

Step 1: Invert before and at collision. For $a \neq 1$, partial fractions give

$$F_a = \frac{1}{a-1} \left(\frac{1}{s+1} - \frac{1}{s+a} \right), \quad \boxed{f_a(t) = \frac{e^{-t} - e^{-at}}{a-1}}.$$

At $a = 1$, $F_1 = (s+1)^{-2}$, so $\boxed{f_1(t) = te^{-t}}$. Both forward transforms verify the original expression. The two individual partial-fraction coefficients diverge as $a \rightarrow 1$; their combined inverse has a finite limit.

Step 2: Check sign and the slowest tail. For $a > 1$, numerator and denominator of f_a are positive when $t > 0$; for $0 < a < 1$, both are negative. Also $te^{-t} > 0$. For unequal poles the slower exponential has a positive, nonzero coefficient, while the other term decays faster. Thus its tail is bounded below by a positive multiple of $e^{-\min(1,a)t}$. Together with direct absolute convergence, this proves the exact domain

$$\boxed{s > -\min(1, a)}.$$

At $a = 1$, the same statement follows from the positive polynomial-exponential tail.

Step 3: Remove the cancellation analytically. Integrating $e^{-(a-1)tv}$ in v gives

$$te^{-t} \int_0^1 e^{-(a-1)tv} dv = \frac{e^{-t} - e^{-at}}{a-1}$$

when $a \neq 1$ and $t > 0$. At $t = 0$ both sides extend to zero; at $a = 1$ the integral is one, yielding te^{-t} . This representation avoids subtracting nearly equal exponentials.

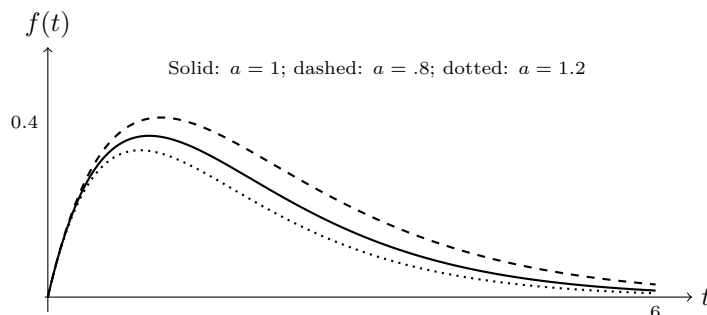
Step 4: Prove uniform half-line convergence. The mean value theorem gives $|e^{-x} - 1| \leq |x|e^{\max(0, -x)}$. With $m = \min(1, a)$, the representation yields

$$|f_a(t) - te^{-t}| \leq \frac{|a-1|}{2} t^2 e^{-mt}.$$

The maximum of $t^2 e^{-mt}$ on $t \geq 0$ occurs at $t = 2/m$ and equals $4/(m^2 e^2)$. Therefore, for $1/2 \leq a \leq 3/2$,

$$\boxed{\sup_{t \geq 0} |f_a(t) - f_1(t)| \leq \frac{8}{e^2} |a-1| \rightarrow 0}.$$

This is uniform on the entire half-line, stronger than checking each fixed time. The plotted nearby curves meet at the same zero initial value.



Question 7

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A parameter-dependent expression is

$$F_\lambda(s) = \frac{s - 2 + \lambda}{(s - 2)(s + 1)}, \quad \lambda \in \mathbb{R}.$$

When a factor cancels, interpret its algebraic value at the canceled point by continuous extension.

Tasks

1. Find the inverse for every real λ and verify its transform.
2. Determine exactly which parameter values give a bounded inverse. State the full real convergence domain in every case.
3. Explain why the printed factor $s - 2$ alone does not establish a growing mode when $\lambda = 0$. What transform value exists at $s = 2$ in that case?
4. For $\lambda = 0.03$, find the time at which the growing and decaying contributions are equal. Explain how a small coefficient can change the eventual behavior.

Question 7 – Solution

Strategy. Residues, including their possible zeros, decide which modes are actually present. An unreduced denominator can mislead.

Step 1: Recover both mode coefficients. Write $F_\lambda = A/(s-2) + B/(s+1)$. Clearing denominators gives $A + B = 1$ and $A - 2B = -2 + \lambda$, so

$$f_\lambda(t) = \frac{\lambda}{3}e^{2t} + (1 - \frac{\lambda}{3})e^{-t}.$$

Forward transformation recombines to $(A+B)s + (A-2B) = s - 2 + \lambda$, as required, for sufficiently large s .

Step 2: Classify boundedness and domains. If $\lambda = 0$, the function is e^{-t} , bounded and decaying, with exact transform domain $s > -1$. If $\lambda \neq 0$, then $e^{-2t}f_\lambda(t) \rightarrow \lambda/3 \neq 0$. Thus it is unbounded and its weighted tail at $s \leq 2$ has an eventual fixed sign and nonintegrable magnitude. For $s > 2$, all remaining terms converge absolutely. Consequently

bounded exactly when $\lambda = 0$;

$$\begin{array}{ll} s > -1, & \lambda = 0, \\ s > 2, & \lambda \neq 0. \end{array}$$

This includes $\lambda = 3$, when the decaying term disappears.

Step 3: Interpret the canceled point. At $\lambda = 0$, cancellation gives $F_0 = 1/(s+1)$. The apparent pole at two was never a pole of the reduced transform. The actual integral at $s = 2$ is

$$\int_0^\infty e^{-2t}e^{-t} dt = 1/3.$$

The continuous algebraic extension agrees with it. The inverse is determined by the whole expression, not by a list of uncanceled denominator factors.

Step 4: Measure the delayed dominance. For $\lambda = .03$, the inverse is $.01e^{2t} + .99e^{-t}$. Equality of the contributions requires $e^{3t} = 99$, so

$$t_c = \frac{1}{3} \ln 99 \approx 1.5317.$$

The ratio of growing to decaying contributions is $e^{3t}/99$, strictly increasing through one. The initially small growing coefficient eventually dominates. At fixed finite time the inverse approaches e^{-t} as $\lambda \rightarrow 0$, but for any nonzero λ its eventual boundedness is different. That distinction does not justify discarding a small nonzero residue.

Question 8

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

An inverse is to be selected from the rational family

$$F_a(s) = \frac{as + 1}{(s + 1)^2}, \quad a \in \mathbb{R}.$$

The design requires a nonnegative continuous function with total integral one. Among such functions, one may also require it to be nonincreasing.

Tasks

1. Find the inverse for all a and verify its total integral and initial value.
2. Determine the exact set of a for which the inverse is nonnegative on $[0, \infty)$. Prove necessity as well as sufficiency.
3. Within that set, classify precisely when it is nonincreasing. For the other admissible parameters, locate the unique maximum.
4. The additional measurement $\int_0^\infty t f_a(t) dt = 3/2$ is exact. Determine a and decide whether all design requirements hold.

Question 8 – Solution

Strategy. Invert first, then turn global sign and monotonicity requirements into inequalities for a linear time factor.

Step 1: Separate the numerator at its pole. Since $as + 1 = a(s + 1) + (1 - a)$,

$$f_a(t) = e^{-t}[a + (1 - a)t].$$

Forward transformation gives $a/(s + 1) + (1 - a)/(s + 1)^2 = F_a$. The exact domain is $s > -1$ for every a : the nonzero linear or constant polynomial tail has eventual fixed sign and diverges at or below -1 . Thus $s = 0$ is allowed, and

$$\int_0^\infty f_a(t) dt = F_a(0) = 1, \quad f_a(0) = a.$$

Unit signed integral alone does not establish nonnegativity.

Step 2: Classify nonnegative inverses. Because $e^{-t} > 0$, the sign is that of $a + (1 - a)t$. Nonnegativity at zero requires $a \geq 0$. If $a > 1$, the negative slope makes the factor negative for sufficiently large t . Conversely, $0 \leq a \leq 1$ makes both its intercept and slope nonnegative. Therefore the exact admissible set is $0 \leq a \leq 1$.

Step 3: Distinguish monotone and peaked designs. For admissible a ,

$$f'_a(t) = e^{-t}[1 - 2a - (1 - a)t].$$

The bracket is nonincreasing in t , so it is nonpositive everywhere exactly when its initial value is nonpositive. Hence the nonincreasing designs have $1/2 \leq a \leq 1$. For $0 \leq a < 1/2$, the derivative changes from positive to negative at

$$t_* = \frac{1 - 2a}{1 - a}, \quad f_a(t_*) = (1 - a) \exp\left(-\frac{1 - 2a}{1 - a}\right).$$

This also covers $a = 0$, whose maximum occurs at one.

Step 4: Use the extra integral to select a design. Direct exponential moments give

$$\int_0^\infty t f_a(t) dt = a + 2(1 - a) = 2 - a.$$

Equating this to $3/2$ yields $a = 1/2$ and $f(t) = (1 + t)e^{-t}/2$. This is nonnegative, has unit integral, and is nonincreasing: $f'(t) = -te^{-t}/2 \leq 0$. Its derivative vanishes only at the initial endpoint. The measurement selects one member, rather than allowing the initial value to be prescribed independently.

Question 9

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$G(s) = \frac{s^2 + 2s + 2}{(s + 1)^2}.$$

Work only with ordinary integrals and continuous functions of exponential order. You may use that such a function has a global bound $|g(t)| \leq Me^{ct}$ on $t \geq 0$ for some $M > 0$ and real c .

Tasks

1. Divide the polynomials to separate G into its polynomial part and a strictly proper rational part.
2. Prove from the stated bound that every allowed ordinary transform tends to zero as $s \rightarrow \infty$. Decide whether G has an inverse in the stated class.
3. Find the inverse h of the strictly proper remainder and verify it directly. Explain why reporting h as the inverse of the whole G would be incorrect.
4. Define $k(t) = h(t)$ except that $k(1) = 7$. Do k and h have the same ordinary transform? Explain why this does not contradict uniqueness in the stated class.

Question 9 – Solution

Strategy. Check whether an ordinary inverse can exist before matching a rational remainder to a table. Keep the continuity hypothesis in any uniqueness claim.

Step 1: Divide before attempting inversion. Because $(s+1)^2 = s^2 + 2s + 1$,

$$G(s) = 1 + \frac{1}{(s+1)^2}.$$

The nonzero constant part cannot be silently discarded. In particular $G(s) \rightarrow 1$, rather than zero, at large positive s .

Step 2: Apply a necessary existence test. For an allowed g and $s > c$, absolute convergence and the global bound yield

$$|\mathcal{L}\{g\}(s)| \leq \int_0^\infty e^{-st} |g(t)| dt \leq M \int_0^\infty e^{-(s-c)t} dt = \frac{M}{s-c} \rightarrow 0.$$

A continuous function of exponential order admits this global bound by enlarging M on an initial compact interval. Since G has limit one, G has no inverse in the stated class. This conclusion concerns ordinary continuous functions under the explicit hypotheses.

Step 3: Invert only the actual remainder. The strictly proper part has inverse $h(t) = te^{-t}$. Direct integration by parts gives

$$\int_0^\infty te^{-(s+1)t} dt = \frac{1}{(s+1)^2}, \quad s > -1.$$

The integral diverges at or below -1 . Its transform is $G - 1$, not G . The missing constant does not arise from changing $h(0)$: changing one finite point value changes no ordinary integral.

Step 4: State uniqueness with its hypothesis. Here $h(1) = e^{-1} \neq 7$. Changing that one value leaves every finite-interval integral, and hence every convergent improper transform, unchanged. Thus k and h have identical transforms on $s > -1$. However, k is discontinuous at one, so it is not in the continuous class specified in the question. There is no contradiction: uniqueness within that class compares two continuous functions. The example separates equality of transforms from unrestricted pointwise equality. No generalized transform is needed to reach any of these conclusions.

Question 10

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A transform has the form

$$F(s) = \frac{As^2 + Bs + C}{(s+1)(s+2)(s+3)}.$$

Its inverse satisfies $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 6$. Assume f is three times continuously differentiable and $f, f', f'', f^{(3)}$ are of exponential order, so repeated integration by parts is available.

Tasks

1. Derive the expansion $F(s) = f(0)/s + f'(0)/s^2 + f''(0)/s^3 + O(s^{-4})$. Use it to determine A, B, C .
2. Find the inverse by partial fractions and verify all three prescribed initial quantities.
3. Factor the inverse to prove its sign on $t \geq 0$ and determine its exact real convergence interval.
4. Find its sharp maximum value and the unique time at which it is attained. Explain why the initial data determine the numerator here but would not determine an arbitrary function.

Question 10 – Solution

Strategy. Initial data determine large-parameter coefficients. The fixed denominator converts those coefficients into a finite algebraic reconstruction.

Step 1: Recover the missing numerator. Three integrations by parts, with vanishing upper boundary terms for sufficiently large s , give

$$F(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{1}{s^3} \mathcal{L}\{f^{(3)}\}(s).$$

The last transform is $O(1/s)$ by an exponential bound. Since the denominator is $s^3 + 6s^2 + 11s + 6$, expansion of the rational expression gives

$$F(s) = \frac{A}{s} + \frac{B - 6A}{s^2} + \frac{C - 6B + 25A}{s^3} + O(s^{-4}).$$

The prescribed values imply $A = 0, B = 0, C = 6$.

Step 2: Resolve the three poles and check. Evaluation of the cleared identity at $s = -1, -2, -3$ gives coefficients 3, -6, 3. Hence

$$F = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}, \quad f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}.$$

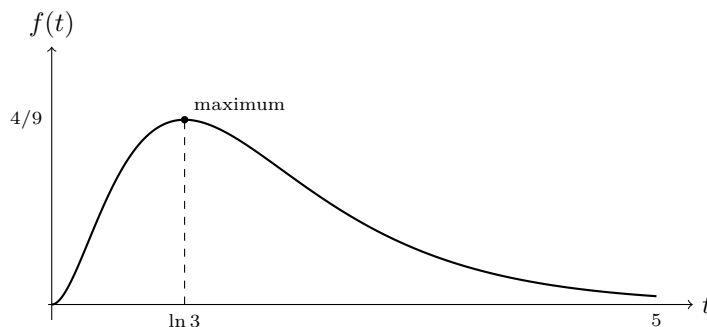
Their forward numerator is six. The initial derivative sums are $3 - 6 + 3 = 0$, $-3 + 12 - 9 = 0$, and $3 - 24 + 27 = 6$, verifying all data.

Step 3: Factor to determine the sign and tail. The useful form is $f(t) = 3e^{-t}(1 - e^{-t})^2$. It is zero at $t = 0$ and strictly positive for $t > 0$. Its tail is asymptotic to $3e^{-t}$, so the exact real convergence interval is $s > -1$: above it all terms converge; at or below it the eventually positive weighted tail has infinite integral.

Step 4: Find the sharp peak and scope of reconstruction. Put $u = e^{-t}$, so $0 < u \leq 1$ and $f = 3u(1 - u)^2$. Its derivative with respect to u is $3(1 - u)(1 - 3u)$, so the unique interior maximum occurs at $u = 1/3$:

$$t_* = \ln 3, \quad \max_{t \geq 0} f(t) = 4/9.$$

The endpoint values are zero (at $t = 0$ and as $t \rightarrow \infty$). The fixed three-parameter numerator made the reconstruction unique. Adding $t^3 e^{-t}$ to this inverse preserves all three initial data but changes its transform and leaves the specified rational family. Thus three initial quantities alone do not determine an arbitrary function.



Question 1

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A signal follows the schedule

$$f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ -1, & 1 \leq t < 3, \\ 4, & 3 \leq t < 5, \\ 0, & t \geq 5. \end{cases}$$

Tasks

1. Write f as an initial level plus a linear combination of u_1, u_3, u_5 . Explain how to choose the coefficients.
2. Find its transform first for $s > 0$, then determine its full real convergence set.
3. Compute $F(0)$ and $F'(0)$ directly from the finite time intervals. Check the removable singularity in your formula.
4. A student uses the new levels themselves as the step coefficients. Diagnose the error by finding the resulting levels after each switch.

Question 1 – Solution

Strategy. A step adds a change to the existing level; it does not replace that level. Compact support must be assessed after the changes are combined.

Step 1: Encode the changes. The changes at times one, three and five are $-3, +5, -4$. Therefore

$$f(t) = 2 - 3u_1(t) + 5u_3(t) - 4u_5(t).$$

The convention $u_a(a) = 1$ gives the prescribed value at every switch. Successively subtracting adjacent levels determines these coefficients uniquely for the specified switch times.

Step 2: Transform and check support. For $s > 0$, $\mathcal{L}\{u_a\} = e^{-as}/s$ by direct integration from a to infinity. Thus

$$F(s) = \frac{2 - 3e^{-s} + 5e^{-3s} - 4e^{-5s}}{s} \quad (s \neq 0).$$

Although the separate step integrals require $s > 0$, the actual signal vanishes after five. Integrating on its three finite intervals proves the same formula for every real $s \neq 0$; the original integral exists for every real s .

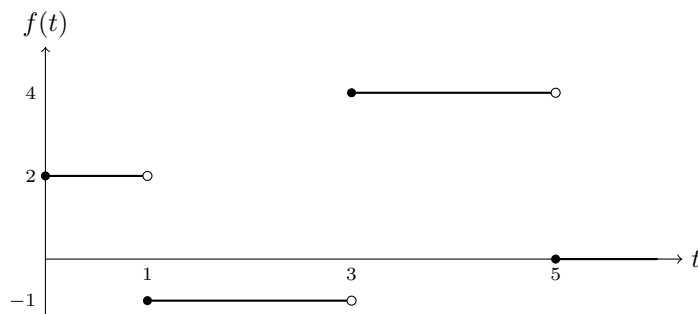
Step 3: Evaluate the removable value and first moment. Directly,

$$F(0) = 2(1) - 1(2) + 4(2) = \boxed{8},$$

$$F'(0) = -\int_0^5 t f(t) dt = -[1 - 4 + 32] = \boxed{-29}.$$

Finite support justifies differentiation under the integral. Expanding the numerator gives $8s - 29s^2 + O(s^3)$, so $F(s) = 8 - 29s + O(s^2)$, confirming both results. Zero is removable, not a divergent transform point.

Step 4: Diagnose replacement versus addition. The mistaken expression $2 - u_1 + 4u_3 + 0u_5$ produces levels 2, 1, 5, 5. It neither reaches the intended levels nor shuts off. Each step must carry the difference between the new and old levels. The graph uses open and filled endpoints to record the specified right-hand values.



Question 2

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Compare the switched-on and delayed signals

$$p(t) = t^2 u_2(t), \quad q(t) = (t - 2)^2 u_2(t).$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{t\} = 1/s^2$ and $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.

Tasks

1. Derive the shift formula for $u_a(t)g(t - a)$ by substitution in the defining integral.
2. Find the transforms of p and q . Rewrite the switched-on polynomial in its local time before using the formula.
3. A student assigns $2e^{-2s}/s^3$ to both functions. Identify the missing terms and check the values immediately after the switch.
4. Give the exact real convergence domains. Determine the leading large- s behavior of $e^{2s}P(s)$ and $e^{2s}Q(s)$ and relate it to the local onset.

Question 2 – Solution

Strategy. The factor e^{-as} shifts the time origin of the entire accompanying inverse, not just its domain of activity.

Step 1: Derive the delayed-shape rule. For parameters where the integral converges, substitute $v = t - a$:

$$\int_a^\infty e^{-st} g(t-a) dt = e^{-as} \int_0^\infty e^{-sv} g(v) dv.$$

Hence $\boxed{\mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}G(s)}$. The effective shape is a function of elapsed time since the switch.

Step 2: Rewrite the unshifted polynomial. For $t \geq 2$, write $t = v + 2$. Then p has local shape $(v+2)^2 = v^2 + 4v + 4$, whereas q has local shape v^2 . Therefore

$$\boxed{P(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right), \quad Q(s) = \frac{2e^{-2s}}{s^3}, \quad s > 0.}$$

Direct substitution in each defining integral verifies these expressions.

Step 3: Identify the omitted local terms. The difference is

$$p - q = [4(t-2) + 4]u_2(t), \quad P - Q = e^{-2s}(4/s^2 + 4/s).$$

At the switch, $p(2) = 4$ while $q(2) = 0$, equal to their respective right limits. The switched-on signal jumps from zero to four; the delayed square starts continuously at zero with zero right slope. The student's expression describes only the latter.

Step 4: Check domains and onset scales. Both signals have nonnegative quadratic tails, so they converge absolutely for $s > 0$ and diverge for $s \leq 0$. Their exact domain is $\boxed{s > 0}$. Removing the delay factor gives

$$e^{2s}P(s) \sim 4/s, \quad e^{2s}Q(s) = 2/s^3 \quad (s \rightarrow \infty).$$

The constant local onset of p produces the $1/s$ term; the quadratic local onset of q produces $2/s^3$. A common switch time does not imply a common transformed shape.

Question 3

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

An oscillator is observed only during the window

$$f(t) = \sin t [u_{\pi/2}(t) - u_{3\pi/2}(t)].$$

Use $\mathcal{L}\{\cos t\} = s/(s^2 + 1)$ for $s > 0$.

Tasks

1. Write the exact piecewise function, including its values at both window endpoints.
2. Express each switched sine using its local clock and derive $F(s)$ for $s > 0$.
3. Determine the full real convergence set, $F(0)$ and $F'(0)$. Verify them by direct finite-interval integration.
4. Explain why $F(s) > 0$ for $s > 0$ even though the signal has both signs and zero unweighted integral. Use the window symmetry to justify the sign.

Question 3 – Solution

Strategy. Re-express the phase at each switch separately. The second switch occurs at a different oscillator phase.

Step 1: Identify the window and endpoint values. The function equals $\sin t$ on $[\pi/2, 3\pi/2]$ and zero elsewhere. In particular $f(\pi/2) = 1$, while $f(3\pi/2) = 0$. At the latter time the left limit is -1 ; the graph distinguishes it from the assigned value.

Step 2: Shift the two phases correctly. With $a = \pi/2$ and $b = 3\pi/2$, we have $\sin(v + a) = \cos v$ and $\sin(v + b) = -\cos v$. Therefore

$$f = u_a(t) \cos(t - a) + u_b(t) \cos(t - b),$$

$$F(s) = \frac{s(e^{-\pi s/2} + e^{-3\pi s/2})}{s^2 + 1}.$$

The plus sign in the transform reflects the phase reversal at shutoff. After b , the two shifted cosine terms cancel exactly, as the original window requires.

Step 3: Use finite support to evaluate zero. The defining integral has finite support, so it converges for every real s ; finite-interval integration gives the displayed expression on the whole real line. Directly,

$$F(0) = \int_a^b \sin t \, dt = 0,$$

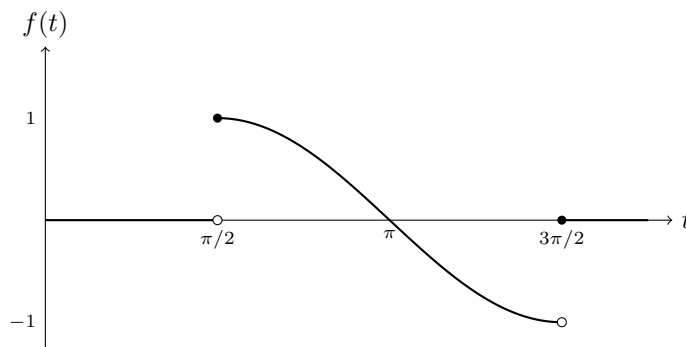
$$F'(0) = - \int_a^b t \sin t \, dt = -[-t \cos t + \sin t]_a^b = \boxed{2}.$$

The formula also gives $F(0) = 0$ and expands as $2s + O(s^2)$, verifying the derivative.

Step 4: Pair the equal and opposite lobes. Write $t = \pi + x$ with $-\pi/2 \leq x \leq \pi/2$. Pair x with $-x$. For $0 < x < \pi/2$, the earlier value is $\sin x > 0$ and the later value is $-\sin x$. Their weighted contribution is

$$e^{-\pi s} \sin x [e^{sx} - e^{-sx}] > 0 \quad (s > 0).$$

Integrating these pairs proves strict positivity, despite cancellation at $s = 0$. This also agrees with the positive numerator and denominator of the formula for $s > 0$.



Question 4

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Let $r_a(t) = (t - a)u_a(t)$. A continuous triangular signal is

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t - 1, & 1 \leq t < 3, \\ (7 - t)/2, & 3 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$$

Tasks

1. Express f as a linear combination of r_1, r_3, r_7 by tracking slope changes.
2. Find $F(s)$ using shifted ramp transforms, and establish its full real convergence set.
3. Explain algebraically why the ramp combination is zero after time seven. State the two cancellation conditions for a general sum $\sum c_j r_{a_j}(t)$ to vanish after its last switch.
4. Compute the total area and the first moment. Confirm the values from the expansion of F at zero.

Question 4 – Solution

Strategy. A ramp changes slope rather than level. Cancel both the final slope and the final intercept to create compact support.

Step 1: Encode slope changes. The slopes are 0, 1, $-1/2$, 0, giving changes 1, $-3/2$, $1/2$. Thus

$$f = r_1 - \frac{3}{2}r_3 + \frac{1}{2}r_7.$$

Each ramp is zero at its own switch, so the combination is continuous. At the peak time three its value is two; it then decreases to zero at seven.

Step 2: Transform the ramps. For $s > 0$, substitution gives $\mathcal{L}\{r_a\} = e^{-as}/s^2$. Hence

$$F(s) = \frac{e^{-s} - \frac{3}{2}e^{-3s} + \frac{1}{2}e^{-7s}}{s^2} \quad (s \neq 0).$$

The actual function has finite support, so its integral exists for all real s . Direct integration of the two linear pieces extends the formula to all $s \neq 0$; zero is removable.

Step 3: Cancel the tail, not just the slope. For $t \geq 7$, the ramp sum is

$$(1 - \frac{3}{2} + \frac{1}{2})t - [1 - \frac{3}{2}(3) + \frac{1}{2}(7)] = 0.$$

In general, after the last switch a finite ramp sum equals $t \sum c_j - \sum c_j a_j$. It vanishes identically there exactly when

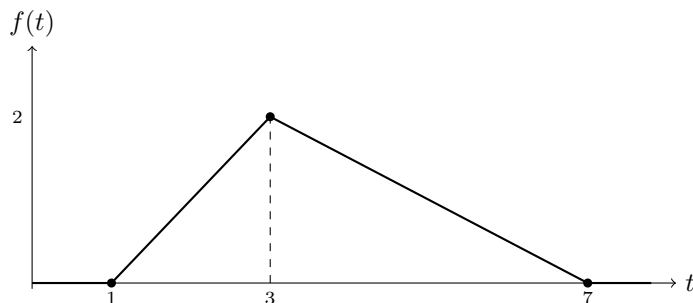
$$\sum c_j = 0, \quad \sum c_j a_j = 0.$$

Canceling only the slope could leave a nonzero constant tail and a different convergence domain.

Step 4: Check area and moment. The triangle has base six and height two, so $F(0) = 6$. Direct integration gives

$$\int_1^3 t(t-1) dt = \frac{14}{3}, \quad \frac{1}{2} \int_3^7 t(7-t) dt = \frac{52}{3}.$$

Thus the first moment is $\frac{22}{3}$ and $F'(0) = -22$. Expanding the numerator of F gives $6s^2 - 22s^3 + O(s^4)$, agreeing with both results. The horizontal time coordinate of the area centroid is $22/6 = 11/3$, consistent with the three triangle vertices.



Question 5

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Find a piecewise continuous inverse of

$$F(s) = (e^{-2s} - e^{-5s}) \frac{s+3}{(s+1)^2 + 4}.$$

Use the step convention above to specify values at switches. Basic shifted sine and cosine pairs may be used.

Tasks

1. Invert the rational factor, then apply the two delays correctly.
2. Write the result on $0 \leq t < 2$, $2 \leq t < 5$ and $t \geq 5$. Find the jumps at two and five.
3. Does the inverse vanish after five? Compare it with genuinely cutting off the first delayed signal at five.
4. Determine the exact real convergence interval and verify the transform on it.

Question 5 – Solution

Strategy. Subtracting a delayed copy restarts that copy's clock. It generally does not switch off the earlier copy.

Step 1: Invert the common shape. Since $s + 3 = (s + 1) + 2$, the rational factor is the transform of

$$g(v) = e^{-v}(\cos 2v + \sin 2v), \quad v \geq 0.$$

Thus one inverse with the stipulated endpoint convention is

$$f(t) = u_2(t)g(t-2) - u_5(t)g(t-5).$$

Step 2: Inspect the clocks and jumps. The piecewise values are zero before two, $g(t-2)$ on $[2, 5)$, and $g(t-2) - g(t-5)$ on $[5, \infty)$. As $g(0) = 1$, the jumps are $\boxed{+1 \text{ at } 2, -1 \text{ at } 5}$. In particular $f(2) = 1$ and $f(5) = g(3) - 1$, while the left limit at five is $g(3)$.

Step 3: Distinguish subtraction from shutoff. At $t = 5$, $|g(3)| \leq \sqrt{2}e^{-3} < 1$, so $g(3) - 1 \neq 0$. In fact the final piece is a nonzero exponentially decaying sinusoid, not zero. A genuinely truncated signal is instead

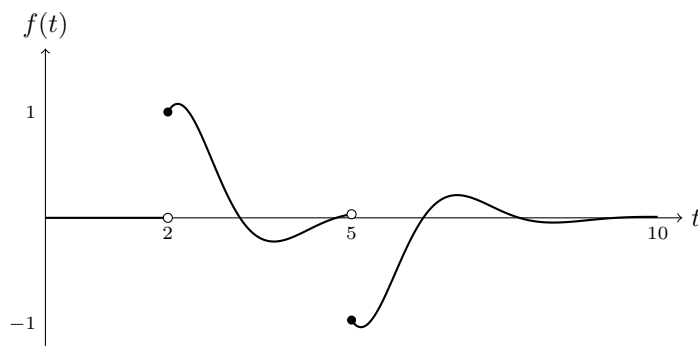
$$h(t) = [u_2(t) - u_5(t)]g(t-2).$$

Its shutoff term has the same global argument $t-2$. To transform that term at time five, its local shape must be $g(v+3)$, not $g(v)$. The graph shows the actual continuing tail of f .

Step 4: Verify the transform and domain. On $t \geq 5$, put $v = t - 5$. The tail is

$$e^{-v}\{[e^{-3}(\cos 6 + \sin 6) - 1]\cos 2v + [e^{-3}(\cos 6 - \sin 6) - 1]\sin 2v\}.$$

The cosine coefficient is nonzero by the preceding bound. Hence absolute convergence holds for $s > -1$; at $s = -1$ a nonzero undamped sinusoid has no convergent primitive, and below it fixed-sign lobe intervals violate the Cauchy criterion. The exact domain is $\boxed{s > -1}$. On that domain each delayed copy converges and transforms to $e^{-2s}G(s)$ or $e^{-5s}G(s)$ by substitution, proving the given expression. The truly truncated h would instead converge for every real s .



Question 6

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A function has a deliberately exceptional point value:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 7, & t = 1, \\ e^{-(t-1)}, & t > 1. \end{cases}$$

Compare it with $g(t) = u_1(t)e^{-(t-1)}$.

Tasks

1. Compare f and g pointwise, including their left and right limits at one.
2. Find their ordinary transforms and exact convergence intervals.
3. Give a pointwise formula for f using g and the indicator of the singleton $\{1\}$. Explain why a finite sum of right-continuous step terms times continuous functions cannot reproduce this exceptional point value.
4. If a step convention instead assigns value $1/2$ at the switch, what changes in the formula for g and what stays the same in its transform? State precisely the uniqueness conclusion justified here.

Question 6 – Solution

Strategy. Separate equality at every point from equality except at isolated points. An ordinary integral cannot recover the latter point values.

Step 1: Locate the discrepancy. For $t \neq 1$, $f(t) = g(t)$. At one, $g(1) = 1$ but $f(1) = 7$. Both functions have left limit zero and right limit one. Thus g is right-continuous, whereas the chosen value of f agrees with neither limit.

Step 2: Transform the interval contribution. A single changed value contributes nothing to an integral. Substituting $v = t - 1$ in the tail gives, for both functions,

$$F(s) = G(s) = \frac{e^{-s}}{s+1}, \quad s > -1.$$

At $s = -1$, the weighted tail is the positive constant e ; below -1 it grows exponentially. Thus the stated convergence interval is exact. The isolated value seven cannot change this conclusion.

Step 3: Give a pointwise correction and prove the limitation. If $\mathbf{1}_{\{1\}}$ is one only at $t = 1$ and zero elsewhere, then

$$f(t) = g(t) + 6\mathbf{1}_{\{1\}}(t).$$

Each u_a under our convention is right-continuous. Multiplying it by a continuous function preserves right-continuity, and finite sums preserve that property. Such a sum must equal its right limit at one, so it cannot have the specified value seven while its right limit is one. This is a representational obstruction, not a transform-calculation error.

Step 4: Change the convention without changing the integral. With switch value $1/2$, the corresponding expression for g equals $1/2$ at one and is unchanged elsewhere. Its ordinary transform is still $e^{-s}/(s+1)$. A pointwise correction to recover f would now be $(13/2)\mathbf{1}_{\{1\}}$. Thus transform data alone do not specify the value at an isolated discontinuity. Here the right-hand continuous representative is fixed by the chosen convention; an unrestricted pointwise uniqueness claim would be false. The usual uniqueness statement at continuity points is consistent with all these examples.

Question 7

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Let $f(t) = u_2(t)[1 + (t - 2)]$. Define its ordinary derivative away from the switch as $h(t) = 0$ for $t < 2$ and $h(t) = 1$ for $t > 2$; its value at two is irrelevant to integration. A student asserts $H(s) = sF(s) - f(0)$.

Tasks

1. Calculate $F(s)$ and $H(s)$ directly for $s > 0$, and test the assertion.
2. Apply integration by parts separately on $[0, 2)$ and $(2, R]$ to identify the missing contribution.
3. For a piecewise C^1 function with finitely many jumps $J_j = f(c_j+) - f(c_j-)$ at positive times, derive the corresponding formula for the transform of its ordinary derivative on the smooth pieces. State the needed boundary assumptions.
4. Apply your formula to the continuous delayed ramp $k(t) = (t - 2)u_2(t)$. Explain why the usual derivative rule works for k despite its corner.

Question 7 – Solution

Strategy. Integration by parts across smooth pieces leaves internal boundary terms. A jump is different from a continuous corner.

Step 1: Compute both ordinary integrals. The local shape of f is $1 + v$, and the local shape of h is one. Thus

$$F(s) = e^{-2s}(1/s + 1/s^2), \quad H(s) = e^{-2s}/s, \quad s > 0.$$

Since $f(0) = 0$, the student's expression is $sF = e^{-2s}(1 + 1/s)$, larger than H by $\boxed{e^{-2s}}$. Both actual integrals have exact domain $s > 0$ because their tails are a positive linear function and a positive constant, respectively.

Step 2: Retain the internal boundary. Only the second piece contributes. For $R > 2$, integration by parts gives

$$\int_2^R e^{-st} f'(t) dt = e^{-sR} f(R) - e^{-2s} f(2+) + s \int_2^R e^{-st} f(t) dt.$$

Here $f(2+) = 1$. The upper term vanishes for $s > 0$, leaving $H = sF - e^{-2s}$. The omitted term records the jump from zero to one, not a value assigned to h at the single point two.

Step 3: Sum the piecewise boundary terms. At an internal time c_j , the left piece contributes $e^{-sc_j} f(c_j-)$ and the right piece contributes $-e^{-sc_j} f(c_j+)$. Their sum is $-J_j e^{-sc_j}$. Consequently

$$\boxed{\mathcal{L}\{f'_{\text{pw}}\}(s) = sF(s) - f(0+) - \sum_j J_j e^{-sc_j}.$$

Use finite one-sided limits, convergence of the piecewise function and derivative integrals, and $e^{-sR} f(R) \rightarrow 0$. Exponential-order bounds on the pieces and their derivatives suffice for large s . If f is right-continuous at zero, $f(0+) = f(0)$. The formula concerns ordinary piecewise derivatives only.

Step 4: Check a continuous corner. For k , both one-sided values at two are zero, so its jump is zero. We have $K = e^{-2s}/s^2$ and its ordinary piecewise derivative is u_2 except possibly at two. Thus

$$\mathcal{L}\{k'_{\text{pw}}\} = e^{-2s}/s = sK - k(0).$$

A corner changes slopes but introduces no internal value jump. Splitting the integral proves the usual rule here without pretending the derivative exists at the corner.

Question 8

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A three-stage pulse is one on $[0, 1)$, has unknown height B on $[1, 2)$ and unknown height C on $[2, 3)$, and is zero for $t \geq 3$. It must satisfy

$$\int_0^\infty f(t) dt = 0, \quad \int_0^\infty t f(t) dt = 0.$$

Tasks

1. Determine B, C and prove that these heights are uniquely determined.
2. Write the resulting pulse using steps and derive its transform in factored form.
3. Find the full real convergence set, the first nonzero term of $F(s)$ near zero, and the second time moment.
4. Explain why the two vanishing moments do not make the pulse or its transform identically zero. Determine the sign of $F(s)$ for $s > 0$.

Question 8 – Solution

Strategy. Moment constraints form a small linear system. The step differences then expose cancellation in the transform numerator.

Step 1: Solve the two moment equations. The unit-width intervals give

$$1 + B + C = 0, \quad \frac{1}{2} + \frac{3}{2}B + \frac{5}{2}C = 0.$$

Subtracting the first equation from twice the second yields $2B + 4C = 0$. Combining this with $B + C = -1$ gives $\boxed{B = -2, C = 1}$. The coefficient determinant is $5/2 - 3/2 = 1 \neq 0$, so the solution is unique.

Step 2: Encode changes and factor the transform. The levels 1, -2, 1, 0 have jumps $-3, +3, -1$. Hence

$$f(t) = 1 - 3u_1(t) + 3u_2(t) - u_3(t),$$

$$\boxed{F(s) = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s} = \frac{(1 - e^{-s})^3}{s} \quad (s \neq 0).}$$

This follows first by transforming steps for $s > 0$, or directly on the three finite intervals for any $s \neq 0$.

Step 3: Interpret the removable zero. Compact support gives convergence for every real s . Since $1 - e^{-s} = s + O(s^2)$,

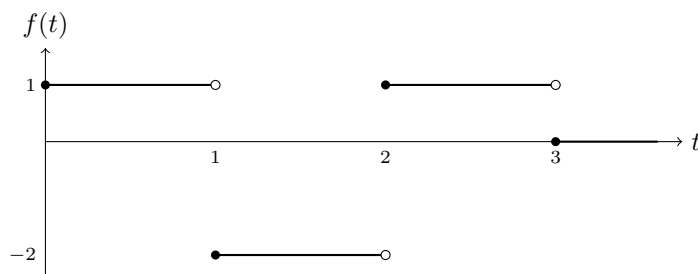
$$\boxed{F(s) = s^2 + O(s^3), \quad F(0) = F'(0) = 0, \quad F''(0) = 2.}$$

Directly, the second moment is

$$\int_0^3 t^2 f(t) dt = \frac{1}{3} - 2\left(\frac{7}{3}\right) + \frac{19}{3} = \boxed{2},$$

consistent with differentiating the finite-support transform twice.

Step 4: Separate two constraints from identity. The pulse is one throughout its first interval, so it is not zero. For $s > 0$, both $1 - e^{-s}$ and s are positive, giving $\boxed{F(s) > 0}$. The zero area and first moment eliminate only the constant and linear terms near $s = 0$. They do not eliminate all weighted integrals. The second moment already witnesses the remaining information. The graph makes the signed cancellation visible without replacing it by a pointwise sign claim.



Question 9

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

For an integer $n \geq 2$, define a right-continuous staircase

$$f_n(t) = \frac{k}{n} \quad \left(\frac{k}{n} \leq t < \frac{k+1}{n}, \quad k = 0, \dots, n-1\right), \quad f_n(t) = 0 \quad (t \geq 1).$$

The target is $r(t) = t$ on $0 \leq t < 1$ and $r(t) = 0$ for $t \geq 1$.

Tasks

1. Express f_n using steps and derive a finite-sum formula for its transform.
2. Find $F_n(0)$ and the target transform $R(s)$, including $R(0)$. Give the full real convergence sets.
3. For $s > 0$, derive a bound on $0 \leq R(s) - F_n(s)$ from the pointwise staircase error.
4. Using that bound, find the least integer n that certifies $R(s) - F_n(s) \leq 0.01$ simultaneously for every $s \geq 1$. Distinguish this certificate from an exact minimal-error calculation.

Question 9 – Solution

Strategy. Encode each upward increment and the final reset. A pointwise error bound becomes an integral certificate valid for a whole range of parameters.

Step 1: Include the final shutoff. There are increments $1/n$ at k/n , $1 \leq k \leq n-1$, followed by a drop $(n-1)/n$ at one. Hence

$$f_n(t) = \frac{1}{n} \sum_{k=1}^{n-1} u_{k/n}(t) - \frac{n-1}{n} u_1(t),$$

$$F_n(s) = \frac{\sum_{k=1}^{n-1} e^{-ks/n} - (n-1)e^{-s}}{ns} \quad (s \neq 0).$$

Transforming steps gives the formula for $s > 0$; finite-interval integration gives it for all $s \neq 0$.

Step 2: Evaluate area and the target. Each stair has width $1/n$, so

$$F_n(0) = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{n-1}{2n}.$$

Integration by parts on $[0, 1]$ gives

$$R(s) = \frac{1 - (1+s)e^{-s}}{s^2} \quad (s \neq 0), \quad R(0) = \frac{1}{2}.$$

Both original functions have finite support, so both transform domains are the whole real line. Their apparent singularities at zero are removable.

Step 3: Convert pointwise error to weighted error. On each half-open stair, $0 \leq r - f_n < 1/n$, and both functions vanish from one onward. For $s > 0$,

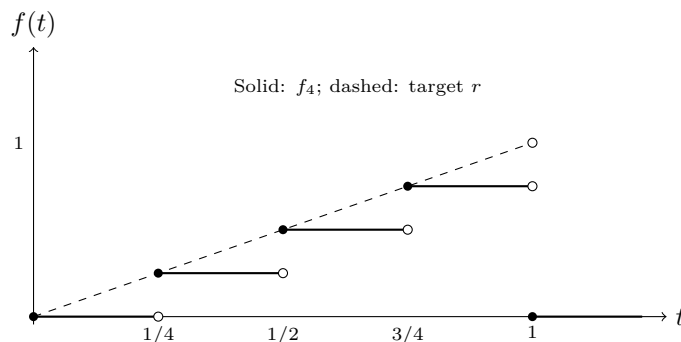
$$0 \leq R(s) - F_n(s) \leq \frac{1}{n} \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{ns}.$$

At zero the actual error is $1/(2n)$, while the continuous limiting bound is $1/n$. The bound is sufficient, not an equality.

Step 4: Certify a simultaneous tolerance. The function $b(s) = \int_0^1 e^{-st} dt$ is decreasing for $s > 0$, since its derivative is $-\int_0^1 te^{-st} dt < 0$. Its largest value on $s \geq 1$ is $1 - e^{-1}$. Thus the certificate requires

$$n \geq 100(1 - e^{-1}) \approx 63.2121, \quad \boxed{n = 64}.$$

For $n = 63$ this bound exceeds .01 at $s = 1$; that does not prove the actual error exceeds .01. The graph shows $n = 4$ only to display the staircase geometry; it is not the certified design.



Question 10

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

An unknown rectangular pulse has height $A > 0$ on $[a, b)$ and is zero elsewhere, where $0 \leq a < b$. Let F be its transform and set $H(s) = sF(s)$. Exact data are

$$H(1) = \frac{3}{8}, \quad H(2) = \frac{15}{64}, \quad H(3) = \frac{63}{512}.$$

Tasks

1. Write the pulse using steps and derive $H(s)$ in terms of A, a, b .
2. Set $x = e^{-a}$ and $y = e^{-b}$. Use the ratios $H(2)/H(1)$ and $H(3)/H(1)$ to determine $x + y$ and xy .
3. Recover A, a, b , prove uniqueness within the stated pulse family, and check all three data.
4. Find the pulse area and full real transform domain. Explain, with a construction or a dimension argument, why these three data do not determine an arbitrary piecewise continuous signal.

Question 10 – Solution

Strategy. Consecutive exponential measurements determine symmetric functions of the two endpoints. The ordering of the endpoints then removes the root ambiguity.

Step 1: Encode the window. The pulse is $f = A(u_a - u_b)$. Integrating on its finite support gives

$$F(s) = A \frac{e^{-as} - e^{-bs}}{s} \quad (s \neq 0), \quad H(s) = A(e^{-as} - e^{-bs}).$$

With $x = e^{-a}$ and $y = e^{-b}$, the restrictions become $0 < y < x \leq 1$.

Step 2: Recover the two symmetric quantities. Since $H(1) = A(x - y) > 0$,

$$\frac{H(2)}{H(1)} = x + y = \frac{5}{8}, \quad \frac{H(3)}{H(1)} = x^2 + xy + y^2 = \frac{21}{64}.$$

Therefore $xy = (5/8)^2 - 21/64 = \boxed{1/16}$. The numbers x, y are the roots of $z^2 - (5/8)z + 1/16 = 0$.

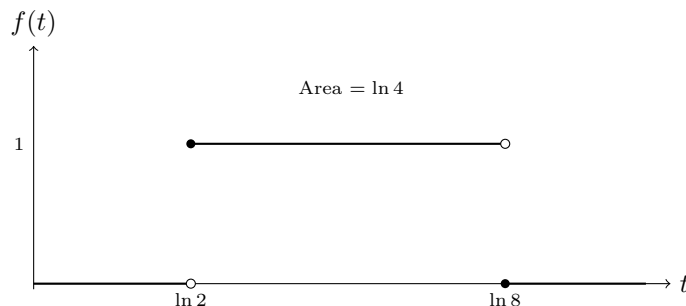
Step 3: Order the roots and find the height. The roots are $1/2$ and $1/8$. The condition $x > y$ gives

$$a = \ln 2, \quad b = \ln 8, \quad A = \frac{3/8}{1/2 - 1/8} = 1.$$

Both endpoint restrictions hold. Direct checks yield $x - y = 3/8$, $x^2 - y^2 = 15/64$ and $x^3 - y^3 = 63/512$. The two roots, their ordering and the nonzero first measurement fix all three parameters, proving uniqueness within this family.

Step 4: State what the data do and do not determine. The area is $\boxed{F(0) = A(b - a) = \ln 4}$.

Finite support gives convergence for every real s , with zero removable in the displayed formula. To see the limitation of three measurements, take four disjoint unit pulses on $[4 + j, 5 + j)$, $j = 0, 1, 2, 3$. Requiring their linear combination to have zero transform at $s = 1, 2, 3$ gives three homogeneous linear equations in four coefficients. A nonzero coefficient vector exists; disjoint supports make its signal nonzero. Adding it preserves all three data but changes the original signal. This signed perturbation need not belong to the rectangular family, so it does not contradict the uniqueness just proved.



Question 1

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider the IVP

$$y' + 2y = 3e^{-t}, \quad y(0) = 1.$$

A student writes $Y(s) = 3/[(s+1)(s+2)]$.

Tasks

1. Derive the correct transformed equation, and identify which IVP the student actually solved.
2. Find the correct inverse using partial fractions.
3. Check the original differential equation and initial value directly. State the exact real convergence interval of Y .
4. Find the unique maximum of y and its time. Explain why a decaying forcing does not force the response to decrease from the start.

Question 1 – Solution

Strategy. The derivative transform carries the initial value into the algebra. Keep that contribution before resolving the two poles.

Step 1: Include the initial term. Using $\mathcal{L}\{y'\} = sY - y(0)$ gives

$$(s+2)Y - 1 = \frac{3}{s+1}, \quad Y = \frac{1}{s+2} + \frac{3}{(s+1)(s+2)}.$$

The student omitted the first term. Their expression solves the same equation with initial value zero, not one.

Step 2: Resolve and invert. Combining the terms yields

$$Y = \frac{s+4}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{2}{s+2}, \quad \boxed{y(t) = 3e^{-t} - 2e^{-2t}}.$$

Each reciprocal factor matches an elementary exponential transform.

Step 3: Verify the IVP and domain. We have $y(0) = 3 - 2 = 1$ and $y' = -3e^{-t} + 4e^{-2t}$. Consequently

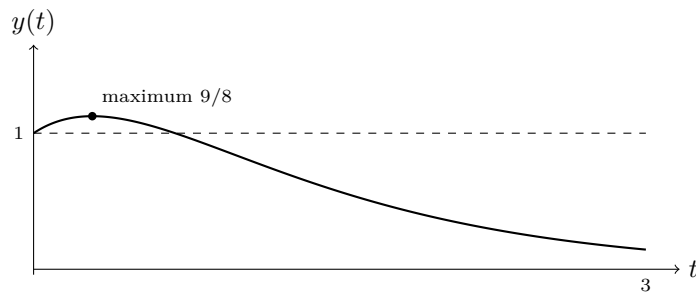
$$y' + 2y = (-3 + 6)e^{-t} + (4 - 4)e^{-2t} = 3e^{-t}.$$

The coefficients and forcing are continuous, so the usual uniqueness theorem for this linear IVP makes the verified solution the unique one. Its tail is asymptotic to $3e^{-t}$ and is positive; hence its exact real transform domain is $\boxed{s > -1}$. The student instead obtains $3e^{-t} - 3e^{-2t}$, whose initial value is zero.

Step 4: Locate the overshoot. Since $y' = e^{-t}(-3 + 4e^{-t})$, the derivative is positive before and negative after $e^{-t} = 3/4$. Thus

$$\boxed{t_* = \ln(4/3), \quad y(t_*) = 9/8.}$$

Here $y'(0) = 1$ although the forcing decreases. The input initially exceeds the loss term $2y$, so the state first rises. It later decays to zero. The graph marks the overshoot above the initial level one.



Question 2

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Solve

$$y'' + 3y' + 2y = 4, \quad y(0) = 1, \quad y'(0) = -1.$$

Call the solution with the same initial data and zero forcing the initial-data response. Call the solution with forcing four and zero initial data the forced response.

Tasks

1. Derive Y and split it into the two responses defined above.
2. Invert both parts and obtain the complete solution.
3. Verify both initial data and the original equation. Check that the two parts satisfy their respective data and forcing.
4. Find the limiting value and the exact real transform domain. Explain why solving only the forced response would not solve the stated IVP.

Question 2 – Solution

Strategy. Linearity separates the numerator contribution of initial data from that of the input, but the final answer needs both.

Step 1: Transform all derivative terms. The derivative transforms are $s^2Y - s + 1$ and $sY - 1$. Thus

$$(s^2 + 3s + 2)Y - s - 2 = 4/s,$$

$$Y = \underbrace{\frac{s+2}{(s+1)(s+2)}}_{Y_{\text{initial}}} + \underbrace{\frac{4}{s(s+1)(s+2)}}_{Y_{\text{forced}}}.$$

The first part simplifies to $1/(s+1)$; the second resolves as $2/s - 4/(s+1) + 2/(s+2)$.

Step 2: Invert and combine. Hence

$$y_{\text{initial}} = e^{-t}, \quad y_{\text{forced}} = 2 - 4e^{-t} + 2e^{-2t},$$

$$\boxed{y(t) = 2 - 3e^{-t} + 2e^{-2t}}.$$

The initial-data response satisfies the homogeneous equation, and the constant term in the forced response supplies the constant forcing.

Step 3: Verify each contribution. For the sum, $y' = 3e^{-t} - 4e^{-2t}$ and $y'' = -3e^{-t} + 8e^{-2t}$. Then

$$y'' + 3y' + 2y = 4 + (-3 + 9 - 6)e^{-t} + (8 - 12 + 4)e^{-2t} = 4.$$

Also $y(0) = 1$ and $y'(0) = -1$. The initial-data part has values 1, -1 and zero equation residual. The forced part has values 0, 0 and residual four. These checks establish the intended decomposition; uniqueness of the linear IVP establishes the final solution.

Step 4: Interpret the steady value and domain. The exponentials vanish, so $\boxed{y(t) \rightarrow 2}$. This agrees with the constant balance $2y = 4$. Its ordinary transform converges exactly for $s > 0$: boundedness gives convergence above zero, while the eventually positive nonzero constant tail causes divergence at or below zero. Although the initial-data response has the wider domain $s > -1$, the complete response does not. Reporting only the forced response gets the limiting value right but misses both prescribed initial data.

Question 3

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Consider a repeated-root operator with matching exponential forcing:

$$y'' + 2y' + y = e^{-t}, \quad y(0) = 2, \quad y'(0) = -2.$$

Tasks

1. Find Y and identify the order of the pole produced by the forcing.
2. Invert Y , keeping the factorial normalization of the repeated pole.
3. Verify the equation and both initial data directly. You may simplify the residual by setting $y = e^{-t}v$.
4. Determine whether y decreases to zero and whether $e^t y$ stays bounded. Explain why these conclusions are compatible and state the transform domain.

Question 3 – Solution

Strategy. A repeated operator root and a matching input pole combine. Pole order controls the polynomial factor, not necessarily growth of the full time response.

Step 1: Form the transformed equation. The initial terms give

$$(s^2Y - 2s + 2) + 2(sY - 2) + Y = \frac{1}{s + 1},$$

so

$$(s + 1)^2Y = 2(s + 1) + \frac{1}{s + 1}, \quad Y = \frac{2}{s + 1} + \frac{1}{(s + 1)^3}.$$

The forcing contributes a third-order pole at $s = -1$, although the differential equation has order two.

Step 2: Invert the repeated pole correctly. Since $\mathcal{L}\{t^2e^{-t}\} = 2/(s + 1)^3$,

$$y(t) = e^{-t}(2 + t^2/2).$$

The factor $1/2$ is essential. Transforming the two terms back gives exactly the displayed Y .

Step 3: Check the residual and data. With $v = 2 + t^2/2$, the product rule gives $y' = e^{-t}(v' - v)$ and $y'' = e^{-t}(v'' - 2v' + v)$. Therefore

$$y'' + 2y' + y = e^{-t}v'' = e^{-t}.$$

At zero, $v(0) = 2$, $v'(0) = 0$, so $y(0) = 2$ and $y'(0) = -2$. These checks and linear-IVP uniqueness verify the solution.

Step 4: Distinguish a scaled response from the actual response. We have

$$y' = e^{-t}(-2 + t - t^2/2) = -\frac{1}{2}e^{-t}[(t - 1)^2 + 3] < 0.$$

Thus y decreases strictly and tends to zero, since exponential decay dominates its quadratic factor. Meanwhile $e^ty = 2 + t^2/2 \rightarrow \infty$. Dividing out the decaying envelope changes the quantity being studied; there is no contradiction. The positive polynomial-exponential tail gives the exact domain $s > -1$, with divergence at and below the boundary.

Question 4

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An undamped oscillator starts at rest:

$$y'' + 4y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 0.$$

Define its energy by $E(t) = [y'(t)^2 + 4y(t)^2]/2$.

Tasks

1. Derive Y and invert it. You may use $\mathcal{L}\{t \cos 2t\} = (s^2 - 4)/(s^2 + 4)^2$.
2. Check the equation and both initial data directly.
3. Prove that the response is unbounded by evaluating a suitable explicit time sequence. State its exact real transform domain.
4. Derive the energy identity and calculate $E(n\pi)$ for positive integers n . Explain why bounded forcing can supply unbounded accumulated energy.

Question 4 – Solution

Strategy. Matching forcing and natural frequencies produce repeated transform poles. Explicit values and the work identity distinguish real growth from a mere upper bound.

Step 1: Transform and invert. Zero initial data give

$$(s^2 + 4)Y = \frac{2}{s^2 + 4}, \quad Y = \frac{2}{(s^2 + 4)^2}.$$

Combining the sine pair with the supplied time-cosine pair yields

$$y(t) = \frac{\sin 2t - 2t \cos 2t}{8}.$$

Indeed the forward numerator is $(s^2 + 4)/4 - (s^2 - 4)/4 = 2$.

Step 2: Verify the IVP. Direct differentiation gives

$$y' = \frac{t}{2} \sin 2t, \quad y'' = \frac{1}{2} \sin 2t + t \cos 2t.$$

Adding $4y = \frac{1}{2} \sin 2t - t \cos 2t$ gives $\sin 2t$. Both initial values are zero, proving the claimed solution by uniqueness.

Step 3: Exhibit unbounded values and the domain. At $t = n\pi$,

$$y(n\pi) = -n\pi/4 \longrightarrow -\infty.$$

The actual response is therefore unbounded; a growing envelope alone would not prove that. Its linear envelope gives absolute transform convergence for $s > 0$. At zero, its finite integral is $[1 - \cos 2R - R \sin 2R]/8$, which has no limit. Below zero, intervals around successive peaks of the $t \cos 2t$ term violate the Cauchy criterion. Hence the exact domain is $s > 0$.

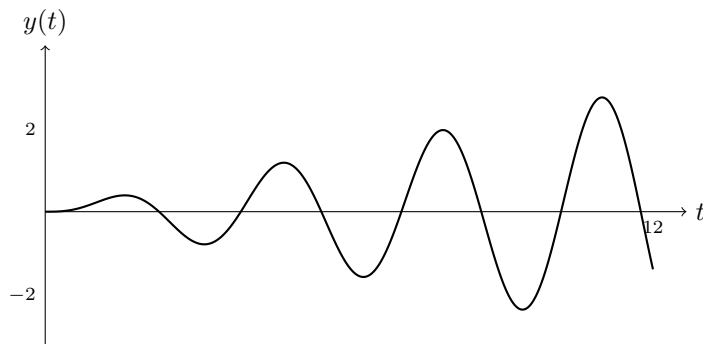
Step 4: Account for the energy input. Multiplying the equation by y' gives

$$E' = y'(y'' + 4y) = y' \sin 2t = \frac{t}{2} \sin^2 2t \geq 0.$$

At $n\pi$, the velocity is zero, so

$$E(n\pi) = 2y(n\pi)^2 = n^2 \pi^2 / 8.$$

Although the forcing amplitude is at most one, its work accumulates over an increasing time interval. Resonance makes that accumulated work unbounded. The graph shows the growing displacement oscillations, not the energy itself.



Question 5

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An unstable equation has adjustable initial velocity:

$$y'' - y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = v.$$

Tasks

1. Derive and invert Y for arbitrary real v .
2. Find the unique v that gives a bounded solution, and verify the resulting IVP.
3. If the chosen velocity is perturbed by δ , derive the exact error in the solution. Determine the largest allowed $|\delta|$ that guarantees an error at most 0.01 for every $0 \leq t \leq 5$.
4. State the exact transform domains for the bounded choice and for every other v . Explain why the uncanceled operator factor alone does not determine the response domain.

Question 5 – Solution

Strategy. Initial data control the coefficient of the unstable mode. Exact cancellation can give a bounded trajectory but does not remove sensitivity to an initial error.

Step 1: Transform and resolve the modes. The transformed equation is

$$(s^2 - 1)Y = s + v + \frac{1}{s + 2}.$$

Partial fractions give

$$Y = \frac{v/2 + 2/3}{s - 1} + \frac{-v/2}{s + 1} + \frac{1/3}{s + 2},$$

$$y(t) = (v/2 + 2/3)e^t - (v/2)e^{-t} + \frac{1}{3}e^{-2t}.$$

Recombining the fractions reproduces the transformed equation.

Step 2: Cancel the growing coefficient. Boundedness requires $v/2 + 2/3 = 0$, so the unique choice is

$$v = -4/3, \quad y_b(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-2t}.$$

Then $y_b(0) = 1$, $y'_b(0) = -2/3 - 2/3 = -4/3$, and $y''_b - y_b = e^{-2t}$. If the growing coefficient is nonzero, it eventually dominates both decaying terms, so no other velocity works.

Step 3: Quantify the initial-error tolerance. Replacing v by $-4/3 + \delta$ changes the solution by

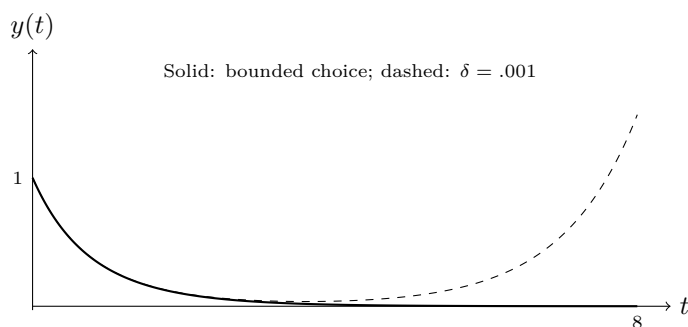
$$y - y_b = \frac{\delta}{2}(e^t - e^{-t}) = \delta \sinh t.$$

Since $\sinh t$ increases on $t \geq 0$, the maximum absolute error on $[0, 5]$ is $|\delta| \sinh 5$. The exact, necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01}{\sinh 5} \approx 1.3477 \times 10^{-4}.$$

Equality is permitted and attains the error limit at time five. The graph compares the bounded choice with a small positive velocity error.

Step 4: Check the actual tails. For the bounded choice, the nonzero slow tail is $(2/3)e^{-t}$, so the exact domain is $s > -1$. For every other v , a nonzero e^t term gives exact domain $s > 1$. The operator factor $s - 1$ cancels from the selected response numerator. Its presence in the operator does not mean it survives in every solution; the initial data determine the residue.



Question 6

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Two parameters are unknown in the IVP

$$y'' + 2y' + 5y = Ae^{-t}, \quad y(0) = 0, \quad y'(0) = v.$$

Exact measurements are $e^{\pi/2}y(\pi/2) = 2$ and $e^{\pi/4}y(\pi/4) = 2$.

Tasks

1. Solve by Laplace transforms in terms of A, v .
2. Use the two measurements to recover A, v and prove uniqueness of this parameter recovery.
3. Verify the calibrated equation and both initial data. State the transform domain of the calibrated solution.
4. Would measurements taken only at $t = n\pi$, for positive integers n , identify these parameters? Justify your answer from the exact response, not from a count of samples.

Question 6 – Solution

Strategy. Solve once with symbolic parameters, then evaluate the resulting basis functions at the measurement times.

Step 1: Transform with unknown initial velocity. The transformed equation is

$$[(s+1)^2 + 4]Y = v + \frac{A}{s+1}.$$

Writing $p = s + 1$ and using $1/[p(p^2 + 4)] = (1/4)[1/p - p/(p^2 + 4)]$ gives

$$y(t) = e^{-t} \left[\frac{v}{2} \sin 2t + \frac{A}{4} (1 - \cos 2t) \right].$$

Transforming the sine and constant-minus-cosine terms verifies the algebraic equation for sufficiently large s .

Step 2: Recover the parameters. At $t = \pi/2$, the local sine is zero and the cosine is -1 , so $A/2 = 2$, giving $A = 4$. At $t = \pi/4$, the sine is one and the cosine is zero, so $v/2 + A/4 = 2$, giving $v = 2$. Thus

$$A = 4, \quad v = 2, \quad y = e^{-t}(1 - \cos 2t + \sin 2t).$$

The first measurement fixes A independently, then the second fixes v with a nonzero coefficient. This proves uniqueness for the specified two-parameter model.

Step 3: Verify calibration and the IVP. Set $w = 1 - \cos 2t + \sin 2t$, so $y = e^{-t}w$. The product rule gives

$$y'' + 2y' + 5y = e^{-t}(w'' + 4w) = 4e^{-t}.$$

Also $w(0) = 0$ and $w'(0) = 2$, so $y(0) = 0$ and $y'(0) = 2$. The exact transform domain is $s > -1$. At its boundary the weighted signal is the nonzero periodic function w , with nonzero mean; below it fixed-sign lobe intervals fail the Cauchy criterion. Above it the exponential envelope gives absolute convergence.

Step 4: Identify blind sampling times. At every $t = n\pi$, both $\sin 2t = 0$ and $1 - \cos 2t = 0$. Therefore

$$y(n\pi) = 0 \quad \text{for every } A, v.$$

Even infinitely many such samples carry no information about either parameter. The issue is where the response basis vanishes, not merely how many measurements are collected. The two given times avoid this degeneracy.

Question 7

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

The data are prescribed at time one, not at zero:

$$y'' + y = t, \quad y(1) = 2, \quad y'(1) = 0, \quad t \geq 1.$$

Use a new time origin before applying a one-sided transform.

Tasks

1. Define $\tau = t - 1$ and $z(\tau) = y(1 + \tau)$. Derive the transformed IVP for z .
2. Find $Z(s) = \mathcal{L}_\tau\{z\}(s)$ and invert it.
3. Convert back to t and verify the equation and the data at time one.
4. A student treats the supplied values as $y(0), y'(0)$ and transforms the forcing as $1/s^2$. Explain both changes needed to repair that calculation. State the exact domain of Z .

Question 7 – Solution

Strategy. Shift the independent variable and the forcing together. A Laplace transform beginning at the new origin must use data at that origin.

Step 1: Move the initial time. With $\tau = t - 1$, the derivatives have the same form but the forcing becomes $\tau + 1$. Thus

$$z'' + z = \tau + 1, \quad z(0) = 2, \quad z'(0) = 0.$$

Writing $Z = \mathcal{L}_\tau\{z\}$ gives

$$(s^2 + 1)Z - 2s = \frac{1}{s^2} + \frac{1}{s}.$$

Step 2: Resolve and invert. A useful decomposition is

$$Z = \frac{2s + 1/s^2 + 1/s}{s^2 + 1} = \frac{1}{s^2} + \frac{1}{s} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.$$

Therefore $\boxed{z(\tau) = \tau + 1 + \cos \tau - \sin \tau}$. Transforming each term back checks the decomposition, including the constant part of the shifted forcing.

Step 3: Return to the original clock. The solution on the requested domain is

$$\boxed{y(t) = t + \cos(t - 1) - \sin(t - 1), \quad t \geq 1.}$$

Its first derivative is $1 - \sin(t - 1) - \cos(t - 1)$ and its second is $-\cos(t - 1) + \sin(t - 1)$. Hence $y'' + y = t$, while $y(1) = 2$ and $y'(1) = 0$. These are the actual specified data, so uniqueness on $[1, \infty)$ verifies the result.

Step 4: Diagnose the two clock errors. The values two and zero belong to $z(0), z'(0)$, not to values of y at an unspecified earlier time. Once that clock change is made, the forcing is $\tau + 1$ and its transform is $1/s^2 + 1/s$, not just $1/s^2$. The transform being inverted is Z , defined using $\tau \geq 0$. Since $z(\tau) = \tau + O(1)$ is eventually positive, it has exact domain $\boxed{s > 0}$. No extension of y to times before one was needed.

Question 8

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An accumulated quantity enters an IVP:

$$y'(t) + y(t) + \int_0^t y(u) du = 1, \quad y(0) = 0.$$

You may use $\mathcal{L}\{\int_0^t y(u) du\} = Y(s)/s$ where the integrals converge.

Tasks

1. Solve directly by Laplace transforms, including the integral term.
2. Find an explicit expression for $I(t) = \int_0^t y(u) du$ and verify the original equation.
3. Differentiate the original equation and determine the extra initial condition needed for the resulting second-order equation. Explain why omitting that condition enlarges the solution set.
4. Find the first positive zero of y , its limit, the limit of I , and the exact transform domain of Y . Explain why a constant right-hand side does not force a nonzero limiting y here.

Question 8 – Solution

Strategy. Transform the accumulated state as well as the instantaneous state. Verify the original integral equation after any differentiation.

Step 1: Transform and complete the square. For sufficiently large s , the initial value gives

$$sY + Y + Y/s = 1/s, \quad Y = \frac{1}{s^2 + s + 1}.$$

Let $b = \sqrt{3}/2$. Completing the square gives

$$y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin(bt).$$

This inverts $Y = 1/[(s + 1/2)^2 + b^2]$ with the required sine normalization.

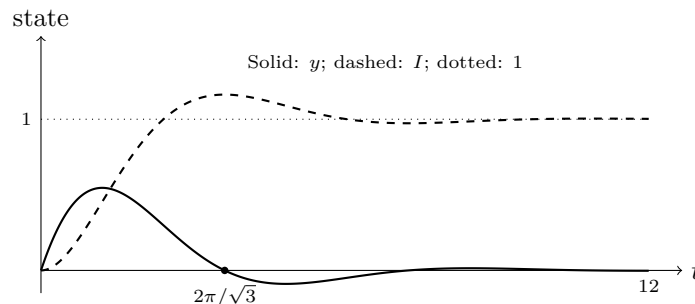
Step 2: Reconstruct the integral and verify. Integration, or differentiation of the following expression, gives

$$I(t) = 1 - e^{-t/2} [\cos(bt) + \frac{1}{\sqrt{3}} \sin(bt)].$$

It satisfies $I(0) = 0$ and $I' = y$. Also $y' = e^{-t/2} [\cos(bt) - \sin(bt)/\sqrt{3}]$. Substitution yields $y' + y + I = 1$ identically, and $y(0) = 0$. This checks the undifferentiated problem.

Step 3: Retain the compatibility condition. Differentiation gives $y'' + y' + y = 0$. Evaluating the original equation at zero gives $y'(0) = 1$. Thus the equivalent second-order IVP has data $y(0) = 0, y'(0) = 1$. If only $y(0) = 0$ is retained, a free initial slope remains; it sets the constant value of $y' + y + I$, which need not be one. Conversely, a solution of the differentiated equation with both correct data makes that quantity constant and equal to one, proving equivalence and uniqueness.

Step 4: Interpret the limiting balance. The first positive zero is $t = \pi/b = 2\pi/\sqrt{3}$. The function changes sign there and tends to zero, while $I(t) \rightarrow 1$. The accumulated state supplies the limiting balance, rather than a nonzero instantaneous state. The exact domain of Y is $s > -1/2$, from the damped sine and its nonconvergent boundary integral. The integral state's transform needs $s > 0$ because $I \rightarrow 1$; the initial derivation used this common domain. The wider domain of Y follows from the recovered function itself.



Question 9

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

Use Laplace transforms for the third-order IVP

$$y^{(3)} + 3y'' + 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Tasks

1. Derive the transformed equation, displaying the contributions of all three initial data.
2. Invert the result at the repeated pole, and check all three data.
3. Verify the third-order differential equation directly, using $y = e^{-t}v$ if useful.
4. Prove positivity and strict decrease for $t > 0$. Compute the total integral and exact transform domain, and explain why the initial flatness does not make the solution constant.

Question 9 – Solution

Strategy. Higher derivatives introduce more initial terms. A repeated transform pole packages the polynomial-exponential solution compactly.

Step 1: Transform the derivative hierarchy. The general third-derivative rule is $\mathcal{L}\{y^{(3)}\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$. Here the first three derivatives transform to $sY - 1$, $s^2Y - s$ and $s^3Y - s^2$. Hence

$$(s^3Y - s^2) + 3(s^2Y - s) + 3(sY - 1) + Y = 0,$$

$$Y = \frac{s^2 + 3s + 3}{(s + 1)^3}.$$

Even the zero initial values must be placed in the correct derivative formulas before simplifying.

Step 2: Expand at the repeated pole. Writing $p = s + 1$ makes the numerator $p^2 + p + 1$, so

$$Y = \frac{1}{s + 1} + \frac{1}{(s + 1)^2} + \frac{1}{(s + 1)^3}, \quad \boxed{y = e^{-t}(1 + t + t^2/2)}.$$

We have $y' = -t^2e^{-t}/2$ and $y'' = e^{-t}(t^2/2 - t)$. Thus the three initial values are exactly 1, 0, 0.

Step 3: Verify the operator identity. For $y = e^{-t}v$, repeated product rules give

$$y^{(3)} + 3y'' + 3y' + y = e^{-t}v^{(3)}.$$

Here $v = 1 + t + t^2/2$ has third derivative zero. This verifies the equation everywhere. The linear third-order IVP with all three data has a unique solution.

Step 4: Check shape, area and domain. The polynomial factor is positive on $t \geq 0$, and $y' < 0$ for every $t > 0$, so y is positive and strictly decreasing after its initially flat point. It tends to zero. Direct exponential moments give

$$\boxed{\int_0^\infty y(t) dt = 1 + 1 + 1 = 3 = Y(0)}.$$

The exact transform domain is $s > -1$; at or below -1 the positive polynomial tail makes the integral diverge. The data $y'(0) = y''(0) = 0$ describe only the initial point. The equation gives $y^{(3)}(0) = -1$, so the first change is cubic rather than absent.

Question 10

Use one-sided Laplace transforms and retain all initial-value terms. Write $Y(s) = \mathcal{L}\{y\}(s)$, with real s sufficiently large during the transformation. Unless stated otherwise, solve on $t \geq 0$.

An input $q(t)$ is to produce a prescribed smooth rise in

$$y'' + 3y' + 2y = q(t), \quad y(0) = 0, \quad y'(0) = 0.$$

The target family is $y_k(t) = 1 - (1 + kt)e^{-kt}$, with $k > 0$. Allowed inputs have the form $q(t) = 2 + Ce^{-kt}$, with no term involving te^{-kt} .

Tasks

1. Find the target transform $Y_k(s)$ and derive the input transform required by the IVP.
2. Invert that input transform and determine exactly which $k > 0$ satisfy the allowed input form.
3. An additional requirement is $y(0) = 4$. Select k, C and verify the complete IVP with the resulting input.
4. Prove that the selected response increases to one without overshoot. Explain why the required input is unique once the full target trajectory is fixed.

Question 10 – Solution

Strategy. Transform the target first, then use the differential operator to recover the necessary input. Check the permitted input family before selecting parameters.

Step 1: Transform the prescribed trajectory. The exponential and time-exponential pairs give

$$Y_k = \frac{1}{s} - \frac{1}{s+k} - \frac{k}{(s+k)^2} = \boxed{\frac{k^2}{s(s+k)^2}}.$$

All targets have $y_k(0) = y'_k(0) = 0$, so their required input transform is

$$Q_k = (s^2 + 3s + 2)Y_k = \frac{k^2(s+1)(s+2)}{s(s+k)^2}.$$

Step 2: Recover the input and impose the restriction. Partial fractions, verified by clearing denominators, give

$$Q_k = \frac{2}{s} + \frac{k^2 - 2}{s+k} - \frac{k(k-1)(k-2)}{(s+k)^2},$$

$$q_k(t) = 2 + (k^2 - 2)e^{-kt} - k(k-1)(k-2)te^{-kt}.$$

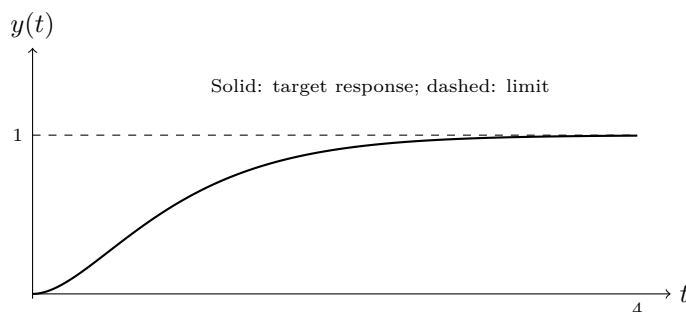
For $k > 0$, the last term vanishes exactly for $\boxed{k=1 \text{ or } k=2}$. Their respective values of C are -1 and 2 . A nonzero te^{-kt} coefficient cannot be absorbed into a constant multiple of e^{-kt} on an interval.

Step 3: Select and verify the desired acceleration. Differentiation gives $y'_k = k^2te^{-kt}$ and $y''_k = k^2(1 - kt)e^{-kt}$, so $y''_k(0) = k^2$. The added requirement selects

$$\boxed{k=2, \quad C=2, \quad q=2+2e^{-2t}, \quad y=1-(1+2t)e^{-2t}}.$$

Here $y' = 4te^{-2t}$, $y'' = (4 - 8t)e^{-2t}$, and $y'' + 3y' + 2y = 2 + 2e^{-2t}$. The initial values are $0, 0$ and the acceleration is four, as required. Both Y and Q have exact domain $s > 0$ because their original functions approach positive constants.

Step 4: Prove the rise and uniqueness of the input. Since $y'(t) > 0$ for $t > 0$, the response increases from zero. Also $1 - y = (1 + 2t)e^{-2t} > 0$ for every finite t , and this difference tends to zero. Thus $\boxed{0 < y(t) < 1 \ (t > 0), \quad y(t) \rightarrow 1}$ with no overshoot. Once the entire twice-differentiable target is fixed, the equation forces $q = y'' + 3y' + 2y$ pointwise. The input is unique, rather than merely one successful choice; linear-IVP uniqueness then verifies that it produces this trajectory.



Question 1

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the regular IVP

$$(1+t)y' + y = 1, \quad y(0) = 0, \quad t \geq 0.$$

An answer in terms of a convergent parameter integral is acceptable for Y .

Tasks

1. Derive the differential equation satisfied by Y . Differentiate the entire transform of y' when handling ty' .
2. Solve that equation using an integrating factor and a condition at $s = \infty$ that a bounded time function must satisfy.
3. Identify the inverse and verify the original IVP. Check your parameter-integral expression directly against its forward transform.
4. State the exact real transform domain and derive the leading large- s behavior from the initial slope.

Question 1 – Solution

Strategy. Multiplication by time makes a differential equation in the transform variable. Its integration constant must be selected using transform behavior.

Step 1: Differentiate the full derivative transform. Here $\mathcal{L}\{y'\} = sY$, so $\mathcal{L}\{ty'\} = -(Y + sY')$. The transformed equation is

$$sY - Y - sY' + Y = 1/s, \quad \boxed{Y' - Y = -1/s^2}, \quad s > 0.$$

Treating $\mathcal{L}\{ty'\}$ as only $-sY'$ would introduce an erroneous extra term.

Step 2: Select the integration constant. Multiply by e^{-s} . A bounded function has $Y = O(1/s)$ as $s \rightarrow \infty$, hence $e^{-s}Y \rightarrow 0$. Integration from s to infinity gives

$$\boxed{Y(s) = e^s \int_s^\infty \frac{e^{-u}}{u^2} du.}$$

The other homogeneous term is Ce^s , incompatible with that bound unless $C = 0$. The inverse identified below is bounded, so this selection is self-consistent and verified rather than assumed without a check.

Step 3: Identify and forward-check the inverse. The original left side is $[(1+t)y]'$, giving $(1+t)y = t$ and

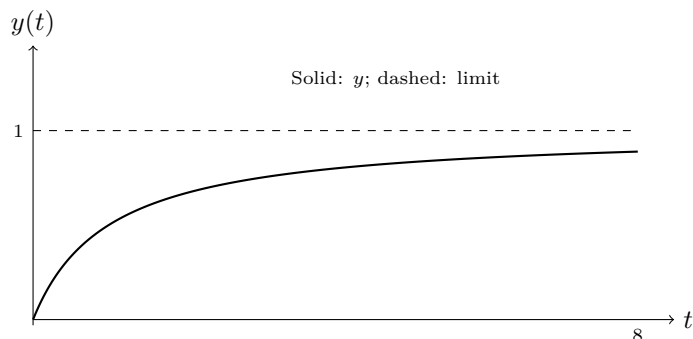
$$\boxed{y(t) = \frac{t}{1+t}.}$$

Its derivative is $(1+t)^{-2}$, so $(1+t)y' + y = 1$ and $y(0) = 0$. For $s > 0$, substitution $u = s(1+t)$ gives $\mathcal{L}\{(1+t)^{-1}\} = e^s \int_s^\infty e^{-u}/u du$. Thus the forward transform is $1/s - e^s \int_s^\infty e^{-u}/u du$, equal to the boxed parameter integral by integration by parts. The regular first-order IVP is unique.

Step 4: Check domain and onset. Since $y \rightarrow 1$ and is nonnegative, its exact domain is $\boxed{s > 0}$. At the origin $y'(0) = 1$, and the substitution $v = st$ gives

$$s^2 Y(s) = \int_0^\infty e^{-v} \frac{v}{1+v/s} dv \rightarrow 1.$$

The integrand is dominated by ve^{-v} , so dominated convergence justifies the limit. Thus $Y \sim 1/s^2$, matching the zero initial value and unit initial slope.



Question 2

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve

$$y' + 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-v^2} dv = \sqrt{\pi}/2$.

Tasks

1. Derive the first-order equation for Y and solve it using an integrating factor.
2. Select the admissible transform branch by its behavior as $s \rightarrow \infty$.
3. Identify the time solution, verify the IVP, and prove that the Gaussian-integral expression is its forward transform for every real s .
4. Evaluate $Y(0)$ and prove $1/s - 2/s^3 \leq Y(s) \leq 1/s$ for $s > 0$. Interpret the resulting initial-value limit.

Question 2 – Solution

Strategy. The variable damping gives a growing homogeneous solution in transform space; a genuine transform selects the complementary integral tail.

Step 1: Transform and apply the integrating factor. The equation becomes $sY - 1 - 2Y' = 0$, or

$$Y' - \frac{s}{2}Y = -\frac{1}{2}, \quad (e^{-s^2/4}Y)' = -\frac{1}{2}e^{-s^2/4}.$$

Step 2: Impose the transform condition at infinity. For a bounded candidate, $Y = O(1/s)$, so $e^{-s^2/4}Y \rightarrow 0$. Integrating to infinity yields

$$Y(s) = \frac{1}{2}e^{s^2/4} \int_s^\infty e^{-u^2/4} du = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

Any additional $Ce^{s^2/4}$ violates the bound. The recovered inverse will verify boundedness and the entire transform calculation.

Step 3: Identify the inverse and its full domain. Separation in time gives $y = e^{-t^2}$, with $y' = -2te^{-t^2}$ and $y(0) = 1$. Completing the square directly gives

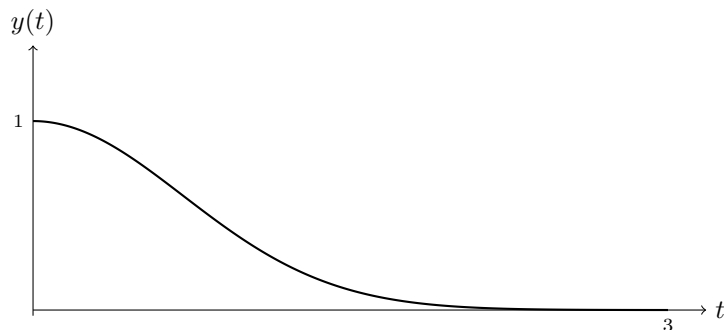
$$\int_0^\infty e^{-st-t^2} dt = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

The Gaussian tail is integrable for every real s , so this is the exact real domain. This direct verification also justifies the selected transformed branch. Uniqueness follows from the regular first-order equation.

Step 4: Check the area and large-parameter bound. At zero, $Y(0) = \sqrt{\pi}/2$. For $t \geq 0$, the elementary inequality $e^{-x} \geq 1 - x$ gives $1 - t^2 \leq e^{-t^2} \leq 1$. Integrating with the positive kernel yields

$$\frac{1}{s} - \frac{2}{s^3} \leq Y(s) \leq \frac{1}{s}, \quad s > 0.$$

The lower bound may be negative for small s but remains valid. Multiplying by s and taking $s \rightarrow \infty$ gives $sY(s) \rightarrow 1 = y(0)$. The graph shows the actual rapidly decaying time solution.



Question 3

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Investigate the validity of the Laplace method for

$$y' - 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

Do not assume that every global smooth IVP solution has an ordinary Laplace transform.

Tasks

1. Solve the time-domain IVP and determine whether its ordinary transform exists for any real s .
2. Formally apply the transform rules and solve the resulting equation for a candidate Y , without yet claiming it is a transform.
3. Show that every member of this formal family satisfies $Y(s) \rightarrow 0$ and $sY(s) \rightarrow 1$ as $s \rightarrow \infty$.
4. Explain why neither those limits nor solving the formal equation validates the calculation. Identify the failed hypothesis and contrast it with the sign-reversed damping equation.

Question 3 – Solution

Strategy. Check the actual time solution first. Necessary transform limits can hold for functions of s that are not transforms of the IVP solution.

Step 1: Establish nonexistence of an ordinary transform. Separation and the initial datum give $y = e^{t^2}$, which is smooth and global. For any fixed real s , however, $t^2 - st \geq t^2/2$ for sufficiently large t . Thus

$$\int_0^\infty e^{-st} y(t) dt = \int_0^\infty e^{t^2-st} dt = \infty.$$

The exact real transform domain is empty.

Step 2: Solve the formal transformed equation. Pretending the rules apply gives $sY - 1 + 2Y' = 0$, so

$$Y' + \frac{s}{2}Y = \frac{1}{2}, \quad Y_C(s) = e^{-s^2/4} \left[C + \frac{1}{2} \int_0^s e^{u^2/4} du \right].$$

This is a valid family of solutions to the differential equation in s . It is not yet a family of Laplace transforms.

Step 3: Check the deceptively correct limits. As $s \rightarrow \infty$, l'Hopital's rule gives

$$\frac{\int_0^s e^{u^2/4} du}{2e^{s^2/4}/s} \rightarrow 1,$$

because the derivative ratio is $1/(1 - 2/s^2) \rightarrow 1$. Hence the integral contribution to Y_C is asymptotic to $1/s$, while $Ce^{-s^2/4}$ decays faster. Every real C therefore satisfies

$$Y_C(s) \rightarrow 0, \quad sY_C(s) \rightarrow 1.$$

These two necessary-looking conditions do not even select a unique formal branch here.

Step 4: Identify the invalid step. The ordinary integrals defining Y , the transform of ty , and the derivative transform do not converge for any real s . The time solution is not of exponential order, and direct growth proves failure, not merely absence of a sufficient hypothesis. Applying transform identities to these divergent integrals was unjustified. The limits in Step3 cannot repair that step. In contrast, changing the sign to $y' + 2ty = 0$ gives e^{-t^2} , whose transform exists for every real s and can be verified directly. A differential equation in transform space is useful only after its relationship to convergent time integrals is established.

Question 4

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider a singular initial point:

$$ty' + 2y = e^{-t}, \quad y(0) = \frac{1}{2}.$$

Seek a solution continuously differentiable on $[0, \infty)$ and satisfying the equation also at zero.

Tasks

1. Check compatibility at zero and determine whether boundedness near zero leaves a free integration constant in the time equation.
2. Derive and solve the equation for Y , selecting a branch that tends to zero at large positive s .
3. Identify and verify the inverse, including its limiting value and derivative at zero.
4. Find the full real convergence set and $Y(0)$. Explain why a finite transform value at zero need not imply a finite derivative there.

Question 4 – Solution

Strategy. A vanishing leading coefficient imposes compatibility. Boundedness at the singular endpoint can replace a free constant that would exist away from it.

Step 1: Inspect the singular endpoint. At zero, the equation requires $2y(0) = 1$, so the given datum is compatible. For $t > 0$, multiplying by t gives $(t^2y)' = te^{-t}$. Its general integral has an additive constant in t^2y . Boundedness of y forces that constant to be zero, hence

$$y(t) = \frac{1 - (1+t)e^{-t}}{t^2} \quad (t > 0).$$

This proves uniqueness among bounded solutions near zero, without invoking a regular-IVP theorem at a singular point.

Step 2: Solve in transform space. The full rule $\mathcal{L}\{ty'\} = -Y - sY'$ gives $Y - sY' = 1/(s+1)$. Therefore

$$(Y/s)' = -\frac{1}{s^2(s+1)}, \quad Y = s \int_s^\infty \frac{du}{u^2(u+1)} = \boxed{1 - s \ln \frac{s+1}{s}}, \quad s > 0.$$

The omitted homogeneous term is Cs and cannot tend to zero. The inverse below verifies boundedness and transform existence.

Step 3: Verify the endpoint and forward transform. A nonsingular representation is $y(t) = \int_0^1 ue^{-ut} du$, including at zero. Thus $y(0) = 1/2$, $y'(0) = -1/3$, and differentiation verifies the original equation (or use $(t^2y)' = te^{-t}$ for $t > 0$ and compatibility at zero). For $s > 0$, integrating first in time gives

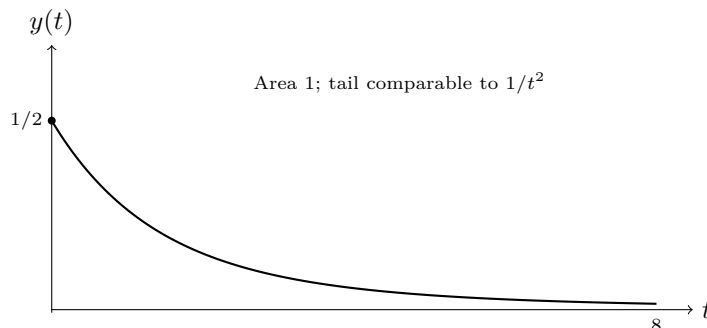
$$Y(s) = \int_0^1 \frac{u}{s+u} du = 1 - s \ln \frac{s+1}{s}.$$

The finite absolute double integral justifies interchange.

Step 4: Check the boundary rather than just the formula. The function is positive and $y(t) \sim 1/t^2$. Therefore the exact real convergence set is $\boxed{[0, \infty)}$, with divergence for negative s . At zero the same

nonnegative double integral gives $\boxed{Y(0) = \int_0^1 1 du = 1}$. But $ty(t) \sim 1/t$, so its first moment diverges.

Indeed $Y'(s) = -\ln[(s+1)/s] + 1/(s+1) \rightarrow -\infty$ as $s \downarrow 0$. A finite boundary area does not imply a finite boundary moment.



Question 5

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Seek a twice continuously differentiable solution of

$$ty'' + 2y' + ty = 0, \quad y(0) = 1,$$

including at $t = 0$. When checking the transform boundary, you may use the estimate $|\int_R^M e^{-st} \sin t/t dt| \leq 2/R$ for $M > R > 0$, uniformly for $s \geq 0$, obtained by integration by parts.

Tasks

1. Determine the initial slope forced by the equation at zero, then derive the differential equation for Y .
2. Solve for $Y(s)$ on $s > 0$ using its decay at infinity and identify the inverse.
3. Use $w(t) = ty(t)$ to verify the equation and prove uniqueness in the stated smooth class. Explain why the initial slope disappeared from the transformed equation.
4. Find the exact real transform domain, including the ordinary boundary value at zero. Use the supplied uniform tail estimate to justify passage to that value.

Question 5 – Solution

Strategy. The singular equation fixes a missing initial slope. Transform differentiation removes that slope algebraically, so compatibility must be checked separately.

Step 1: Recover compatibility and transform. At zero the equation requires $y'(0) = 0$. Since

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2Y' + 1,$$

adding $2\mathcal{L}\{y'\}$ and $\mathcal{L}\{ty\}$ gives

$$(s^2 + 1)Y' = -1.$$

Step 2: Select and identify the inverse. The condition $Y \rightarrow 0$ as $s \rightarrow \infty$ gives

$$Y(s) = \int_s^\infty \frac{du}{1+u^2} = \arctan(1/s), \quad s > 0.$$

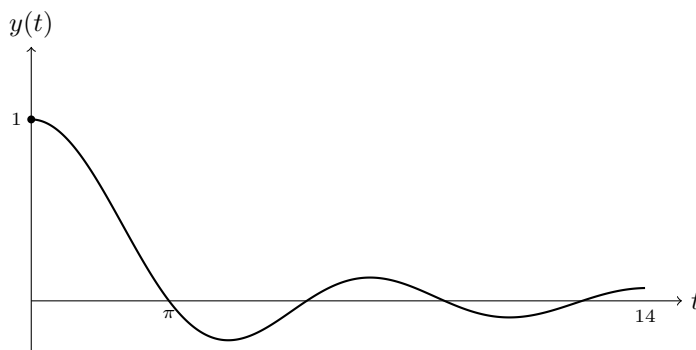
The time function $y(t) = \sin t/t$ ($t > 0$), $y(0) = 1$ has representation $y = \int_0^1 \cos(ut) du$. Transforming this integral for $s > 0$ yields $\int_0^1 s/(s^2 + u^2) du = \arctan(1/s)$. Absolute double integrability justifies the interchange.

Step 3: Verify smoothness and uniqueness. The expansion $y = 1 - t^2/6 + O(t^4)$ supplies a smooth extension with slope zero. Set $w = ty$. Then $w'' + w = ty'' + 2y' + ty = 0$, with $w(0) = 0$, $w'(0) = 1$. The regular oscillator IVP has the unique solution $w = \sin t$, proving the original equation and uniqueness of its smooth extension. The term $-y'(0)$ is constant in s , so it disappears when differentiating the transform of y'' ; arbitrary initial slopes are nevertheless not admissible.

Step 4: Establish the boundary value. The supplied estimate makes the tails uniformly small for $s \geq 0$. On each fixed finite interval the damped integrals converge to the undamped one as $s \downarrow 0$. Taking the interval long first and then taking $s \downarrow 0$ proves

$$Y(0) = \int_0^\infty \frac{\sin t}{t} dt = \pi/2.$$

The boundary integral is conditional: intervals where $|\sin t| \geq 1/2$ give a divergent harmonic lower bound for its absolute integral. For $s < 0$, fixed-length positive lobe intervals have weighted integrals growing like $e^{|s|t}/t$, violating the Cauchy criterion. The exact real domain is $[0, \infty)$.



Question 6

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve the regular second-order IVP

$$(1+t)y'' + y' = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

A convergent parameter integral may be retained in the formula for Y .

Tasks

1. Derive the differential equation for Y , including all initial contributions.
2. Solve it with an integrating factor and explain the condition that removes its homogeneous term.
3. Identify and verify the time solution, then check the parameter-integral transform directly.
4. Prove $0 < Y(s) < 1/s^2$ for $s > 0$ and determine the exact real convergence interval. Explain why a growing time solution can still be transformable.

Question 6 – Solution

Strategy. Keep the initial derivative term from the unmultiplied second derivative. The coefficient $1 + t$ never vanishes on the time domain.

Step 1: Transform with both data. The three contributions are

$$\mathcal{L}\{ty''\} = -2sY - s^2Y', \quad \mathcal{L}\{y''\} = s^2Y - 1, \quad \mathcal{L}\{y'\} = sY.$$

Hence $-s^2Y' + (s^2 - s)Y = 1$, or

$$Y' - (1 - 1/s)Y = -1/s^2, \quad s > 0.$$

Step 2: Select the parameter integral. The integrating factor is se^{-s} , so $(se^{-s}Y)' = -e^{-s}/s$. An exponential-order continuous time solution has $Y = O(1/s)$ for large s , and thus $se^{-s}Y \rightarrow 0$. Therefore

$$Y(s) = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du.$$

An additional term Ce^s/s violates the large-parameter bound unless $C = 0$. Existence and the growth condition are verified by the recovered solution.

Step 3: Identify and check the inverse. In time, the left side is $[(1+t)y']'$. Integrating with the data gives $y' = 1/(1+t)$ and

$$y(t) = \ln(1+t).$$

It has $y(0) = 0$, $y'(0) = 1$, $y'' = -1/(1+t)^2$, and zero equation residual. The coefficient is regular, so uniqueness applies. Integrating its transform by parts gives

$$Y(s) = \frac{1}{s} \int_0^\infty \frac{e^{-st}}{1+t} dt = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du,$$

where the upper boundary vanishes for $s > 0$.

Step 4: Compare growth rates and domains. For $t > 0$, $0 < \ln(1+t) < t$. Multiplying by the positive kernel and integrating gives

$$0 < Y(s) < \frac{1}{s^2}, \quad s > 0.$$

The exact domain is $s > 0$: logarithmic growth is dominated by the exponential kernel above zero, whereas its positive time integral diverges at zero and for negative parameters. Unboundedness by itself does not prevent a Laplace transform; growth relative to the kernel is what matters.

Question 7

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the local IVP

$$(t-1)y' + y = 0, \quad y(0) = 1.$$

For $0 < T < 1$, define the finite transform $F_T(s) = \int_0^T e^{-st}y(t) dt$.

Tasks

1. Solve the IVP up to its first singularity and decide whether a global classical solution on $[0, \infty)$ exists.
2. Formally applying the infinite-transform rules gives $Y' + Y = 1/s$. Identify why this equation cannot be used as an ordinary transform solution of the IVP.
3. Derive the correct differential equation for F_T by retaining the upper boundary term during integration by parts.
4. Check that finite-transform equation directly, including its value at $s = 0$. Explain why a finite cutoff repairs the calculation but does not remove the singularity.

Question 7 – Solution

Strategy. An interior zero of the leading coefficient can stop the classical solution. A finite transform before that point must retain its endpoint term.

Step 1: Locate the obstruction in time. Since $[(t-1)y]' = 0$, the datum gives $(t-1)y = -1$. Thus

$$y(t) = \frac{1}{1-t}, \quad 0 \leq t < 1.$$

This diverges as $t \uparrow 1$ and admits no continuous classical continuation through one. There is no global classical IVP solution on the full half-line. The integral up to one already diverges for every real s , because its positive weight stays bounded below near one while $1/(1-t)$ has a logarithmic divergence.

Step 2: Diagnose the formal infinite equation. The formal substitutions $\mathcal{L}\{ty'\} = -Y - sY'$ and $\mathcal{L}\{y'\} = sY - 1$ would give $Y' + Y = 1/s$. But the required infinite time integrals do not exist and no global classical solution satisfies the starting hypotheses. Solving this formal equation cannot provide an ordinary transform of the local IVP solution.

Step 3: Retain the actual cutoff boundary. For $T < 1$, integration by parts gives

$$0 = [e^{-st}(t-1)y(t)]_0^T + s \int_0^T e^{-st}(t-1)y(t) dt.$$

The first term is $1 - e^{-sT}$, using $(T-1)y(T) = -1$. Since $\int_0^T te^{-st}y = -F'_T$, the result is

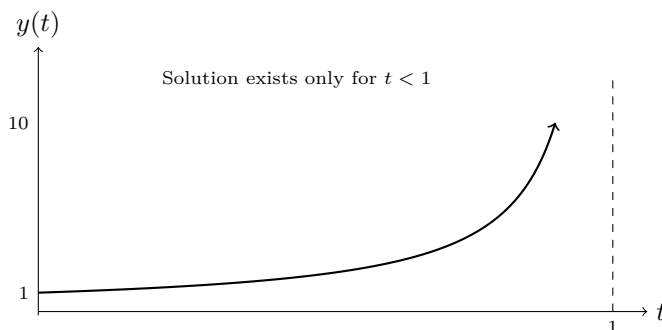
$$F'_T(s) + F_T(s) = \frac{1 - e^{-sT}}{s} \quad (s \neq 0).$$

Discarding the upper endpoint would lose the term $-e^{-sT}/s$.

Step 4: Verify the finite identity and its limits. Direct differentiation on the compact interval gives

$$F'_T + F_T = \int_0^T (1-t)e^{-st}y(t) dt = \int_0^T e^{-st} dt.$$

This is the stated expression for $s \neq 0$ and equals $\boxed{T}$ at $s = 0$. Also $F_T(0) = -\ln(1-T)$ and $F'_T(0) = T + \ln(1-T)$, whose sum is T . Every finite F_T exists for all real s , but it diverges as $T \uparrow 1$ for each fixed s . A valid finite-window calculation does not supply a missing continuation through the pole. The plot stops short of the singular time.



Question 8

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Classify the data for which a twice continuously differentiable solution on $[0, \infty)$ can satisfy

$$t^2 y'' + 2ty' - 2y = 0, \quad y(0) = a, \quad y'(0) = b.$$

For a smooth exponential-order function with $a = 0$, you may use $\lim_{s \rightarrow \infty} s^2 Y(s) = b$.

Tasks

1. Find the compatibility condition at zero and solve the time equation for $t > 0$ using trial powers t^m .
2. Derive the second-order differential equation for Y , accounting for the effect of two parameter derivatives on the initial terms.
3. Solve the transformed equation using trial powers s^m . Determine what the condition $Y \rightarrow 0$ fixes and what initial information is still needed.
4. Classify all admissible pairs (a, b) , verify the resulting solutions, and state their exact transform domains. Is specifying only $y(0)$ enough for uniqueness?

Question 8 – Solution

Strategy. Singular coefficients can erase initial data from the transformed equation. Recover admissibility and the remaining amplitude separately.

Step 1: Check compatibility and time powers. At zero, the equation requires $\boxed{a = 0}$. For $t > 0$, trial powers give $m(m-1) + 2m - 2 = (m-1)(m+2) = 0$. The two independent solutions are t and t^{-2} , so $y = Ct + Dt^{-2}$. They span the solutions on $(0, \infty)$ because the normalized equation is regular there. Continuity at zero forces $D = 0$; the initial slope then sets $C = b$.

Step 2: Transform the polynomial coefficients. For sufficiently large s ,

$$\mathcal{L}\{t^2 y''\} = \frac{d^2}{ds^2}[s^2 Y - sa - b] = 2Y + 4sY' + s^2 Y'',$$

while $2\mathcal{L}\{ty'\} = -2Y - 2sY'$. Adding $-2Y$ gives

$$\boxed{s^2 Y'' + 2sY' - 2Y = 0.}$$

Both a and b vanished under differentiation; their disappearance is not permission to ignore the time-domain data.

Step 3: Select the transform and its amplitude. Trial powers of s have exponents one and -2 , yielding $Y = As + B/s^2$ on $s > 0$. A genuine transform of an exponential-order continuous function tends to zero at large s , so $A = 0$. This still leaves B arbitrary. The supplied initial-slope limit gives $\boxed{B = b}$ and hence $Y = b/s^2$. The initial-value limit $sY \rightarrow 0$ also agrees with the necessary datum $a = 0$.

Step 4: Verify and classify uniqueness. All and only the data $\boxed{a = 0, b \in \mathbb{R}}$ are admissible, with unique solution $\boxed{y = bt}$. Substitution gives $0 + 2tb - 2bt = 0$, and both data hold. For $b \neq 0$, the exact domain is $s > 0$; the signed linear tail diverges at or below zero. For $b = 0$, the zero function has transform zero for every real s . Specifying only $y(0) = 0$ leaves the entire family bt , so it does not determine a unique solution at this singular point.

Question 9

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

An unknown parameter $\alpha > 0$ appears in

$$(1+t)y' + \alpha y = 0, \quad y(0) = 1.$$

An exact measurement gives $\int_0^\infty y(t) dt = 1/2$. A second proposed measurement is $\int_0^\infty ty(t) dt = 1/3$.

Tasks

1. Derive the differential equation for Y and solve it as a convergent parameter integral for $s > 0$.
2. Identify and verify the time solution for general $\alpha > 0$.
3. Determine which parameters admit a finite area, then recover α from the first measurement.
4. Test the proposed first moment for consistency, and state the exact real transform domain of the recovered solution.

Question 9 – Solution

Strategy. Solve the parameter family before fitting measurements. A boundary transform value is meaningful only when the actual time integral converges.

Step 1: Derive and solve the transform equation. Using $\mathcal{L}\{y'\} = sY - 1$ and $\mathcal{L}\{ty'\} = -Y - sY'$ gives

$$-sY' + (s + \alpha - 1)Y = 1.$$

The integrating factor for the divided equation is $e^{-s}s^{1-\alpha}$, so its product with Y has derivative $-e^{-s}s^{-\alpha}$. The transform bound at infinity selects

$$Y(s) = e^s s^{\alpha-1} \int_s^\infty e^{-u} u^{-\alpha} du, \quad s > 0.$$

The alternative homogeneous term $Ce^s s^{\alpha-1}$ violates decay as $s \rightarrow \infty$.

Step 2: Verify the parameter family in time. Separation gives $y(t) = (1+t)^{-\alpha}$. Its derivative is $-\alpha(1+t)^{-\alpha-1}$, so the equation and initial value hold. The coefficient $1+t$ is nonzero on $t \geq 0$, establishing regular-IVP uniqueness. Substitution $u = s(1+t)$ in the forward integral gives exactly the parameter expression above, verifying its inversion.

Step 3: Use the convergent area to identify the parameter. The positive power tail has finite area exactly when $\alpha > 1$, and then

$$\int_0^\infty (1+t)^{-\alpha} dt = \frac{1}{\alpha-1}.$$

Equating this to $1/2$ gives the unique value $\alpha = 3$. This uses convergence of the integral, not a formal substitution into an expression initially derived only for $s > 0$.

Step 4: Test the second datum rather than refit. A finite first moment requires $\alpha > 2$. Substituting $x = 1+t$ gives

$$\int_0^\infty \frac{t}{(1+t)^\alpha} dt = \frac{1}{\alpha-2} - \frac{1}{\alpha-1} = \frac{1}{(\alpha-1)(\alpha-2)}.$$

At $\alpha = 3$, it is $1/2$, not $1/3$, so the proposed data are inconsistent with this family. The recovered positive tail is $(1+t)^{-3}$; its exact transform domain is $[0, \infty)$. Positive s gives exponential damping, zero gives the finite area $1/2$, and negative s makes the weighted power tail diverge.

Question 10

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Choose constants A, B so that a twice continuously differentiable solution on $[0, \infty)$ satisfies

$$ty' + (t+2)y = (At+B)e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

The equation must hold at the singular initial point as well as for $t > 0$.

Tasks

1. Derive the differential equation for Y without inserting unsupported free initial terms.
2. Solve it for general A, B and select the branch that can be an ordinary transform.
3. Use the recovered time solution and the two initial conditions to determine A, B . Verify the resulting equation at and away from zero.
4. Prove uniqueness in the stated smooth class, determine the exact transform domain, and describe the sign and monotonicity of the selected response.

Question 10 – Solution

Strategy. The data lost from the transform equation return as compatibility conditions on the forcing. Boundedness removes the singular time-domain homogeneous mode.

Step 1: Transform the time multipliers. The three left-hand terms transform to $-Y - sY'$, $-Y'$ and $2Y$. Thus

$$-(s+1)Y' + Y = \frac{A}{(s+1)^2} + \frac{B}{s+1}.$$

The initial value in $sY - y(0)$ differentiates to zero in $\mathcal{L}\{ty'\}$. It must still be imposed on the inverse, not added back as a fictitious term.

Step 2: Solve and remove the inadmissible branch. Writing $p = s + 1$, divide by p^2 to obtain

$$(Y/p)' = -A/p^4 - B/p^3.$$

Integration gives

$$Y = \frac{A/3}{(s+1)^2} + \frac{B/2}{s+1} + C(s+1).$$

Decay requires $C = 0$. The remaining terms invert to $y = e^{-t}(At/3 + B/2)$, whose transform can now be verified directly.

Step 3: Recover the forcing constants and verify. The initial value requires $B/2 = 1$, and the initial slope requires $A/3 - B/2 = 0$. Hence

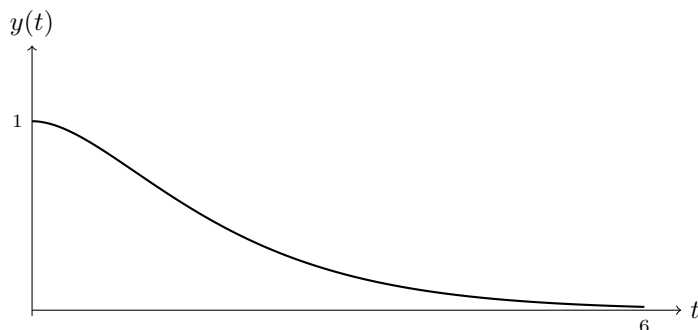
$$A = 3, \quad B = 2, \quad y = (1+t)e^{-t}.$$

Its derivative is $-te^{-t}$, so

$$ty' + (t+2)y = e^{-t}[-t^2 + (t+2)(1+t)] = (3t+2)e^{-t}.$$

At zero the equation gives $2y(0) = B$, and both specified data hold. The smooth extension also has $y''(0) = -1$.

Step 4: Prove uniqueness and check the response. For $t > 0$, multiplying the original first-order equation by te^t gives $(t^2e^ty)' = At^2 + Bt$. Thus every local solution differs from the selected one by Ke^{-t}/t^2 . This mode is unbounded at zero unless $K = 0$, proving uniqueness in the stated smooth class. The selected function is positive, with $y' < 0$ for $t > 0$, and decreases from one to zero after an initially horizontal tangent. Its positive polynomial-exponential tail gives exact transform domain $s > -1$. The parameter restrictions arise from compatibility, not from a regular existence theorem where the leading coefficient vanishes.



Question 1

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A first-order system starts above zero and receives a finite pulse:

$$y' + 2y = 3[H_1(t) - H_3(t)], \quad y(0) = 1, \quad t \geq 0.$$

Tasks

1. Find $Y(s)$ and invert it in step-function form, retaining the initial response.
2. Write the solution on the three time intervals and verify the equation and matching values.
3. Find the jumps in y' at both switches. Explain whether y jumps.
4. Locate the global maximum on $[0, \infty)$, prove it is global, and state the exact real transform domain.

Question 1 – Solution

Strategy. Keep the initial response separate from the two delayed step responses, then compare the three monotonic pieces.

Step 1: Transform and invert. The initial term gives $(s+2)Y - 1 = 3(e^{-s} - e^{-3s})/s$, hence

$$Y = \frac{1}{s+2} + \frac{3(e^{-s} - e^{-3s})}{s(s+2)}, \quad s > 0.$$

Since $\mathcal{L}^{-1}\{1/[s(s+2)]\} = (1 - e^{-2t})/2$,

$$y = e^{-2t} + \frac{3}{2}H_1(t)(1 - e^{-2(t-1)}) - \frac{3}{2}H_3(t)(1 - e^{-2(t-3)}).$$

Step 2: Match the pieces. Set $M = \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-4}$. The same solution is

$$y(t) = \begin{cases} e^{-2t}, & 0 \leq t < 1, \\ \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-2(t-1)}, & 1 \leq t < 3, \\ Me^{-2(t-3)}, & t \geq 3. \end{cases}$$

The matching values are $y(1) = e^{-2}$ and $y(3) = M$. Each piece satisfies $y' + 2y = 0, 3, 0$, respectively, and $y(0) = 1$. These matching conditions also give uniqueness interval by interval.

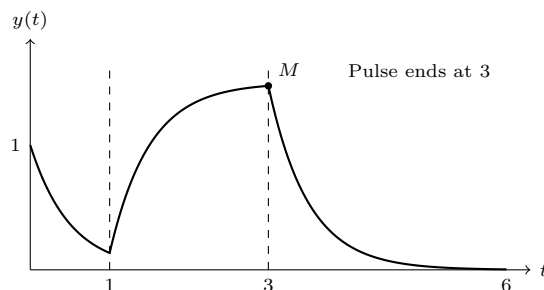
Step 3: Identify the corners. Writing $[v]_a = v(a+) - v(a-)$, the equation and continuity imply

$$[y']_1 = 3, \quad [y']_3 = -3, \quad [y]_1 = [y]_3 = 0.$$

The step factors multiply responses that vanish at their activation times, so no jump in y is introduced.

Step 4: Compare all possible maxima. The solution decreases on $[0, 1]$, increases on $[1, 3]$ toward $3/2$, then decreases to zero. Moreover $M > \frac{3}{2} - \frac{3}{2}e^{-4} > 1$, since $e^4 > 3$. Thus the unique global maximum is

$y(3) = M$. Its nonzero tail is exactly a positive multiple of e^{-2t} , so the exact real transform domain is $s > -2$; the displayed formula extends removably across $s = 0$.



Question 2

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A ramp is switched on without resetting its clock, then switched off:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} t, & 1 \leq t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

A proposed input transform is $(e^{-s} - e^{-3s})/s^2$.

Tasks

1. Explain the error in the proposed transform and derive the correct $F(s)$.
2. Find $Y(s)$ and invert it, rewriting each active input in its own delayed time.
3. Give the piecewise solution, its maximum, and the two jumps in y' .
4. Compute $\int_0^\infty y(t) dt$ directly and from the differential equation. Explain the apparent singularities in the transform formula.

Question 2 – Solution

Strategy. A window does not reset the ramp's clock. At each endpoint rewrite t as delayed time plus the endpoint.

Step 1: Shift the whole input. The input is $H_1(t)[(t-1)+1] - H_3(t)[(t-3)+3]$, so

$$F = e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) - e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right).$$

The proposed expression instead transforms $H_1(t)(t-1) - H_3(t)(t-3)$, which has a constant tail of 2 after 3. It describes neither the given ramp window nor its shutoff.

Step 2: Invert the response. The zero datum gives $Y = F/(s+1)$. For a delayed input $\tau + c$, the zero-start response solves $r'_c + r_c = \tau + c$, and is

$$r_c(\tau) = \tau + c - 1 + (1 - c)e^{-\tau}.$$

Its transform is $(1/s^2 + c/s)/(s+1)$, as direct transformation confirms. In particular $r_1 = \tau$ and $r_3 = \tau + 2 - 2e^{-\tau}$. Thus

$$y = H_1(t)(t-1) - H_3(t)[(t-3) + 2 - 2e^{-(t-3)}].$$

Step 3: Read the intervals and switches. Simplifying after the second switch gives

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 3, \\ 2e^{-(t-3)}, & t \geq 3. \end{cases}$$

The pieces meet at 0 and 2, and satisfy $y' + y = 0, t, 0$. Their slopes are $0, 1, -2e^{-(t-3)}$, so $[y']_1 = 1$ and $[y']_3 = -3$. The unique global maximum is $y(3) = 2$, by increase during the window and strict decay afterward.

Step 4: Check the total response. Directly,

$$\int_0^\infty y \, dt = \int_1^3 (t-1) \, dt + \int_3^\infty 2e^{-(t-3)} \, dt = 2 + 2 = 4.$$

Integrating the equation gives the same answer because $y(\infty) - y(0) = 0$ and $\int_1^3 t \, dt = 4$. The compact input has a transform for every real s , with its apparent $s = 0$ poles canceling. The output's nonzero exponential tail gives exact domain $s > -1$, and the removable value $Y(0) = 4$ agrees with the area.

Question 3

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An undamped oscillator receives a rectangular force of selectable duration:

$$y'' + \omega^2 y = F[1 - H_T(t)], \quad y(0) = y'(0) = 0, \quad F > 0, \omega > 0, T > 0.$$

Use the energy $E = (y'^2 + \omega^2 y^2)/2$.

Tasks

1. Derive $Y(s)$ and the delayed step-response formula for y .
2. Find the displacement and velocity at shutoff, and the residual oscillation amplitude.
3. Classify every duration T for which the system remains at rest after shutoff; explain why $y(T) = 0$ alone is generally insufficient.
4. Compute the residual energy and classify the exact real transform domain, including the rest-producing durations.

Question 3 – Solution

Strategy. Removing the force subtracts a delayed step response. The two surviving state coordinates determine whether motion persists.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{F(1 - e^{-Ts})}{s(s^2 + \omega^2)}, \quad \boxed{y = \frac{F}{\omega^2} [1 - \cos \omega t - H_T(t)(1 - \cos \omega(t - T))].}$$

Before T this has zero initial data and satisfies the forced equation. The subtracted response and its first derivative vanish at activation, so y, y' match at T ; after T the equation is homogeneous.

Step 2: Recover the shutoff state and amplitude. The matching state is

$$y(T) = \frac{F}{\omega^2}(1 - \cos \omega T), \quad y'(T) = \frac{F}{\omega} \sin \omega T.$$

For $t \geq T$, subtraction and the cosine difference identity give

$$y(t) = \frac{2F}{\omega^2} \sin(\omega T/2) \sin(\omega(t - T/2)).$$

Its amplitude is therefore $\boxed{R = 2F|\sin(\omega T/2)|/\omega^2}$.

Step 3: Classify complete return to rest. A homogeneous oscillator remains zero exactly when both state coordinates are zero. Here that happens precisely for

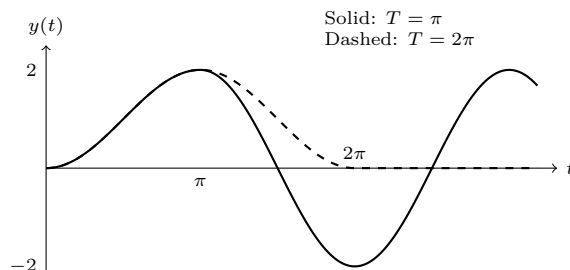
$$\boxed{T = \frac{2\pi n}{\omega}, \quad n = 1, 2, \dots}$$

In this particular pulse family, $y(T) = 0$ forces $\cos \omega T = 1$ and hence also $y'(T) = 0$. For a general forcing history, zero displacement alone is insufficient: a nonzero velocity launches a new oscillation.

Step 4: Compute energy and convergence. After shutoff,

$$\boxed{E = \frac{1}{2}\omega^2 R^2 = \frac{2F^2}{\omega^2} \sin^2(\omega T/2)}.$$

If $R > 0$, the oscillatory tail has an ordinary Laplace transform exactly for $s > 0$: at zero its primitive does not converge, and negative s gives growing integrals over fixed sign intervals. If $R = 0$, the output has compact support, so its transform exists for every real s . Apparent poles in the displayed formula then cancel. The graph compares $F = \omega = 1$ at two durations.



Question 4

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A critically damped system receives a unit pulse:

$$y'' + 2y' + y = 1 - H_2(t), \quad y(0) = y'(0) = 0.$$

The force is already zero for $t > 2$.

Tasks

1. Find $Y(s)$ and derive the unit-step response needed to invert it.
2. Write the post-shutoff response explicitly and verify continuity of y and y' at 2.
3. Find the unique global peak time and height. Prove whether the peak occurs before or after shutoff.
4. Determine the jump in acceleration at shutoff and the total area under y . State the exact real transform domain.

Question 4 – Solution

Strategy. A decaying stable system can continue moving upward after its input stops. Differentiate the actual tail to locate the peak.

Step 1: Transform the pulse. The zero data give

$$Y = \frac{1 - e^{-2s}}{s(s+1)^2}, \quad \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Thus $r(t) = 1 - (1+t)e^{-t}$ is the unit-step response, with $r(0) = r'(0) = 0$ and $r'' + 2r' + r = 1$. The solution is $y = r(t) - H_2(t)r(t-2)$.

Step 2: Simplify the tail and match. For $t \geq 2$,

$$y(t) = e^{-t}[(e^2 - 1)t - (e^2 + 1)].$$

At 2 this gives $1 - 3e^{-2} = r(2)$; its derivative there is $2e^{-2} = r'(2)$. Hence y, y' are continuous, with positive velocity even after the forcing is removed.

Step 3: Locate and certify the peak. For $0 < t < 2$, $y' = te^{-t} > 0$. On the tail,

$$y' = e^{-t}[2e^2 - (e^2 - 1)t].$$

This derivative changes sign exactly once, at

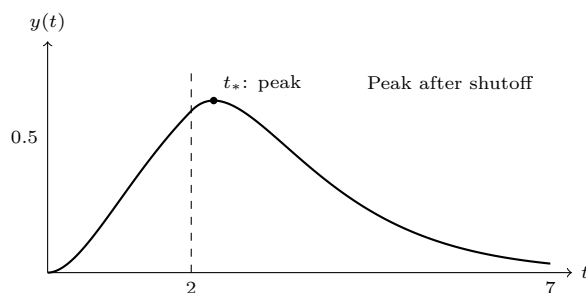
$$t_* = \frac{2e^2}{e^2 - 1} > 2, \quad y(t_*) = (e^2 - 1)e^{-t_*}.$$

Thus the peak is unique and global. The tail remains positive because its linear factor is already positive at 2 and has positive slope, while $y(t) \rightarrow 0$. Damping does not imply an immediate drop in displacement at shutoff.

Step 4: Check the switch, area and domain. Since y, y' are continuous, we have $[y'']_2 = -1$. Both y, y' start at zero and tend to zero. Integrating the equation over the half-line, including the matching boundary terms, yields

$$\int_0^\infty y \, dt = \int_0^2 1 \, dt = 2.$$

The nonzero te^{-t} tail gives exact domain $s > -1$. The $s = 0$ singularity in the transform expression is removable, with $Y(0) = 2$.



Question 5

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A resonant input is windowed without restarting its phase:

$$y'' + y = \sin t [H_a(t) - H_b(t)], \quad y(0) = y'(0) = 0, \quad a = \frac{\pi}{2}, \quad b = \frac{3\pi}{2}.$$

Tasks

1. Rewrite the forcing in delayed coordinates and derive its transform. Pay particular attention to the sign at the second switch.
2. Find and invert $Y(s)$, deriving the required repeated-quadratic transform pair.
3. Find the exact motion after b and its displacement, velocity and energy at shutoff.
4. Determine the acceleration jumps and the exact real transform domain. Explain why a zero displacement at shutoff does not mean rest.

Question 5 – Solution

Strategy. The two endpoints have opposite sine phases. Rewriting both phases first prevents a sign error in the delayed inverse.

Step 1: Shift the phases. For delayed time τ , $\sin(\tau + a) = \cos \tau$ and $\sin(\tau + b) = -\cos \tau$. Therefore

$$f = H_a(t) \cos(t - a) + H_b(t) \cos(t - b), \quad F = \frac{s(e^{-as} + e^{-bs})}{s^2 + 1}.$$

The plus sign at b cancels the continuing negative-phase input; it does not add a second physical forcing window.

Step 2: Derive and invert the response pair. The zero-data equation gives

$$Y = \frac{s(e^{-as} + e^{-bs})}{(s^2 + 1)^2}, \quad s > 0.$$

Differentiating $\mathcal{L}\{\sin t\} = 1/(s^2 + 1)$ with respect to s gives $\mathcal{L}\{t \sin t/2\} = s/(s^2 + 1)^2$. With $q(\tau) = \tau \sin \tau/2$,

$$\boxed{y = H_a(t)q(t - a) + H_b(t)q(t - b).}$$

Here $q(0) = q'(0) = 0$ and $q'' + q = \cos \tau$, directly verifying the equation, the zero initial data and both matching conditions.

Step 3: Evaluate the residual motion. For $t = b + v$, $v \geq 0$, the two ages are $v + \pi$ and v . Hence

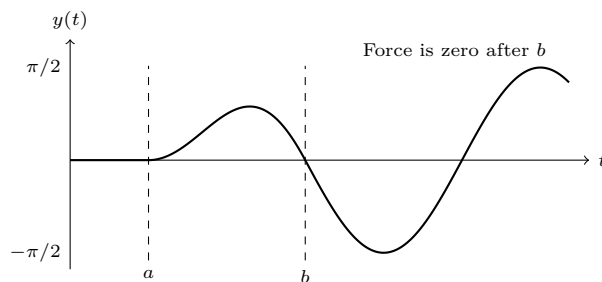
$$y(b + v) = q(v + \pi) + q(v) = \boxed{-\frac{\pi}{2} \sin v}.$$

In particular $y(b) = 0$, $y'(b) = -\pi/2$, and the post-shutoff energy is

$$\boxed{E = \frac{1}{2}(y'^2 + y^2) = \frac{\pi^2}{8}}.$$

The pulse ends precisely at a zero crossing with nonzero velocity, so the system continues oscillating with amplitude $\pi/2$.

Step 4: Inspect the switches and convergence. The input jumps from 0 to 1 at a , and from -1 to 0 at b . Continuity of y gives $\boxed{[y'']_a = [y'']_b = 1}$. There are no jumps in y or y' . The nonzero sinusoidal tail has exact real transform domain $\boxed{s > 0}$: its undamped primitive fails to converge at zero, and exponential growth of fixed-sign tail integrals rules out negative s .



Question 6

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Choose two constant force levels to bring an oscillator exactly to rest:

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \quad f(t) = \begin{cases} A, & 0 \leq t < a, \\ B, & a \leq t < T, \\ 0, & t \geq T, \end{cases} \quad a = \frac{\pi}{4}, \quad T = \frac{\pi}{2}.$$

Require $y(t) = 0$ for every $t \geq T$. The levels A, B may have either sign.

Tasks

1. Express f using steps and derive $Y(s)$, including the initial response.
2. Invert $Y(s)$ and derive the two linear equations that the terminal conditions impose on A, B .
3. Solve for both force levels and prove that they give a unique motion which remains at rest after T .
4. Decide whether nonnegative levels could meet the target. Describe the smoothness at both switches and the exact real transform domain of the designed output.

Question 6 – Solution

Strategy. Exact rest requires two terminal state conditions. Use the step response to convert them into two equations for the two force levels.

Step 1: Transform with the initial displacement. The level changes give $f = A + (B - A)H_a(t) - BH_T(t)$, so

$$(s^2 + 1)Y - s = \frac{A + (B - A)e^{-as} - Be^{-Ts}}{s}.$$

Writing $r(t) = 1 - \cos t$, the inverse is

$$y = \cos t + Ar(t) + (B - A)H_a(t)r(t - a) - BH_T(t)r(t - T).$$

The unforced term $\cos t$ supplies the initial data. Since $r(0) = r'(0) = 0$, each added response preserves y, y' at activation, and $r'' + r = 1$ verifies the forcing on each open interval.

Step 2: Impose both terminal conditions. Set $c = \cos a = \sin a = 1/\sqrt{2}$. At $T = \pi/2$ the last response and its derivative still vanish. The terminal values are

$$y(T) = cA + (1 - c)B, \quad y'(T) = -1 + (1 - c)A + cB.$$

Thus rest requires

$$cA + (1 - c)B = 0, \quad (1 - c)A + cB = 1.$$

Step 3: Solve and prove complete rest. The determinant is $c^2 - (1 - c)^2 = 2c - 1 = \sqrt{2} - 1 > 0$. Consequently

$$A = -\frac{1 - c}{2c - 1} = \boxed{-\frac{1}{\sqrt{2}}}, \quad B = \frac{c}{2c - 1} = \boxed{1 + \frac{1}{\sqrt{2}}}.$$

These are the unique levels in the stated two-stage family. The formula satisfies the original IVP and has zero displacement and velocity at T . For $t > T$ the forcing is zero, so uniqueness of the homogeneous oscillator with these terminal data implies $y(t) = 0$ there. Matching gives a unique solution on the entire half-line.

Step 4: Check feasibility and regularity. Both weights in $cA + (1 - c)B = 0$ are positive. If $A, B \geq 0$, that equation forces $A = B = 0$, contradicting the velocity equation. A negative first level is therefore necessary within this family. The motion is C^1 , while the acceleration jumps are

$$[y'']_a = B - A = 1 + \sqrt{2}, \quad [y'']_T = -B = -1 - \frac{1}{\sqrt{2}}.$$

After T the output is identically zero, so its transform exists for $\boxed{\text{every real } s}$, despite the apparent denominators in Y .

Question 7

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An unstable first-order system is to be stopped by a two-stage input:

$$y' - y = f_B(t), \quad y(0) = 1, \quad f_B(t) = \begin{cases} -1, & 0 \leq t < 1, \\ B, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Choose $B = B_*$ so that $y(t) = 0$ for all $t \geq 2$. A realized level may instead be $B_* + \delta$, where δ is a constant error.

Tasks

1. Find $Y(s)$ and invert it using the unstable system's step response.
2. Determine B_* and the resulting piecewise output. Verify the stopping condition.
3. Derive the exact output error for $t \geq 2$ and the sharp bound on $|\delta|$ that ensures $|y(t)| \leq 0.01$ for every $2 \leq t \leq 5$.
4. Compare the exact real transform domains for perfect and imperfect levels. Explain why exact cancellation does not make the system stable.

Question 7 – Solution

Strategy. A finite input can cancel one unstable trajectory exactly. Keep the cancellation coefficient explicit to measure its sensitivity.

Step 1: Transform and invert. Here $f_B = -1 + (B + 1)H_1(t) - BH_2(t)$ and

$$Y = \frac{1 + [-1 + (B + 1)e^{-s} - Be^{-2s}]/s}{s - 1}, \quad s > 1.$$

The zero-start response to a unit step for $y' - y = 1$ is $r(t) = e^t - 1$. Adding the initial response gives

$$y = e^t - r(t) + (B + 1)H_1(t)r(t - 1) - BH_2(t)r(t - 2).$$

This formula satisfies $y(0) = 1$ and is continuous at both switches.

Step 2: Select the exact cancellation. Before 1, $y = 1$. On $[1, 2]$, $y = (B + 1)e^{t-1} - B$, so $y(2) = e + B(e - 1)$. Consequently

$$B_* = -\frac{e}{e - 1}, \quad y_*(t) = \begin{cases} 1, & 0 \leq t < 1, \\ (e - e^{t-1})/(e - 1), & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Direct substitution gives $y'_* - y_* = -1, B_*, 0$ on the three intervals, and the values match at 1 and 2. The zero terminal state remains zero under the homogeneous equation after 2.

Step 3: Quantify the error sharply. The error is zero before 1, then equals $\delta(e^{t-1} - 1)$ until 2. After 2 the homogeneous equation amplifies its terminal value:

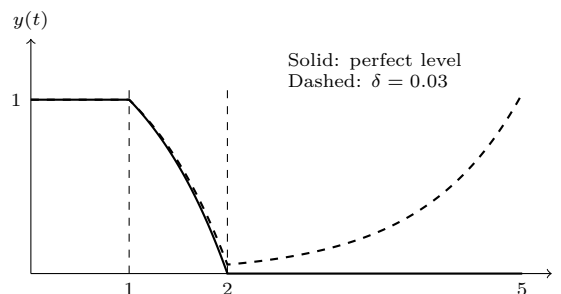
$$y(t) - y_*(t) = \delta(e - 1)e^{t-2}, \quad t \geq 2.$$

Its magnitude increases on $[2, 5]$ when $\delta \neq 0$. Thus the required necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01 e^{-3}}{e - 1}.$$

Equality attains the limit at $t = 5$.

Step 4: Distinguish trajectory cancellation from stability. The perfect output has compact support and a transform for every real s . For any $\delta \neq 0$, a nonzero e^t tail remains and the exact real domain is $s > 1$. At the perfect setting the apparent singularity at $s = 1$ is removable, as is the $s = 0$ expression after cancellations. Small level errors eventually grow without bound; selecting one compact response does not change the unstable homogeneous equation.



Question 8

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An initially unexcited system is driven by one unknown positive step:

$$y' + 2y = AH_a(t), \quad y(0) = 0, \quad A > 0, \quad a > 0.$$

Exact measurements give $y(1) = 0$, $y(2) = 1$, and $y(3) = 2$. A fourth proposed measurement is $y(4) = 3$.

Tasks

1. Derive the transformed and time-domain response for arbitrary A, a .
2. Use positivity and the measurements to locate the switch interval, then determine A and a exactly.
3. Verify that the recovered parameters satisfy all three original measurements and are unique in this one-step family.
4. Test the fourth measurement and find the limiting response. Explain the scope of the identification.

Question 8 – Solution

Strategy. The zero measurement locates the delay. Once the input is on, equally spaced samples obey a simple exponential approach to the same level.

Step 1: Solve the parameterized IVP. For $s > 0$,

$$Y = \frac{Ae^{-as}}{s(s+2)}, \quad \boxed{y(t) = \frac{A}{2}H_a(t)(1 - e^{-2(t-a)}).$$

The response is zero through a , positive for $t > a$, and continuous at a . Its derivative jumps by A , as the equation requires.

Step 2: Locate and recover the switch. Since $A > 0$, the first two measurements imply $1 \leq a < 2$. Put $r = e^{-2}$ and $C = A/2$. On each full unit interval after activation,

$$y(t+1) = ry(t) + C(1-r).$$

Thus $2 = r + C(1-r)$, and

$$\boxed{A = \frac{2(2-r)}{1-r}}.$$

Using $y(2) = 1$ gives $e^{-2(2-a)} = 1 - 1/C = 1/(2-r)$, hence

$$\boxed{a = 2 - \frac{1}{2} \ln(2 - e^{-2})}.$$

Step 3: Verify admissibility and uniqueness. Because $1 < 2 - r < 2 < e^2$, one has $0 < \ln(2 - r) < 2$, so $1 < a < 2$. Also $C = (2 - r)/(1 - r) > 0$. The recovered values give $y(1) = 0$ and

$$y(2) = C \left(1 - \frac{1}{2-r} \right) = 1, \quad y(3) = r + C(1-r) = 2.$$

The sample recurrence fixes C uniquely. Then the exponential expression for $y(2)$ is strictly monotone in a , fixing the delay uniquely. Direct substitution into the response verifies the IVP on both sides of the switch.

Step 4: Test the extra datum and interpret. The same recurrence predicts

$$\boxed{y(4) = 2r + C(1-r) = 2 + e^{-2} < 3.}$$

The fourth datum is inconsistent with the stated model. The limiting level is $\boxed{C = (2 - e^{-2})/(1 - e^{-2})}$, approached from below after activation. Its positive constant tail gives exact real transform domain $s > 0$. Uniqueness here concerns the specified single positive step with known system coefficient; finitely many samples do not identify an arbitrary forcing history.

Question 9

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Consider two qualitatively different input changes:

$$y'' + 3y' + 2y = H_1(t) + (t - 2)H_2(t), \quad y(0) = y'(0) = 0.$$

One claim says that the jump at 1 makes y jump; another says that because the ramp is continuous at 2, the solution must be smooth to every order there.

Tasks

1. Find $Y(s)$ and invert it explicitly using the step and ramp responses.
2. Verify the original IVP and determine the jumps in y, y', y'', y''' at 1 and 2.
3. Derive the leading small- τ behavior of each newly activated response. Relate the orders of onset to the jump calculations.
4. Evaluate both claims and explain whether a different assigned value of $H_a(a)$ changes the response.

Question 9 – Solution

Strategy. A second-order system smooths a step more than the input itself, but only by a finite number of derivatives. Check one-sided limits directly.

Step 1: Transform and invert. With zero data,

$$Y = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{e^{-2s}}{s^2(s+1)(s+2)}.$$

Partial fractions give the two zero-start responses

$$r(\tau) = \frac{1}{2} - e^{-\tau} + \frac{1}{2}e^{-2\tau}, \quad p(\tau) = \frac{\tau}{2} - \frac{3}{4} + e^{-\tau} - \frac{1}{4}e^{-2\tau},$$

so $y = H_1(t)r(t-1) + H_2(t)p(t-2)$. Their transforms are the two unshifted rational factors displayed above.

Step 2: Verify the equation and all jumps. Direct differentiation gives $r'' + 3r' + 2r = 1$ and $p'' + 3p' + 2p = \tau$. Moreover $r(0) = r'(0) = p(0) = p'(0) = 0$, verifying the initial data and C^1 matching. For the next derivatives,

$$r''(0) = 1, \quad r'''(0) = -3, \quad p''(0) = 0, \quad p'''(0) = 1.$$

The previously active response is smooth at the next switch. Thus

switch	$[y]$	$[y']$	$[y'']$	$[y''']$
1	0	0	1	-3
2	0	0	0	1

These are jumps of one-sided ordinary derivatives, not assigned values at the switch.

Step 3: Relate onset orders to regularity. Taylor expansion of the newly added responses gives

$$r(\tau) = \frac{1}{2}\tau^2 - \frac{1}{2}\tau^3 + O(\tau^4), \quad p(\tau) = \frac{1}{6}\tau^3 + O(\tau^4), \quad \tau \downarrow 0.$$

Joining the first to zero preserves one derivative but creates a second- derivative jump. Joining the second to zero preserves two derivatives but creates a third-derivative jump. These leading coefficients reproduce the jump sizes above.

Step 4: Resolve both claims and the convention. The first claim is false: y and y' stay continuous at 1, while y'' jumps. The second is also false: y is C^2 near 2 but not C^3 there. Changing an input at one isolated time does not change its Laplace integral or the solution obtained by continuous state matching. At switches the equation is interpreted through its one-sided limits, not by demanding an unavailable classical derivative. Away from them the solution is smooth. The growing linear tail gives exact real transform domain $s > 0$.

Question 10

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A unit input is switched on and off repeatedly:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 1, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2, \end{cases} \quad k = 0, 1, 2, \dots$$

Use only step functions and a geometric series; no periodic-transform formula is needed.

Tasks

1. Represent f by a locally finite sum of steps, derive $F(s)$ for $s > 0$, and obtain $Y(s)$ and a delayed-response series.
2. For $x_k = y(2k)$, derive and solve a recurrence for the samples.
3. Find the attracting two-periodic response explicitly on one cycle and prove the exact transient error for all $t \geq 0$.
4. Find its cycle minimum, maximum and mean. Does $y(t)$ have a limit? Find the earliest $T \geq 0$ for which the error from the periodic response is at most 0.01 for every $t \geq T$.

Question 10 – Solution

Strategy. The step series gives the transient solution, while the cycle map identifies a periodic state and its exact attraction rate.

Step 1: Transform the repeated switches. At each fixed time only finitely many terms are active:

$$f = \sum_{k=0}^{\infty} (H_{2k}(t) - H_{2k+1}(t)), \quad F = \frac{1 - e^{-s}}{s} \sum_{k=0}^{\infty} e^{-2ks} = \frac{1}{s(1 + e^{-s})}, \quad s > 0.$$

Integration of the nonnegative disjoint pulses justifies the sum. With $r(t) = 1 - e^{-t}$ for $t \geq 0$,

$$Y = \frac{1}{s(s+1)(1+e^{-s})}, \quad y = \sum_{k=0}^{\infty} [H_{2k}(t)r(t-2k) - H_{2k+1}(t)r(t-2k-1)].$$

This time series is locally finite; each activated response starts at zero and satisfies the required first-order equation on its active intervals.

Step 2: Solve the cycle recurrence. During the on-half, $y(2k+1) = 1 + (x_k - 1)e^{-1}$. During the off-half this value is multiplied by e^{-1} , so

$$x_{k+1} = e^{-2}x_k + e^{-1}(1 - e^{-1}), \quad \boxed{x_k = m(1 - e^{-2k}), \quad m = \frac{1}{e+1}},$$

using $x_0 = 0$. The cycle map has the unique fixed point m .

Step 3: Build the periodic state and exact error. Write $t = 2k + \tau$, $0 \leq \tau < 2$, and put $M = e/(e+1)$. Then

$$y_{\text{per}}(t) = \begin{cases} 1 - (1 - m)e^{-\tau}, & 0 \leq \tau < 1, \\ Me^{-(\tau-1)}, & 1 \leq \tau < 2. \end{cases}$$

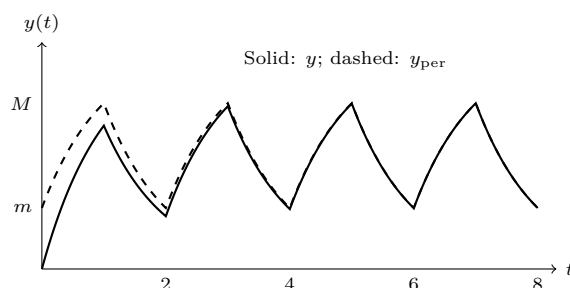
The pieces match at M , and the value at the end of the cycle is m . Both solutions have the same forcing, so their difference satisfies $d' + d = 0$ with $d(0) = m$. Continuity propagates this identity across every switch:

$$\boxed{y_{\text{per}}(t) - y(t) = me^{-t}, \quad t \geq 0.}$$

Step 4: Interpret the limiting cycle and tolerance. The periodic minimum is m and maximum is M . Integrating its equation over a cycle gives $\int_0^2 y_{\text{per}} dt = 1$, so its mean is $1/2$. The even and odd integer samples of y tend to the different values m, M ; there is no single limit. The error decreases strictly, giving the sharp time

$$\boxed{T = \ln\left(\frac{100}{e+1}\right)}.$$

Both responses have exact real transform domain $s > 0$, since their positive persistent cycles prevent convergence at or below zero.



Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Let $a, J > 0$ and $\varepsilon > 0$. A finite pulse and its impulse idealization drive

$$y'_\varepsilon + 2y_\varepsilon = \frac{J}{\varepsilon}[H_a(t) - H_{a+\varepsilon}(t)], \quad y' + 2y = J\delta(t-a), \quad y_\varepsilon(0) = y(0) = 0.$$

Tasks

1. Prove that the finite pulse tends to $J\delta(t-a)$ when integrated against a continuous test function. Derive the limiting input transform.
2. Find the impulsive response, including its jump, and solve the finite-pulse IVP explicitly.
3. For $t \geq a + \varepsilon$, prove $0 \leq y_\varepsilon(t) - y(t) \leq J\varepsilon$.
4. Describe pointwise convergence, including the exceptional time a , and decide whether convergence can be uniform on an interval containing a in its interior.

Question 1 – Solution

Strategy. Preserve pulse area rather than height. Compare the two exact responses before making any claim about convergence.

Step 1: Test the pulse and transform. Its action on a continuous ϕ is

$$\frac{J}{\varepsilon} \int_a^{a+\varepsilon} \phi(t) dt \longrightarrow J\phi(a).$$

The error is at most $J \sup_{[a, a+\varepsilon]} |\phi(t) - \phi(a)|$, which tends to zero. Taking $\phi = e^{-st}$ gives $F_\varepsilon = J e^{-as}(1 - e^{-\varepsilon s})/(\varepsilon s) \rightarrow J e^{-as}$, with the removable value $F_\varepsilon(0) = J$.

Step 2: Solve both equations. The impulse gives $Y = J e^{-as}/(s+2)$, so $y = J H_a(t) e^{-2(t-a)}$ and $[y]_a = J$. The finite-pulse solution is continuous and equals

$$y_\varepsilon(t) = \begin{cases} 0, & t < a, \\ \frac{J}{2\varepsilon} (1 - e^{-2(t-a)}), & a \leq t < a + \varepsilon, \\ \frac{J}{2\varepsilon} (1 - e^{-2\varepsilon}) e^{-2(t-a-\varepsilon)}, & t \geq a + \varepsilon. \end{cases}$$

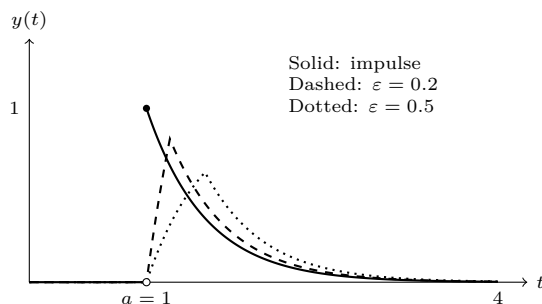
Each piece satisfies its ordinary equation, and the endpoints match.

Step 3: Bound the tail error. Put $\tau = t - a \geq \varepsilon$. The explicit tail can be rewritten as $y_\varepsilon = (J/\varepsilon) \int_0^\varepsilon e^{-2(\tau-u)} du$. Since $x \mapsto e^{-2x}$ is decreasing and has derivative magnitude at most 2 for $x \geq 0$,

$$0 \leq y_\varepsilon - y = \frac{J}{\varepsilon} \int_0^\varepsilon [e^{-2(\tau-u)} - e^{-2\tau}] du \leq \frac{J}{\varepsilon} \int_0^\varepsilon 2u du = J\varepsilon.$$

This representation follows directly from the finite-pulse formula.

Step 4: Distinguish the kinds of convergence. For each fixed $t \neq a$, $y_\varepsilon(t) \rightarrow y(t)$. At a , $y_\varepsilon(a) = 0$ but the right-hand representative has $y(a) = J$. Uniform convergence on an interval containing a is impossible: the approximants are continuous and the limit has a jump. Changing only $y(a)$ does not remove that jump or change its transform. The graph uses $a = J = 1$; open and filled dots mark the impulsive response.



Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Consider a weighted impulse and a reversed impulse argument:

$$y' + y = (t^2 - 1)\delta(2t - 4) + 3\delta(3 - t), \quad y(0) = 2.$$

You may use $\delta(c(t-a)) = \delta(t-a)/|c|$ for $c \neq 0$.

Tasks

1. Reduce the forcing to impulses at named times with constant weights. Explain the absolute value in the scaling rule.
2. Find $Y(s)$ and the complete time response, including the initial state.
3. Determine both jumps and verify the ordinary equation on each interval.
4. Compute the response area and its exact real transform domain. Explain why the response area is not just the sum of the impulse weights.

Question 2 – Solution

Strategy. Scale the delta first, then evaluate its smooth multiplier at the impulse time. The initial state contributes its own response area.

Step 1: Normalize and sift. Here $\delta(2t - 4) = \delta(t - 2)/2$ and $\delta(3 - t) = \delta(t - 3)$. The sampling property gives

$$f(t) = \frac{3}{2}\delta(t - 2) + 3\delta(t - 3).$$

The first weight is $(2^2 - 1)/2 = 3/2$. Reversing an argument does not turn a positive unit mass into a negative one: changing integration variables reverses the limits as well as introducing the negative derivative. Thus the scale factor is $1/|c|$.

Step 2: Transform and invert. The initial term is retained:

$$(s + 1)Y - 2 = \frac{3}{2}e^{-2s} + 3e^{-3s}, \quad Y = \frac{2 + \frac{3}{2}e^{-2s} + 3e^{-3s}}{s + 1}.$$

Consequently

$$y(t) = 2e^{-t} + \frac{3}{2}H_2(t)e^{-(t-2)} + 3H_3(t)e^{-(t-3)}.$$

Step 3: Verify the jumps and intervals. Every active exponential satisfies $y' + y = 0$ away from 2 and 3. The newly activated terms give

$$[y]_2 = \frac{3}{2}, \quad [y]_3 = 3, \quad y(0) = 2.$$

Equivalently, integrating the equation across either impulse makes the bounded y integral vanish as the interval shrinks, leaving the specified jump. Ordinary uniqueness on each interval and these jumps determine the whole response.

Step 4: Account for the initial stored response. Each shifted exponential of coefficient C has integral C . Hence

$$\int_0^\infty y(t) dt = 2 + \frac{3}{2} + 3 = \frac{13}{2}.$$

Integrating the impulsive equation over the half-line gives $y(\infty) - y(0) + \int_0^\infty y dt = 9/2$ and the same result. The derivative contribution is -2 , not zero. The positive nonzero e^{-t} tail gives exact real domain $s > -1$.

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An oscillator moving through equilibrium receives an impulse:

$$y'' + 4y = 6\delta(t - a), \quad y(0) = 1, \quad y'(0) = 0, \quad a = \frac{\pi}{4}.$$

For unit mass, use $E = (y'^2 + 4y^2)/2$. Ordinary impulses in a second-order equation keep displacement continuous but may jump velocity.

Tasks

1. Derive the velocity jump and find $Y(s)$ and $y(t)$.
2. Compute the states immediately before and after the impulse and simplify the subsequent motion.
3. Find the energy change and derive the general energy-change formula for an impulse of strength J at this same time.
4. For $J > 0$, classify when the impulse lowers the energy, preserves it, or raises it. Identify the unique strength that brings this oscillator completely to rest.

Question 3 – Solution

Strategy. The impulse changes velocity, not displacement. Its work depends on the velocity immediately before the kick as well as its strength.

Step 1: Jump, transform and invert. Integrating across a gives $[y']_a = 6$, because the integral of $4y$ tends to zero. With $[y]_a = 0$, the transformed equation is

$$Y = \frac{s + 6e^{-as}}{s^2 + 4}, \quad \boxed{y(t) = \cos 2t + 3H_a(t) \sin(2(t-a))}.$$

The new term vanishes at activation, while its derivative starts at 6. Both pieces satisfy $y'' + 4y = 0$ away from a and retain the initial data.

Step 2: Evaluate the state at impact. Since $2a = \pi/2$,

$$(y(a-), y'(a-)) = (0, -2), \quad (y(a+), y'(a+)) = \boxed{(0, 4)}.$$

For $t = a + \tau$, the pre-existing response is $-\sin 2\tau$, so the total post-impact motion is $\boxed{y(a + \tau) = 2 \sin 2\tau}$. Thus the impulse reverses the direction and doubles the amplitude.

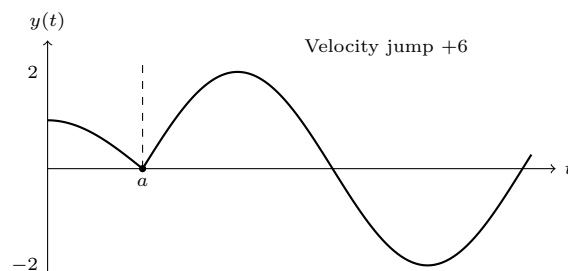
Step 3: Compute the energy transfer. The energies are $E_- = 2$ and $E_+ = 8$, giving $\boxed{\Delta E = 6}$. For any impulse J , continuity of position and $v_+ = v_- + J$ imply

$$\Delta E = \frac{1}{2}[(v_- + J)^2 - v_-^2] = Jv_- + \frac{1}{2}J^2.$$

At this impact time $v_- = -2$, hence

$$\boxed{\Delta E = \frac{1}{2}J(J - 4), \quad E_+ = \frac{1}{2}(J - 2)^2}.$$

Step 4: Classify the positive kicks. For $0 < J < 4$ the energy decreases; at $J = 4$ it is unchanged; for $J > 4$ it increases. A positive impulse can remove energy when it opposes the incoming velocity. Complete rest requires both post-impact state coordinates to vanish. Position is already zero here, so the unique strength is $\boxed{J = 2}$, which makes velocity zero. A zero velocity kick at a nonzero displacement would not in general leave an oscillator at rest. The plot shows the stated $J = 6$ case, with continuous displacement and a corner.



Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A critically damped system starts from rest and receives two impulses:

$$y'' + 2y' + y = \delta(t-1) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad b > 1.$$

You may choose b and K . The energy is $E = (y'^2 + y^2)/2$.

Tasks

1. Find $Y(s)$ and the response in terms of the age since each impulse.
2. Determine whether any finite $b > 1$ and K can make $y(t) = 0$ for every $t \geq b$.
3. For a fixed b , find the unique K minimizing the energy immediately after the second impulse, and find that minimum.
4. Derive the motion after this minimizing kick and explain why zero post-impact velocity does not imply complete rest.

Question 4 – Solution

Strategy. One impulse can adjust velocity only. Inspect the displacement at the chosen impact time before attempting to cancel the entire state.

Step 1: Find the impulse response. The zero-data transform is

$$Y = \frac{e^{-s} + Ke^{-bs}}{(s+1)^2}.$$

Since $\mathcal{L}\{te^{-t}\} = 1/(s+1)^2$, set $h(t) = te^{-t}$ for $t \geq 0$. Then

$$y = H_1(t)h(t-1) + KH_b(t)h(t-b).$$

Here $h(0) = 0$, $h'(0) = 1$, and $h'' + 2h' + h = 0$ for positive age. Thus displacement is continuous, and velocity jumps by 1 and K .

Step 2: Test the terminal displacement. Put $d = b - 1 > 0$. Immediately before the second kick,

$$y_b = de^{-d} > 0, \quad v_- = (1-d)e^{-d}.$$

The second impulse leaves y_b unchanged. A solution identically zero after b must have zero displacement there, so no finite $b > 1$ and K can produce complete rest. The strict positivity of $h(d)$ is the obstruction, not a shortage of algebraic manipulation.

Step 3: Minimize the post-impact energy. For fixed b ,

$$E_+(K) = \frac{1}{2}[(v_- + K)^2 + y_b^2].$$

This strictly convex quadratic has the unique minimizer and value

$$K_* = (d-1)e^{-d}, \quad E_{\min} = \frac{1}{2}d^2e^{-2d} > 0.$$

The minimizing kick makes $v_+ = 0$ but leaves positive stored spring energy.

Step 4: Follow the resulting motion. For $\tau = t - b \geq 0$, the critical homogeneous solution with state $(y_b, 0)$ is

$$y(b+\tau) = de^{-d}(1+\tau)e^{-\tau}.$$

Its derivative is $-de^{-d}\tau e^{-\tau}$: it starts horizontal and then moves strictly downward. Its acceleration at $\tau = 0+$ is $-y_b \neq 0$. Substitution verifies the equation and the terminal state. Thus the system eventually approaches rest, but is not at rest throughout the post-impact interval. The positive polynomial-exponential tail has transform domain $s > -1$.

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Two impulses act on an initially resting undamped oscillator:

$$y'' + \omega^2 y = I\delta(t-a) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad I, \omega > 0, \quad 0 < a < b.$$

Require complete rest after the second impulse.

Tasks

1. Find the transformed response and the time-domain solution.
2. Classify all delays $b-a$ and strengths K that meet the rest condition.
3. If K must be positive, find the earliest possible second impulse. Explain why the net impulse need not be zero.
4. Keep this earliest positive strength, but deliver it at $b+\eta$, where $|\eta| < \pi/\omega$. Find the residual amplitude and its leading behavior as $\eta \rightarrow 0$.

Question 5 – Solution

Strategy. The second kick must occur at a zero of displacement, then cancel the incoming velocity. Timing errors leave a measurable residual oscillation.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{Ie^{-as} + Ke^{-bs}}{s^2 + \omega^2}, \quad \boxed{y = \frac{I}{\omega}H_a(t) \sin \omega(t - a) + \frac{K}{\omega}H_b(t) \sin \omega(t - b).}$$

Each activated sine is zero initially, with derivative equal to its impulse strength. Thus y is continuous and $[y']_a = I$, $[y']_b = K$.

Step 2: Impose both state conditions. For $d = b - a$, the pre-impact state is

$$y(b-) = \frac{I}{\omega} \sin \omega d, \quad y'(b-) = I \cos \omega d.$$

Displacement cannot jump. Therefore complete rest requires and is ensured by

$$\boxed{d = \frac{n\pi}{\omega}, \quad K = (-1)^{n+1}I, \quad n = 1, 2, \dots}$$

After these jumps the homogeneous oscillator has zero data, so it remains zero. The corresponding output has compact support and a transform for every real s .

Step 3: Restrict the second impulse to be positive. Positive K requires odd n . The earliest choice is $\boxed{b = a + \pi/\omega, \ K = I}$. The net applied impulse is $2I$, yet the oscillator stops: during the intervening motion the restoring force also changes momentum. Indeed $\int_a^b \omega^2 y \, dt = 2I$, balancing the applied impulses in the integrated equation. Ignoring this force would give an incorrect momentum test.

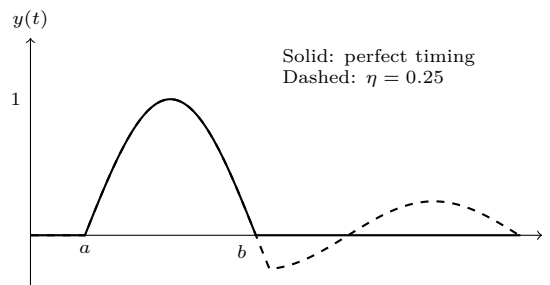
Step 4: Quantify a timing error. At the actual second kick the age is $\pi/\omega + \eta > 0$. The state immediately afterward is

$$y_+ = -\frac{I}{\omega} \sin \omega \eta, \quad v_+ = I(1 - \cos \omega \eta).$$

The subsequent homogeneous amplitude is

$$R = \sqrt{y_+^2 + (v_+/\omega)^2} = \boxed{\frac{2I}{\omega} |\sin(\omega\eta/2)|} = I|\eta| + O(|\eta|^3).$$

For every nonzero η in the stated interval this amplitude is positive, so the tail's exact real transform domain is $s > 0$. The plot compares perfect timing and $\eta = 0.25$ for $I = \omega = a = 1$.



Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A single unknown impulse drives a first-order system:

$$y' + \lambda y = J\delta(t-a), \quad y(0) = 0, \quad \lambda, J > 0, \quad 0 < a < 2.$$

Exact observations give $y(2) = 4$ and $y(3) = 2$. An additional measurement is the total response area $\int_0^\infty y(t) dt = 8$.

Tasks

1. Derive the transform and response for general λ, J, a .
2. Use only the two point observations to determine what is identifiable. Parameterize every admissible triple consistent with them.
3. Use the area measurement to recover all three parameters uniquely, and verify the delay restriction.
4. Check all measurements directly and explain why the extra area measurement resolves an ambiguity that another later point sample would not.

Question 6 – Solution

Strategy. Post-impulse samples determine a decay rate and an effective amplitude. The area determines the impulse strength independently of its delay.

Step 1: Solve the parameterized problem. The zero datum gives

$$Y = \frac{J e^{-as}}{s + \lambda}, \quad \boxed{y = J H_a(t) e^{-\lambda(t-a)}}.$$

The response has jump J at a , solves $y' + \lambda y = 0$ afterward, and has area J/λ . Its exact real transform domain is $s > -\lambda$.

Step 2: Identify the point-sample information. Both observations occur after the impulse. Taking their ratio gives $e^{-\lambda} = 1/2$, so $\lambda = \ln 2$. The first observation then gives $J 2^{a-2} = 4$, or

$$\boxed{\lambda = \ln 2, \quad J = 16 \cdot 2^{-a}, \quad 0 < a < 2.}$$

This is a full one-parameter family, not a unique impulse time. The strength ranges over $4 < J < 16$.

Step 3: Use the area and prove admissibility. The area condition now fixes $J/\ln 2 = 8$. Thus

$$\boxed{J = 8 \ln 2, \quad a = \frac{\ln(2/\ln 2)}{\ln 2}, \quad \lambda = \ln 2.}$$

Since $1/2 < \ln 2 < 1$, we have $1 < 2/\ln 2 < 4$. Taking logarithms to base 2 shows $0 < a < 2$. The area fixes J uniquely, and $J = 16 \cdot 2^{-a}$ is strictly decreasing in a , so this recovered triple is unique in the stated family.

Step 4: Verify and interpret. The recovered delay satisfies $2^a = 2/\ln 2$. Therefore

$$y(2) = (8 \ln 2) \frac{2/\ln 2}{4} = 4, \quad y(3) = \frac{1}{2} y(2) = 2, \quad \int_0^\infty y \, dt = \frac{8 \ln 2}{\ln 2} = 8.$$

For every member of the two-sample family, all later values are $y(t) = 16 \cdot 2^{-t}$ for $t \geq 2$. Thus another exact sample after 2 would either agree with every member or reject every member; it could not separate strength from delay. The total area includes the unsampled interval after the unknown impulse and supplies the missing information.

Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An unstable second-order system is given one corrective impulse:

$$y'' - y = J\delta(t-a), \quad y(0) = 1, \quad y'(0) = 0, \quad a > 0.$$

Choose J to make the response decay to zero after the impulse. A realized impulse may differ by a constant error Δ .

Tasks

1. Derive $Y(s)$ and $y(t)$ for arbitrary J , including the initial response.
2. Find the unique decaying choice J_* , its explicit tail and the global maximum of the selected response.
3. Derive the response error for strength $J_* + \Delta$ and a sharp bound on $|\Delta|$ ensuring error at most $\epsilon > 0$ throughout $a \leq t \leq a + L$, where $L > 0$.
4. Compare exact real transform domains for the selected and perturbed strengths. Explain the effect of cancellation on the system's stability.

Question 7 – Solution

Strategy. Cancel the growing mode's coefficient, then inspect the remaining mode and the error produced by imperfect cancellation.

Step 1: Transform and invert. For sufficiently large s ,

$$(s^2 - 1)Y - s = Je^{-as}, \quad Y = \frac{s + Je^{-as}}{s^2 - 1}, \quad \boxed{y = \cosh t + JH_a(t) \sinh(t - a)}.$$

The added term starts at zero with derivative J , so it preserves displacement and produces the correct velocity jump. Away from a both terms solve the homogeneous equation.

Step 2: Select and interpret the decaying mode. After a , the e^t coefficient is $(1 + Je^{-a})/2$. Decay requires

$$\boxed{J_* = -e^a, \quad y_*(t) = \begin{cases} \cosh t, & 0 \leq t < a, \\ \cosh a e^{-(t-a)}, & t \geq a. \end{cases}}$$

The pre-impact velocity is $\sinh a$ and the post-impact velocity is $-\cosh a$; their difference is $-e^a$, as required. The response increases strictly before a and decreases strictly after it. Its unique global maximum is $\boxed{y_*(a) = \cosh a}$.

Step 3: Quantify the sensitivity. Linearity or direct subtraction gives

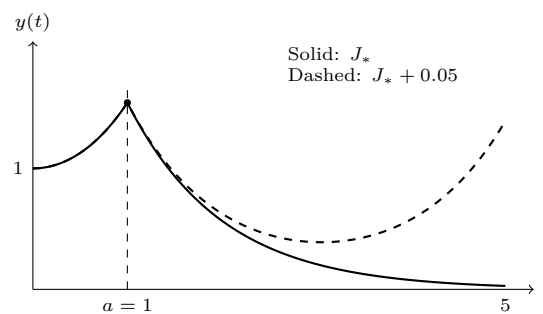
$$\boxed{y(t) - y_*(t) = \Delta H_a(t) \sinh(t - a)}.$$

Because $\sinh \tau$ increases strictly on $[0, L]$, the desired error bound is equivalent to

$$\boxed{|\Delta| \leq \frac{\epsilon}{\sinh L}}.$$

This is necessary as well as sufficient, and equality attains the bound at $a + L$. The selected solution still approaches zero only asymptotically; the impulse does not put it at the zero state.

Step 4: Check convergence and stability. The selected tail is a positive multiple of e^{-t} , so its exact real transform domain is $\boxed{s > -1}$. The apparent singularity at $s = 1$ in its transform is removable. For every $\Delta \neq 0$, the error contains a nonzero growing mode and the actual domain becomes $\boxed{s > 1}$. The differential equation's unstable homogeneous mode is unchanged. The plot uses $a = 1$, with a dashed curve for $\Delta = 0.05$.



Question 8

Interpret the equation through ordinary motion away from the impulse and its jump conditions. For this question only, the initial-time transform convention is specified below; write $[v]_0 = v(0^+) - v(0^-)$ for a jump.

Use a *full-mass initial impulse* convention: the transform starts at 0^- , $\mathcal{L}\{\delta(t)\} = 1$, and

$$\mathcal{L}\{y'\} = sY - y(0^-), \quad \mathcal{L}\{y''\} = s^2Y - sy(0^-) - y'(0^-).$$

Consider

$$2y'' + 4y' + 2y = J\delta(t), \quad y(0^-) = 1, \quad y'(0^-) = 2.$$

These are pre-impact data, not post-impact data.

Tasks

1. Find $y(0^+)$ and $y'(0^+)$ by the impulse jump conditions.
2. Use the stated 0^- convention to find $Y(s)$ and the response for $t \geq 0$.
3. Reproduce the same response by solving a regular homogeneous IVP from 0^+ . Explain why keeping the impulse again would double-count it.
4. Classify every J for which the post-impact response is nonnegative and nonincreasing for all $t \geq 0$. Describe the boundary cases.

Question 8 – Solution

Strategy. An impulse at the initial time requires a declared convention. Either transform pre-impact data with the impulse or restart after the jump.

Step 1: Compute the post-impact data. There is no displacement jump under an ordinary force impulse. Integrating the equation across the origin gives $2[y']_0 = J$; the damping contribution is $4[y]_0 = 0$. Thus

$$y(0^+) = 1, \quad y'(0^+) = 2 + \frac{J}{2}.$$

The factor 2 is the mass and must not be omitted.

Step 2: Transform the pre-impact formulation. The stipulated derivative rules give

$$2(s^2Y - s - 2) + 4(sY - 1) + 2Y = J,$$

so

$$Y = \frac{2s + 8 + J}{2(s + 1)^2} = \frac{1}{s + 1} + \frac{3 + J/2}{(s + 1)^2}.$$

Writing $c = 3 + J/2$, the inverse is $y(t) = e^{-t}(1 + ct)$, $t \geq 0$. It has $y(0^+) = 1$ and $y'(0^+) = c - 1 = 2 + J/2$, exactly as the jump calculation.

Step 3: Check the equivalent restart. For $t > 0$ the equation is $y'' + 2y' + y = 0$. Using the post-impact data $1, 2 + J/2$ gives the critical solution $e^{-t}[1 + (3 + J/2)t]$ again. This restart has no remaining initial impulse: its effect is already encoded in the velocity. Inserting $J\delta(t)$ again with those data would add a second velocity change $J/2$.

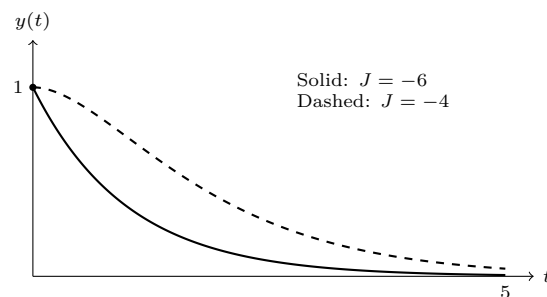
Step 4: Classify positivity and monotonicity. Since the exponential is positive, nonnegativity for every $t \geq 0$ requires and is ensured by $c \geq 0$. Also

$$y' = e^{-t}(c - 1 - ct).$$

For $c \geq 0$, this is nonpositive for all $t \geq 0$ exactly when $c \leq 1$. Therefore

$$0 \leq c \leq 1 \iff -6 \leq J \leq -4.$$

At $J = -6$, $y = e^{-t}$; at $J = -4$, $y = (1 + t)e^{-t}$ starts horizontal and then decreases strictly. For $J < -6$ the response eventually becomes negative, and for $J > -4$ it initially increases. All these nonzero polynomial-exponential responses have exact real transform domain $s > -1$.



Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An actuator may use ordinary impulses at 1 and 3, with no other force:

$$y'' + 2y' + y = J_1\delta(t-1) + J_3\delta(t-3), \quad y(0) = y'(0) = 0.$$

The prescribed response has the form

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ (t-1)e^{-(t-1)}, & 1 \leq t < 3, \\ Ce^{-(t-3)}, & t \geq 3. \end{cases}$$

An ordinary impulse in this equation can jump velocity but not displacement.

Tasks

1. Determine the only possible value of C under the permitted actuator.
2. Recover J_1, J_3 from the one-sided states, verifying that no ordinary forcing is needed between impulses.
3. Find $Y(s)$, invert it, and confirm the prescribed tail. Decide whether choosing $C = 0$ could shut the motion off instantaneously with an ordinary impulse.
4. Verify the total response area from the time formula and the integrated impulsive equation, and state the exact real transform domain.

Question 9 – Solution

Strategy. Infer the required impulses from velocity jumps after enforcing displacement continuity. Then independently check the recovered forcing by inversion.

Step 1: Enforce continuity at the second impulse. The target displacement just before 3 is $2e^{-2}$, while the right-hand value is C . Hence the permitted actuator requires

$$\boxed{C = 2e^{-2}}.$$

The first junction is already continuous because $(t-1)e^{-(t-1)}$ vanishes at 1.

Step 2: Recover the impulse strengths. For $h(\tau) = \tau e^{-\tau}$, $h'(0) = 1$ and $h'(2) = -e^{-2}$. Thus the first velocity jump is 1. At the second switch the new tail has velocity $-C = -2e^{-2}$, so

$$\boxed{J_1 = 1, \quad J_3 = -2e^{-2} - (-e^{-2}) = -e^{-2}}.$$

Because y is continuous, integrating the equation across a junction gives $J_j = [y']_j$. Direct differentiation of each piece gives $y'' + 2y' + y = 0$ on all three open intervals.

Step 3: Verify by transformation and inversion. The recovered input gives

$$Y = \frac{e^{-s} - e^{-2}e^{-3s}}{(s+1)^2}, \quad y = H_1(t)(t-1)e^{-(t-1)} - e^{-2}H_3(t)(t-3)e^{-(t-3)}.$$

For $t = 3 + \tau$, the last expression reduces to

$$e^{-2}[(\tau+2) - \tau]e^{-\tau} = 2e^{-2}e^{-\tau},$$

which is the prescribed tail. Setting $C = 0$ instead would require a jump of $-2e^{-2}$ in displacement at 3, forbidden for the stated impulse-only actuator. Canceling velocity cannot instantaneously erase nonzero displacement.

Step 4: Check the area balance. Direct integration gives

$$\int_0^\infty y \, dt = \int_0^2 \tau e^{-\tau} \, d\tau + 2e^{-2} \int_0^\infty e^{-\tau} \, d\tau = 1 - 3e^{-2} + 2e^{-2} = \boxed{1 - e^{-2}}.$$

The integrated derivative terms are zero when their impulse jumps are included: initial and limiting displacement and velocity are zero. Thus $\int y = J_1 + J_3 = 1 - e^{-2}$ independently confirms the result. The nonzero e^{-t} tail gives exact real domain $\boxed{s > -1}$.

Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A leaky system receives a sequence of equal positive impulses:

$$y' + \kappa y = I \sum_{n=1}^{\infty} \delta(t - nT), \quad y(0) = 0, \quad I, \kappa, T > 0.$$

The first impulse is at T , not at zero. Use right-hand response values at each impulse.

Tasks

1. Derive $Y(s)$ by a geometric series and give the locally finite time-domain sum.
2. For $x_n = y(nT^+)$ and $x_0 = 0$, derive and solve the recurrence for post-impulse values.
3. Find the attracting periodic profile and the exact transient error. State its peak, pre-impulse trough and cycle mean, distinguishing a left limit from an attained value.
4. For $I = \kappa = 1$, classify all T for which the steady peak is at most 2 while the steady cycle mean is at least 1. State whether $y(t)$ has a single long-time limit.

Question 10 – Solution

Strategy. Each impulse adds I to the state, while the gap between impulses multiplies it by an exponential. The cycle map determines the limiting profile.

Step 1: Sum the transforms and invert. For $s > 0$, the impulse series converges geometrically:

$$(s + \kappa)Y = I \sum_{n=1}^{\infty} e^{-nTs} = \frac{Ie^{-Ts}}{1 - e^{-Ts}}, \quad Y = \frac{Ie^{-Ts}}{(s + \kappa)(1 - e^{-Ts})}.$$

Thus

$$y(t) = I \sum_{n=1}^{\infty} H_{nT}(t) e^{-\kappa(t-nT)}.$$

Only finitely many terms are active at each time. The formula gives zero before T , jumps of I , and $y' + \kappa y = 0$ between impulses.

Step 2: Solve the post-impulse recurrence. Put $q = e^{-\kappa T} \in (0, 1)$. Decay followed by a jump gives

$$x_n = qx_{n-1} + I, \quad x_n = \frac{I(1 - q^n)}{1 - q}, \quad n \geq 0.$$

The steady post-impulse level is $P = I/(1 - q)$.

Step 3: Construct the profile and compare. For $t = nT + \tau$, $0 \leq \tau < T$, define $y_{\text{per}}(t) = Pe^{-\kappa\tau}$. Then $y(t) = x_n e^{-\kappa\tau}$, including $n = 0$, and

$$y_{\text{per}}(t) - y(t) = Pe^{-\kappa t}.$$

The profile's peak P is attained just after each impulse. Its pre-impulse trough is the left limit qP ; under the right-hand convention this infimum is not attained within a cycle. Its mean is

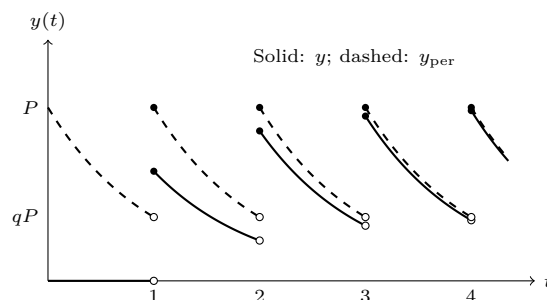
$$\frac{1}{T} \int_0^T Pe^{-\kappa\tau} d\tau = \frac{I}{\kappa T}.$$

The jump $P - qP = I$ is preserved in the limit, not smoothed away.

Step 4: Design a feasible spacing and interpret. With $I = \kappa = 1$, the peak constraint gives $1/(1 - e^{-T}) \leq 2$, equivalent to $T \geq \ln 2$. The mean constraint gives $1/T \geq 1$, equivalent to $T \leq 1$. Therefore

$$\ln 2 \leq T \leq 1.$$

There is no single long-time limit: samples at nT tend to P , while samples at $nT + T/2$ tend to the different value $Pe^{-\kappa T/2}$. Both responses have exact real transform domain $s > 0$ because their persistent positive cycles prevent convergence at or below zero. The plot uses $I = \kappa = T = 1$ and marks the jump endpoints explicitly.



Question 1

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ for $t \geq 0$. A student claims that multiplying their transforms gives the transform of their pointwise product.

Tasks

1. Compute $c = f * g$ directly and prove $f * g = g * f$ by a change of variable.
2. Find the pointwise product $p = fg$. Compare $c(0)$, $p(0)$ and their transforms to assess the claim.
3. Derive the convolution theorem for these functions by changing variables in a double integral. State a real range of s that justifies the interchange of integrals.
4. Find the unique global maximum of c , its total area, and its exact real transform domain. Explain why the convolution initially vanishes even though both factors are positive at zero.

Question 1 – Solution

Strategy. Convolution integrates over a moving interval; pointwise multiplication does not.

Step 1: Integrate and reverse the factors. For $t \geq 0$,

$$c(t) = e^{-t} \int_0^t e^{-u} du = \boxed{e^{-t} - e^{-2t}}.$$

The substitution $v = t - u$ changes $\int_0^t f(t-u)g(u) du$ to $\int_0^t g(t-v)f(v) dv$, proving commutativity directly.

Step 2: Test the student's claim. Pointwise multiplication gives $p(t) = e^{-3t}$, so $p(0) = 1$ whereas $c(0) = 0$. Their transforms are

$$P(s) = \frac{1}{s+3}, \quad C(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}.$$

The product $F(s)G(s)$ equals $C(s)$, not $P(s)$. For example, at $s = 0$ they have values $1/2$ and $1/3$ respectively.

Step 3: Derive the product formula. In the triangular region $0 \leq u \leq t < \infty$, put $v = t - u$. Its image is the quadrant $u, v \geq 0$, and its Jacobian has absolute value 1. For real $s > -1$, the resulting integrand has a finite absolute integral:

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty \int_0^t e^{-st} e^{-(t-u)} e^{-2u} du dt \\ &= \int_0^\infty \int_0^\infty e^{-(s+1)v} e^{-(s+2)u} du dv = F(s)G(s). \end{aligned}$$

Thus the interchange is justified by absolute integrability; it is not a formal rule for pointwise products.

Step 4: Locate the peak and inspect the tail. Since $c' = e^{-t}(2e^{-t} - 1)$, the derivative changes from positive to negative only at $t = \ln 2$. Consequently

$$\boxed{\max_{t \geq 0} c(t) = \frac{1}{4}, \quad \int_0^\infty c(t) dt = 1 - \frac{1}{2} = \frac{1}{2}.$$

The positive tail is asymptotic to e^{-t} , giving exact real domain $\boxed{s > -1}$; at $s = -1$ the transformed integrand tends to 1. At zero the integration interval has length zero. Finite endpoint values cannot produce a nonzero integral over that interval.

Question 2

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two delayed rectangular pulses are

$$f = H_1 - H_3, \quad g = H_2 - H_5,$$

and $w = f * g$.

Tasks

1. Express $w(t)$ as the length of an intersection of intervals, then derive a complete piecewise formula.
2. Find its transform and invert it as a sum of delayed ramps. Check agreement with the geometric formula.
3. Determine the support, all maximizing times, continuity, and the jumps of the first derivative.
4. Compute the area in two independent ways and give the exact real transform domain, including the value at $s = 0$.

Question 2 – Solution

Strategy. For unit-height rectangles, the integral measures overlap length.

Step 1: Track the overlap. Using commutativity, integrate $f(u)g(t-u)$. Both factors are 1 on $[1, 3] \cap [t-5, t-2]$, so

$$w = \max\{0, \min(3, t-2) - \max(1, t-5)\} = \begin{cases} 0, & t < 3, \\ t-3, & 3 \leq t < 5, \\ 2, & 5 \leq t < 6, \\ 8-t, & 6 \leq t < 8, \\ 0, & t \geq 8. \end{cases}$$

Endpoints of the intervals do not change their length.

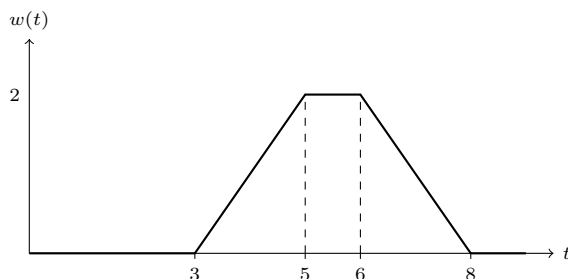
Step 2: Transform and recover the ramps. For $s \neq 0$,

$$W = \frac{e^{-3s}(1 - e^{-2s})(1 - e^{-3s})}{s^2}.$$

The inverse is $w = (t-3)H_3 - (t-5)H_5 - (t-6)H_6 + (t-8)H_8$. Activating these four ramps successively gives slopes 1, 0, -1, 0, exactly as in the piecewise formula.

Step 3: Describe the geometry and regularity. The support (closure of the nonzero set) is $[3, 8]$; $w > 0$ precisely on $(3, 8)$. Its maximum is 2, attained at every $t \in [5, 6]$. It is continuous everywhere. The first derivative has jumps $+1, -1, -1, +1$ at 3, 5, 6, 8, respectively, and is undefined there in the ordinary two-sided sense.

Step 4: Check area and convergence. Two triangles and the plateau have total area $2 + 2 + 2 = 6$. Independently, the convolution area is $(\int f)(\int g) = 2 \cdot 3 = 6$, by integrating over the rectangle in the two integration variables. Compact support makes the transform finite for every real s . The displayed expression has removable value $W(0) = 6$.



Question 3

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

An undamped oscillator receives a rapidly vanishing force:

$$y'' + y = e^{-t^2}, \quad y(0) = 2, \quad y'(0) = -1.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2 < 1$.

Tasks

1. Find $Y(s)$ in terms of $G(s) = \mathcal{L}\{e^{-t^2}\}$ and write a convolution formula for y .
2. Verify the equation and both initial conditions by differentiating your convolution.
3. Find a limiting sinusoidal expression $y_\infty(t)$ with constant coefficients and prove $|y - y_\infty| \leq e^{-t^2}/(2t)$ for $t > 0$.
4. Prove that the response does not tend to zero, despite the vanishing force. Identify which feature of the homogeneous equation makes this possible.

Question 3 – Solution

Strategy. Keep the Gaussian force inside an integral and separate the permanent oscillation from the remaining tail integral.

Step 1: Transform and invert. The initial terms give $(s^2 + 1)Y - 2s + 1 = G$, hence

$$Y = \frac{2s - 1 + G}{s^2 + 1}, \quad \boxed{y = 2 \cos t - \sin t + z(t), \quad z(t) = \int_0^t \sin(t - u)e^{-u^2} du.}$$

These transform operations are valid for real $s > 0$.

Step 2: Differentiate with the endpoint terms. The sine kernel vanishes at its endpoint, so $z' = \int_0^t \cos(t - u)e^{-u^2} du$. Differentiating again gives $z'' = e^{-t^2} - z$ because $\cos 0 = 1$. Also $z(0) = z'(0) = 0$. Thus the formula satisfies the full IVP.

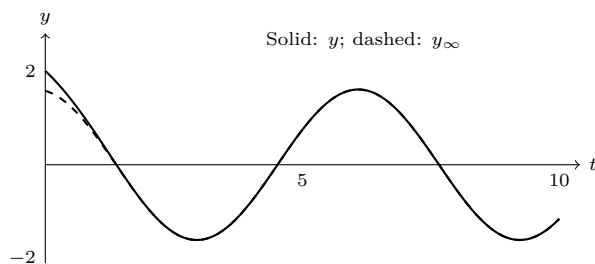
Step 3: Isolate the asymptotic oscillation. Set $A = \int_0^\infty e^{-u^2} \cos u du$ and $B = \int_0^\infty e^{-u^2} \sin u du$, both absolutely convergent. Expanding $\sin(t - u)$ gives

$$y_\infty = (2 - B) \cos t + (A - 1) \sin t, \quad y - y_\infty = - \int_t^\infty \sin(t - u)e^{-u^2} du.$$

For $t > 0$, use $1 \leq u/t$ on the tail:

$$|y - y_\infty| \leq \int_t^\infty e^{-u^2} du \leq \frac{1}{t} \int_t^\infty ue^{-u^2} du = \boxed{\frac{e^{-t^2}}{2t}}.$$

Step 4: Establish persistent motion. Since $|A| \leq \sqrt{\pi}/2 < 1$, the coefficient $A - 1$ is nonzero. The limiting sinusoid therefore has positive amplitude and has recurring positive and negative extrema. Along their times the error tends to zero, so y cannot tend to zero (or to any single constant). The homogeneous modes $\cos t, \sin t$ have no damping. The figure compares the exact integral solution with y_∞ ; the two quickly become indistinguishable.



Question 4

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let f be real and piecewise continuous on each finite interval, with $|f(t)| \leq 1$. Consider

$$y'' + 2y' + 2y = f(t), \quad y(0) = y'(0) = 0.$$

We seek the smallest constant M such that $|y(t)| \leq M$ for every admissible input and every $t \geq 0$.

Tasks

1. Find the impulse-response kernel h and express y by convolution. Compute $\int_0^\infty h$.
2. Compute $\int_0^\infty |h|$ exactly by summing half-periods, and prove it is a uniform response bound.
3. Construct an input attaining $\int_0^T |h|$ at any prescribed $T > 0$. Prove that your uniform bound is sharp, and decide whether it can be attained at a finite time.
4. Give an explicit two-level input on $[0, 2\pi]$ whose response at 2π exceeds $1/2$. Explain the error in using the signed kernel integral as the universal bound.

Question 4 – Solution

Strategy. A sign-changing kernel can reinforce a suitably chosen sign-changing input. Its signed area loses that information.

Step 1: Recover the kernel. The zero-data transform gives $Y = F/[(s+1)^2 + 1]$, so

$$h(t) = e^{-t} \sin t, \quad y = h * f, \quad \int_0^\infty h(t) dt = \frac{1}{2}.$$

For example, the last integral follows from the antiderivative $-e^{-t}(\sin t + \cos t)/2$.

Step 2: Sum the absolute areas. Put $q = e^{-\pi}$. The first positive lobe has area $A_0 = (1+q)/2$. Each later absolute lobe is q times its predecessor, so

$$M = \int_0^\infty |h| = \frac{1+q}{2(1-q)}.$$

For every admissible input, $|y(t)| \leq \int_0^t |h(t-u)| du \leq M$.

Step 3: Prove sharpness and distinguish attainment. For any $T > 0$, set

$$f_T(u) = \operatorname{sgn}(\sin(T-u)) \quad (0 \leq u \leq T), \quad f_T(u) = 0 \quad (u > T),$$

with arbitrary values in $[-1, 1]$ at zeros. This input is admissible and $y(T) = \int_0^T |h(T-u)| du = \int_0^T |h(v)| dv$. These values approach M as $T \rightarrow \infty$, so any smaller proposed constant fails for some finite T . However, the absolute kernel has a strictly positive tail after every finite T , so $|y(T)| < M$ for every input at every finite time. The universal supremum is sharp but is not attained.

Step 4: Refute the signed-area bound explicitly. Take $f = -1$ on $(0, \pi)$, $f = 1$ on $(\pi, 2\pi)$ and zero afterward. This is the preceding construction for $T = 2\pi$. Therefore

$$y(2\pi) = A_0(1+q) = \frac{(1+q)^2}{2} > \frac{1}{2}.$$

The signed integral describes cancellation within the kernel, as for a constant input's limiting response. An arbitrary bounded input can reverse its own sign so that both lobes contribute positively; the absolute area is the appropriate worst-case quantity.

Question 5

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

For a real parameter λ , solve the Volterra integral equation

$$u(t) = 1 + \lambda \int_0^t e^{-(t-\tau)} u(\tau) d\tau, \quad t \geq 0.$$

Seek a continuous solution on the entire half-line.

Tasks

1. Use transforms to derive the solution, treating every exceptional parameter value explicitly.
2. Derive an equivalent first-order IVP and prove the converse implication, establishing uniqueness among continuous solutions.
3. Classify boundedness, positivity and long-time behavior for all real λ , including $\lambda = 0$.
4. Give the exact real transform domain in every case. Explain why exponential decay of the memory kernel does not by itself guarantee a bounded solution.

Question 5 – Solution

Strategy. Feedback changes the effective growth rate. Verify the transformed candidate through an equivalent IVP so no growth assumption is needed for uniqueness.

Step 1: Solve the transformed equation. Initially for sufficiently large s ,

$$U = \frac{1}{s} + \frac{\lambda U}{s+1}, \quad U = \frac{s+1}{s(s+1-\lambda)}.$$

Partial fractions give

$$u(t) = \frac{1 - \lambda e^{(\lambda-1)t}}{1 - \lambda} \quad (\lambda \neq 1), \quad u(t) = 1 + t \quad (\lambda = 1).$$

In particular $\lambda = 0$ gives $u = 1$; the factor $s + 1$ then cancels in U .

Step 2: Prove equivalence without assuming a transform. For continuous u , define $v = e^{-t} * u$. Differentiation gives $v' = u - v$, $v(0) = 0$. The integral equation makes u continuously differentiable and yields

$$u' = (\lambda - 1)u + 1, \quad u(0) = 1.$$

Conversely, let u solve this IVP and form v as above. The residual $w = u - 1 - \lambda v$ satisfies $w' = -u + 1 + \lambda v = -w$ and $w(0) = 0$. Hence $w = 0$ and the integral equation holds, including at $\lambda = 0$. The linear IVP has a unique global solution, which proves the asserted existence and uniqueness.

Step 3: Classify all parameters. For $\lambda < 1$, the solution is bounded and tends to $1/(1 - \lambda) > 0$. For $\lambda = 1$ it grows linearly; for $\lambda > 1$ it grows exponentially with positive leading coefficient $\lambda/(\lambda - 1)$. For $\lambda \neq 1$, $u' = \lambda e^{(\lambda-1)t}$: it decreases from 1 to a positive limit when $\lambda < 0$, is constant when $\lambda = 0$, and increases when $\lambda > 0$. Thus it is positive for every real λ and every $t \geq 0$.

Step 4: Inspect the actual transform domain. For $\lambda < 1$ the positive nonzero limiting constant forces $s > 0$; for $\lambda = 1$ the linear term also requires $s > 0$. For $\lambda > 1$ the nonzero exponential requires $s > \lambda - 1$. Consequently

$$s > 0 \quad (\lambda \leq 1), \quad s > \lambda - 1 \quad (\lambda > 1).$$

Positive feedback can offset or exceed the kernel's decay. The rate in the equivalent equation is $\lambda - 1$, not simply the kernel's rate -1 .

Question 6

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Three identical smoothing stages each have causal kernel $k_a(t) = ae^{-at}$, where $a > 0$. Their cascade has kernel $r_a = k_a * k_a * k_a$.

Tasks

1. Compute the two-stage and three-stage kernels by direct integration.
2. Find the transform of r_a and justify why the grouping of the convolutions does not matter.
3. Find the unique peak of r_a , its height and its initial value and slope. Compare these with a single stage.
4. Compute the mass and mean time of both kernels. Explain what is preserved by cascading and what changes. You may derive needed exponential moments by integration by parts.

Question 6 – Solution

Strategy. Equal exponential factors leave polynomial integrals inside the convolution. Unit mass is preserved while the response is spread over time.

Step 1: Integrate the cascade. Directly,

$$(k_a * k_a)(t) = a^2 e^{-at} \int_0^t 1 \, du = a^2 t e^{-at}, \quad r_a(t) = a^3 e^{-at} \int_0^t u \, du = \boxed{\frac{1}{2} a^3 t^2 e^{-at}}.$$

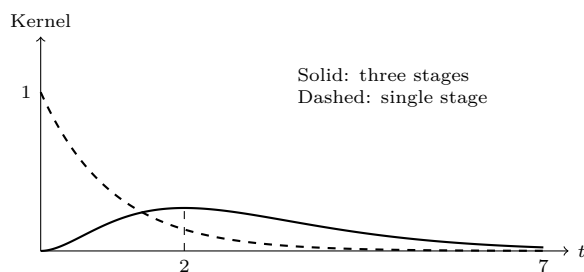
Step 2: Check transforms and grouping. For real $s > -a$, $R_a(s) = [a/(s+a)]^3$. Both groupings integrate the same product $k_a(t-u-v)k_a(u)k_a(v)$ over $u, v \geq 0$, $u+v \leq t$. The continuous integrand is integrable on this bounded triangle, so reversing the integration order proves associativity directly.

Step 3: Compare the peak and onset. For $t > 0$, $r'_a = \frac{1}{2} a^3 e^{-at} t(2-at)$. Thus the unique maximum is at $t = 2/a$, with height $2ae^{-2}$. Also $r_a(0) = r'_a(0) = 0$. A single stage has maximum a at zero and right derivative $-a^2$ there. The cascade begins more smoothly, reaches its peak later and has a lower peak.

Step 4: Compute mass and mean time. Integration by parts yields $\int_0^\infty t^n e^{-at} dt = n!/a^{n+1}$ for nonnegative integers n . Hence

$$\int k_a = 1, \quad \int r_a = 1, \quad \int t k_a(t) dt = \frac{1}{a}, \quad \int t r_a(t) dt = \frac{3}{a}.$$

Since both masses are one, these first moments are their mean times. The total area is preserved; the mean time triples. The cascade remains positive at every $t > 0$, so its later peak is not a literal waiting interval with zero output. The figure uses $a = 1$ and the same vertical scale for both.



Question 7

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let the unstable kernel be $h(t) = e^t$. First allow a piecewise continuous input f supported in $[0, T]$, where $T > 0$. Then specialize to

$$f_c(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -c, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad c > 0,$$

with output $y = h * f_c$.

Tasks

1. For the general input, derive a necessary and sufficient integral condition for the output to vanish for every $t \geq T$.
2. For f_c , compare the choice that makes the ordinary input area zero with the choice that eliminates the output tail.
3. For the tail-eliminating choice, derive the complete piecewise output and find its maximum.
4. Find the transformed output for general c and compare its exact real transform domain before and after cancellation. Explain why the common transform domain of the factors need not be the output's full domain.

Question 7 – Solution

Strategy. An unstable kernel remembers a weighted input area. Equal positive and negative ordinary areas need not cancel that memory.

Step 1: Factor the growing exponential. For an input supported in $[0, T]$,

$$(h * f)(t) = e^t \int_0^{\min(t, T)} e^{-u} f(u) du.$$

Thus the output is zero for every $t \geq T$ if and only if $\int_0^T e^{-u} f(u) du = 0$, since e^t never vanishes.

Step 2: Distinguish the two cancellations. The ordinary area is $1 - c$, so it vanishes at $c = 1$. The weighted area is $(1 - e^{-1}) - c(e^{-1} - e^{-2})$; its unique zero is $c = e$. At $c = 1$ it equals $(1 - e^{-1})^2 > 0$, leaving a growing tail.

Step 3: Recover the selected response. Using the partial weighted integrals with $c = e$ gives

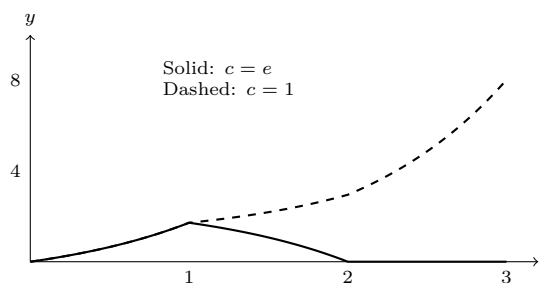
$$y_*(t) = \begin{cases} e^t - 1, & 0 \leq t < 1, \\ e - e^{t-1}, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

The pieces agree at both junctions. It increases up to 1 and decreases from 1 to 2, so its unique maximum is $e - 1$ at $t = 1$.

Step 4: Track the transform domain. The convolution theorem initially gives, for $s > 1$,

$$Y(s) = \frac{1 - (1 + c)e^{-s} + ce^{-2s}}{s(s - 1)}.$$

If $c \neq e$, the output has a nonzero e^t tail and exact real domain $s > 1$. If $c = e$, the output is compactly supported, so its transform exists for every real s ; both apparent singularities at 0 and 1 are removable. The product formula agrees with it on $s > 1$ and extends by removal of these singularities. Cancellation changes this output, not the unstable kernel. The dashed comparison uses $c = 1$, which cancels only the ordinary input area.



Question 8

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Suppose an ordinary continuous input f produces $z = e^{-t} * f$. No impulses and no nonzero initial state are allowed. Consider the desired output $z_0(t) = te^{-t}$ and the perturbed outputs

$$z_{\varepsilon,N}(t) = te^{-t} + \varepsilon e^{-t} \sin(Nt), \quad \varepsilon > 0, \quad N \in \{1, 2, \dots\}.$$

Tasks

1. Characterize all continuously differentiable outputs obtainable from such inputs, and prove uniqueness of the recovered input.
2. Recover the input for z_0 . Decide whether the target $z(t) = e^{-t}$ is obtainable, explaining why differentiating alone can give a misleading answer.
3. Recover the input for $z_{\varepsilon,N}$ and bound its output error in the uniform norm on $[0, \infty)$. Compute the input error at zero.
4. Choose a sequence $\varepsilon = \varepsilon_N \rightarrow 0$ for which the output errors tend uniformly to zero but the input errors grow without bound. Interpret what this means for reconstruction from approximate measurements.

Question 8 – Solution

Strategy. Inversion differentiates the output. Check the initial-value restriction before using that derivative formula.

Step 1: Derive and prove the inverse rule. Differentiation under the integral gives $z' = f - z$, $z(0) = 0$. Therefore a C^1 target is obtainable exactly when it starts at zero, and its only possible continuous input is $\boxed{f = z' + z}$. Conversely, for this input, the target and the convolution solve the same IVP $w' + w = f$, $w(0) = 0$. Uniqueness proves they agree.

Step 2: Check two targets. For $z_0 = te^{-t}$, $z'_0 + z_0 = e^{-t}$, so the input is $f_0 = e^{-t}$. The target e^{-t} is impossible because its value at zero is 1. Although its derivative plus itself is zero, a zero input with zero initial state produces the zero output. Differentiation discards the missing initial condition.

Step 3: Differentiate the perturbation. The two exponential-derivative terms cancel, leaving

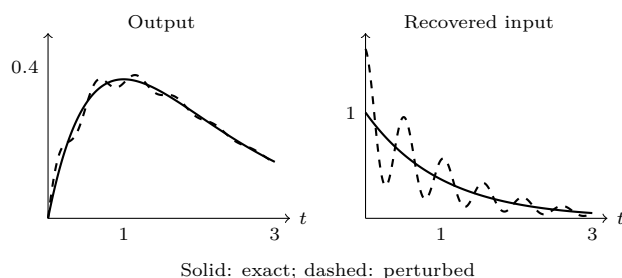
$$\boxed{f_{\varepsilon,N} = e^{-t} + \varepsilon N e^{-t} \cos(Nt)}.$$

All these inputs are continuous, and the target outputs vanish at zero. The uniform output error is at most ε because $e^{-t}|\sin Nt| \leq 1$. The input error at zero is εN ; in fact this is its uniform norm since $e^{-t}|\cos Nt| \leq 1$.

Step 4: Exhibit the instability. Take $\varepsilon_N = N^{-1/2}$. Then

$$\|z_{\varepsilon_N,N} - z_0\|_{\infty} \leq N^{-1/2} \rightarrow 0, \quad \|f_{\varepsilon_N,N} - f_0\|_{\infty} = \sqrt{N} \rightarrow \infty.$$

Thus the inverse is not continuous in the uniform output norm, even on smooth attainable outputs. Small high-frequency measurement errors can produce large reconstruction errors. The figure uses $\varepsilon = 0.05$, $N = 12$; each panel compares its own exact and perturbed functions.



Question 9

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two normalized stages $k_\alpha(t) = \alpha e^{-\alpha t}$ and $k_\beta(t) = \beta e^{-\beta t}$, with $\alpha, \beta > 0$, are followed by a delay $d \geq 0$:

$$r(t) = H_d(t)(k_\alpha * k_\beta)(t-d).$$

The observed response is positive precisely for $t > 1$, has mass 1, mean $\mu = 5/2$ and variance $v = 5/4$. For a nonnegative unit-mass kernel p , $\mu_p = \int_0^\infty tp(t) dt$ and $v_p = \int_0^\infty (t - \mu_p)^2 p(t) dt$.

Tasks

1. Prove that convolution adds the means and variances of two nonnegative unit-mass kernels with finite second moments.
2. Compute the mean and variance of k_α , and explain the effect of a delay on mass, mean and variance.
3. Recover d and the two rates from the observations. State precisely what is and is not uniquely identifiable.
4. For general prescribed delay d , mean μ and variance v , give necessary and sufficient conditions for two finite positive rates to exist. Interpret the equality case and the excluded upper boundary.

Question 9 – Solution

Strategy. Work with reciprocal rates. Convolution moment identities reduce identification to a quadratic equation.

Step 1: Prove the moment identities. In the convolution integral use $t = u + w$, $u, w \geq 0$. Nonnegativity permits interchange, and finite second moments make the following values finite:

$$\begin{aligned}\int (p * q) &= \left(\int p\right)\left(\int q\right) = 1, \\ \mu_{p*q} &= \iint (u + w)p(u)q(w) du dw = \mu_p + \mu_q, \\ \int t^2(p * q)(t) dt &= \int u^2 p(u) du + 2\mu_p \mu_q + \int w^2 q(w) dw.\end{aligned}$$

Subtracting $(\mu_p + \mu_q)^2$ from the last line proves $\boxed{v_{p*q} = v_p + v_q}$.

Step 2: Calculate the stage statistics. Integration by parts gives mass 1, first moment $1/\alpha$ and second moment $2/\alpha^2$ for k_α . Its variance is therefore $1/\alpha^2$. A delay replaces time by $t = d + u$: mass stays 1, the mean increases by d , and the centered second moment is unchanged. Hence

$$\mu = d + \frac{1}{\alpha} + \frac{1}{\beta}, \quad v = \frac{1}{\alpha^2} + \frac{1}{\beta^2}.$$

Step 3: Recover the parameters. Both stage kernels are positive for positive times, so their convolution is positive at every positive time and zero at zero. The observed onset thus forces $d = 1$. Put $x = 1/\alpha$, $z = 1/\beta$. Then

$$x + z = \frac{3}{2}, \quad x^2 + z^2 = \frac{5}{4}, \quad xz = \frac{1}{2}, \quad (X - x)(X - z) = X^2 - \frac{3}{2}X + \frac{1}{2}.$$

Its roots are 1 and $1/2$, giving $\boxed{\{\alpha, \beta\} = \{1, 2\}}$. The delay and unordered rate pair are unique. Stage order cannot be identified even from the entire output because convolution is commutative.

Step 4: Give the full feasibility condition. Write $m = \mu - d$. Positive x, z require and are possible exactly when

$$\boxed{m > 0, \quad \frac{m^2}{2} \leq v < m^2.}$$

Indeed $xz = (m^2 - v)/2 > 0$ and $(x - z)^2 = 2v - m^2 \geq 0$; conversely these conditions make $(m \pm \sqrt{2v - m^2})/2$ real and strictly positive. At the lower equality the rates coincide. At the upper boundary one reciprocal rate would be zero, requiring an infinite rate, excluded here.

Question 10

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f \in C^1[0, \infty)$ satisfy $|f'(t)| \leq L$, with $L \geq 0$. For $\varepsilon > 0$ define

$$k_\varepsilon(t) = \frac{1}{\varepsilon} e^{-t/\varepsilon}, \quad y_\varepsilon = k_\varepsilon * f.$$

This causal averaging kernel becomes concentrated near zero as ε decreases.

Tasks

1. Derive an IVP for y_ε and use integration by parts to express $y_\varepsilon - f$ in terms of $f(0)$ and f' .
2. Prove an explicit error bound containing both the initial transient and a term at most $L\varepsilon$.
3. Distinguish uniform convergence on $[0, \infty)$ from uniform convergence on $[t_0, \infty)$ for $t_0 > 0$. Show the $L\varepsilon$ bound is sharp when $f(0) = 0$.
4. For $f(t) = 1 + t$, decide separately whether $\varepsilon = 1/20$ and $\varepsilon = 1/40$ ensure $|y_\varepsilon(t) - f(t)| \leq 1/20$ for every $t \geq 1$. Justify the strict inequalities involved.

Question 10 – Solution

Strategy. Separate the error due to starting from rest from the error due to variation of the input.

Step 1: Obtain the equation and exact error identity. Differentiation gives $\varepsilon y'_\varepsilon + y_\varepsilon = f$, $y_\varepsilon(0) = 0$. Integrating the convolution by parts in u gives

$$y_\varepsilon(t) - f(t) = -e^{-t/\varepsilon} f(0) - \int_0^t e^{-(t-u)/\varepsilon} f'(u) du.$$

Step 2: Bound the two contributions. Taking absolute values yields

$$|y_\varepsilon - f| \leq |f(0)|e^{-t/\varepsilon} + L\varepsilon(1 - e^{-t/\varepsilon}).$$

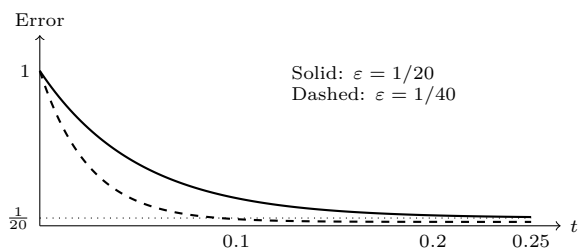
This accounts for the kernel mass missing before time zero as well as the variation over its recent history.

Step 3: Distinguish the convergence statements. If $f(0) = 0$, the error is uniformly at most $L\varepsilon$ on the half-line. For the ramp $f = Lt$ it equals $L\varepsilon(1 - e^{-t/\varepsilon})$ in magnitude, whose supremum is $L\varepsilon$ (approached as $t \rightarrow \infty$ when $L > 0$). Thus the constant is sharp. If $f(0) \neq 0$, the error at zero is always $|f(0)|$, preventing uniform convergence on $[0, \infty)$. On $[t_0, \infty)$ it is bounded by $|f(0)|e^{-t_0/\varepsilon} + L\varepsilon \rightarrow 0$ for every fixed $t_0 > 0$.

Step 4: Test the two proposed averaging scales. For $f = 1 + t$, $y_\varepsilon = t + 1 - \varepsilon - (1 - \varepsilon)e^{-t/\varepsilon}$. For $0 < \varepsilon < 1$ the error magnitude is $E_\varepsilon = \varepsilon + (1 - \varepsilon)e^{-t/\varepsilon}$. At $\varepsilon = 1/20$ this is strictly greater than $1/20$ at every finite time, so that choice fails. At $\varepsilon = 1/40$ and $t \geq 1$,

$$E_\varepsilon \leq \frac{1}{40} + \frac{39}{40}e^{-40} < \frac{1}{20},$$

since $e^{40} > 1 + 40 = 41 > 39$. Thus the smaller choice succeeds. The figure shows the initial error layer and the $1/20$ tolerance.



Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A draft transform table contains these three proposed pairs, with $a > 0$:

$$t^3 e^{-2t} \longleftrightarrow \frac{1}{(s+2)^4}, \quad e^{-at} \sin(3t) \longleftrightarrow \frac{s+a}{(s+a)^2+9}, \quad f(2t) \longleftrightarrow F(s/2).$$

In the third pair $F = \mathcal{L}\{f\}$, where f is continuous and of exponential order.

Tasks

1. Identify and correct every error. Derive the power entry by integration by parts and the scaling entry by substitution.
2. State the exact real convergence domains for the first two corrected entries. State the sufficient transformed domain for the third if F converges absolutely for $s > c$.
3. Use the corrected rules to find the transform of $t^3 e^{-2t} + e^{-at} \sin(3t)$, without a fresh integration.
4. Check your answer independently using the leading large- s behavior and the value at $s = 0$. Explain what each check can and cannot establish.

Question 1 – Solution

Strategy. A table entry includes its normalization and domain, not only its denominator.

Step 1: Repair the entries. For $p > 0$, integration by parts gives $I_n(p) = nI_{n-1}(p)/p$, $I_0(p) = 1/p$, with vanishing boundary terms. Thus $I_3(p) = 6/p^4$. The sine entry has frequency in its numerator. Substitution $u = 2t$ in the transform supplies the missing Jacobian. The corrected pairs are

$$t^3 e^{-2t} \leftrightarrow \frac{6}{(s+2)^4}, \quad e^{-at} \sin 3t \leftrightarrow \frac{3}{(s+a)^2 + 9}, \quad f(2t) \leftrightarrow \frac{1}{2}F(s/2).$$

The proposed sine numerator instead represents $e^{-at} \cos 3t$.

Step 2: Include convergence information. The first domain is $s > -2$. At its boundary the integrand is t^3 . The second is $s > -a$; at the boundary the sine integral has no improper limit, and below it the oscillations grow. For the scaled general function, $s > 2c$ ensures absolute convergence. This is sufficient and need not be exact if the supplied c is merely an exponential-order bound.

Step 3: Combine the corrected entries. Linearity gives

$$Y(s) = \frac{6}{(s+2)^4} + \frac{3}{(s+a)^2 + 9}, \quad s > \max(-2, -a).$$

The polynomial-exponential and sinusoidal tails cannot cancel one another.

Step 4: Make independent checks. Near zero, the sum is $3t + O(t^2)$, so its leading transform behavior is $3/s^2$, consistent with the displayed expression. This would immediately reject a cosine numerator, which would introduce a $1/s$ term. Direct areas are $6/2^4 = 3/8$ and $3/(a^2 + 9)$, giving $Y(0) = 3/8 + 3/(a^2 + 9)$. This catches the missing factorial. These checks are necessary consistency tests, not substitutes for the integration and operational derivations: many different transforms can share one asymptotic term and one value.

Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $a > 0$, define the delay and time-multiplication operations

$$(D_a f)(t) = H_a(t)f(t-a), \quad (Mf)(t) = tf(t).$$

Take $f(t) = e^{-t}$, and compare $p = D_a(Mf)$ with $q = M(D_a f)$.

Tasks

1. Derive the transform rules for D_a and M from their defining integrals. State a domain where the derivations are justified for this f .
2. Find p, q and their transforms. Explain exactly why applying the two operations in opposite orders changes the answer.
3. Prove the general identity $MD_a f - D_a Mf = aD_a f$ whenever the expressions are defined, and verify its transform.
4. For $a = 2$, invert $e^{-2s}/(s+1)^2$ and $-\frac{d}{ds}[e^{-2s}/(s+1)]$. Compute the area of each inverse and use it to distinguish them.

Question 2 – Solution

Strategy. A delayed clock measures $t - a$; multiplication after the delay still uses the original clock t .

Step 1: Derive the rules. The substitution $u = t - a$ gives $\mathcal{L}\{D_a f\} = e^{-as}F(s)$. Differentiating the transform integral with respect to s gives $\mathcal{L}\{Mf\} = -F'(s)$. For $f = e^{-t}$ and its delayed polynomial multiples, $s > -1$ makes these integrals absolutely convergent and permits this differentiation, locally dominated by an integrable exponential times a polynomial.

Step 2: Keep the clocks explicit. Here

$$p = H_a(t)(t - a)e^{-(t-a)}, \quad q = H_a(t)te^{-(t-a)},$$

so

$$P = \frac{e^{-as}}{(s+1)^2}, \quad Q = -\frac{d}{ds} \frac{e^{-as}}{s+1} = e^{-as} \left[\frac{a}{s+1} + \frac{1}{(s+1)^2} \right].$$

The extra term comes from differentiating the delay factor itself.

Step 3: Prove the operation identity. For $t < a$ both sides vanish. For $t \geq a$, subtraction gives $tf(t-a) - (t-a)f(t-a) = af(t-a)$. Therefore

$$\boxed{MD_a f - D_a Mf = aD_a f.}$$

In transform language, $-(e^{-as}F)' - e^{-as}(-F') = ae^{-as}F$, the same identity by the product rule.

Step 4: Invert and distinguish the two expressions. For $a = 2$ the first inverse is $H_2(t)(t-2)e^{-(t-2)}$, while the second is $H_2(t)te^{-(t-2)}$. Their areas, after putting $u = t - 2$, are

$$\int_0^\infty ue^{-u} du = \boxed{1}, \quad \int_0^\infty (u+2)e^{-u} du = \boxed{3}.$$

Their transforms at zero have these same values. The first starts continuously from zero at the delay; the second has right-hand value 2 there. Isolated endpoint conventions do not change either area.

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A short table gives only $\mathcal{L}\{\sin(at)\} = a/(s^2 + a^2)$, where $a > 0$. You need inverses of

$$A_a(s) = \frac{s}{(s^2 + a^2)^2}, \quad B_a(s) = \frac{1}{(s^2 + a^2)^2}.$$

Tasks

1. Derive the inverse of A_a by differentiating a transform with respect to s .
2. Derive the inverse of B_a by differentiating the given sine entry with respect to a . Explain why the parameter differentiation is justified for real $s > 0$.
3. Check the first nonzero Taylor term at $t = 0$ against the leading large- s term of each transform.
4. Find both pointwise limits as $a \rightarrow 0^+$ and explain why substituting $a = 0$ into an unsimplified inverse formula is invalid.

Question 3 – Solution

Strategy. Generate missing table rows through operations whose normalization can be checked independently.

Step 1: Differentiate with respect to s . Since $-\frac{d}{ds}[a/(s^2 + a^2)] = 2as/(s^2 + a^2)^2$,

$$\boxed{\mathcal{L}^{-1}\{A_a\} = \frac{t \sin(at)}{2a}.$$

The factor $2a$ must be divided out.

Step 2: Differentiate with respect to a . The parameter derivative gives $\mathcal{L}\{t \cos(at)\} = (s^2 - a^2)/(s^2 + a^2)^2$. For fixed $s > 0$, te^{-st} dominates the differentiated integrand uniformly in a , justifying differentiation under the integral. Subtracting this row from $\mathcal{L}\{\sin(at)/a\}$ yields

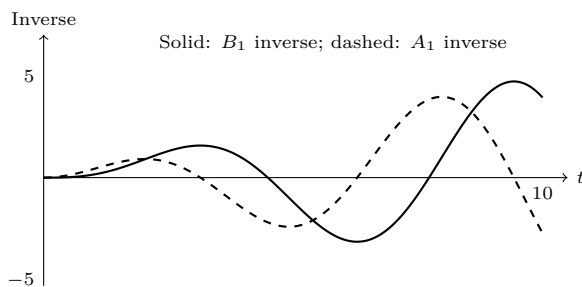
$$\frac{1}{s^2 + a^2} - \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{2a^2}{(s^2 + a^2)^2}.$$

Thus

$$\boxed{\mathcal{L}^{-1}\{B_a\} = \frac{\sin(at) - at \cos(at)}{2a^3}.$$

Step 3: Check the onset powers. The inverses start as $t^2/2 + O(t^4)$ and $t^3/6 + O(t^5)$. Their leading transforms are respectively $1/s^3$ and $1/s^4$, matching A_a and B_a . This checks both factorials and the cancellation of the linear term in the second numerator.

Step 4: Remove the apparent parameter singularities. At each fixed t , the Taylor expansions give limits $t^2/2$ and $t^3/6$. Their transforms are $1/s^3$ and $1/s^4$, agreeing with the limits of the rational functions for $s > 0$. Directly inserting $a = 0$ into the inverse quotients produces $0/0$; the limits require cancellation first. The figure uses $a = 1$ and shows that the repeated denominator can produce oscillations with increasing size, unlike a single sine entry.



Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

An unknown function has transform

$$F(s) = \frac{As + B}{s^2 + 4}, \quad s > 0,$$

where A, B are real. Measurements give $f(0) = 1$ and $f'(0) = -2$.

Tasks

1. Invert the general transform by matching numerators to table entries, then recover A, B from the data.
2. For the recovered function, find its amplitude, all positive zeros, and the conserved quantity $f'(t)^2 + 4f(t)^2$.
3. Classify all real pairs (A, B) for which the inverse is nonnegative for every $t \geq 0$. Prove the classification.
4. A student argues that $\lim_{s \rightarrow 0^+} sF(s) = 0$ proves $\lim_{t \rightarrow \infty} f(t) = 0$. Refute the conclusion with two sequences of times and identify the missing hypothesis in that use of the final-value theorem.

Question 4 – Solution

Strategy. Matching a denominator is insufficient: the numerator determines the phase, initial state and sign changes.

Step 1: Match and use the data. The table gives $f = A \cos 2t + (B/2) \sin 2t$, so $f(0) = A$ and $f'(0) = B$. Therefore $A = 1$, $B = -2$ and

$$f(t) = \cos 2t - \sin 2t = \sqrt{2} \cos(2t + \pi/4).$$

Step 2: Determine the motion. The amplitude is $\sqrt{2}$. The positive zeros are $t = \pi/8 + k\pi/2$, $k = 0, 1, 2, \dots$. Direct differentiation gives $f'' + 4f = 0$, so $(f'^2 + 4f^2)' = 2f'(f'' + 4f) = 0$. Its initial value is $(-2)^2 + 4(1)^2 = \boxed{8}$.

Step 3: Prove the positivity classification. If $(A, B) \neq (0, 0)$, the amplitude $\sqrt{A^2 + B^2/4}$ is positive. The sinusoid attains both that amplitude and its negative at arbitrarily large nonnegative times. It cannot be nonnegative on the whole half-line. Only $\boxed{A = B = 0}$ works. An alternative proof uses zero integral over one period: a continuous nonnegative function with zero integral must vanish.

Step 4: Check the claimed final value. Although $sF(s) \rightarrow 0$ as $s \rightarrow 0^+$, the selected solution obeys $f(n\pi) = 1$ and $f(n\pi + \pi/2) = -1$. Hence it has no time limit. The usual rational final-value theorem requires the poles of $sF(s)$ to lie strictly in the left half-plane after cancellation. Here the uncanceled poles are $s = \pm 2i$, supporting persistent oscillations. An algebraic limit of the transform does not by itself establish a limit of the function.

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A table contains the infinite-tail entry $te^{-t} \leftrightarrow 1/(s+1)^2$. A finite experiment instead uses

$$w(t) = \begin{cases} te^{-t}, & 0 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

A student subtracts $e^{-2s}/(s+1)^2$ to remove the tail.

Tasks

1. Explain why the proposed subtraction is incorrect, and express the actual tail as a delayed function with its internal clock made explicit.
2. Derive $W(s)$ from table operations and verify it by integration over the finite experiment.
3. Find the exact real transform domain and the removable value at $s = -1$. Explain why the infinite-tail table domain does not restrict this finite-window transform.
4. Find the area, the global maximum and the one-sided values at the cutoff. Draw the finite-window response accurately on the solution page.

Question 5 – Solution

Strategy. Removing an old tail requires its original age, not a fresh copy starting at zero.

Step 1: Identify the tail that must be removed. For $t = 2 + u$, the old signal is $e^{-2}(u + 2)e^{-u}$. Thus

$$w = te^{-t} - e^{-2}H_2(t)[(t - 2) + 2]e^{-(t-2)}.$$

The proposed subtraction instead removes $H_2(t)(t - 2)e^{-(t-2)}$, which has both the wrong age polynomial and the wrong exponential scale.

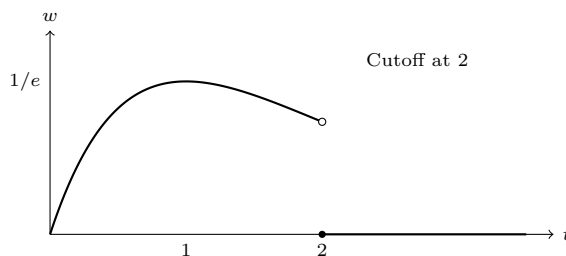
Step 2: Transform and check directly. For $s \neq -1$, the correct expression is

$$W(s) = \frac{1 - e^{-2(s+1)}[1 + 2(s + 1)]}{(s + 1)^2}.$$

Initially table operations justify it for $s > -1$. Independently put $p = s + 1$: $W = \int_0^2 te^{-pt} dt$. Integration by parts gives $[1 - e^{-2p}(1 + 2p)]/p^2$, agreeing with the table calculation and extending its formula to $p \neq 0$ of either sign.

Step 3: Remove the singularity and identify the domain. A bounded finite interval makes this transform finite for every real s . At $s = -1$ the defining integral is $\int_0^2 t dt = [2]$. The numerator's Taylor expansion is $2p^2 + O(p^3)$, so the apparent pole is removable. The two infinite-tail terms need only be transformed separately on their common domain; their difference has a larger actual domain.

Step 4: Record the geometric checks. The area is $W(0) = [1 - 3e^{-2}]$. Since $(te^{-t})' = (1 - t)e^{-t}$, the unique global maximum is $1/e$ at $t = 1$. At the cutoff, $w(2^-) = 2e^{-2}$ while $w(2) = w(2^+) = 0$. The open dot marks the excluded left-hand value and the filled dot the actual value at 2; no slanted segment bridges the jump.



Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Let p be the continuous periodic triangular signal of period 2 with

$$p(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2-t, & 1 \leq t < 2, \end{cases} \quad p(t+2) = p(t).$$

A standard table lists delayed ramps but does not list this periodic function.

Tasks

1. Derive the periodic-transform formula by splitting the half-line into cycles. Apply it to this signal.
2. Write a locally finite series of delayed ramps for p and use it to obtain the same transform independently.
3. Find the exact real transform domain. Compute $\lim_{s \rightarrow 0^+} sP(s)$ and interpret it using the mean over one period.
4. Explain why the cycle mean is not a pointwise final value, and determine the first two leading terms of $P(s)$ as $s \rightarrow 0^+$.

Question 6 – Solution

Strategy. A one-cycle transform and a geometric series supply the periodic row; delayed ramps independently encode its slope changes.

Step 1: Sum the cycles. For $s > 0$, boundedness gives absolute convergence. Splitting into intervals $[2n, 2n+2]$ and translating yields

$$P(s) = \frac{\int_0^2 e^{-st} p(t) dt}{1 - e^{-2s}} = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-2s})} = \frac{1 - e^{-s}}{s^2(1 + e^{-s})}.$$

The one-cycle integral is $(1 - e^{-s})^2/s^2$, obtained by integration by parts on $[0, 1]$ and $[1, 2]$.

Step 2: Recover the slope changes. Write $R_a = (t - a)H_a(t)$. The series

$$p(t) = \sum_{n=0}^{\infty} [R_{2n} - 2R_{2n+1} + R_{2n+2}]$$

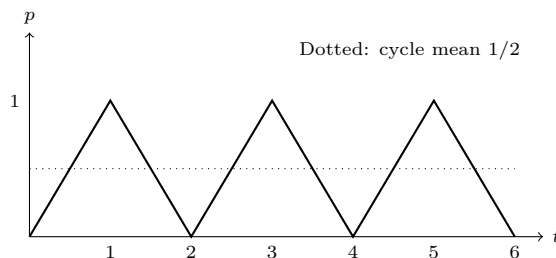
is locally finite; each bracket is a unit triangular pulse on $[2n, 2n+2]$. Its transform is $e^{-2ns}(1 - 2e^{-s} + e^{-2s})/s^2$. Summing for $s > 0$ gives the same P . These brackets, rather than the individual growing ramps across all cycles, also make absolute convergence clear.

Step 3: Check convergence and the mean. The signal is bounded, nonnegative and has area 1 per period. Thus the exact real domain is $s > 0$: at $s = 0$ the areas sum to infinity, and for $s < 0$ the weighted cycle contributions grow. Using $(1 - e^{-s})/(1 + e^{-s}) = \tanh(s/2)$, $\lim_{s \rightarrow 0^+} sP(s) = \boxed{1/2}$, the cycle mean $1/2$.

Step 4: Distinguish the limit notions. The samples $p(2n) = 0$ and $p(2n+1) = 1$ rule out a pointwise final value. Taylor expansion of the last quotient gives

$$P(s) = \frac{1}{2s} - \frac{s}{24} + O(s^3).$$

The leading pole records the mean, while the constant term vanishes. The figure displays continuous corners and equal successive cycles.



Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $t > 0$, let

$$g(t) = \frac{e^{-t} - e^{-3t}}{t}.$$

A rational-only table offers no obvious row for this function.

Tasks

1. Extend g continuously to zero and represent it as an integral of exponential functions over a finite parameter interval.
2. Use that representation to derive its transform, carefully justifying interchange and specifying a real domain.
3. Prove positivity and strict decrease on $[0, \infty)$, and compute the total area.
4. Find the exact real transform domain, including a proof of divergence at its boundary. Explain why an algebraically real continuation of a logarithm is not sufficient evidence of transform convergence.

Question 7 – Solution

Strategy. Integrating a familiar exponential row over its rate parameter generates a nonrational row.

Step 1: Remove the time singularity. Direct integration in a gives

$$g(t) = \int_1^3 e^{-at} da, \quad g(0) = 2.$$

This formula also proves continuity at zero without dividing by t .

Step 2: Integrate the table row. For real $s > -1$, the nonnegative double integral is finite, and interchange gives

$$G(s) = \int_1^3 \int_0^\infty e^{-(s+a)t} dt da = \int_1^3 \frac{da}{s+a} = \ln \frac{s+3}{s+1}.$$

Both logarithm arguments are positive in this domain. Finiteness follows also from $s+a \geq s+1 > 0$ on the finite parameter interval.

Step 3: Establish shape and area. The integrand is positive, so $g(t) > 0$. Parameter integration on a bounded interval permits differentiation at every $t \geq 0$:

$$g'(t) = - \int_1^3 a e^{-at} da < 0.$$

Thus the extension decreases strictly from 2, approaching zero. Since 0 lies in the transform domain, its area is $G(0) = \ln 3$.

Step 4: Check the boundary from the original integral. At $s = -1$, the transformed integrand is $(1 - e^{-2t})/t$ for $t > 0$. For $t \geq 1$ it is at least $(1 - e^{-2})/t$, whose integral diverges. For $s < -1$ the positive integrand is even larger on the tail. Therefore $s > -1$ is the exact real domain. For example, the logarithm of $(s+3)/(s+1)$ is real again when $s < -3$, but that algebraic expression cannot represent the divergent defining integral. The table entry must travel with its domain and the continuous value at zero.

Question 8

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Consider the transform $F(s) = s/(s+1)$. Allow a full-mass impulse at the origin, with $\mathcal{L}\{\delta(t)\} = 1$ and pre-impact data at 0^- . For ordinary candidates, restrict to functions continuous near zero and of exponential order. The input with transform F drives

$$y' + 2y = f, \quad y(0^-) = 0.$$

Tasks

1. Explain why F cannot be the transform of an ordinary candidate in the stated class. Recover its distributional inverse.
2. Find $Y(s)$ and the complete response for $t \geq 0$, including the jump at zero.
3. Find the positive zero, the global minimum, and the total response area. Explain how a nonzero response can have zero area.
4. Verify the post-impact equation and the equivalent ordinary restart. State the exact real transform domain of the response.

Question 8 – Solution

Strategy. Divide an improper rational expression before consulting ordinary-function rows. Its polynomial part may represent an impulse.

Step 1: Identify the input class. For an ordinary candidate continuous near zero and of exponential order, its Laplace transform tends to zero as $s \rightarrow \infty$: the bounded initial piece has integral $O(1/s)$ and the exponentially bounded tail tends to zero. But $s/(s+1) \rightarrow 1$. Polynomial division gives

$$F = 1 - \frac{1}{s+1}, \quad \boxed{f = \delta(t) - e^{-t}}.$$

The impulse supplies the nonvanishing constant term.

Step 2: Transform the equation and account for the jump. With the stated pre-impact convention, $(s+2)Y = F$, so

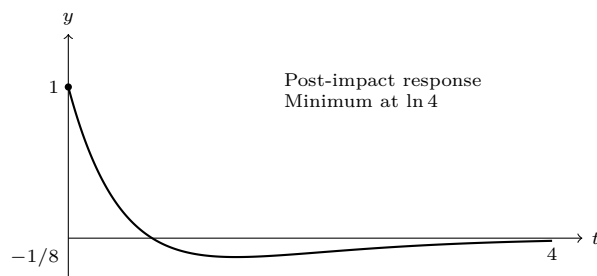
$$Y = \frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2}, \quad \boxed{y = 2e^{-2t} - e^{-t} \quad (t \geq 0)}.$$

Write $[y]_0 = y(0^+) - y(0^-)$. Integrating across zero gives $[y]_0 = 1$, consistent with $y(0^+) = 1$. The negative exponential input has no impulse mass.

Step 3: Find the sign changes and area. The unique positive zero is $t = \ln 2$. The derivative is $y' = e^{-t} - 4e^{-2t}$, negative before $\ln 4$ and positive afterward. Thus the global minimum is $\boxed{y(\ln 4) = -1/8}$.

The area is $2/2 - 1 = \boxed{0}$: positive and negative parts cancel. For comparison, the signed input area is $1 - 1 = 0$ when the full initial impulse is included.

Step 4: Verify the ordinary restart and convergence. For $t > 0$, direct substitution gives $y' + 2y = -e^{-t}$. The same response solves this ordinary equation with $y(0^+) = 1$. Adding another initial impulse to that restart would count it twice. The tail is asymptotic to $-e^{-t}$, so the exact real transform domain is $\boxed{s > -1}$. The figure shows only the post-impact branch on $t \geq 0$; the filled initial point records the prescribed right-hand value.



Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A measured response is known to lie in the three-mode family

$$f(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

The available data are $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 2$.

Tasks

1. Recover the three coefficients and prove uniqueness within the stated family.
2. Construct $F(s)$ by table lookup and simplify it to one rational expression. Check its leading large- s term against the data.
3. Prove that the recovered function is positive for every $t > 0$ despite a negative coefficient. Find its unique global maximum.
4. Compute its area and exact real transform domain. Explain why the three measurements would not identify a unique response without the three-mode assumption.

Question 9 – Solution

Strategy. The assumed modes turn derivative data into a finite linear system. A negative coefficient need not make their sum negative.

Step 1: Solve and prove uniqueness. The data give

$$c_1 + c_2 + c_3 = 0, \quad -c_1 - 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 2.$$

Adding the first two equations gives $c_2 = -2c_3$, and the first then gives $c_1 = c_3$. Substitution into the third gives $2c_3 = 2$. Every coefficient is forced, proving uniqueness: $(c_1, c_2, c_3) = (1, -2, 1)$.

Step 2: Assemble the rational transform. The table yields

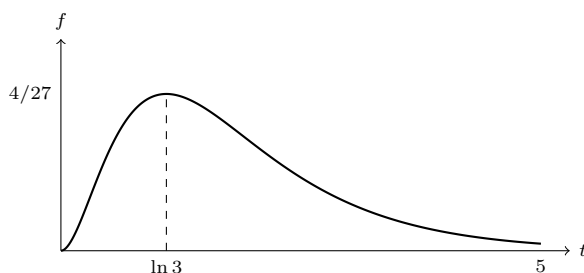
$$F = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} = \frac{2}{(s+1)(s+2)(s+3)}.$$

Near zero, $f = t^2 + O(t^3)$, so its leading transform is $2/s^3$, matching the quotient. The $1/s$ and $1/s^2$ terms cancel exactly.

Step 3: Establish positivity and the peak. Factor the function as $f = e^{-t}(1 - e^{-t})^2$. It is positive for every $t > 0$ and zero at 0. Put $x = e^{-t} \in (0, 1]$. The function $x(1-x)^2$ has derivative $(1-x)(1-3x)$ with respect to x ; along increasing t its unique interior maximum is at $x = 1/3$. Therefore

$$t_{\max} = \ln 3, \quad f(t_{\max}) = 4/27.$$

Step 4: Check area, domain and identifiability. The area is $1 - 2/2 + 1/3 = 1/3$. The nonzero e^{-t} tail gives exact real domain $s > -1$. Without the mode restriction, adding any multiple of $t^3 e^{-t}$ would preserve all three measured derivatives but change the response. Thus the identification is unique within the stated model, not among all smooth functions.



Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Use the chapter's table and operational rules to solve

$$y'' + 2y' + y = H_1(t)(t-1)e^{-(t-1)} + 2\delta(t-3), \quad y(0) = 1, \quad y'(0) = -1.$$

Ordinary force impulses preserve displacement and may jump velocity. All figures belong on the solution page.

Tasks

1. Find $Y(s)$, carefully retaining both initial terms and the two different forcing operations.
2. Invert the transform and give the response on the three intervals separated by 1 and 3.
3. Verify the initial state, the equation between switches, and the one-sided displacement and velocity changes at each switch.
4. Compute the total response area in two ways and determine the exact real transform domain. Explain why this finite list of table operations gives a unique solution.

Question 10 – Solution

Strategy. Separate the initial response, delayed ordinary forcing and impulse response before inversion.

Step 1: Transform with all data. The left side transforms to $(s^2Y - s + 1) + 2(sY - 1) + Y = (s + 1)^2Y - s - 1$. Consequently

$$Y = \frac{1}{s + 1} + \frac{e^{-s}}{(s + 1)^4} + \frac{2e^{-3s}}{(s + 1)^2}.$$

The ordinary forcing has transform $e^{-s}/(s + 1)^2$; the impulse has transform $2e^{-3s}$.

Step 2: Invert and make the intervals explicit. Define $v(\tau) = \tau^3 e^{-\tau}/6$ and $h(\tau) = \tau e^{-\tau}$ for $\tau \geq 0$. The factorial in the fourth-power table row gives

$$y(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-t} + v(t - 1), & 1 \leq t < 3, \\ e^{-t} + v(t - 1) + 2h(t - 3), & t \geq 3. \end{cases}$$

Step 3: Verify each kind of activation. The initial branch gives $(y(0), y'(0)) = (1, -1)$. For $D = d/d\tau$ and any polynomial P , $(D + 1)^2[e^{-\tau}P(\tau)] = e^{-\tau}P''(\tau)$. Thus v produces $\tau e^{-\tau}$ and h produces zero between switches. At 1, $v(0) = v'(0) = 0$, so displacement and velocity remain continuous. At 3, $h(0) = 0$, $h'(0) = 1$, so $[y]_3 = 0$ and $[y']_3 = 2$. Here $[z]_a = z(a^+) - z(a^-)$. These are exactly the required jump conditions.

Step 4: Check area, convergence and uniqueness. Direct areas are 1, $3!/6 = 1$, and 2, giving $\int_0^\infty y = 4$. Independently, integrating the equation, including the velocity jump, gives $1 - 2 + \int y = 1 + 2$, again yielding $\int y = 4$. The tail is a nonzero cubic polynomial times e^{-t} , so the exact real domain is $s > -1$. Linear IVP uniqueness on each ordinary interval, together with the forced matching and jump conditions, fixes the whole solution.

