

Question 1

Consider the following differential equations for a real function $y(t)$:

$$\begin{aligned} \text{A : } (ty')' + y &= \sin t, & \text{B : } y'' + ty' + y^2 &= 0, \\ \text{C : } y'' + (y')^2 &= 0, & \text{D : } y'' + t^2y &= 0. \end{aligned}$$

An equation is linear in y if y and its derivatives occur to the first power, are not multiplied together, and have coefficients depending only on t . The terms homogeneous and nonhomogeneous here refer to linear equations with zero and nonzero forcing, respectively.

Tasks

1. Expand equation A. Classify all four equations by order and linearity; classify the linear ones by homogeneity.
2. Explain why t^2y does not destroy linearity, whereas y^2 and $(y')^2$ do. Does a zero right-hand side make B or C a homogeneous linear equation?
3. Put A in the form $y'' + p(t)y' + q(t)y = g(t)$. Find the largest open interval containing $t = 1$ on which these coefficients are continuous.
4. Suppose a C^2 solution of the undivided equation A passes through $t = 0$. Derive the necessary relation between $y(0)$ and $y'(0)$. Can $y(0) = 1$, $y'(0) = 0$ be prescribed there?

Question 1 – Solution

Strategy. Expand derivatives before classifying, and distinguish an equation from its normalized form at a zero leading coefficient.

Step 1: Expand and classify. The product rule gives $(ty')' = ty'' + y'$. Hence A is

$$ty'' + y' + y = \sin t.$$

The classification is

equation	order	type
A	2	linear, nonhomogeneous
B	2	nonlinear
C	2	nonlinear
D	2	linear, homogeneous

A is a second-order equation whose leading coefficient vanishes at one point; its order as an equation is not reclassified solely at that point.

Step 2: Identify what is being squared. In D, t^2 is a known coefficient, while y occurs to the first power. In B and C the unknown function or its derivative is squared. A zero right-hand side does not remove that nonlinearity. Under the stated convention, neither B nor C is a homogeneous linear equation.

Step 3: Normalize on a regular interval. For $t \neq 0$, division by t yields

$$y'' + \frac{1}{t}y' + \frac{1}{t}y = \frac{\sin t}{t}.$$

All three normalized coefficients are continuous on $(0, \infty)$, the largest open interval containing 1 with this property. Although $\sin t/t$ has a removable singularity at zero, $1/t$ does not.

Step 4: Test the original equation at zero. For a C^2 solution, the undivided equation at zero requires

$$0 \cdot y''(0) + y'(0) + y(0) = 0, \quad y'(0) + y(0) = 0.$$

The proposed data give $0 + 1 \neq 0$, so no such solution exists. The condition is necessary; deriving it alone is not an existence or uniqueness proof for every compatible pair. Division by the leading coefficient must not hide this compatibility requirement.

Question 2

An unknown trajectory satisfies

$$y'' = 6t, \quad y(1) = 2, \quad y'(1) = -1.$$

A student writes $y = t^3 + C$ as the general solution because differentiating twice gives $6t$.

Tasks

1. Find every C^2 solution of the differential equation by integrating twice. Explain the role of each integration constant.
2. Apply both initial conditions and verify the resulting solution in the equation and the data.
3. Diagnose the student's proposed general solution. Which initial slopes at $t = 1$ can that smaller family realize?
4. Retain only $y(1) = 2$. Describe all resulting solutions and their slopes at $t = 1$. Prove directly that restoring the slope condition selects exactly one solution.

Question 2 – Solution

Strategy. Integrate each derivative level separately, then examine which independent data the constants control.

Step 1: Recover the full family. One integration gives $y' = 3t^2 + A$; another gives

$$\boxed{y = t^3 + At + B}, \quad A, B \in \mathbb{R}.$$

These are all solutions on any interval: if two functions have second derivative $6t$, their difference has zero second derivative and therefore is affine. The constant A changes velocity; B changes position without changing velocity.

Step 2: Fit and verify both data. At $t = 1$, the conditions become $1 + A + B = 2$ and $3 + A = -1$. Thus $A = -4$, $B = 5$, giving

$$\boxed{y = t^3 - 4t + 5}.$$

Indeed, $y' = 3t^2 - 4$, $y'' = 6t$, $y(1) = 2$, and $y'(1) = -1$. The polynomial is defined on all of $\mathbb{R}$.

Step 3: Locate the missing freedom. Every function $t^3 + C$ solves the differential equation, but every one has derivative $3t^2$. In particular, its slope at $t = 1$ is always 3. It cannot realize the required slope -1 , regardless of C . Verification of a family does not by itself prove that the family is general.

Step 4: Separate position from velocity. With only $y(1) = 2$, we obtain $B = 1 - A$, so all possibilities are

$$\boxed{y = t^3 + A(t - 1) + 1}, \quad y'(1) = 3 + A.$$

The initial slope ranges over every real number as A varies. Setting it to -1 forces $A = -4$ and hence the single solution above.

Alternatively, the difference w of any two solutions with both prescribed data satisfies $w'' = 0$, $w(1) = w'(1) = 0$. Writing $w = at + b$ gives $a = 0$ and $b = 0$. This proves uniqueness directly, rather than merely counting constants.

Question 3

The functions e^t and e^{-t} are supplied as candidate solutions of

$$y'' - y = 0.$$

You may use this theorem: if p, q, g are continuous on an open interval I , then $y'' + py' + qy = g$ with $y(t_0) = a$, $y'(t_0) = b$, $t_0 \in I$, has exactly one solution on I .

Tasks

1. Verify that every $y = Ae^t + Be^{-t}$ solves the equation, without using a characteristic equation.
2. Determine A, B when $y(\ln 2) = 5/2$ and $y'(\ln 2) = 3/2$. Verify both data.
3. For arbitrary real data $y(\ln 2) = a$, $y'(\ln 2) = b$, solve for A, B . Show that every pair of initial data can be realized.
4. Use the theorem to prove that the displayed family contains every solution on $\mathbb{R}$. Explain why having two named constants would not be sufficient if the proposed family were $Ce^t + De^t$ instead.

Question 3 – Solution

Strategy. Check the functions, solve the two data equations, then use uniqueness to establish completeness.

Step 1: Verify the entire family. Differentiation gives

$$y' = Ae^t - Be^{-t}, \quad y'' = Ae^t + Be^{-t} = y.$$

Therefore $y'' - y = 0$ for every A, B . No root-solving method is required.

Step 2: Fit the stated data at their actual base point. Since $e^{\ln 2} = 2$ and $e^{-\ln 2} = 1/2$, the equations are

$$2A + B/2 = 5/2, \quad 2A - B/2 = 3/2.$$

Adding and subtracting yields $A = 1$, $B = 1$. Thus

$$\boxed{y = e^t + e^{-t}}.$$

Its value at $\ln 2$ is $2 + 1/2 = 5/2$, and its derivative there is $2 - 1/2 = 3/2$.

Step 3: Solve the general data problem. For arbitrary a, b the same equations give

$$4A = a + b, \quad B = a - b, \quad \boxed{A = (a + b)/4, \quad B = a - b}.$$

There is exactly one coefficient pair for each $(a, b) \in \mathbb{R}^2$. Substitution returns $y(\ln 2) = a$ and $y'(\ln 2) = b$, so the claim includes all data, not only the numerical example.

Step 4: Establish completeness, not just verification. Here $p = 0$, $q = -1$, $g = 0$ are continuous on $\mathbb{R}$. Any solution has some data a, b at $\ln 2$. Step 3 constructs a member of the displayed family with those same data; uniqueness forces agreement on $\mathbb{R}$.

By contrast, $Ce^t + De^t = (C + D)e^t$ has only one effective parameter. Every such function satisfies $y' = y$, so at the base point it can realize only data with $b = a$. Two symbols for constants do not necessarily provide two independent freedoms.

Question 4

For $y'' + y = 0$, you may use the complete family

$$y(t) = A \cos t + B \sin t.$$

Consider two endpoint conditions $y(0) = 0$ and $y(L) = c$, where $L > 0$ is specified. Compare these with initial data $y(0) = 0$, $y'(0) = b$.

Tasks

1. Solve the initial-data problem for an arbitrary real b . Explain why it always determines exactly one member of the family.
2. Determine all solutions of the endpoint problem for $L = \pi/2$ and arbitrary c .
3. Determine all solutions for $L = \pi$, treating $c = 0$ and $c \neq 0$ separately. Explain why two conditions need not select one solution.
4. For $L = \pi$, $c = 0$, sketch the solutions with $B = -1, 0, 1, 2$. Explain why their common endpoint values do not violate uniqueness for initial-value problems.

Question 4 – Solution

Strategy. Translate each condition into a constraint on A, B and check whether the second constraint supplies new information.

Step 1: Use value and slope at the same point. From $y(0) = A$ and $y'(0) = B$, initial data $0, b$ give

$$y = b \sin t.$$

Both constants are fixed, for every real b . The coefficients of the normalized equation are continuous everywhere, consistent with initial-value uniqueness.

Step 2: Move the second datum to $\pi/2$. The first condition gives $A = 0$. At the other endpoint,

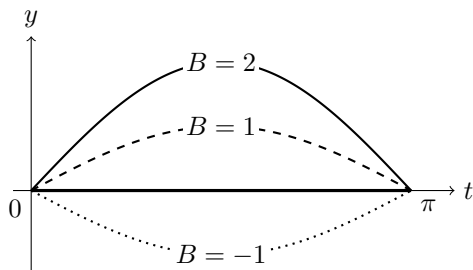
$$y(\pi/2) = B = c, \quad y = c \sin t.$$

This particular pair of endpoint conditions also determines one solution.

Step 3: Detect a redundant or incompatible condition. At $L = \pi$, every function $B \sin t$ has $y(\pi) = 0$. Consequently,

$$c = 0 : y = B \sin t \ (B \in \mathbb{R}), \quad c \neq 0 : \text{no solution}.$$

When $c = 0$, the second condition adds no restriction on B . When $c \neq 0$, it contradicts the first condition and the equation. Merely counting two stated conditions misses this distinction.



The horizontal solution is $B = 0$.

Step 4: Compare endpoint agreement with initial-data agreement. The four curves share their endpoint positions but have different initial slopes B . At $t = \pi$ their slopes are $-B$, also different. The initial-value theorem requires agreement of both value and first derivative at the same point. Agreement of values at two separate points is a different requirement, so the sketch creates no contradiction.

Question 5

For a normalized linear equation $y'' + p(t)y' + q(t)y = g(t)$, continuity of p, q, g on an open interval containing the initial time guarantees a unique solution throughout that interval for any two initial data.

Consider

$$t(t-2)y'' + (t+1)y' + \ln(t+3)y = \frac{1}{t-4}.$$

All functions and solutions are real.

Tasks

1. Find the domain of the displayed coefficients before normalization. List the additional points excluded when solving for y'' .
2. Find the largest theorem-guaranteed interval for initial time $t_0 = 1$, and then for $t_0 = 3$, with arbitrary finite initial value and slope.
3. Does a finite endpoint of such an interval prove that every solution blows up there? Explain precisely what the theorem does and does not say.
4. Test that distinction on the separate equation $ty'' + y' = 0$ with data $y(1) = a, y'(1) = b$. Solve on $(0, \infty)$ and determine exactly which resulting solutions extend as C^2 solutions of the undivided equation across zero.

Question 5 – Solution

Strategy. Locate coefficient obstructions before applying the theorem, then compare its guaranteed interval with the behavior of actual solutions.

Step 1: Separate domain from degeneracy. The logarithm requires $t > -3$ and the forcing excludes $t = 4$. Thus the original coefficient domain is $(-3, 4) \cup (4, \infty)$. Normalization additionally excludes $t = 0, 2$, where the leading coefficient vanishes. For other allowed t ,

$$p = \frac{t+1}{t(t-2)}, \quad q = \frac{\ln(t+3)}{t(t-2)}, \quad g = \frac{1}{(t-4)t(t-2)}.$$

Step 2: Choose the connected interval containing the data. The regular intervals are $(-3, 0)$, $(0, 2)$, $(2, 4)$ and $(4, \infty)$. Therefore the largest applicable intervals are

$$\boxed{t_0 = 1 : (0, 2)}, \quad \boxed{t_0 = 3 : (2, 4)}.$$

The initial values do not change these coefficient-based guarantees.

Step 3: State the limitation accurately. The theorem guarantees existence and uniqueness within the selected interval. It does not classify the behavior at a singular endpoint or prove that every solution becomes unbounded there. Whether a particular solution extends requires examination of that solution and the original equation. A coefficient-domain boundary is also distinct from a pole in the solution itself.

Step 4: Solve the comparison equation completely. Since $(ty')' = ty'' + y'$, integration on $t > 0$ gives $ty' = C$, then $y = C \ln t + D$. The data at 1 imply $C = b$, $D = a$, so

$$\boxed{y = a + b \ln t \quad (t > 0)}.$$

If $b \neq 0$, y is unbounded as $t \downarrow 0$, so no continuous, hence no C^2 , extension exists. If $b = 0$, the constant solution $y = a$ satisfies the undivided equation on all of $\mathbb{R}$.

Thus extension occurs exactly when $\boxed{b = 0}$. In fact it is unique among C^2 extensions: on the negative side every solution has form $C_- \ln |t| + D_-$; continuity forces $C_- = 0$, $D_- = a$. The normalized theorem alone did not provide that endpoint conclusion.

Question 6

Let $L[y] = y'' + y$ and consider the forced equation $L[y] = t$. The functions t , $\sin t$ and $\cos t$ are supplied for direct verification.

Tasks

1. Compute $L[t]$, $L[\sin t]$ and $L[\cos t]$. Determine exactly which constants a, b, c make $at + b\sin t + c\cos t$ solve the forced equation on $\mathbb{R}$.
2. Let $u = t + \sin t$ and $v = t + \cos t$. Determine the equations satisfied by $u + v$, $u - v$ and $(u + v)/2$.
3. For arbitrary solutions u, v of $L[y] = t$, determine exactly which real weights α, β ensure $\alpha u + \beta v$ is again a solution. Explain the difference from homogeneous superposition.
4. Find a solution with $y(0) = 2$, $y'(0) = 0$. Verify it and justify uniqueness using the continuous-coefficient initial-value theorem.

Question 6 – Solution

Strategy. Apply the linear operator to the proposed combination and track the forcing as carefully as the unknown function.

Step 1: Identify the forced and zero-forcing pieces. Direct differentiation yields

$$L[t] = t, \quad L[\sin t] = 0, \quad L[\cos t] = 0.$$

Linearity then gives $L[at + b \sin t + c \cos t] = at$. Equality with t for every real t requires $a = 1$; b, c are arbitrary. Hence all such candidates are

$$\boxed{y = t + b \sin t + c \cos t}.$$

The forcing fixes the coefficient of t , leaving the two homogeneous freedoms.

Step 2: Track sums, differences and averages. Because $L[u] = L[v] = t$,

$$L[u + v] = 2t, \quad L[u - v] = 0, \quad L[(u + v)/2] = t.$$

The sum is generally not a solution of the original forced equation. The difference solves the associated homogeneous equation, and the average preserves the original forcing.

Step 3: Determine the allowed weights. For any such u, v ,

$$L[\alpha u + \beta v] = (\alpha + \beta)t.$$

This equals t on $\mathbb{R}$ exactly when $\boxed{\alpha + \beta = 1}$. The weights need not be nonnegative. For homogeneous solutions, by contrast, the right-hand side remains zero for every pair of weights. Also, the zero function is not a solution of $L[y] = t$; the forced solution set does not have all the closure properties of the homogeneous one.

Step 4: Fit and verify the initial data. For the candidate family, $y(0) = c$ and $y'(0) = 1 + b$. The data give $c = 2$, $b = -1$, so

$$\boxed{y = t - \sin t + 2 \cos t}.$$

Its derivative is $1 - \cos t - 2 \sin t$, its initial value and slope are 2 and 0, and $y'' + y = t$. Since $p = 0$, $q = 1$, $g = t$ are continuous on $\mathbb{R}$, the initial-value theorem proves uniqueness there. No general method for finding particular solutions was assumed; the necessary candidate was supplied.

Question 7

Consider $y'' + 4y = 0$ with $y(0) = 1$, $y'(0) = 2$. Define

$$E(t) = (y'(t))^2 + 4(y(t))^2.$$

For direct verification you may use the candidate $y(t) = \cos(2t) + \sin(2t)$.

Tasks

1. Prove that E is constant for every solution and calculate its value for the stated data.
2. Deduce a bound on $|y(t)|$ without first substituting a solution formula. Verify the candidate and determine a time when the bound is attained.
3. In coordinates (y, v) with $v = y'$, sketch the curve forced by the energy identity, mark the initial state and indicate the direction of motion there.
4. Prove uniqueness directly: if two solutions share both initial data, apply the same identity to their difference. Explain why knowing the energy alone does not specify a unique solution.

Question 7 – Solution

Strategy. Differentiate a sum of nonnegative terms, then use its conservation both to bound a solution and to eliminate a zero-data difference.

Step 1: Derive the conserved quantity. For any solution,

$$E' = 2y'y'' + 8yy' = 2y'(y'' + 4y) = 0.$$

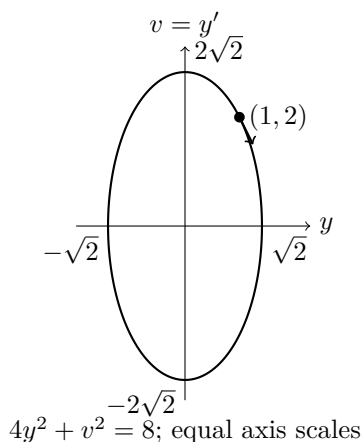
Thus $E(t) = E(0)$ throughout its interval. The prescribed data give

$$E = 2^2 + 4(1)^2 = 8.$$

Step 2: Obtain and attain the displacement bound. Since $(y')^2 \geq 0$, $4y^2 \leq 8$, whence $|y| \leq \sqrt{2}$. For the supplied candidate,

$$y' = -2\sin(2t) + 2\cos(2t), \quad y'' = -4y.$$

It has the required data, and at $t = \pi/8$ it has $y = \sqrt{2}$, $y' = 0$. Therefore the bound is sharp, not just a convenient overestimate.



Step 3: Interpret the state curve. Writing $v = y'$ gives $4y^2 + v^2 = 8$, an ellipse with intercepts $y = \pm\sqrt{2}$ and $v = \pm 2\sqrt{2}$. The initial point is $(1, 2)$. Its velocity in these coordinates is

$$(y', v') = (v, -4y) = (2, -4),$$

so it moves rightward and downward there. The axes represent state variables, not time versus displacement.

Step 4: Prove uniqueness and distinguish energy from data. The difference w of two solutions with the same initial data satisfies $w'' + 4w = 0$ and $w(0) = w'(0) = 0$. Its conserved quantity is $(w')^2 + 4w^2 = 0$. Both terms are nonnegative, so $w = 0$ everywhere on their common interval.

Energy alone only fixes an ellipse. For example, $\sqrt{2}\cos(2t)$ and $\sqrt{2}\sin(2t)$ both have energy 8 but different initial states. Initial position and velocity identify a point and its motion; the scalar energy value does not replace them.

Question 8

Suppose both $y_1(t) = 1$ and $y_2(t) = t^2$ solve an unknown normalized homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on $(0, \infty)$.

Tasks

1. Determine p and q directly from the two supplied solutions. Check your result by substitution.
2. Can any normalized equation of this form, with continuous p, q on an interval containing zero, have both supplied functions as solutions there? Prove your answer without a determinant formula.
3. Multiply the equation from Task 1 by t . Show that the undivided equation admits $y = C_1 + C_2t^2$ on all of $\mathbb{R}$. Determine all members of this family satisfying $y(0) = a$, $y'(0) = b$.
4. Sketch $y = t^2$ and $y = 2t^2$ near zero. Explain how distinct solutions can share the same tangent there without contradicting the regular linear initial-value theorem.

Question 8 – Solution

Strategy. Insert simple known solutions to recover coefficients, then inspect what changes when the leading coefficient vanishes.

Step 1: Recover the equation on its stated interval. Inserting $y_1 = 1$ gives $q(t) = 0$. Inserting $y_2 = t^2$ then gives $2 + 2tp(t) = 0$, so

$$\boxed{p(t) = -1/t, \quad q(t) = 0}, \quad \boxed{y'' - \frac{1}{t}y' = 0 \quad (t > 0)}.$$

The constant solution has both derivatives zero, and $2 - (1/t)(2t) = 0$ verifies the quadratic solution.

Step 2: Rule out a regular equation through zero. At $t = 0$, the function t^2 has value 0, first derivative 0 and second derivative 2. In any normalized equation with finite coefficients at zero it would require

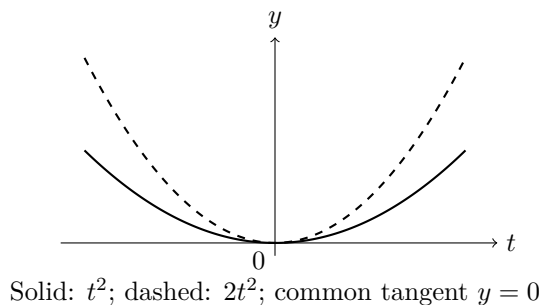
$$2 + p(0) \cdot 0 + q(0) \cdot 0 = 0,$$

which is impossible. Thus no continuous-coefficient normalized equation on an interval containing zero can have both supplied functions as solutions. The obstruction is already visible from the quadratic alone.

Step 3: Retain the degenerate leading coefficient. The undivided equation is $ty'' - y' = 0$. For $y = C_1 + C_2t^2$, its left-hand side is $t(2C_2) - 2C_2t = 0$, including at zero. The family has $y(0) = C_1$, $y'(0) = 0$, so

$$\boxed{b = 0 : y = a + C_2t^2 \ (C_2 \in \mathbb{R})}, \quad \boxed{b \neq 0 : \text{no member}}.$$

Indeed, the undivided equation itself at zero forces $y'(0) = 0$, so nonzero b permits no C^2 solution at all.



Step 4: Explain the shared tangent. The two plotted functions share value 0 and slope 0 at zero but have different curvatures. The coefficient of y'' vanishes there, so the equation does not determine the second derivative from those data. Its normalized coefficient $-1/t$ is not continuous at the initial point. The regular linear theorem therefore does not apply, and the multiple zero-data solutions do not contradict it.

Question 9

Consider the nonlinear equation

$$y'' = \sqrt{|y|}, \quad y(0) = 0, \quad y'(0) = 0.$$

For a real parameter $a \geq 0$, define

$$y_a(t) = \begin{cases} 0, & t \leq a, \\ (t-a)^4/144, & t > a. \end{cases}$$

A classical solution here means a real C^2 function satisfying the equation at every point of its interval.

Tasks

1. Verify the equation on each open side of $t = a$.
2. Check the value, first derivative and second derivative at the joining point. Does the equation hold there as well?
3. Verify both initial conditions for every $a \geq 0$, and show that different finite choices of a give different solutions. Identify one additional solution.
4. Sketch y_0 , y_1 and the zero solution on $[0, 3]$. Explain why continuity of the right-hand side does not justify applying the linear initial-value uniqueness theorem to this example. Does an assigned position and slope always determine a unique trajectory?

Question 9 – Solution

Strategy. Verify the join at the derivative order required by the equation, then use the family as a counterexample to an overgeneralized uniqueness claim.

Step 1: Verify away from the join. For $t < a$, both sides of the equation are zero. For $t > a$,

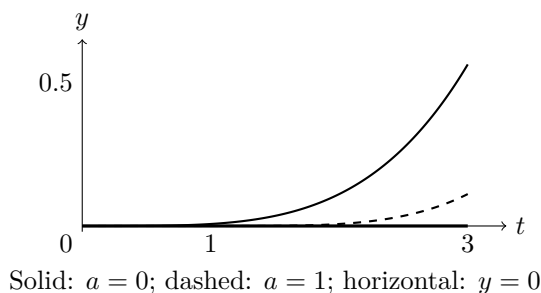
$$y'_a = \frac{(t-a)^3}{36}, \quad y''_a = \frac{(t-a)^2}{12}, \quad \sqrt{|y_a|} = \sqrt{\frac{(t-a)^4}{144}} = \frac{(t-a)^2}{12}.$$

The coefficient 144 is essential to this identity.

Step 2: Verify classical regularity at the join. The right-hand limits of y_a, y'_a, y''_a as $t \downarrow a$ are all zero, matching the values on the left. The first and second derivatives at the join are also zero, as follows directly from their difference quotients. Hence y_a is C^2 on $\mathbb{R}$.

At $t = a$, the equation reads $0 = \sqrt{|0|}$, so it holds there, not merely away from the join. There is no hidden corner in position or velocity.

Step 3: Check the common data and distinctness. Since $a \geq 0$, every family member has $y_a(0) = y'_a(0) = 0$, including $a = 0$ by the join calculation. If $a_1 < a_2$, choose $a_1 < t < a_2$. Then $y_{a_1}(t) > 0$ but $y_{a_2}(t) = 0$, proving they are different solutions. The function $\boxed{y \equiv 0}$ is another solution, different from every finite-delay member.



Step 4: State exactly what has failed. The right-hand side $\sqrt{|y|}$ is continuous but depends nonlinearly on the unknown function. This is not an equation $y'' + p(t)y' + q(t)y = g(t)$ with prescribed continuous coefficients, so that linear theorem cannot be invoked. The verified family proves that continuity alone does not ensure uniqueness for a general nonlinear second-order equation.

Two initial data determine one solution under appropriate hypotheses, not for every differential equation. This example exhibits infinitely many solutions; it does not claim to classify all solutions of the nonlinear equation.

Question 10

A prescribed acceleration changes abruptly at $t = 1$:

$$y'' = \begin{cases} 0, & 0 \leq t < 1, \\ 2, & t > 1, \end{cases} \quad y(0) = 0, \quad y'(0) = 1.$$

Seek a function that is C^1 on $[0, \infty)$ and C^2 on each open side of 1, satisfying the equation there. No value for y'' at 1 is assumed. Thus position and velocity must be continuous at the switch.

Tasks

1. Integrate on the first phase and find the position and velocity arriving at $t = 1$.
2. Integrate on the second phase with two new constants, determine them using the matching conditions, and give the complete piecewise trajectory.
3. Compute the one-sided derivatives of velocity at the switch. Can any assigned value of the forcing at $t = 1$ make this trajectory a classical C^2 solution through the switch?
4. Sketch position and velocity. A student adds $k(t - 1)$ to the second-phase position while leaving the first phase unchanged. Determine which k satisfy the stated requirements, and explain why position matching alone is insufficient.

Question 10 – Solution

Strategy. Carry both position and velocity across the switch, and distinguish a piecewise classical solution from one that is C^2 everywhere.

Step 1: Propagate the first phase. With zero acceleration and initial velocity 1, $y' = 1$ and $y = t$ for $0 \leq t \leq 1$. The arriving data are

$$\boxed{y(1) = 1, \quad y'(1) = 1}.$$

Step 2: Use both matching conditions. Writing the second-phase antiderivative about the switch gives

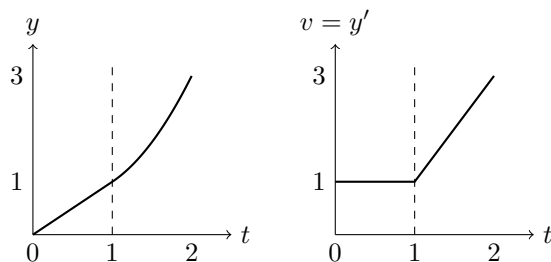
$$y = (t - 1)^2 + A(t - 1) + B \quad (t > 1).$$

Continuity of position sets $B = 1$; continuity of velocity sets $A = 1$. Thus

$$\boxed{y(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ t + (t - 1)^2, & t > 1, \end{cases}} \quad y'(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 1 + 2(t - 1), & t > 1. \end{cases}$$

These formulas satisfy all the stated requirements and uniquely determine both phases.

Step 3: Check the actual regularity. The derivative of velocity at 1 has left-hand value 0 and right-hand value 2. Therefore $y''(1)$ does not exist. Assigning a forcing value at that single point cannot change the unequal one-sided limits. The trajectory is C^1 and piecewise C^2 , but is not a classical C^2 solution across the switch.



Position has a continuous tangent; velocity has a corner.

Step 4: Reject an unmatched velocity. Adding $k(t - 1)$ on $t > 1$ keeps the position continuous and does not change the second derivative on that open phase. However, the right-hand velocity at 1 becomes $1 + k$, while the left-hand velocity stays 1. Hence the C^1 requirement forces $\boxed{k = 0}$.

A second-order problem carries two pieces of state information. Position matching alone would admit a jump in velocity that is excluded by the problem's stated regularity; checking only the differential equation away from the switch would miss it.

Question 1

Consider

$$6y'' + y' - 2y = 0, \quad y(1) = 7, \quad y'(1) = 0.$$

For a constant-coefficient homogeneous second-order equation with distinct real characteristic roots r_1, r_2 , the general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

Tasks

1. Substitute $y = e^{rt}$ to derive the characteristic equation. Find and verify both roots.
2. Write the general solution using exponentials based at $t = 1$, then determine its constants from the initial data.
3. Verify the resulting function in the original equation and both initial conditions. Explain why it is the unique solution on $\mathbb{R}$.
4. Classify the stationary point at $t = 1$ and determine the limits as $t \rightarrow \infty$ and $t \rightarrow -\infty$. Explain why zero initial slope does not mean a constant solution.

Question 1 – Solution

Strategy. Keep the leading coefficient in the characteristic polynomial and choose a shifted exponential basis to simplify the data.

Step 1: Derive, factor and check the polynomial. Substitution gives $(6r^2 + r - 2)e^{rt} = 0$. Since $e^{rt} \neq 0$,

$$6r^2 + r - 2 = (2r - 1)(3r + 2) = 0, \quad \boxed{r_1 = 1/2, \quad r_2 = -2/3}.$$

Direct evaluation gives $6(1/2)^2 + 1/2 - 2 = 0$ and $6(-2/3)^2 - 2/3 - 2 = 0$.

Step 2: Use the actual initial time. A shifted basis only rescales the arbitrary constants, so write

$$y = Ae^{(t-1)/2} + Be^{-2(t-1)/3}.$$

The data give $A + B = 7$ and $A/2 - 2B/3 = 0$. Thus $A = 4$, $B = 3$, and

$$\boxed{y = 4e^{(t-1)/2} + 3e^{-2(t-1)/3}}.$$

Step 3: Verify the equation and uniqueness. Each exponential contributes its coefficient times $6r^2 + r - 2$ to the residual, so their sum solves the original equation. At 1, its value is $4 + 3 = 7$ and its derivative is $2 - 2 = 0$. After division by 6, all coefficients are continuous on $\mathbb{R}$. The linear initial-value theorem therefore gives uniqueness there.

Step 4: Interpret the initial balance of modes. Differentiating twice gives

$$y'' = e^{(t-1)/2} + \frac{4}{3}e^{-2(t-1)/3} > 0.$$

Thus y' is strictly increasing, crosses zero at 1, and $\boxed{y(1) = 7}$ is the unique global minimum. The positive-root mode gives $y \rightarrow +\infty$ as $t \rightarrow \infty$; the negative-root mode gives $y \rightarrow +\infty$ as $t \rightarrow -\infty$.

Zero slope at one instant is a balance between two changing modes. Indeed the equation at 1 gives $y''(1) = 7/3$, not zero. A constant solution would have to satisfy $-2y = 0$ and could not have value 7.

Question 2

Consider the family of initial-value problems

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = v.$$

Call a solution forward bounded if $|y(t)|$ is bounded for $t \geq 0$.

Tasks

1. Find the roots and express the solution coefficients in terms of v .
2. Determine exactly which initial slope produces a forward-bounded solution. State its limit.
3. Replace that slope by $v = -2 + \varepsilon$, where $0 < \varepsilon < 1$. Find when the growing and decaying terms have equal magnitude, and determine the eventual behavior.
4. For $\varepsilon = 0.01$, calculate that time to three decimal places and sketch the perturbed and unperturbed solutions on $[0, 2]$. Explain why a small initial error can change the long-time conclusion.

Question 2 – Solution

Strategy. Express the initial data in the two modes and isolate the coefficient of the growing exponential.

Step 1: Resolve the data into modes. The characteristic polynomial is $(r - 3)(r + 2)$, so $y = Ae^{3t} + Be^{-2t}$. From $A + B = 1$ and $3A - 2B = v$,

$$A = (v + 2)/5, \quad B = (3 - v)/5.$$

The nonzero root separation makes this data system invertible.

Step 2: Remove the growing mode exactly. If $A \neq 0$, the term Ae^{3t} dominates and $|y| \rightarrow \infty$. Thus forward boundedness holds exactly when $A = 0$, or $v = -2$. The selected solution is $y = e^{-2t}$ and tends to zero.

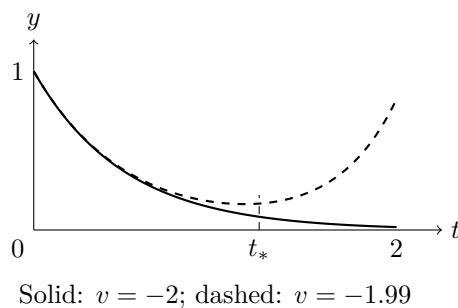
Step 3: Track a small slope perturbation. For $v = -2 + \varepsilon$,

$$y_\varepsilon = \frac{\varepsilon}{5}e^{3t} + \frac{5 - \varepsilon}{5}e^{-2t}.$$

Both terms are positive. They are equal precisely when $e^{5t} = (5 - \varepsilon)/\varepsilon$, giving

$$t_* = \frac{1}{5} \ln \frac{5 - \varepsilon}{\varepsilon}.$$

This is positive for the stated range of ε . Eventually $y_\varepsilon \rightarrow +\infty$, although its initial slope is negative.



Step 4: Quantify the delayed disagreement. For $\varepsilon = 0.01$, $t_* = (\ln 499)/5 \approx 1.243$. The exact difference is

$$y_\varepsilon - e^{-2t} = \frac{\varepsilon}{5}(e^{3t} - e^{-2t}).$$

It vanishes initially and is small with ε on any fixed finite interval, but grows without bound as $t \rightarrow \infty$ for every fixed positive ε . Forward boundedness here requires exact cancellation of a mode; it is not robust to a nonzero slope error. The time t_* compares mode magnitudes and is not the time of the trajectory's minimum.

Question 3

A transient response satisfies

$$y'' + 5y' + 6y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Only $t \geq 0$ is considered in the qualitative questions.

Tasks

1. Solve the initial-value problem using its real distinct roots.
2. Prove that $y(t) > 0$ for $t > 0$, even though $y(t) \rightarrow 0$ as $t \rightarrow \infty$.
3. Find the unique maximum, including its time and height, and sketch the response with that point marked.
4. Calculate $\int_0^\infty y(t) dt$ in two ways: from the explicit formula and by integrating the differential equation. Explain why negative roots alone do not imply monotone decay from the initial value.

Question 3 – Solution

Strategy. Separate eventual decay from the shape created by two modes with opposite coefficients.

Step 1: Solve for the two coefficients. The characteristic equation is $(r + 2)(r + 3) = 0$, so $y = Ae^{-2t} + Be^{-3t}$. The data give $A + B = 0$ and $-2A - 3B = 1$. Hence

$$y = e^{-2t} - e^{-3t}.$$

Its value and derivative at zero are 0 and 1, and both exponential terms satisfy the equation.

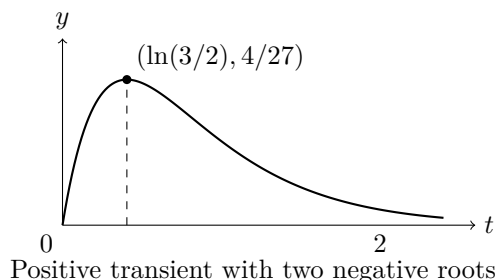
Step 2: Prove positivity and decay. For $t > 0$, $y = e^{-3t}(e^t - 1) > 0$. Both exponential terms tend to zero, so the positive response eventually decays to zero. Positivity does not require both mode coefficients to be positive.

Step 3: Locate and evaluate the only peak. The derivative factors as

$$y' = e^{-3t}(3 - 2e^t).$$

It is positive before $t_m = \ln(3/2)$ and negative afterward. At that time $e^{-t_m} = 2/3$, so

$$t_m = \ln(3/2), \quad y(t_m) = 4/9 - 8/27 = 4/27.$$



Step 4: Check the total area independently. Direct integration gives $\int_0^\infty y \, dt = 1/2 - 1/3 = \boxed{1/6}$. Alternatively, integrate the equation on $[0, T]$:

$$y'(T) - y'(0) + 5(y(T) - y(0)) + 6 \int_0^T y \, dt = 0.$$

The explicit formula verifies $y(T), y'(T) \rightarrow 0$ and convergence of the integral. Passing to the limit gives $-1 + 6 \int_0^\infty y \, dt = 0$, confirming the result.

Negative roots control the eventual limit of each mode. Their combination can initially increase: this solution starts at zero with positive slope, reaches a peak, and only then decreases.

Question 4

For the equation

$$y'' + 3y' + 2y = 0, \quad y(0) = a > 0, \quad y'(0) = b,$$

classify the entire range of real initial slopes b on the time interval $t \geq 0$.

Tasks

1. Find the solution in terms of a, b .
2. Determine exactly when $y(t) \geq 0$ for every $t \geq 0$. Prove both necessity and sufficiency.
3. Determine exactly when the solution is both nonnegative and nonincreasing on $[0, \infty)$, including boundary slopes.
4. For the remaining slopes, find the unique zero when one occurs, or the unique positive-time maximum when one occurs. Summarize the cases on a diagram of the ratio b/a .

Question 4 – Solution

Strategy. Factor out a positive exponential and reduce sign questions to affine expressions in $z = e^{-t} \in (0, 1]$.

Step 1: Fit the real-root solution. The roots are $-1, -2$. Solving $A + B = a$, $-A - 2B = b$ gives

$$y = (2a + b)e^{-t} - (a + b)e^{-2t}.$$

Write $A = 2a + b$, $B = -(a + b)$, so $A + B = a$.

Step 2: Classify nonnegativity. Since $y = e^{-t}(A + Bz)$, its sign is the sign of an affine function with values A at $z = 0$ and $a > 0$ at $z = 1$. If $A \geq 0$, it is positive for all $0 < z \leq 1$. If $A < 0$, it is negative for sufficiently small positive z . Therefore

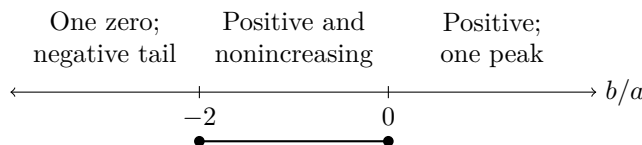
$$y(t) \geq 0 \text{ for all } t \geq 0 \iff b \geq -2a.$$

At $b = -2a$, the solution is the strictly positive ae^{-2t} at every finite time.

Step 3: Impose monotonicity as well. We have $y' = -e^{-t}(A + 2Bz)$. Its bracket has endpoint values A and $A + 2B = -b$. Thus it is nonnegative on $0 < z \leq 1$ exactly when $A \geq 0$ and $b \leq 0$. Combining the conditions yields

$$-2a \leq b \leq 0.$$

At $b = 0$, the derivative vanishes initially but is negative for every $t > 0$. Both boundary slopes are included.



Both boundary slopes belong to the middle case.

Step 4: Resolve the two remaining regimes. If $b < -2a$, solving $A + Be^{-t} = 0$ gives the unique zero

$$t_z = \ln \frac{a + b}{2a + b} > 0.$$

Both numerator and denominator are negative, and their ratio exceeds 1. The solution crosses from positive to negative and tends to zero from below.

If $b > 0$, it remains positive but initially rises. Solving $y' = 0$ gives the unique positive-time maximum at

$$t_m = \ln \frac{2(a + b)}{2a + b} > 0.$$

The derivative changes from positive to negative there. These cases exhaust all real b ; every solution still tends to zero because both roots are negative.

Question 5

An unknown monic equation $y'' + py' + qy = 0$ has distinct real roots r_1, r_2 . A solution is known to contain both modes with nonzero coefficients. Its exact samples are

$$S_n = y(n \ln 2), \quad S_0 = 3, \quad S_1 = 1, \quad S_2 = 3/8, \quad S_3 = 5/32.$$

Set $z_j = e^{r_j \ln 2} > 0$, so $S_n = Az_1^n + Bz_2^n$.

Tasks

1. Derive a recurrence $S_{n+2} = sS_{n+1} - kS_n$ in terms of z_1, z_2 .
2. Use the four samples to determine s, k . Explain why this pair is uniquely determined.
3. Recover the real roots, the coefficients p, q , the solution $y(t)$ and its initial slope. Verify all four samples.
4. Explain what would fail if the observed solution contained only one mode. Could arbitrarily many exact samples of one pure exponential identify both roots of an unknown second-order equation?

Question 5 – Solution

Strategy. Recover the two exponential multipliers from a sample recurrence, then convert those multipliers back into differential-equation roots.

Step 1: Derive the recurrence. Each z_j satisfies $z_j^2 - (z_1 + z_2)z_j + z_1z_2 = 0$. Multiplying by its mode coefficient and z_j^n , then summing, gives

$$\boxed{S_{n+2} = sS_{n+1} - kS_n}, \quad s = z_1 + z_2, \quad k = z_1z_2.$$

Step 2: Solve the sample equations exactly. For $n = 0, 1$,

$$s - 3k = 3/8, \quad (3/8)s - k = 5/32.$$

Substitution of $s = 3/8 + 3k$ into the second equation gives $k/8 = 1/64$. Hence

$$\boxed{k = 1/8, \quad s = 3/4}.$$

The nonzero coefficient $1/8$ in this elimination proves uniqueness. Equivalently, $S_0S_2 - S_1^2 = 1/8 \neq 0$.

Step 3: Recover and verify the continuous model. The multipliers solve $z^2 - (3/4)z + 1/8 = (z - 1/2)(z - 1/4) = 0$. Since $r = \ln z / \ln 2$, the roots are $-1, -2$. Thus $p = 3, q = 2$.

Writing $y = Ae^{-t} + Be^{-2t}$, the first two samples give $A + B = 3, A/2 + B/4 = 1$, so $A = 1, B = 2$. Therefore

$$\boxed{y = e^{-t} + 2e^{-2t}}, \quad \boxed{y'(0) = -5}.$$

The four sample values are $3, 1/2 + 2/4 = 1, 1/4 + 2/16 = 3/8$, and $1/8 + 2/64 = 5/32$, as required. Both coefficients are nonzero.

Step 4: Identify the missing-information case. For a pure observed mode $S_n = Cz^n$, the recurrence only imposes $z^2 - sz + k = 0$. Any positive second multiplier different from z gives another pair s, k with exactly the same samples. Thus the active root can be identified, but an unexcited root remains undetermined, even with arbitrarily many noiseless samples of that one trajectory. Exact data do not reveal a mode whose coefficient is zero.

Question 6

Consider homogeneous constant-coefficient equations with two distinct real roots $r_1 < r_2$ and endpoint data $y(0) = a$, $y(L) = b$, where $L > 0$.

Tasks

1. Prove that every pair $a, b \in \mathbb{R}$ determines exactly one solution by solving for the two mode coefficients. Identify the quantity that cannot vanish.
2. Solve $y'' - 3y' + 2y = 0$ with $y(0) = y(\ln 2) = 1$.
3. Prove that a nonzero combination $Ae^{r_1 t} + Be^{r_2 t}$ has at most one real zero. Treat vanishing coefficients explicitly, and deduce positivity between strictly positive endpoints.
4. For the solution in Task 2, find its maximum on $[0, \ln 2]$ and sketch it. Does positivity between positive endpoints mean the solution stays below its endpoint values?

Question 6 – Solution

Strategy. Use the strict inequality of real exponentials to solve the endpoint problem and control the number of zeros.

Step 1: Invert the endpoint constraints. For $y = Ae^{r_1 t} + Be^{r_2 t}$, the conditions give $A + B = a$ and $Ae^{r_1 L} + Be^{r_2 L} = b$. Therefore

$$B = \frac{b - ae^{r_1 L}}{e^{r_2 L} - e^{r_1 L}}, \quad A = a - B.$$

Because $r_2 > r_1$ and $L > 0$, the denominator is strictly positive. There is exactly one coefficient pair for all endpoint data, unlike some oscillatory endpoint problems.

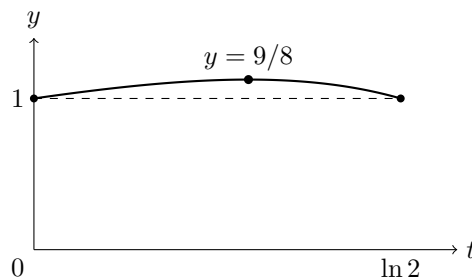
Step 2: Fit the concrete equation. Here the roots are 1 and 2. At $L = \ln 2$, the equations are $A + B = 1$ and $2A + 4B = 1$. Hence

$$y = \frac{3}{2}e^t - \frac{1}{2}e^{2t}.$$

Both endpoints have value 1, and each mode satisfies the original equation.

Step 3: Bound the number of zeros and deduce positivity. Factor the general solution as $e^{r_1 t}(A + Be^{(r_2 - r_1)t})$. If $B \neq 0$, the bracket is strictly monotone, so it has at most one zero. If $B = 0$, a nonzero solution is a nonvanishing pure exponential; the case $A = 0$ is likewise nonvanishing.

If a solution with positive endpoints became negative inside, continuity would force two distinct zeros. If it merely touched zero inside while staying nonnegative, that interior minimum would have $y = y' = 0$; initial-value uniqueness would force the identically zero solution, contradicting the endpoints. Thus it is strictly positive throughout the interval.



Positive endpoints do not bound the interior maximum.

Step 4: Distinguish positivity from an upper bound. For the concrete solution, $y' = e^t(3/2 - e^t)$ changes from positive to negative at $t = \ln(3/2)$, which lies between 0 and $\ln 2$. Its value there is

$$y_{\max} = \frac{3}{2} \frac{3}{2} - \frac{1}{2} \frac{9}{4} = \frac{9}{8} > 1.$$

The curve stays positive but exceeds both endpoint values. A restriction on zeros is not a maximum principle bounding the response by its endpoints.

Question 7

For a real parameter α , consider

$$y'' + 2\alpha y' + (\alpha^2 - 4)y = 0.$$

A solution is forward bounded if it is bounded on $[0, \infty)$. Distinguish statements about every solution from statements about selected nonzero solutions.

Tasks

1. Find both roots and show that they are real and distinct for every real α .
2. Determine exactly when every solution tends to zero as $t \rightarrow \infty$.
3. Determine exactly when every solution is forward bounded. Treat $\alpha = 2$ explicitly.
4. For all $\alpha \leq 2$, classify the individual forward-bounded solutions, including $\alpha = -2$. Give a complete parameter table and explain why a zero root must not be treated as a negative root.

Question 7 – Solution

Strategy. Track the two root signs and retain the distinction between a mode being available and its coefficient actually being present.

Step 1: Keep the root separation visible. The characteristic polynomial is $(r + \alpha)^2 - 4$, so

$$\boxed{r_+ = 2 - \alpha, \quad r_- = -2 - \alpha}, \quad r_+ - r_- = 4.$$

Consequently $y = Ae^{(2-\alpha)t} + Be^{(-2-\alpha)t}$ for every real α ; no repeated or nonreal roots occur.

Step 2: Require both modes to decay. Every solution tends to zero exactly when both roots are negative. Since r_+ is the larger root, this is equivalent to

$$\boxed{\alpha > 2}.$$

If $\alpha \leq 2$, choosing $A = 1$, $B = 0$ gives a constant or growing mode, disproving universal decay.

Step 3: Allow a constant mode for boundedness. Every solution is forward bounded exactly when both roots are nonpositive, or $\boxed{\alpha \geq 2}$. At $\alpha = 2$,

$$y = A + Be^{-4t} \rightarrow A.$$

All solutions are bounded, but only those with $A = 0$ tend to zero. A zero root supplies a constant mode rather than a decaying one.

Step 4: Classify the selected bounded solutions. Using the coefficients A, B from Step 1, the full classification is

α	forward-bounded solutions	their limits
$\alpha < -2$	$y = 0$	0
$\alpha = -2$	$y = B \quad (A = 0)$	B
$-2 < \alpha < 2$	$y = Be^{(-2-\alpha)t} \quad (A = 0)$	0
$\alpha = 2$	$y = A + Be^{-4t}$	A
$\alpha > 2$	all solutions	0

For $\alpha < -2$, both roots are positive. If $A \neq 0$, the faster mode forces unbounded magnitude; if $A = 0$ and $B \neq 0$, the remaining positive mode still grows. Distinct growth rates cannot cancel for all large times.

At $\alpha = -2$ the roots are 4 and 0, so only constant solutions are bounded. For $-2 < \alpha < 2$, one positive mode must be removed, leaving a decaying mode. These boundary cases show why a classification based only on whether roots are real would be incomplete.

Question 8

The positive response

$$y(t) = e^{-t} + 9e^{-3t} \quad (t \geq 0)$$

solves a homogeneous second-order equation with real distinct roots. Define its instantaneous decay rate by $R(t) = -y'(t)/y(t)$.

Tasks

1. Recover the monic differential equation and verify the initial value and slope.
2. Express R as a weighted average of the two decay rates. Find $R(0)$, its limit, and the unique time when $R = 2$.
3. Differentiate R using the differential equation to prove that it is strictly decreasing. Sketch R with its limiting level marked.
4. A constant-rate extrapolation from the initial data predicts $\hat{y}(t) = 10e^{-(14/5)t}$. Prove whether this estimate lies above or below the actual response for every $t > 0$, using the curvature of $\ln y$.

Question 8 – Solution

Strategy. A mixture of two exponential modes need not have a constant effective decay rate; use its logarithmic derivative to quantify the change.

Step 1: Recover and check the equation. The roots are $-1, -3$, so the monic characteristic polynomial is $r^2 + 4r + 3$. Hence $y'' + 4y' + 3y = 0$. Direct differentiation gives $y' = -e^{-t} - 27e^{-3t}$, so

$$y(0) = 10, \quad y'(0) = -28.$$

The response is positive because both mode terms are positive.

Step 2: Read the rate as a changing weighted average. Let $z = 9e^{-2t} > 0$. Then

$$R = \frac{e^{-t} + 27e^{-3t}}{e^{-t} + 9e^{-3t}} = \frac{1 + 3z}{1 + z}.$$

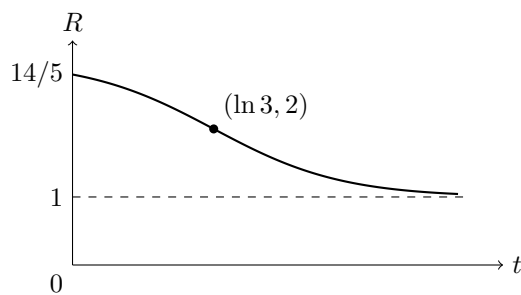
This averages 1 and 3 with positive weights 1, z , so $1 < R < 3$. In particular,

$$R(0) = 14/5, \quad R \rightarrow 1, \quad R = 2 \iff z = 1 \iff t = \ln 3.$$

Step 3: Derive the rate equation. Using $y'' = -4y' - 3y$ and $y'/y = -R$,

$$R' = -\frac{y''}{y} + (y'/y)^2 = R^2 - 4R + 3 = (R - 1)(R - 3) < 0.$$

Thus R is strictly decreasing; its long-time limit is the slower mode's rate, rather than an unchanging average of the roots.



The effective decay rate decreases toward 1.

Step 4: Compare with a constant-rate prediction. Set $f = \ln y$. Then $f' = -R$ and $f'' = -R' > 0$. Since f' is strictly increasing, integration on $[0, t]$ for $t > 0$ gives

$$f(t) > f(0) + tf'(0) = \ln 10 - \frac{14}{5}t.$$

Exponentiation preserves this inequality, so $y(t) > 10e^{-(14/5)t}$ for every $t > 0$. The extrapolation matches the initial value and slope but decays too quickly afterward. The changing mode weights, not an error in the initial data, cause the discrepancy.

Question 9

Let D denote differentiation with respect to t . Consider

$$y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 4.$$

A student reasons that $(D - 3)(D - 1)y = 0$ means every solution must satisfy either $(D - 1)y = 0$ or $(D - 3)y = 0$.

Tasks

1. Verify the operator factorization by expanding it on a twice-differentiable function.
2. Set $z = y' - y$. Derive and solve its first-order initial-value problem.
3. Solve $y' - y = z$ using an integrating factor and the original value $y(0) = 2$. Verify the resulting solution and both initial data.
4. Test the student's either/or claim on your solution. Explain why a product of differential operators must not be interpreted as a product of two numbers acting separately on y .

Question 9 – Solution

Strategy. Factorization creates a sequence of two first-order equations, not a rule requiring one of two first-order residuals to vanish.

Step 1: Expand the operator composition. Applying $(D - 3)$ to $(D - 1)y = y' - y$ gives

$$(y' - y)' - 3(y' - y) = y'' - 4y' + 3y.$$

Thus the factorization is valid and its characteristic roots are 1 and 3.

Step 2: Solve for the intermediate quantity. With $z = y' - y$, the equation is $z' - 3z = 0$. The data imply $z(0) = 4 - 2 = 2$. Separating or using an integrating factor yields

$$\boxed{z = 2e^{3t}}.$$

The intermediate residual is nonzero, yet it is annihilated by $D - 3$.

Step 3: Recover the original solution. We must solve $y' - y = 2e^{3t}$. Multiplication by e^{-t} gives

$$(e^{-t}y)' = 2e^{2t}.$$

Integrating from 0 to t and using $y(0) = 2$ gives $e^{-t}y - 2 = e^{2t} - 1$. Hence

$$\boxed{y = e^t + e^{3t}}.$$

Indeed $y(0) = 2$, $y'(0) = 1 + 3 = 4$, and

$$y'' - 4y' + 3y = (1 - 4 + 3)e^t + (9 - 12 + 3)e^{3t} = 0.$$

The construction recovers both modes by first-order methods and gives the unique solution with the prescribed data.

Step 4: Refute the incorrect either/or inference. For this solution,

$$(D - 1)y = 2e^{3t} \neq 0, \quad (D - 3)y = -2e^t \neq 0.$$

Nevertheless, $(D - 3)((D - 1)y) = (D - 3)(2e^{3t}) = 0$. An operator composition first produces a new function and then applies another operator to it. A nonzero function can be sent to zero by differentiation combined with multiplication. The scalar zero-product rule therefore does not imply the proposed either/or claim. Sums of the two modes must be retained, not just the two separate pure-mode families.

Question 10

For $0 < \varepsilon \leq 1$, consider

$$y'' - 2y' + (1 - \varepsilon^2)y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

All parameter values in the problem have real distinct roots. You may use, for $z \geq 0$, the Taylor bound

$$0 \leq \frac{e^z - e^{-z}}{2} - z \leq \frac{z^3 e^z}{6}.$$

Tasks

1. Find the roots and the exact solution y_ε , including both mode coefficients.
2. For fixed $t \geq 0$, find the limit of $y_\varepsilon(t)$ as $\varepsilon \downarrow 0$. Explain how diverging mode coefficients can still produce a finite limit.
3. Prove an explicit bound on $|y_\varepsilon(t) - te^t|$ for $0 \leq t \leq T$, where $T > 0$ is fixed. Deduce uniform convergence on that interval.
4. Explain the numerical danger in subtracting the two close exponentials when εt is small. Derive an equivalent expression using $H(z) = (e^z - e^{-z})/(2z)$, with $H(0) = 1$, and a small- z expansion that avoids that subtraction. Do not solve the limiting differential equation as a separate root case.

Question 10 – Solution

Strategy. Keep the roots distinct while studying cancellation in the data-to-coefficient representation and a controlled limit of its solutions.

Step 1: Solve the distinct-root IVP. The characteristic polynomial is $(r - 1)^2 - \varepsilon^2$, with roots $1 + \varepsilon$, $1 - \varepsilon$. Writing their coefficients as A, B gives $A + B = 0$, $(1 + \varepsilon)A + (1 - \varepsilon)B = 1$. Thus

$$\boxed{A = 1/(2\varepsilon), \quad B = -1/(2\varepsilon)}, \quad \boxed{y_\varepsilon = \frac{e^{(1+\varepsilon)t} - e^{(1-\varepsilon)t}}{2\varepsilon}}.$$

Both modes satisfy the equation; the value and derivative at zero are 0 and 1.

Step 2: Take a function limit with cancellation intact. Factoring out e^t and using the derivative of the exponential at zero gives

$$y_\varepsilon = e^t \frac{e^{\varepsilon t} - e^{-\varepsilon t}}{2\varepsilon} \longrightarrow \boxed{te^t}.$$

At $t = 0$ the identity and limit are both zero. The coefficients grow without bound, but the two modes approach each other and their leading contributions cancel. Large coefficients in this basis do not alone imply large solution values on a fixed bounded interval.

Step 3: Control the limit on the whole finite interval. Apply the supplied bound with $z = \varepsilon t$ and multiply by e^t/ε :

$$0 \leq y_\varepsilon - te^t \leq \frac{\varepsilon^2 t^3 e^{(1+\varepsilon)t}}{6} \leq \boxed{\frac{\varepsilon^2 T^3 e^{2T}}{6}} \quad (0 \leq t \leq T).$$

The last expression tends to zero independently of t , proving uniform convergence on each fixed $[0, T]$. It is not a uniform statement over all $t \geq 0$.

Step 4: Separate algebraic cancellation from its numerical evaluation. The direct numerator subtracts two nearly equal numbers, potentially losing relative accuracy in rounded arithmetic. An equivalent expression is

$$\boxed{y_\varepsilon = te^t H(\varepsilon t)}, \quad H(z) = 1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6) \quad (z \rightarrow 0).$$

This expansion follows by subtracting the Taylor series of e^z and e^{-z} before numerical evaluation. Evaluating H by this series for small z avoids the close-number subtraction; merely renaming the original quotient would not. The limit concerns this distinct-root family and does not require a general solution method for a repeated root.

Question 1

Consider

$$2y'' + 4y' + 10y = 0, \quad y(1) = 2, \quad y'(1) = 2.$$

You may use Euler's identity $e^{i\theta} = \cos \theta + i \sin \theta$ and the continuous-coefficient linear initial-value uniqueness theorem.

Tasks

1. Find the characteristic roots. Use their complex exponential solutions to construct two real solutions.
2. Write a real solution family based at $t = 1$ and determine the coefficients from the initial data.
3. Verify the solution in the original equation and both data. Show that the real family can realize arbitrary initial value and slope at 1, and hence is complete.
4. In the complex form $c_+e^{(-1+2i)(t-1)} + c_-e^{(-1-2i)(t-1)}$, determine c_+, c_- for this IVP. Explain why taking both coefficients real and equal would lose some real solutions.

Question 1 – Solution

Strategy. Pair conjugate exponentials to obtain real functions, retaining two independent real data freedoms.

Step 1: Find the roots and real building blocks. The characteristic equation is $2r^2 + 4r + 10 = 0$, or $(r + 1)^2 + 4 = 0$, giving

$$r = -1 \pm 2i.$$

The half-sum and the difference divided by $2i$ of $e^{(-1 \pm 2i)t}$ give, respectively, $e^{-t} \cos 2t$ and $e^{-t} \sin 2t$. These solve the real equation because it is linear with real coefficients.

Step 2: Fit the shifted family. Set $x = t - 1$ and write $y = e^{-x}(A \cos 2x + B \sin 2x)$. At $x = 0$, $y = A$ and $y' = -A + 2B$. Thus $A = 2$ and $B = 2$, so

$$y = 2e^{-(t-1)}(\cos(2(t-1)) + \sin(2(t-1))).$$

Step 3: Verify the equation and completeness. Writing $y = e^{-x}u$ gives $y' = e^{-x}(u' - u)$ and $y'' = e^{-x}(u'' - 2u' + u)$. Hence

$$2y'' + 4y' + 10y = 2e^{-x}(u'' + 4u) = 0.$$

The value and slope at 1 are 2 and $-2 + 4 = 2$. More generally, data $y(1) = a$, $y'(1) = b$ give $A = a$, $B = (a + b)/2$, uniquely. Any solution shares its data with one family member; the linear theorem forces equality on $\mathbb{R}$.

Step 4: Interpret the complex constants correctly. The real combination satisfies

$$A \cos 2x + B \sin 2x = \frac{A - iB}{2} e^{2ix} + \frac{A + iB}{2} e^{-2ix}.$$

Thus $\boxed{c_+ = 1 - i, \quad c_- = 1 + i}$ for this IVP. The constants are conjugates, so the imaginary parts cancel. Equal real constants would force $B = 0$, permitting only initial slopes $b = -a$ in the shifted basis. Real solutions require conjugate coefficients, not necessarily real coefficients in the complex basis.

Question 2

Consider

$$y'' + 2y' + 5y = 0, \quad y(0) = -1, \quad y'(0) = 3.$$

Use the phase convention $y = Re^{-t} \cos(2t - \delta)$ with $R > 0$ and $0 \leq \delta < 2\pi$.

Tasks

1. Solve the IVP in sine/cosine form.
2. Determine R, δ in the stated convention and verify the initial value and derivative in phase form.
3. Find every zero for $t \geq 0$, the sign immediately after the first zero, and the spacing between successive zeros.
4. Explain why solving only $\tan \delta = B/A$ can give the wrong phase quadrant. Diagnose the choice $\delta = -\pi/4$ with positive amplitude for this IVP.

Question 2 – Solution

Strategy. Determine the sine and cosine coefficients before choosing the phase quadrant; then use the phase to locate zeros.

Step 1: Solve the real IVP. The roots are $-1 \pm 2i$. For $y = e^{-t}(A \cos 2t + B \sin 2t)$, the data give $A = -1$ and $-A + 2B = 3$. Thus $B = 1$ and

$$y = e^{-t}(-\cos 2t + \sin 2t).$$

Step 2: Choose the correct quadrant. Expansion of $R \cos(2t - \delta)$ gives $A = R \cos \delta$, $B = R \sin \delta$. Therefore

$$R = \sqrt{A^2 + B^2} = \sqrt{2}, \quad \cos \delta = -1/\sqrt{2}, \quad \sin \delta = 1/\sqrt{2}.$$

The prescribed convention selects $\delta = 3\pi/4$, yielding

$$y = \sqrt{2}e^{-t} \cos(2t - 3\pi/4).$$

At zero the value is -1 , while the derivative is $\sqrt{2}[-\cos(-3\pi/4) - 2\sin(-3\pi/4)] = 3$, as required.

Step 3: Enumerate the nonnegative zeros. The exponential is never zero. The first zero occurs when $2t - 3\pi/4 = -\pi/2$, so all nonnegative zeros are

$$t_n = \frac{\pi}{8} + \frac{n\pi}{2}, \quad n = 0, 1, 2, \dots$$

At t_0 , the derivative is $2\sqrt{2}e^{-\pi/8} > 0$, so the solution crosses from negative to positive. Successive zeros are separated by $\pi/2$, half the trigonometric period π . The decaying solution itself is not periodic.

Step 4: Resolve the tangent ambiguity. The equation $\tan \delta = -1$ does not distinguish quadrants II and IV. With positive $R = \sqrt{2}$, the phase $-\pi/4$ gives $A = 1$, $B = -1$, which is the negative of the required solution. It has value 1 and slope -3 at zero. The signs of both sine and cosine, together with the amplitude convention, determine the phase; a tangent value alone does not.

Question 3

For the response $y(t) = e^{-t} \cos 2t$ on $t \geq 0$, the curves $y = \pm e^{-t}$ are called its exponential envelopes because $|y(t)| \leq e^{-t}$.

Tasks

1. Find a monic homogeneous equation with this solution and verify its initial value and slope.
2. Find all positive-time stationary points, classify them, and identify the first minimum and the first positive-time maximum.
3. Determine where the response touches an envelope. Are these times stationary points? Find the ratio of each stationary-point magnitude to the envelope height there.
4. Sketch the response and envelopes on $[0, 2\pi]$, marking the first minimum. Determine the ratio of consecutive positive local maxima and explain why envelope contacts and extrema must be distinguished.

Question 3 – Solution

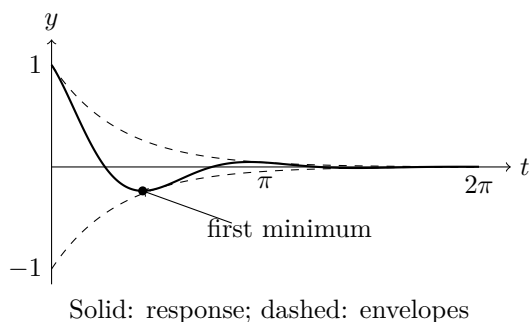
Strategy. Differentiate the entire product, including the exponential factor, before locating peaks.

Step 1: Verify the complex-root model. The roots $-1 \pm 2i$ give $y'' + 2y' + 5y = 0$. Differentiation yields $y' = e^{-t}(-\cos 2t - 2\sin 2t)$, so $y(0) = 1$ and $y'(0) = -1$. Substituting a second derivative verifies the equation.

Step 2: Locate and classify the extrema. Let $\theta = \arctan(1/2) \in (0, \pi/2)$. The stationary condition is $\cos 2t + 2\sin 2t = 0$, giving

$$t_k = (k\pi - \theta)/2, \quad k = 1, 2, \dots$$

At these times $\cos 2t_k = (-1)^k \cos \theta = (-1)^k 2/\sqrt{5}$. Since $y' = 0$, the ODE gives $y'' = -5y$. Thus odd k give negative minima and even k give positive maxima. The first minimum is at $(\pi - \theta)/2$; the first positive-time maximum is at $\pi - \theta/2$.



Step 3: Compare extrema with envelope contacts. Equality $|y| = e^{-t}$ requires $|\cos 2t| = 1$, or $t = n\pi/2$ for $n \geq 0$. At those times $y' = -y \neq 0$, so they are not stationary points. The response does share the tangent of the corresponding envelope there.

At every stationary point,

$$|y(t_k)| = \frac{2}{\sqrt{5}} e^{-t_k} < e^{-t_k}.$$

The decreasing envelope shifts the extrema away from cosine peaks.

Step 4: Compare like-signed peaks. Consecutive positive local maxima have indices k and $k + 2$, so their time separation is π . Their common trigonometric factor cancels, giving the height ratio $e^{-\pi}$. This is the envelope decay over a full cycle, even though the maxima occur strictly below the envelopes. At $t = 0$ the solution has a one-sided endpoint maximum; it is not a positive-time stationary maximum.

Question 4

A nonzero solution of an unknown monic equation $y'' + py' + qy = 0$ has characteristic roots $\alpha \pm i\beta$, where $\beta > 0$. Two consecutive positive local maxima occur at $t = 1$ and $t = 3$, with respective heights 4 and 1. The equation has real constant coefficients, and there is no forcing.

Tasks

1. Determine β and α from the peak times and heights. Justify why positive maxima are separated by a full cycle even when $\alpha \neq 0$.
2. Recover p, q and write the equation exactly.
3. Reconstruct the unique solution from the peak data at $t = 1$. Verify that the point at $t = 3$ is also a positive local maximum of height 1.
4. If the word consecutive were removed, would the two peak observations still uniquely determine the equation? Describe the resulting possibilities for β and explain the ambiguity.

Question 4 – Solution

Strategy. Use the interval between like-signed extrema for frequency and their height ratio for the real part of the roots.

Step 1: Extract frequency and decay. Write $y = Re^{\alpha t} \cos(\beta t - \delta)$. Stationarity requires $\alpha \cos(\beta t - \delta) - \beta \sin(\beta t - \delta) = 0$. Successive extrema have phase separation π and alternate sign, so consecutive positive maxima are separated by $2\pi/\beta$.

The observed separation is 2, giving $\beta = \pi$. The trigonometric factors at those maxima agree, so $1/4 = e^{2\alpha}$ and

$$\alpha = -\ln 2.$$

Step 2: Recover the real coefficients. Let $k = \ln 2 > 0$. The characteristic polynomial is $(r + k)^2 + \pi^2$, hence

$$y'' + 2ky' + (k^2 + \pi^2)y = 0.$$

Step 3: Reconstruct from value and zero slope. Using $x = t - 1$, write $y = e^{-kx}(A \cos \pi x + B \sin \pi x)$. The peak gives $A = 4$ and $-kA + \pi B = 0$, so

$$y = e^{-k(t-1)} \left[4 \cos(\pi(t-1)) + \frac{4k}{\pi} \sin(\pi(t-1)) \right].$$

At $t = 3$, $x = 2$, and $y = 4e^{-2k} = 1$, $y' = 0$. At either point the equation gives $y'' = -(k^2 + \pi^2)y < 0$, verifying strict positive local maxima. The solution is unique because its value and slope at 1 are specified for a regular linear equation.

Step 4: Identify what consecutiveness contributes. Without that condition, the time interval 2 could contain m full cycles, where m is any positive integer. Then $\beta = m\pi$ while the height ratio still fixes $\alpha = -\ln 2$. Each choice gives a different equation and a solution with the stated peaks, but for $m > 1$ there are additional positive maxima between them. The observed heights alone do not resolve the missing cycle count.

Question 5

For a real parameter a , consider

$$y'' + 2ay' + (a^2 + 9)y = 0.$$

All qualitative claims concern $t \geq 0$. A solution is nontrivial if it is not identically zero.

Tasks

1. Find the roots and the real general solution for every a .
2. Classify when every solution tends to zero and when every solution is bounded. Describe the nontrivial behavior at $a = 0$.
3. Prove that every nontrivial solution has infinitely many simple zeros, with the same spacing for all a . Can any such solution be eventually of one strict sign?
4. For $a < 0$, distinguish unbounded magnitude from the assertion $|y(t)| \rightarrow \infty$. Give explicit sequences of times that prove the correct conclusions.

Question 5 – Solution

Strategy. Separate the exponential amplitude from the trigonometric zero pattern; growth of the former does not erase zeros of the latter.

Step 1: Find the universal complex-root form. The polynomial is $(r + a)^2 + 9$, so the roots are $-a \pm 3i$. Hence

$$y = e^{-at}(A \cos 3t + B \sin 3t).$$

For a nontrivial solution, $R = \sqrt{A^2 + B^2} > 0$ and some phase δ give $y = Re^{-at} \cos(3t - \delta)$.

Step 2: Classify decay and boundedness. If $a > 0$, $|y| \leq Re^{-at} \rightarrow 0$. At $a = 0$, all solutions are bounded, and every nontrivial one is periodic with least positive period $2\pi/3$ and does not tend to zero. If $a < 0$, the magnitudes at trigonometric peaks grow without bound. Therefore

$$\boxed{\text{all solutions decay} \iff a > 0}, \quad \boxed{\text{all solutions are bounded} \iff a \geq 0}.$$

The zero solution is bounded and decays for every parameter value.

Step 3: Track the zeros and their slopes. Zeros occur when $3t - \delta = \pi/2 + n\pi$, so

$$t_n = \frac{\delta + \pi/2 + n\pi}{3},$$

with all integers n giving nonnegative times retained. There are infinitely many, spaced by $\pi/3$. At each zero,

$$y'(t_n) = -3Re^{-at_n} \sin(\pi/2 + n\pi) \neq 0.$$

Thus every zero is simple and the sign changes there. No nontrivial solution is eventually strictly positive or strictly negative, even when its amplitude decays.

Step 4: State growth without an incorrect limit. For $a < 0$, choose large integers n and set $s_n = (\delta + 2n\pi)/3$. Then $y(s_n) = Re^{-as_n} \rightarrow +\infty$. At $u_n = (\delta + (2n+1)\pi)/3$, $y(u_n) = -Re^{-au_n} \rightarrow -\infty$. But along the zero sequence, $|y(t_n)| = 0$.

Consequently the solution has unbounded magnitude, $\limsup y = +\infty$ and $\liminf y = -\infty$, while $\liminf |y| = 0$. The limit $|y(t)| \rightarrow \infty$ is false. A growing envelope describes arbitrarily large excursions, not a lower bound on the response at every late time.

Question 6

For constants $a \in \mathbb{R}$, $b > 0$, consider

$$y'' + 2ay' + (a^2 + b^2)y = 0.$$

Define transformed coordinates

$$u = e^{at}y, \quad v = \frac{e^{at}(y' + ay)}{b}.$$

The axes in a plot of (u, v) are these transformed quantities, not time and displacement.

Tasks

1. Differentiate u, v and prove $u' = bv$, $v' = -bu$.
2. Deduce a conserved quantity and express its value using $y(0) = Y$, $y'(0) = V$. Relate it to the amplitude of the trigonometric factor in y .
3. For $a = 1$, $b = 2$, $Y = 1$, $V = -1$, find y, u, v . Sketch the transformed curve with its initial point, direction and quarter-cycle points.
4. Explain why the circle in these transformed coordinates does not imply constant amplitude of the original response. Determine exactly which original initial data make the conserved quantity zero and what solution follows.

Question 6 – Solution

Strategy. Remove the exponential factor and scale the remaining derivative so the two trigonometric coordinates have equal amplitudes.

Step 1: Derive the transformed equations. Directly, $u' = e^{at}(y' + ay) = bv$. Also,

$$v' = \frac{e^{at}}{b}(y'' + 2ay' + a^2y) = \frac{e^{at}}{b}(-b^2y) = -bu.$$

The original equation is essential in the second equality.

Step 2: Identify the invariant and amplitude. Differentiating gives $(u^2 + v^2)' = 2u(bv) + 2v(-bu) = 0$. At zero,

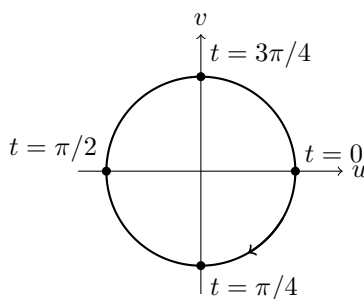
$$u^2 + v^2 = Y^2 + \frac{(V + aY)^2}{b^2} = R^2.$$

The real solution has trigonometric coefficients $A = Y$, $B = (V + aY)/b$, so $R = \sqrt{A^2 + B^2}$ is exactly its trigonometric amplitude.

Step 3: Trace the concrete circle. For the stated data, $A = 1$, $B = 0$. Thus

$$y = e^{-t} \cos 2t, \quad u = \cos 2t, \quad v = -\sin 2t.$$

The initial point is $(1, 0)$ and its velocity is $(0, -2)$, so motion is clockwise. At times $0, \pi/4, \pi/2, 3\pi/4$ the points are $(1, 0), (0, -1), (-1, 0), (0, 1)$, respectively; the cycle closes at $t = \pi$.



$$u^2 + v^2 = 1; \text{ equal axis scales}$$

Step 4: Translate back to the original quantity. The transformation removed e^{-at} : the original response satisfies $|y| \leq Re^{-at}$. Its amplitude decays for $a > 0$, is constant for $a = 0$, and grows for $a < 0$. A circle in (u, v) describes a normalized oscillation, not constant original displacement amplitude or a claim about physical energy.

Because $b > 0$, the sum of squares R^2 is zero exactly when $Y = 0$ and $V + aY = 0$, hence exactly when $Y = V = 0$. These data give $y \equiv 0$, consistent with initial-value uniqueness.

Question 7

An unknown frequency $\beta > 0$ appears in

$$y'' + \beta^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Exact observations at all nonnegative integer times report $y(n) = 0$.

Tasks

1. Solve the IVP in terms of β and determine every frequency compatible with the integer observations.
2. Explain why the observations, even with the known initial slope, do not imply the zero solution and do not identify a unique frequency.
3. An additional exact observation gives $y(1/2) = 1/\pi$. Determine which of the previously possible frequencies remain. Prove that no larger candidate frequency can fit it.
4. Sketch the candidates with $\beta = \pi$ and $\beta = 3\pi$ on $[0, 2]$, marking the integer sampling times. Explain what feature of the observation schedule hides their different oscillations.

Question 7 – Solution

Strategy. Solve before interpreting the data; a sampling grid can repeatedly land on zeros of a nonzero oscillation.

Step 1: Determine all admissible frequencies. The roots are $\pm i\beta$, and the data give

$$y(t) = \frac{\sin \beta t}{\beta}.$$

In particular, $y(1) = 0$ implies $\sin \beta = 0$, so $\beta = m\pi$ for a positive integer m . Conversely, each such frequency makes every integer observation zero. The complete candidate set is therefore

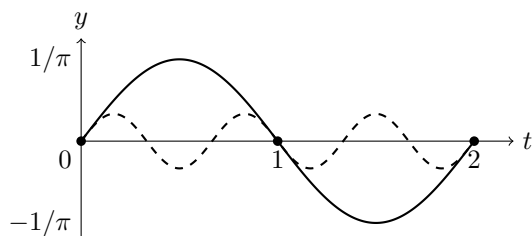
$$\beta = m\pi, \quad m = 1, 2, 3, \dots$$

Step 2: Retain the slope information correctly. Each candidate has $y'(t) = \cos(m\pi t)$ and hence $y'(0) = 1$, so each is nonzero. The zero function would violate that slope. The slope sets the sine coefficient to $1/(m\pi)$, but still allows infinitely many frequencies. A known initial slope does not rescue a schedule that misses all intermediate oscillations.

Step 3: Use the extra observation to select a frequency. For the candidate indexed by m ,

$$y(1/2) = \frac{\sin(m\pi/2)}{m\pi}.$$

For $m = 1$ this equals $1/\pi$. For every $m > 1$, its absolute value is at most $1/(m\pi) < 1/\pi$, making agreement impossible. Thus the new datum selects exactly $\beta = \pi$ and $y = \sin(\pi t)/\pi$.



Solid: $\beta = \pi$; dashed: $\beta = 3\pi$; dots: samples

Step 4: Explain the indistinguishable samples. The integer grid samples an integer number of half-cycles for every candidate, so it always observes a zero. The two plotted curves differ between those times, including in amplitude, yet share all grid values and the initial slope. This is an explicit sampling ambiguity, not a violation of uniqueness: each fixed equation has one solution for its data, while different frequencies define different equations.

Question 8

Consider the endpoint problem

$$y'' + 2y' + 5y = 0, \quad y(0) = A, \quad y(L) = B, \quad L > 0.$$

The numbers A, B are endpoint values, not mode coefficients.

Tasks

1. Derive the endpoint equation for the remaining sine coefficient. Determine all lengths L for which every pair of endpoint values gives a unique solution.
2. At each excluded length, give the exact compatibility condition for existence and classify the number of solutions.
3. For $A = 0, B = 1$, solve when $L = \pi/4$ and determine what happens at $L = \pi/2$.
4. Keep $A = 0, B = 1$ but take $L = \pi/2 + \varepsilon$, where $0 < \varepsilon < \pi/4$. Find $y(\pi/4)$ and its leading behavior as $\varepsilon \downarrow 0$. Explain how unique solvability for every such ε can coexist with arbitrarily large interior values.

Question 8 – Solution

Strategy. An exponential prefactor never vanishes, so solvability and sensitivity are governed by the sine factor in the endpoint data map.

Step 1: Reduce to one coefficient. The roots are $-1 \pm 2i$. The first endpoint fixes the cosine coefficient, giving $y = e^{-t}(A \cos 2t + C \sin 2t)$. The second requires

$$C \sin 2L = Be^L - A \cos 2L.$$

If $\sin 2L \neq 0$, there is exactly one C , so unique solvability for arbitrary data holds exactly when

$$L \neq n\pi/2 \quad (n = 1, 2, 3, \dots).$$

Step 2: Classify the exceptional lengths. At $L = n\pi/2$, the condition becomes $0 = Be^L - (-1)^n A$. Thus

$$B = (-1)^n e^{-L} A$$

is necessary and sufficient for existence. If it holds, every real C works, giving infinitely many solutions. If it fails, there is no solution. The initial-value uniqueness theorem does not promise endpoint uniqueness at these lengths.

Step 3: Apply the two specified lengths. For $A = 0$, $B = 1$, $L = \pi/4$, we have $C = e^{\pi/4}$ and

$$y = e^{\pi/4-t} \sin 2t.$$

At $L = \pi/2$, compatibility would require $B = 0$, contradicting $B = 1$, so no solution exists.

Step 4: Quantify sensitivity near an exceptional length. For $L = \pi/2 + \varepsilon$, $\sin 2L = -\sin 2\varepsilon \neq 0$. Hence

$$y(t) = -\frac{e^{L-t} \sin 2t}{\sin 2\varepsilon}, \quad y(\pi/4) = -\frac{e^{\pi/4+\varepsilon}}{\sin 2\varepsilon}.$$

Using $\sin 2\varepsilon \sim 2\varepsilon$ and $e^\varepsilon \rightarrow 1$ gives

$$y(\pi/4) \sim -\frac{e^{\pi/4}}{2\varepsilon} \longrightarrow -\infty.$$

For each positive ε there is one finite coefficient, but the denominator approaches zero as the length approaches the incompatible endpoint problem. Uniqueness at each parameter value is not a uniform bound on the dependence on that parameter.

Question 9

For the decaying oscillation $y(t) = e^{-t} \sin t$, define its signed total area and its total absolute area by

$$I = \int_0^{\infty} y(t) dt, \quad J = \int_0^{\infty} |y(t)| dt.$$

Tasks

1. Find the corresponding monic homogeneous equation and initial data. State the signs on consecutive half-cycles.
2. Find an antiderivative of $e^{-t} \sin t$ and calculate I .
3. Calculate J exactly by comparing the absolute areas of successive half-cycles. Explain why $J > I$ even though both are finite.
4. Sketch the first two half-cycles with their signed areas indicated. Bound the absolute tail after an arbitrary $T \geq 0$, and give its exact value when $T = n\pi$ for a nonnegative integer n .

Question 9 – Solution

Strategy. Separate cancellation between lobes from decay of each lobe, then sum the absolute areas as a geometric series.

Step 1: Identify the equation and signs. The roots are $-1 \pm i$, so $y'' + 2y' + 2y = 0$. The derivative is $e^{-t}(\cos t - \sin t)$, giving $y(0) = 0$, $y'(0) = 1$. On $(n\pi, (n+1)\pi)$, the sign is $(-1)^n$: positive first, negative next, and alternating thereafter.

Step 2: Integrate the signed response. Integration by parts twice, or differentiation of the result, gives

$$F(t) = -\frac{1}{2}e^{-t}(\sin t + \cos t), \quad F' = e^{-t} \sin t.$$

Since $F(t) \rightarrow 0$ as $t \rightarrow \infty$ and $F(0) = -1/2$,

$$I = 1/2.$$

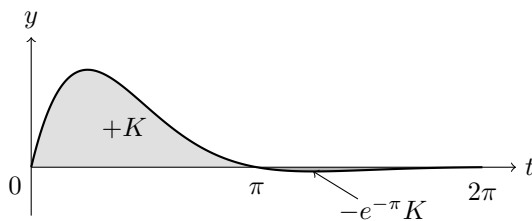
Step 3: Sum the magnitudes without cancellation. The first positive lobe has area

$$K = F(\pi) - F(0) = \frac{1 + e^{-\pi}}{2}.$$

The substitution $t = n\pi + s$ shows that the absolute area of the n th lobe is $e^{-n\pi}K$. Thus the convergent geometric sum gives

$$J = \frac{K}{1 - e^{-\pi}} = \frac{1 + e^{-\pi}}{2(1 - e^{-\pi})} > \frac{1}{2}.$$

The negative lobes reduce I but contribute positively to J . Infinitely many oscillations do not prevent absolute convergence because the lobe magnitudes decrease geometrically.



Signed lobe areas; both magnitudes contribute to J .

Step 4: Control the unobserved tail. Since $|\sin t| \leq 1$,

$$\int_T^\infty |y(t)| dt \leq \int_T^\infty e^{-t} dt = e^{-T}.$$

At a half-cycle boundary, substitution by $n\pi$ preserves $|\sin|$, so the exact absolute tail is $e^{-n\pi}J$. The signed area of the second plotted lobe is $-e^{-\pi}K$, while its contribution to J is $+e^{-\pi}K$.

Question 10

Consider

$$y'' + 4y' + 13y = 0, \quad y(0) = 3, \quad y'(0) = 6.$$

A settling certificate at time $T \geq 0$ means a proof that $|y(t)| \leq 0.1$ for every $t \geq T$. Distinguish a certificate for this specific solution from one valid for every phase with the same trigonometric amplitude.

Tasks

1. Find the solution, its amplitude R , and its phase $\delta \in [0, 2\pi)$ in the form $Re^{-2t} \cos(3t - \delta)$.
2. Use the envelope to find the smallest certificate time valid for every phase with amplitude R . Prove the claimed optimality over phases.
3. For the stated IVP, show that $T = 1.9$ is already a certificate. Use the value at 1.9 and the magnitudes of all later stationary points, justifying their ordering. Numerical evaluations should be supported by exact expressions.
4. Sketch the response for $t \geq 1.5$ with the tolerance levels and envelope certificate marked. Explain why the envelope certificate need not be the earliest settling time for one fixed phase, and whether Task 3 identifies that exact earliest time.

Question 10 – Solution

Strategy. First obtain a phase-independent bound, then sharpen it for one solution by controlling every later extremum.

Step 1: Resolve the data and phase. The roots are $-2 \pm 3i$. Writing $y = e^{-2t}(A \cos 3t + B \sin 3t)$ gives $A = 3$ and $-2A + 3B = 6$, hence $B = 4$. Thus

$$y = e^{-2t}(3 \cos 3t + 4 \sin 3t) = 5e^{-2t} \cos(3t - \delta), \quad \delta = \arctan(4/3) \in (0, \pi/2).$$

Step 2: Obtain the best certificate for every phase. The envelope condition is $5e^{-2t} \leq 0.1$, so

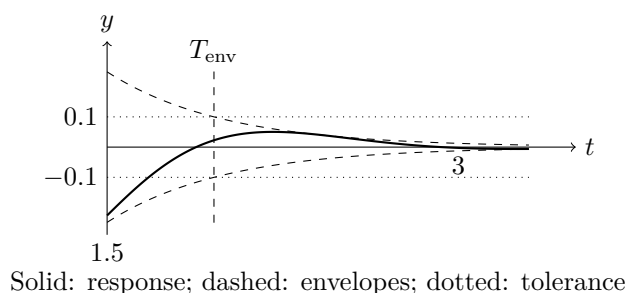
$$T_{\text{env}} = \frac{1}{2} \ln 50 \approx 1.956012.$$

It guarantees the tolerance at every later time. For any proposed $0 \leq T < T_{\text{env}}$, choose a phase with $3T - \delta$ an integer multiple of 2π . That solution has $|y(T)| = 5e^{-2T} > 0.1$. Thus no earlier time works for every phase with amplitude 5.

Step 3: Check the fixed phase more sharply. Let $\theta = \arctan(2/3)$. Stationary times are $t_k = (\delta - \theta + k\pi)/3$, and their magnitudes are $(15/\sqrt{13})e^{-2t_k}$. In particular,

$$t_1 \approx 1.160295 < 1.9 < t_2 \approx 2.207493, \quad |y(t_2)| \approx 0.050317 < 0.1.$$

Also $y(1.9) = e^{-3.8}(3 \cos 5.7 + 4 \sin 5.7) \approx 0.006742$. Between 1.9 and t_2 there is no stationary point, so the response is monotone and its absolute value is bounded by the larger endpoint magnitude. Each subsequent stationary magnitude is multiplied by $e^{-2\pi/3} < 1$. Monotonicity between those extrema then proves $|y(t)| \leq 0.1$ for all $t \geq 1.9$.



Step 4: Distinguish a certificate from the exact last crossing. The envelope treats all phases, whereas the specified phase can have small values while the envelope is still above 0.1. The preceding negative minimum has magnitude greater than 0.1, so a final tolerance crossing occurs between t_1 and 1.9. Task 3 proves a sufficient time, not that 1.9 is the earliest one; locating the exact last crossing requires solving the corresponding trigonometric-exponential level equation.

Question 1

Consider

$$4y'' - 12y' + 9y = 0, \quad y(2) = -1, \quad y'(2) = 1/2.$$

A student proposes $y = C_1 e^{3t/2} + C_2 e^{3t/2}$ as the general solution because the characteristic root occurs twice.

Tasks

1. Find the characteristic root and explain precisely which initial-data freedom the proposed family lacks.
2. Put $x = t - 2$ and $y = e^{3x/2}u(x)$. Differentiate to reduce this particular equation to an equation for u , and obtain the full solution family.
3. Fit the initial data and verify the resulting solution in the original equation and both data.
4. Show that the derived family can represent arbitrary value and slope at $t = 2$. Justify completeness and explain why multiplying one exponential by $t - 2$ creates a new freedom while merely naming a second coefficient does not.

Question 1 – Solution

Strategy. Remove the common exponential in this constant-coefficient equation and determine what equation remains for its multiplier.

Step 1: Identify the repeated root and lost freedom. The characteristic polynomial is $(2r - 3)^2$, so $r = 3/2$ is repeated. The student's family equals $(C_1 + C_2)e^{3t/2}$ and has only one effective constant. It always satisfies $y' = (3/2)y$, so at $t = 2$ a value of -1 would force slope $-3/2$, not $1/2$.

Step 2: Derive the missing multiplier. With $x = t - 2$ and $y = e^{3x/2}u$,

$$y' = e^{3x/2}(u' + 3u/2), \quad y'' = e^{3x/2}(u'' + 3u' + 9u/4).$$

Substitution gives $4y'' - 12y' + 9y = 4e^{3x/2}u''$. Thus $u'' = 0$ and $u = A + Bx$, yielding

$$y = (A + B(t - 2))e^{3(t-2)/2}.$$

Step 3: Apply and verify the data. At $t = 2$, $y = A$ and $y' = B + 3A/2$. Hence $A = -1$, $B = 2$, so

$$y = (-1 + 2(t - 2))e^{3(t-2)/2}.$$

The multiplier has zero second derivative, making the original residual zero by Step 2. At 2, the value is -1 and the derivative is $2 - 3/2 = 1/2$.

Step 4: Establish the two independent data freedoms. For arbitrary data $y(2) = a$, $y'(2) = b$, we obtain $A = a$, $B = b - 3a/2$, uniquely. The transformation is invertible because the exponential never vanishes, and integrating $u'' = 0$ gives all multipliers. Alternatively, continuous-coefficient IVP uniqueness proves completeness after the data map is verified.

The function $(t - 2)e^{3(t-2)/2}$ has zero value but nonzero derivative at the base point, allowing slope to change without changing the prescribed value. A second copy of the same exponential cannot do that.

Question 2

Consider

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -3.$$

All qualitative conclusions below concern $t \geq 0$.

Tasks

1. Solve the IVP using the repeated-root family and verify both data.
2. Find every nonnegative zero and determine the sign on either side. Does a repeated negative root prevent the response from changing sign?
3. Find every positive-time stationary point, classify it and calculate its value. Determine the limit and the side from which it is approached.
4. Sketch the response, marking the zero and minimum. Explain why the extra polynomial factor can change the shape without defeating eventual exponential decay.

Question 2 – Solution

Strategy. Use the nonvanishing exponential to separate zeros of the affine multiplier from zeros of the derivative.

Step 1: Fit the repeated-root solution. The characteristic equation is $(r+2)^2 = 0$, so $y = (A+Bt)e^{-2t}$. The data give $A = 1$, $B - 2A = -3$, hence $B = -1$. Therefore

$$y = (1 - t)e^{-2t}.$$

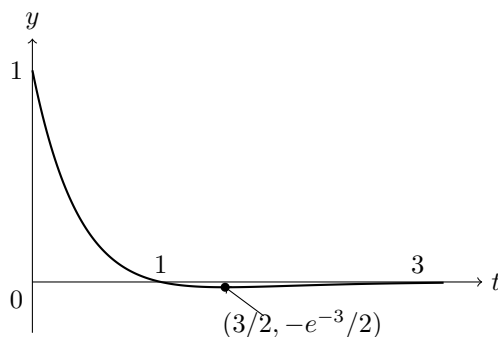
Its derivative is $(2t - 3)e^{-2t}$, with value -3 at zero, and substitution verifies $y'' + 4y' + 4y = 0$.

Step 2: Locate the sign change. Since $e^{-2t} > 0$, the sign is that of $1 - t$. The unique nonnegative zero is $t = 1$, with positive values before and negative values after. At the zero, $y'(1) = -e^{-2} \neq 0$, so this is a crossing rather than a tangency.

Step 3: Locate the minimum and the limiting behavior. The derivative vanishes only at $t = 3/2$. It is negative before and positive afterward, so the point is the unique global minimum on $[0, \infty)$:

$$y(3/2) = -\frac{1}{2}e^{-3}.$$

The factor t grows more slowly than e^{2t} , so $y \rightarrow 0$ as $t \rightarrow \infty$, approaching from below.



One crossing, then a minimum and approach from below.

Step 4: Interpret the roles of the two factors. The negative repeated root ensures decay of both e^{-2t} and te^{-2t} . It does not determine the sign of their combination or prevent a finite-time crossing. The affine multiplier supplies that sign change, while the derivative's affine multiplier determines a later turning point. The response crosses zero at 1, reaches its minimum at $3/2$, and then rises toward zero without crossing again.

Question 3

For $\lambda > 0$ and $a > 0$, consider

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = a, \quad y'(0) = b.$$

Classify all real slopes b on $t \geq 0$.

Tasks

1. Find the solution in terms of a, b, λ .
2. Prove exactly when the response remains nonnegative for every $t \geq 0$.
3. Prove exactly when it is both nonnegative and nonincreasing, treating the boundary slopes separately.
4. For the remaining slopes, find the unique zero or positive-time maximum, as appropriate. Summarize the cases on a diagram of $b/(\lambda a)$ and explain how the repeated root enters the classification.

Question 3 – Solution

Strategy. Reduce both sign and monotonicity to affine functions of time after factoring out a positive exponential.

Step 1: Resolve the data. The root is $-\lambda$ twice. Writing $y = (A + Bt)e^{-\lambda t}$ gives $A = a$, $B = b + \lambda a$. Thus

$$y = (a + (b + \lambda a)t)e^{-\lambda t}.$$

For brevity let $B = b + \lambda a$.

Step 2: Classify nonnegativity. The multiplier starts at $a > 0$. It stays positive at every finite nonnegative time exactly when $B \geq 0$; if $B < 0$, it crosses zero and becomes negative. Therefore

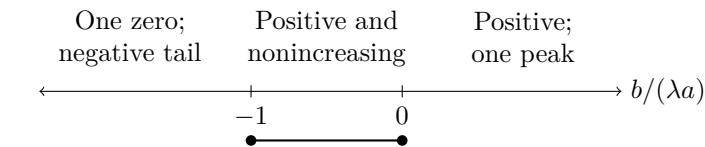
$$y(t) \geq 0 \text{ for all } t \geq 0 \iff b \geq -\lambda a.$$

At equality, the polynomial factor is constant and $y = ae^{-\lambda t}$.

Step 3: Impose nonincrease as well. Differentiation gives $y' = e^{-\lambda t}(b - \lambda Bt)$. With $B \geq 0$, this derivative is nonpositive for all $t \geq 0$ exactly when $b \leq 0$. Hence

$$-\lambda a \leq b \leq 0.$$

At $b = -\lambda a$, the response is a decreasing pure exponential. At $b = 0$, it has zero initial slope but negative derivative for every $t > 0$ because $B = \lambda a > 0$.



Both boundary slopes are included in the middle case.

Step 4: Classify the exterior ranges. If $b < -\lambda a$, then $B < 0$, and the unique zero is

$$t_z = -\frac{a}{b + \lambda a} > 0.$$

The response crosses from positive to negative and tends to zero from below. If $b > 0$, then $B > 0$ and the unique maximum occurs at

$$t_m = \frac{b}{\lambda(b + \lambda a)} > 0.$$

Here y' changes from positive to negative, and the response remains positive. The repeated root produces an affine multiplier; the zeros of that multiplier and its differentiated counterpart govern these thresholds. All cases still decay to zero.

Question 4

An unknown monic homogeneous second-order equation has a repeated real root r . A solution has exact samples

$$S_n = y(n), \quad S_0 = 1, \quad S_1 = 4/e, \quad S_2 = 7/e^2.$$

Write $y(t) = (A + Bt)e^{rt}$ and set $z = e^r > 0$.

Tasks

1. Derive the recurrence $S_{n+2} = 2zS_{n+1} - z^2S_n$.
2. Determine every positive z compatible with the three observations, then find the corresponding r, A, B . Explain why the observations do not yet identify a unique equation.
3. An additional sample gives $S_3 = 10/e^3$. Determine which candidate survives and recover the monic equation and initial slope.
4. Verify all four observations for the selected solution and calculate the rejected candidate's prediction at time 3. Contrast their long-time behavior and explain why three exact observations were insufficient here.

Question 4 – Solution

Strategy. Use the repeated-root sample recurrence, but retain every admissible root of the nonlinear identification equation.

Step 1: Derive the recurrence. Since $S_n = (A + Bn)z^n$,

$$2zS_{n+1} - z^2S_n = (2(A + B(n+1)) - (A + Bn))z^{n+2} = S_{n+2}.$$

Step 2: Find both admissible models. The observed values imply $7/e^2 = 8z/e - z^2$, or

$$(z - 1/e)(z - 7/e) = 0.$$

Both roots are positive. Since $A = S_0 = 1$ and $B = S_1/z - A$, the candidates are

z	r	A	B
$1/e$	-1	1	3
$7/e$	$\ln 7 - 1$	1	$-3/7$

Each gives a different repeated-root equation $y'' - 2ry' + r^2y = 0$, and each fits the original three observations. Selecting only one positive root of the identification quadratic would be unjustified.

Step 3: Use the fourth observation. For the first candidate, $S_3 = (1 + 9)e^{-3} = 10/e^3$, matching the new datum. The other gives

$$S_3 = (1 - 9/7)(7/e)^3 = \boxed{-98/e^3},$$

so it is rejected. The identified equation and data are

$$\boxed{y'' + 2y' + y = 0}, \quad \boxed{y = (1 + 3t)e^{-t}}, \quad \boxed{y'(0) = 2}.$$

Step 4: Verify and interpret the ambiguity. At $n = 0, 1, 2, 3$, the selected multiplier $1 + 3n$ gives 1, 4, 7, 10, with the required exponential factors. This solution tends to zero. The rejected candidate has $r = \ln 7 - 1 > 0$ and negative leading multiplier coefficient, so it eventually tends to $-\infty$.

The initial three sample constraints lead to a quadratic equation with two admissible positive multipliers, despite being exact. The fourth observation distinguishes them. Counting unknown parameters and observations alone does not prove a nonlinear inverse problem has a unique solution.

Question 5

For a real parameter r , consider

$$y'' - 2ry' + r^2y = 0.$$

A solution is forward bounded if bounded on $[0, \infty)$ and two-sided bounded if bounded on all of $\mathbb{R}$.

Tasks

1. Give the complete solution family for every real r , including $r = 0$.
2. Classify exactly which solutions are forward bounded and which tend to zero as $t \rightarrow \infty$, for each sign of r .
3. Classify all two-sided bounded solutions. Justify why a polynomial factor cannot cancel exponential growth in the growing time direction.
4. Explain why a repeated zero root behaves differently from a single zero root paired with a negative root. Give a counterexample to the statement that nonpositive characteristic roots always make every solution bounded.

Question 5 – Solution

Strategy. Treat the zero root separately and distinguish decay in one time direction from boundedness in both directions.

Step 1: Retain both repeated-root freedoms. For every r , the family is $y = (A + Bt)e^{rt}$. At $r = 0$ this becomes the affine family $y = A + Bt$ solving $y'' = 0$, not merely the constant family.

Step 2: Classify forward behavior. For $r < 0$, both e^{rt} and te^{rt} tend to zero, so every solution decays and is forward bounded. For $r = 0$, boundedness requires $B = 0$, while decay to zero requires $A = B = 0$. For $r > 0$, only the zero solution is forward bounded or tends to zero.

r	forward-bounded solutions	solutions tending to zero
$r < 0$	all solutions	all solutions
$r = 0$	$y = A$	$y = 0$
$r > 0$	$y = 0$	$y = 0$

For negative r , boundedness follows from the finite limit and continuity on every finite initial interval.

Step 3: Inspect the other time direction. If $B \neq 0$, the magnitude of $A + Bt$ is at least $|B||t|/2$ for sufficiently large $|t|$. If $B = 0$ but $A \neq 0$, it is a nonzero constant. Thus neither nonzero case can cancel the exponential in its growing direction.

When $r > 0$, that direction is $t \rightarrow \infty$; when $r < 0$, it is $t \rightarrow -\infty$. Hence the complete two-sided classification is

$$\boxed{r \neq 0 : y = 0 \text{ only}}, \quad \boxed{r = 0 : y = A \text{ for any constant } A}.$$

Step 4: Explain the exceptional repeated zero. A single zero root paired with a negative root gives a constant mode plus a decaying exponential, so every solution is forward bounded. A repeated zero root additionally gives the unbounded mode t .

For example, $y(t) = t$ solves $y'' = 0$, whose roots are both zero and therefore nonpositive, but is unbounded forward. Root signs alone are insufficient at a repeated zero; the polynomial multiplier must be included. For a strictly negative repeated root, the polynomial affects the rate and shape but not eventual decay.

Question 6

The repeated-root response $y(t) = te^{-t}$ solves $y'' + 2y' + y = 0$ with $y(0) = 0$, $y'(0) = 1$. Consider exponential bounds on $t \geq 0$.

Tasks

1. Find the maximum of y and verify that it tends to zero.
2. Determine whether any finite constant C can satisfy $y(t) \leq Ce^{-t}$ for every $t \geq 0$.
3. For a fixed $0 < \varepsilon < 1$, find the smallest C_ε such that $y(t) \leq C_\varepsilon e^{-(1-\varepsilon)t}$ for every $t \geq 0$. Prove optimality and sketch the bound for $\varepsilon = 1/2$ with its contact point.
4. Extend the argument to $w(t) = (A + Bt)e^{-t}$ by giving an explicit finite bound with rate $1 - \varepsilon$. Explain why eventual decay does not automatically give a bound with the exact exponent 1.

Question 6 – Solution

Strategy. Divide by the proposed exponential bound and maximize the resulting ratio, rather than judging only the root's sign.

Step 1: Find the transient peak. We have $y' = (1 - t)e^{-t}$, so the response rises until $t = 1$ and falls thereafter. Hence

$$\max_{t \geq 0} y(t) = 1/e.$$

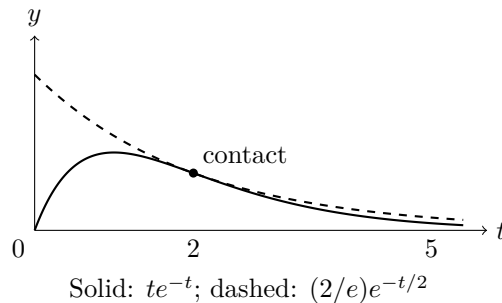
Also $t/e^t \rightarrow 0$, for example by l'Hôpital's rule, proving eventual decay.

Step 2: Test the exact root exponent. The proposed bound is equivalent to $t \leq C$ for all $t \geq 0$, which no finite C can satisfy. The exponential decay is modified by an unbounded polynomial prefactor.

Step 3: Find the sharp slightly slower bound. Dividing by $e^{-(1-\varepsilon)t}$ gives the ratio $te^{-\varepsilon t}$. Its derivative is $e^{-\varepsilon t}(1 - \varepsilon t)$, so its unique maximum occurs at $t = 1/\varepsilon$ and equals $1/(e\varepsilon)$. Thus

$$C_\varepsilon = \frac{1}{e\varepsilon}.$$

This constant is sufficient by the maximum calculation and necessary because equality holds at that time.



For $\varepsilon = 1/2$, the bound is $(2/e)e^{-t/2}$ and contact occurs at $t = 2$, where both values are $2e^{-2}$. Both derivatives equal $-e^{-2}$, confirming tangency rather than a crossing.

Step 4: Bound every fixed repeated-root solution. The triangle inequality gives

$$|w(t)|e^{(1-\varepsilon)t} \leq |A|e^{-\varepsilon t} + |B|te^{-\varepsilon t} \leq |A| + \frac{|B|}{e\varepsilon}.$$

Therefore $|w(t)| \leq (|A| + |B|/(e\varepsilon))e^{-(1-\varepsilon)t}$. This displayed constant need not be optimal for arbitrary A, B . A bound with the exact exponent 1 exists precisely when $B = 0$; for $B \neq 0$, the ratio $|A + Bt|$ is unbounded even though w itself decays.

Question 7

Consider the parameter family

$$y'' + 2y' + (1 - \mu)y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad -1 < \mu < 1.$$

For convenience, $\sinh z = (e^z - e^{-z})/2$. You may use $\sin z/z \rightarrow 1$ and $\sinh z/z \rightarrow 1$ as $z \rightarrow 0$.

Tasks

1. Solve the IVP separately for $\mu > 0$, $\mu = 0$ and $\mu < 0$, identifying the root type in each case.
2. Prove that the formulas from both sides tend to the repeated-root solution for each fixed $t \geq 0$ as $\mu \rightarrow 0$.
3. Determine which of these solutions have positive-time zeros. For $\mu < 0$, find the first positive zero and its limit as $\mu \uparrow 0$.
4. Prove that every solution of every equation in the stated parameter range tends to zero as $t \rightarrow \infty$. Explain why convergence of the displayed IVP solutions on any fixed time window does not preserve their number of positive-time zeros.

Question 7 – Solution

Strategy. Center the characteristic equation at its repeated root and compare the resulting real, repeated and complex formulas without conflating their global zero patterns.

Step 1: Solve each root case. The characteristic polynomial is $(r+1)^2 - \mu$. If $\mu > 0$, set $s = \sqrt{\mu} \in (0, 1)$. The roots are $-1 \pm s$ and the data give

$$y_\mu = e^{-t} \frac{\sinh(st)}{s}.$$

At $\mu = 0$, the root is -1 twice, and the solution is $y_0 = te^{-t}$. If $\mu < 0$, set $b = \sqrt{-\mu} > 0$; the roots are $-1 \pm ib$, giving

$$y_\mu = e^{-t} \frac{\sin(bt)}{b}.$$

Each formula has value 0 and derivative 1 at zero and solves its stated equation.

Step 2: Recover the repeated-root limit from either side. For $t > 0$, factor the first formula as $te^{-t} \sinh(st)/(st)$ and the third as $te^{-t} \sin(bt)/(bt)$. The supplied limits yield $y_\mu(t) \rightarrow te^{-t}$ from either side. At $t = 0$, all values are zero directly. Thus the repeated-root solution is the common fixed-time limit, rather than a missing special value.

Step 3: Locate zeros before taking their limit. For $\mu \geq 0$, the displayed solution is strictly positive for every $t > 0$, since $t > 0$ and $\sinh(st) > 0$. For $\mu < 0$, its positive zeros are

$$t_n = \frac{n\pi}{\sqrt{-\mu}}, \quad n = 1, 2, \dots$$

In particular, $t_1 \rightarrow \infty$ as $\mu \uparrow 0$. The infinitely many oscillations move beyond every fixed finite observation window.

Step 4: Separate decay from zero counts. For $0 < \mu < 1$, both roots $-1 \pm \sqrt{\mu}$ are negative; at zero, both repeated-root modes decay; for $\mu < 0$, the exponential factor e^{-t} bounds every fixed sine/cosine combination. Hence every solution decays throughout the stated parameter range.

On any fixed $[0, T]$, the ratios in Step 2 converge uniformly because their arguments lie in intervals shrinking to zero and both ratios extend continuously with value 1 there. Yet for every negative μ there are infinitely many later zeros, while y_0 has none for $t > 0$. Convergence on fixed windows does not control zeros that escape to infinity.

Question 8

For the dimensionless equation

$$y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1,$$

define the nonnegative quantity $E(t) = ((y'(t))^2 + (y(t))^2)/2$.

Tasks

1. Solve the IVP and find the time and height of the displacement maximum.
2. Differentiate E using the equation. Prove that E is strictly decreasing between any two distinct nonnegative times, even though its derivative can vanish at an isolated time.
3. Compute $E(0)$ and its value at the displacement maximum. Sketch y and E , and explain why increasing displacement is consistent with decreasing E .
4. Use the energy identity to calculate $\int_0^\infty (y')^2 dt$, and confirm the answer by integrating the explicit derivative.

Question 8 – Solution

Strategy. Distinguish a coordinate's amplitude from a quantity combining position and derivative, then integrate the exact dissipation identity.

Step 1: Solve and locate the transient maximum. The repeated root is -1 . The data give $y = (1 + 2t)e^{-t}$ and $y' = (1 - 2t)e^{-t}$. Hence the unique displacement maximum is

$$t_m = 1/2, \quad y(t_m) = 2e^{-1/2} > 1.$$

Step 2: Derive and interpret dissipation. Differentiating E and substituting $y'' = -2y' - y$ gives

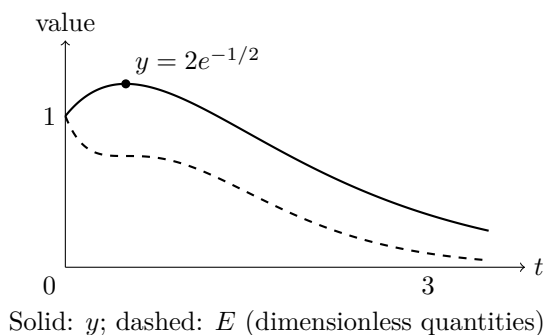
$$E' = y'y'' + yy' = y'(y'' + y) = -2(y')^2 \leq 0.$$

The derivative vanishes only at $t = 1/2$. On any interval $[s, t]$ with $s < t$, $(y')^2$ is positive except possibly at that one point, so its integral is strictly positive. Thus $E(t) - E(s) = -2 \int_s^t (y')^2 d\tau < 0$, proving strict decrease between distinct times.

Step 3: Compare the two quantities. Initially $E(0) = 1$. At the displacement peak, $y' = 0$, so

$$E(1/2) = \frac{1}{2}(2e^{-1/2})^2 = 2/e < 1.$$

The explicit expression is $E = (1 + 4t^2)e^{-2t}$.



Displacement initially increases, but the derivative contribution to E decreases enough for their sum to fall. The two curves represent the defined dimensionless quantities, not two competing claims about the same amplitude. Negative repeated roots ensure eventual decay without prohibiting an initial increase in y .

Step 4: Check the total integral in two ways. Since $y, y' \rightarrow 0$, also $E \rightarrow 0$. Integrating the dissipation identity gives $0 - 1 = -2 \int_0^\infty (y')^2 dt$, so the integral is $1/2$.

Directly, $(y')^2 = (1 - 4t + 4t^2)e^{-2t}$. Integration by parts gives the moments $\int_0^\infty e^{-2t} dt = 1/2$, $\int_0^\infty te^{-2t} dt = 1/4$ and $\int_0^\infty t^2 e^{-2t} dt = 1/4$. The resulting value $1/2 - 4(1/4) + 4(1/4) = 1/2$ agrees.

Question 9

For $L > 0$ and strictly positive endpoint values a, b , consider

$$y'' + 2y' + y = 0, \quad y(0) = a, \quad y(L) = b.$$

The same solution is defined beyond the observed interval.

Tasks

1. Find the unique solution and justify uniqueness for every $L > 0$.
2. Prove that it is strictly positive throughout $[0, L]$ using the transformed quantity $u = e^t y$.
3. Determine exactly which endpoint pairs make the response nonnegative for every future time $t \geq 0$, not just between observations.
4. For $L = 1$, $a = 1$, $b = e^{-2}$, find the first zero after the observed interval and determine the sign for all later times. Explain why positive endpoint observations did not guarantee permanent positivity.

Question 9 – Solution

Strategy. The exponential transformation turns the repeated-root endpoint problem into linear interpolation, but its continuation beyond the interval is extrapolation.

Step 1: Solve the endpoint equations. Write $y = (A + Bt)e^{-t}$. The first datum sets $A = a$, and the second gives $a + BL = be^L$. Thus

$$y(t) = \left[a + \frac{be^L - a}{L}t \right] e^{-t}.$$

The two constants are uniquely determined because $L \neq 0$. This also proves existence by substitution into the repeated-root family.

Step 2: Prove positivity between the data points. For $0 \leq t \leq L$,

$$u(t) = e^t y(t) = (1 - t/L)a + (t/L)be^L.$$

The weights are nonnegative and sum to 1, while both endpoint values are positive. Hence $u > 0$ and therefore $y > 0$ throughout the closed interval.

Step 3: Determine when positivity persists. The affine function $u = a + Bt$ starts positive. It remains nonnegative for all $t \geq 0$ exactly when $B \geq 0$. Equivalently,

$$b \geq ae^{-L}.$$

At equality, u is constant and $y = ae^{-t}$. If $b < ae^{-L}$, the slope of u is negative and its eventual zero is unavoidable despite both observed values being positive.

Step 4: Check the concrete continuation. Here $B = e^{-1} - 1 < 0$, so

$$y(t) = [1 - (1 - e^{-1})t]e^{-t}.$$

Its unique zero is

$$t_z = \frac{1}{1 - e^{-1}} = \frac{e}{e - 1} > 1.$$

The response is negative for every $t > t_z$ and tends to zero from below. The given endpoint at 1 is indeed $[e^{-1}]e^{-1} = e^{-2}$. Linear interpolation of positive transformed data stays positive between them; extending that same negative-slope line beyond the interval eventually crosses zero. The endpoint data determine the continuation uniquely, but do not force it to preserve the observed sign forever.

Question 10

A dimensionless family of repeated-root responses satisfies

$$y'' + 2\lambda y' + \lambda^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad \lambda > 0.$$

The design requirement is $y(t) \leq 0.05$ for every $t \geq 2$, with the additional constraint $|y''(0)| \leq 4$.

Tasks

1. Solve the IVP and prove that the response is positive and strictly decreasing for $t > 0$.
2. Show that the first time at level 0.05 is z_*/λ , where $z_* > 0$ is the unique root of $(1+z)e^{-z} = 0.05$. Locate z_* between two consecutive four-decimal-place numbers by a justified numerical calculation.
3. Translate both the settling requirement and the initial-acceleration constraint into restrictions on λ . Determine whether they can be satisfied simultaneously.
4. Find the earliest achievable settling time under the acceleration constraint alone. Sketch the responses for $\lambda = 1, 2$ with the tolerance and required time marked, and explain why simply increasing λ is not an admissible unlimited improvement.

Question 10 – Solution

Strategy. Scale time by the repeated decay rate, then compare the performance requirement with the independent constraint on the initial acceleration.

Step 1: Solve and establish monotonicity. The repeated root is $-\lambda$. The data give

$$y = (1 + \lambda t)e^{-\lambda t}, \quad y' = -\lambda^2 t e^{-\lambda t}.$$

The response is positive, strictly decreasing for $t > 0$, starts at 1 and tends to zero. Therefore crossing the tolerance once is enough to guarantee it thereafter.

Step 2: Reduce the crossing calculation to one dimensionless number. Let $f(z) = (1 + z)e^{-z}$ for $z \geq 0$. Then $f(0) = 1$, $f(z) \rightarrow 0$ and $f'(z) = -ze^{-z} < 0$ for $z > 0$. The intermediate value theorem and strict decrease give exactly one positive root $f(z_*) = 0.05$.

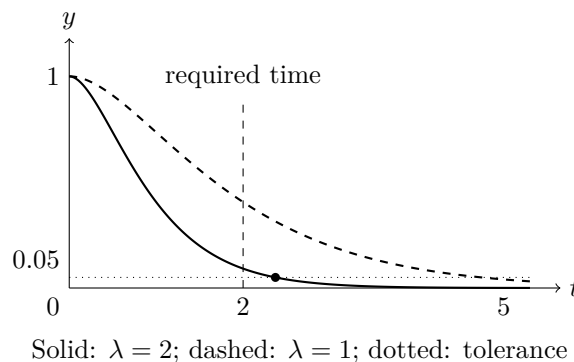
Direct evaluation gives $f(4.7438) > 0.05$ and $f(4.7439) < 0.05$. Thus

$$4.7438 < z_* < 4.7439, \quad t_{\text{settle}} = z_*/\lambda.$$

Numerically $z_* \approx 4.743865$; the bracket, rather than a rounded equality, supports the feasibility comparisons below.

Step 3: Test simultaneous feasibility. Settling by time 2 requires $z_*/\lambda \leq 2$, or $\lambda \geq z_*/2 > 2.3719$. But the equation at zero gives $y''(0) = -\lambda^2$, so the acceleration constraint requires $\lambda \leq 2$. These ranges do not intersect: the two requirements are incompatible.

At the largest admissible value $\lambda = 2$, the response at the deadline is $y(2) = 5e^{-4} \approx 0.091578 > 0.05$, confirming the failure directly.



Step 4: Optimize within the allowed family. The settling time z_*/λ strictly decreases as λ increases. Under $0 < \lambda \leq 2$, it is minimized at $\lambda = 2$, giving

$$t_{\min} = z_*/2 \approx 2.371932.$$

This is later than the required time 2. Larger λ would improve the settling time but violate the initial-acceleration bound. The conclusion optimizes this explicitly specified family and constraint, not every possible differential-equation model or control law.

Question 1

Let p, q be continuous on an open interval I , let $t_0 \in I$, and let y_1 be a known solution of

$$y'' + p(t)y' + q(t)y = 0$$

that never vanishes on I . Write $P(t) = \int_{t_0}^t p(s) ds$.

Tasks

1. Derive the equation for $w = v'$ after substituting $y = y_1 v$.
2. Solve the first-order equation for w and give an integral formula for y .
3. Construct a solution z with $z(t_0) = 0$ and $z'(t_0) = 1$. Prove that z is not a constant multiple of y_1 .
4. Express the solution with $y(t_0) = a$, $y'(t_0) = b$ in terms of y_1, z . Explain why the nonvanishing hypothesis matters.

Question 1 – Solution

Strategy. Factor out the known solution, then normalize the new solution by its initial slope.

Step 1: Reduce the order. Product differentiation gives

$$y' = y_1'v + y_1v', \quad y'' = y_1''v + 2y_1'v' + y_1v''.$$

The coefficient of v is zero because y_1 solves the equation. Hence

$$y_1v'' + (2y_1' + py_1)v' = 0, \quad w' + (2y_1'/y_1 + p)w = 0.$$

Step 2: Integrate twice. An integrating factor is $y_1^2e^P$, so

$$w = C \frac{e^{-P}}{y_1^2}, \quad y = y_1(t)[A + C \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds].$$

Both integrations contribute a constant. The formula includes $w = 0$.

Step 3: Normalize the companion. Set

$$z(t) = y_1(t_0)y_1(t) \int_{t_0}^t \frac{e^{-P(s)}}{y_1(s)^2} ds.$$

The integral vanishes at t_0 . Since $P(t_0) = 0$, differentiation gives $z'(t_0) = 1$. If $z = ky_1$, its zero value forces $k = 0$, contradicting this slope.

Step 4: Match arbitrary data. The two constants are fixed by

$$y = \frac{a}{y_1(t_0)}y_1 + [b - \frac{ay_1'(t_0)}{y_1(t_0)}]z.$$

This is the unique solution on I by the regular linear initial-value theorem. Division by y_1 and the integral kernel require $y_1 \neq 0$. A zero of the seed may obstruct this substitution without obstructing the original equation.

Question 2

On $t > 0$, consider

$$t^2 y'' - 3ty' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

A proposed known solution is $y_1 = t^2$.

Tasks

1. Verify y_1 and use reduction of order to find the general solution.
2. Solve the initial-value problem and verify both data.
3. Find every zero and stationary point of this solution on $t > 0$, and classify the stationary point.
4. Determine whether this solution has a C^1 extension and whether it has a C^2 extension through $t = 0$. Justify using one-sided limits.

Question 2 – Solution

Strategy. Normalize the equation before reduction, then use exact derivatives to study the singular endpoint.

Step 1: Find the missing solution. For t^2 , the residual is $(2 - 6 + 4)t^2 = 0$. With $y = t^2v$, the divided equation reduces to

$$v'' + \frac{1}{t}v' = 0, \quad v' = D/t, \quad \boxed{y = t^2(C + D \ln t)}.$$

Step 2: Impose the data. At 1, $y = C$ and $y' = 2C + D$. Thus $C = D = 1$ and

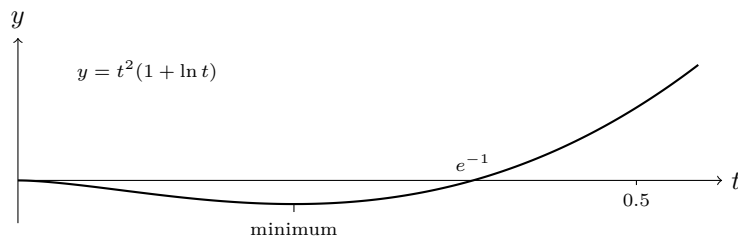
$$\boxed{y = t^2(1 + \ln t)}, \quad y' = t(3 + 2 \ln t), \quad y'' = 5 + 2 \ln t.$$

These give $y(1) = 1$ and $y'(1) = 3$.

Step 3: Locate the features. The sole zero is $t = e^{-1}$. The derivative changes from negative to positive at $t = e^{-3/2}$, giving the unique global minimum

$$(t, y) = (e^{-3/2}, -\frac{1}{2}e^{-3}).$$

Step 4: Test endpoint regularity. As $t \downarrow 0$, both y and y' tend to zero, but $y'' \rightarrow -\infty$. Setting $y = 0$ for $t \leq 0$ gives a C^1 extension (the derivative at zero is also $\lim y(t)/t = 0$). No C^2 extension is possible. The leading coefficient vanishes at zero, so the regular normalized theorem does not apply there.



Question 3

Consider the equation on $\mathbb{R}$

$$y'' - 2ty' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

with known solution $y_1 = e^{t^2}$. An integral is an acceptable exact answer.

Tasks

1. Verify the seed and derive the reduced first-order equation.
2. Find the initial-value solution as a definite integral and verify its residual without evaluating the integral.
3. Prove that the solution is odd and strictly increasing on $\mathbb{R}$.
4. Prove $t \leq y(t) \leq te^{t^2}$ for $t \geq 0$ and compute $y(1)$ to six decimal places. Explain why an unevaluated integral still defines a valid solution.

Question 3 – Solution

Strategy. Keep the definite integral and use the fundamental theorem of calculus for verification and bounds.

Step 1: Reduce. The seed has derivatives $2te^{t^2}$ and $(2 + 4t^2)e^{t^2}$, so its residual vanishes. With $y = e^{t^2}v$,

$$w' + 2tw = 0, \quad w = v' = Ce^{-t^2}.$$

Step 2: Fit and verify. The zero initial value sets the constant in v to zero; the initial slope sets $C = 1$. Therefore

$$y(t) = e^{t^2} \int_0^t e^{-s^2} ds.$$

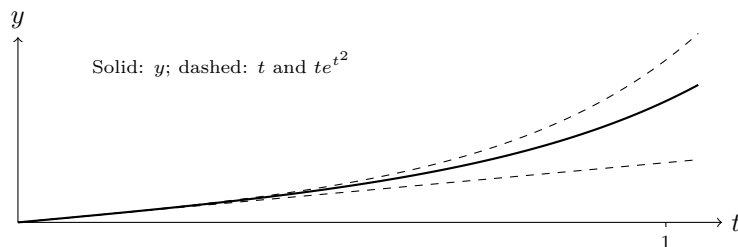
Differentiation gives $y' = 2ty + 1$, then $y'' = 2y + 2ty'$. The residual is zero, and at zero the value and slope are 0, 1.

Step 3: Use symmetry and sign. The integrand is positive and even, so its integral from zero is odd and has the sign of t . Multiplication by e^{t^2} preserves both properties. Consequently $ty \geq 0$ and $y' = 2ty + 1 \geq 1$ everywhere: y is strictly increasing.

Step 4: Bound and evaluate. For $0 \leq s \leq t$,

$$e^{-t^2} \leq e^{-s^2} \leq 1 \implies te^{-t^2} \leq \int_0^t e^{-s^2} ds \leq t.$$

Multiplying by e^{t^2} proves the bounds. Numerical integration gives $y(1) \approx 2.030078$. The integrand is smooth, so the definite integral defines a smooth function on all of $\mathbb{R}$; the exact residual check proves it solves the equation.



Question 4

For $y'' + y = 0$, suppose the known solution is $y_1 = \sin t$. First work on $(0, \pi)$, where the seed does not vanish.

Tasks

1. Use $y = (\sin t)v$ to construct a second solution on $(0, \pi)$.
2. Explain whether the poles of the quotient v prevent the second solution from extending to $\mathbb{R}$.
3. Use the extended solutions to solve $y(0) = 1$, $y'(0) = 0$, and justify uniqueness.
4. A proposed continuation is $\cos t + a \sin t$ for $t < 0$ and $\cos t + b \sin t$ for $t > 0$, with $y(0) = 1$. Find the conditions for continuity, for C^1 regularity, and for being a classical solution through zero.

Question 4 – Solution

Strategy. Separate a failure of division by the seed from a singularity of the original equation.

Step 1: Reduce on the valid interval. Substitution gives

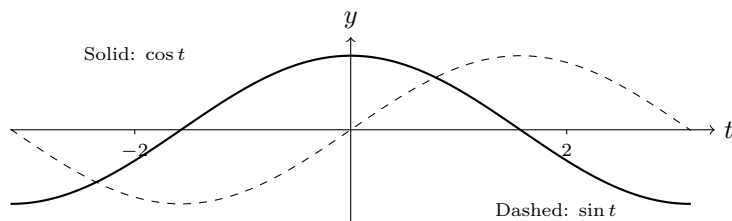
$$v'' + 2 \cot t v' = 0, \quad v' = C \csc^2 t, \quad v = -C \cot t + D.$$

Thus $y = -C \cos t + D \sin t$; a second solution is $\cos t$.

Step 2: Extend the product. Although $\cos t / \sin t = \cot t$ has poles at multiples of π , its product with $\sin t$ is $\cos t$ on each valid interval. The latter is smooth and directly satisfies $y'' + y = 0$ on all of $\mathbb{R}$. The poles belong to the chosen quotient.

Step 3: Solve at the seed's zero. In $y = A \cos t + B \sin t$, the data give $A = 1$, $B = 0$. Thus $\boxed{y = \cos t}$. Continuous coefficients and nonzero leading coefficient ensure global uniqueness for the data at zero.

Step 4: Match derivatives. Both one-sided values are 1, so continuity holds for all real a, b . The one-sided slopes are a, b , so C^1 regularity requires $a = b$. In that case one smooth formula holds on both sides, and it is a classical solution. If $a \neq b$, a derivative at zero does not exist; matching only position is insufficient.



Question 5

On $t > 0$, consider $t^2 y'' - 2ty' + 2y = 0$ with seed $y_1 = t$. Use the following *unnormalized* companion convention:

$$Y_a(t) = y_1(t) \int_a^t \frac{s^2}{y_1(s)^2} ds, \quad a > 0.$$

The numerator s^2 is held fixed when changing a or rescaling the seed.

Tasks

1. Verify the seed and derive the kernel in this convention from the normalized equation.
2. Compute Y_1 , Y_2 , and the companion obtained with seed $3t$ and lower limit 1.
3. Explain why these choices give the same solution space. Why does an arbitrary additive integration constant not create a third solution freedom?
4. Use $y = At + BY_1(t)$ to solve $y(1) = 2$, $y'(1) = 1$. Verify the equation and data.

Question 5 – Solution

Strategy. Compare the actual functions produced by changes in normalization, rather than counting written constants.

Step 1: Identify the kernel. The seed residual is $-2t + 2t = 0$. After division by t^2 , $p = -2/t$ and

$$P(t) = \int_1^t -2/s \, ds = -2 \ln t, \quad e^{-P(t)} = t^2.$$

Reduction gives $v' = Ct^2/y_1^2$, which yields the specified convention.

Step 2: Compute three companions. For $y_1 = t$, the integrand equals 1, so

$$Y_1 = t(t-1) = t^2 - t, \quad Y_2 = t(t-2) = t^2 - 2t.$$

For seed $3t$, the integrand is $1/9$, hence

$$\tilde{Y}_1 = 3t(t-1)/9 = (t^2 - t)/3.$$

Step 3: Compare freedoms. We have $Y_2 = Y_1 - t$ and $\tilde{Y}_1 = Y_1/3$. Adding a seed multiple or multiplying a companion by a nonzero constant leaves the family unchanged: every choice spans $\{At + Bt^2 : A, B \in \mathbb{R}\}$. An additive constant inside the antiderivative produces only another multiple of the seed, already present in the family.

Step 4: Fit and verify. Since $Y_1(1) = 0$, $Y_1'(1) = 1$, the data give $A = 2$ and $A + B = 1$. Therefore

$$\boxed{y = 2t - (t^2 - t) = 3t - t^2.}$$

Its derivatives are $3 - 2t$ and -2 , and

$$t^2(-2) - 2t(3 - 2t) + 2(3t - t^2) = 0.$$

At $t = 1$ the value is 2 and the slope is 1. Two effective constants are sufficient and necessary for arbitrary regular initial data.

Question 6

On $t > 0$, consider

$$ty'' - (t+1)y' + y = 0, \quad y_1 = e^t.$$

A student inserts the undivided coefficient $-(t+1)$ into the usual normalized reduction formula and obtains $v' = e^{t^2/2-t}$.

Tasks

1. Verify the seed, normalize the equation, and derive the correct equation for $w = v'$.
2. Find the full solution family and solve $y(1) = 0$, $y'(1) = 1$.
3. Compute the residual of the student's candidate $y = e^t v$, with $v' = e^{t^2/2-t}$. Explain why testing only $t = 1$ would miss the error.
4. For the original undivided equation at $t = 0$, find the necessary relation between value and slope. Exhibit two distinct smooth solutions with zero value and slope there.

Question 6 – Solution

Strategy. The coefficient in the standard formula is the coefficient after division by the leading term.

Step 1: Normalize. For e^t , the residual is $[t - (t + 1) + 1]e^t = 0$. Division gives $p = -1 - 1/t$, so

$$w' + (1 - 1/t)w = 0, \quad w = Cte^{-t}.$$

Step 2: Integrate and fit. Since $\int te^{-t} dt = -(t + 1)e^{-t}$,

$$y = Ae^t + B(t + 1).$$

At 1 the data give $Ae + 2B = 0$ and $Ae + B = 1$. Thus $B = -1$, $A = 2/e$, and

$$y = 2e^{t-1} - (t + 1).$$

Both basis functions have zero residual, and substitution at 1 gives 0, 1.

Step 3: Diagnose the error. For any v , the undivided residual of $e^t v$ is

$$e^t[tw' + (t - 1)w].$$

The student's w satisfies $w' = (t - 1)w$, giving residual

$$e^t(t^2 - 1)e^{t^2/2-t}.$$

This is zero at $t = 1$ but equals $3e^2$ at $t = 2$. A single vanishing residual does not establish a differential identity.

Step 4: Examine the singular initial point. At zero the original equation requires $y'(0) = y(0)$. Both $y = 0$ and $y = e^t - (t + 1)$ are smooth solutions through zero with value and slope zero; the second is nonzero, for example because its second derivative at zero is 1. The regular initial-value theorem fails here because the leading coefficient vanishes. Compatibility does not imply uniqueness.

Question 7

On $\mathbb{R}$, consider $(1+t^2)y'' - 2y = 0$ with known solution $y_1 = 1+t^2$. Let y_2 denote the solution with $y_2(0) = 0$, $y_2'(0) = 1$, and define $I = y_2/(1+t^2)$.

Tasks

1. Use reduction of order to find y_2 explicitly. Evaluate the required integral.
2. Verify the equation and the two normalization data directly.
3. Find the exact range and monotonicity of $I(t)$ on $\mathbb{R}$, including the limiting values at infinity.
4. For every real C , classify the finite zeros and the sign of $y_C = C(1+t^2) - y_2$. Include the two boundary parameter values.

Question 7 – Solution

Strategy. Divide the solution family by its positive seed so that zeros become level crossings of a monotone function.

Step 1: Construct the companion. The seed residual is $2(1+t^2) - 2(1+t^2) = 0$. Since $p = 0$,

$$I(t) = \int_0^t \frac{ds}{(1+s^2)^2} = \frac{t}{2(1+t^2)} + \frac{1}{2} \arctan t.$$

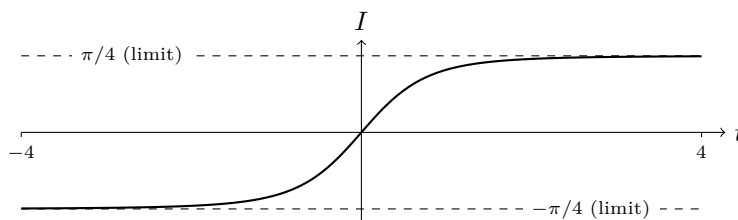
The integral follows by setting $s = \tan u$ and integrating $\cos^2 u$. Hence

$$y_2 = \frac{1}{2}t + \frac{1}{2}(1+t^2) \arctan t.$$

Step 2: Verify. We obtain $y_2' = 1 + t \arctan t$ and $y_2'' = \arctan t + t/(1+t^2) = 2y_2/(1+t^2)$. The equation holds and the data are 0, 1.

Step 3: Determine the ratio range. Since $I' = 1/(1+t^2)^2 > 0$, the ratio is strictly increasing. Its limits are $-\pi/4$ and $\pi/4$; neither is attained. Its range is exactly $(-\pi/4, \pi/4)$ by continuity.

Step 4: Classify all parameters. The sign of y_C is the sign of $C - I(t)$. If $|C| < \pi/4$, there is exactly one finite zero, with positive values before it and negative values after it. If $C \geq \pi/4$, the solution is strictly positive everywhere; if $C \leq -\pi/4$, it is strictly negative everywhere. Equality in these last two cases does not create a finite zero because the limiting ratio levels are excluded.



Question 8

On $\mathbb{R}$, solve the boundary-at-infinity problem

$$y'' + \frac{2t}{1+t^2}y' = 0, \quad \lim_{t \rightarrow -\infty} y(t) = -1, \quad \lim_{t \rightarrow \infty} y(t) = 3.$$

The constant function $y_1 = 1$ is a known homogeneous solution.

Tasks

1. Use reduction of order to obtain the full solution family.
2. Find the unique solution satisfying the two limiting conditions and verify its equation.
3. Find its value and slope at zero, monotonicity and exact range. Is every solution of this equation bounded?
4. Compute $\lim_{t \rightarrow \infty} t[3 - y(t)]$. Decide whether constants $M, c > 0$ can satisfy $3 - y(t) \leq Me^{-ct}$ for all sufficiently large t .

Question 8 – Solution

Strategy. A constant seed reduces the equation directly to a first-order equation for the slope.

Step 1: Integrate the slope. Set $w = y'$. An integrating factor is $1 + t^2$, so

$$((1 + t^2)w)' = 0, \quad w = B/(1 + t^2), \quad y = A + B \arctan t.$$

Step 2: Fit the two limits. The equations $A - B\pi/2 = -1$ and $A + B\pi/2 = 3$ give

$$y = 1 + \frac{4}{\pi} \arctan t.$$

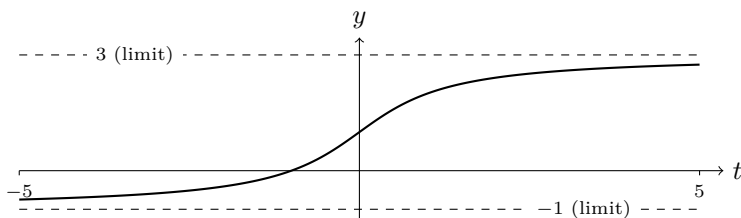
This is the unique choice of the two constants. Its derivatives $4/[\pi(1 + t^2)]$ and $-8t/[\pi(1 + t^2)^2]$ cancel in the original equation.

Step 3: Describe the trajectory. The value and slope at zero are 1 and $4/\pi$. The slope is positive everywhere; continuity and the limiting values give range $(-1, 3)$. Every member $A + B \arctan t$ is bounded, since $|\arctan t| < \pi/2$.

Step 4: Determine the tail rate. For $t > 0$,

$$3 - y(t) = \frac{4}{\pi} \arctan(1/t), \quad \lim_{t \rightarrow \infty} t[3 - y(t)] = \frac{4}{\pi}.$$

Here $\arctan u/u \rightarrow 1$ as $u \rightarrow 0$. An exponential bound would force this positive limit to be at most $\lim Mte^{-ct} = 0$, a contradiction. The approach to the limit has an algebraic $1/t$ tail.



Question 9

Consider the original undivided equation $t^2 y'' - 6y = 0$. First work on $t > 0$ with known solution $y_1 = t^3$. A classical solution through zero means a C^2 function satisfying the undivided equation on an open interval containing zero.

Tasks

1. Use reduction of order to find every solution on $t > 0$.
2. Which right-hand solutions admit a C^2 extension through zero? Find the forced value, slope and second derivative at zero.
3. For each admissible right-hand solution, classify all classical continuations to $t < 0$. Explain whether the zero initial data determine a unique continuation.
4. Which of those continuations are C^3 at zero? Explain why the usual uniqueness theorem does not settle this problem.

Question 9 – Solution

Strategy. Solve on each side of the singular coefficient, then compare one-sided derivatives at the join.

Step 1: Reduce on the positive side. The seed has residual $t^2(6t) - 6t^3 = 0$. In normalized form $p = 0$, so

$$v' = Ct^{-6}, \quad v = D - \frac{1}{5}Ct^{-5}, \quad \boxed{y = At^3 + Bt^{-2} \quad (t > 0)}.$$

These two powers each satisfy the original equation by direct substitution.

Step 2: Exclude the divergent mode. If $B \neq 0$, then y diverges at zero, so even a continuous extension is impossible. If $B = 0$, the limits of y, y', y'' are $0, 0, 0$. These are the required values for a C^2 extension.

Step 3: Classify continuations. On $t < 0$ the same calculation gives $at^3 + bt^{-2}$; continuity forces $b = 0$. Thus all classical continuations of the right branch At^3 are

$$y(t) = \begin{cases} at^3, & t < 0, \\ 0, & t = 0, \\ At^3, & t > 0, \end{cases} \quad a \in \mathbb{R}.$$

The first and second derivatives match at zero for every a . At zero the undivided residual is $0 - 6(0) = 0$. Therefore these are sufficient as well as necessary, and infinitely many continuations share $y(0) = y'(0) = 0$ even with the right branch fixed.

Step 4: Require one more derivative. The one-sided third derivatives are $6a$ and $6A$; a C^3 join occurs exactly when $a = A$. In that case the single polynomial At^3 holds on both sides. The leading coefficient is zero at the joining point, and the normalized coefficient $-6/t^2$ is undefined there, so the regular uniqueness theorem has no applicable interval containing zero.

Question 10

On $t > 0$, consider the forced equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t, \quad y(1) = 0, \quad y'(1) = 0.$$

The function $y_1 = t$ is supplied as a solution of the *associated homogeneous equation*.

Tasks

1. Verify the stated role of y_1 and explain why it is not a solution of the forced equation.
2. Substitute $y = tv$ and derive a reduced equation. Find the full forced solution family, retaining both integration constants.
3. Solve the given initial-value problem.
4. Check the original residual and data directly. Explain the error in applying the homogeneous reduction formula without carrying along the forcing term.

Question 10 – Solution

Strategy. The known homogeneous solution still cancels the term containing v , but the nonzero right-hand side remains.

Step 1: Verify the seed's role. For $y_1 = t$, the left-hand side is $0 - 2/t + 2/t = 0$. This satisfies the homogeneous equation, but cannot equal the positive forcing t on the specified domain.

Step 2: Reduce with the forcing present. With $y = tv$,

$$y' = v + tv', \quad y'' = 2v' + tv''.$$

Substitution cancels the terms in v and v' :

$$2v' + tv'' - \frac{2}{t}(v + tv') + \frac{2}{t}v = tv'' = t.$$

Since $t > 0$, $v'' = 1$. Equivalently the first-order reduced equation is $w' = 1$, with $w = v'$. Integrating twice gives

$$v = \frac{1}{2}t^2 + At + B, \quad \boxed{y = \frac{1}{2}t^3 + At^2 + Bt.}$$

Step 3: Impose the data. At 1 the equations are $1/2 + A + B = 0$ and $3/2 + 2A + B = 0$. Thus $A = -1$, $B = 1/2$, yielding

$$\boxed{y = \frac{1}{2}t(t-1)^2.}$$

Step 4: Verify and diagnose. In expanded form the solution is $t^3/2 - t^2 + t/2$, with $y' = 3t^2/2 - 2t + 1/2$ and $y'' = 3t - 2$. Its left-hand side is

$$(3t - 2) + (-3t + 4 - 1/t) + (t - 2 + 1/t) = t.$$

At 1 its value and slope are zero. Discarding the forcing would instead give $w' = 0$, producing only $At^2 + Bt$, whose left-hand side is zero. Those two terms provide the homogeneous freedom, but the cubic term is essential to match the forcing.

Question 1

A fundamental set for a homogeneous linear equation on an interval is a pair of solutions whose constant linear combinations represent every solution uniquely. Let p, q be continuous on an open interval I . For two solutions of $y'' + py' + qy = 0$, define

$$D(t_0) = u(t_0)v'(t_0) - u'(t_0)v(t_0), \quad t_0 \in I.$$

Tasks

1. Using existence and uniqueness for initial data, prove that the pair is fundamental on I if and only if $D(t_0) \neq 0$.
2. For $y'' + 2y' + 5y = 0$ on $\mathbb{R}$, verify $u = e^{-t} \cos 2t$, $v = e^{-t} \sin 2t$ and apply the criterion at zero.
3. Construct solutions ϕ, ψ with initial data $(\phi(0), \phi'(0)) = (1, 0)$ and $(\psi(0), \psi'(0)) = (0, 1)$.
4. Use this normalized pair to solve $y(0) = 2$, $y'(0) = -3$, and verify the data.

Question 1 – Solution

Strategy. A fundamental pair must supply every value-and-slope pair, so test the two-by-two initial-data system.

Step 1: Prove the criterion. If $D(t_0) \neq 0$, the equations

$$Au(t_0) + Bv(t_0) = a, \quad Au'(t_0) + Bv'(t_0) = b$$

have unique constants A, B for every a, b . The resulting combination solves the equation; uniqueness makes it equal to every solution with those data. Conversely, if $D(t_0) = 0$, some nonzero constant pair (A, B) gives zero value and slope. By uniqueness, $Au + Bv$ is identically zero, contradicting unique representation. Thus the criterion is necessary and sufficient.

Step 2: Verify the candidates. Their derivatives are

$$u' = e^{-t}(-\cos 2t - 2\sin 2t), \quad u'' = e^{-t}(-3\cos 2t + 4\sin 2t),$$

$$v' = e^{-t}(-\sin 2t + 2\cos 2t), \quad v'' = e^{-t}(-3\sin 2t - 4\cos 2t).$$

Both residuals vanish. At zero their data are $(1, -1)$ and $(0, 2)$, so $D(0) = 2 \neq 0$ and the pair is fundamental on $\mathbb{R}$.

Step 3: Normalize the data. The required choices are

$$\phi = u + \frac{1}{2}v, \quad \psi = \frac{1}{2}v.$$

Their data are $(1, 0)$ and $(0, 1)$ respectively, so they too form a fundamental set.

Step 4: Fit the solution. The normalized coefficients equal the data:

$$y = 2\phi - 3\psi = e^{-t}(2\cos 2t - \frac{1}{2}\sin 2t).$$

At zero its value is 2; its slope is $-2 - 1 = -3$. It satisfies the equation by linearity and is unique on $\mathbb{R}$.

Question 2

For the equation $y'' - y = 0$ on $\mathbb{R}$, consider the parameter-dependent pair

$$u = e^t + e^{-t}, \quad v_\alpha = e^t + \alpha e^{-t}, \quad \alpha \in \mathbb{R}.$$

Coefficients in a fundamental set must be constant in t .

Tasks

1. Verify both solutions and find all α for which the pair is fundamental.
2. For those parameters, find the coefficients of the solution with $y(0) = 0$, $y'(0) = 1$ in this pair.
3. As $\alpha \rightarrow 1$, compare the behavior of these coefficients with the behavior of the represented solution.
Can large coefficients alone establish large solution values?
4. Construct from u, v_α a companion with data $(0, 1)$ that extends to a well-defined function at $\alpha = 1$.
Explain why simply setting $\alpha = 1$ in the original pair fails.

Question 2 – Solution

Strategy. Track the initial-data vectors as the two proposed basis elements approach dependence.

Step 1: Test independence. Each exponential solves $y'' - y = 0$, so both combinations do. The data vectors at zero are $(2, 0)$ and $(1 + \alpha, 1 - \alpha)$, with determinant $2(1 - \alpha)$. The pair is fundamental exactly when $\alpha \neq 1$. At 1, both functions equal u .

Step 2: Fit the data. In $y = Au + Bv_\alpha$, the data require

$$2A + (1 + \alpha)B = 0, \quad (1 - \alpha)B = 1.$$

Thus

$$A = -\frac{1 + \alpha}{2(1 - \alpha)}, \quad B = \frac{1}{1 - \alpha}.$$

Step 3: Simplify before taking a limit. Substitution cancels the large terms exactly:

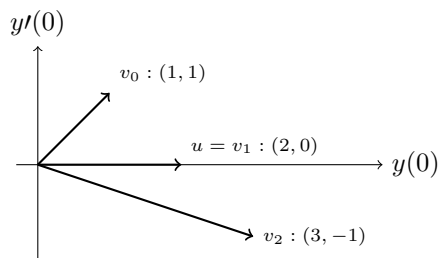
$$Au + Bv_\alpha = \frac{1}{2}(e^t - e^{-t}) = \sinh t.$$

Both coefficients grow without bound in magnitude near $\alpha = 1$, while the function is independent of α . Large coordinates in a nearly dependent pair do not imply large values of the function on a fixed interval.

Step 4: Replace the collapsing direction. Define, for $\alpha \neq 1$,

$$z_\alpha = \frac{v_\alpha - (1 + \alpha)u/2}{1 - \alpha} = \sinh t.$$

This extends by $z_1 = \sinh t$ and has data $(0, 1)$. The pair u, z_1 remains fundamental. The original pair at $\alpha = 1$ contains only one effective direction and cannot supply the requested nonzero slope.



Question 3

For $y'' - 3y' + 2y = 0$ on $\mathbb{R}$, let

$$f = e^t, \quad g = e^{2t}, \quad h = f + g, \quad k = 2f - g.$$

A student writes $y = Af + Bg + Ch + Dk$ and claims this gives four independent solution freedoms.

Tasks

1. Verify that all four functions solve the equation and that f, g form a fundamental set.
2. Determine which of the six unordered pairs chosen from f, g, h, k are fundamental. Justify every pair.
3. Find every quadruple (A, B, C, D) that represents the zero function, and explain the error in the student's claim.
4. Find all quadruples representing the solution with $y(0) = 1$, $y'(0) = 3$. Give its unique representation using only h, k .

Question 3 – Solution

Strategy. Express every candidate in the already verified pair f, g , then count effective coefficients.

Step 1: Verify the underlying pair. Substitution of e^{rt} gives residual $(r^2 - 3r + 2)e^{rt}$, which vanishes for $r = 1, 2$. Linearity verifies h, k . The data determinant for f, g at zero is $2 - 1 = 1$, so this pair is fundamental.

Step 2: Check all six pairs. The coefficient columns relative to f, g are $(1, 0)$, $(0, 1)$, $(1, 1)$ and $(2, -1)$. Their determinants, in the listed order, are

pair	(f, g)	(f, h)	(f, k)	(g, h)	(g, k)	(h, k)
determinant	1	1	-1	-1	-2	-3

Every determinant is nonzero, so every pair is an invertible change of the fundamental pair f, g and is itself fundamental.

Step 3: Identify redundancy. The proposed expression is

$$y = (A + C + 2D)f + (B + C - D)g.$$

It is zero exactly when $A = -C - 2D$ and $B = -C + D$, where C, D are arbitrary. These nontrivial constant relations show that the four functions are not independent. Only two effective coefficients are present.

Step 4: Fit the specified data. Writing $y = af + bg$ gives $a + b = 1$, $a + 2b = 3$, hence $a = -1, b = 2$. All four-coefficient representations are

$$(A, B, C, D) = (-1 - C - 2D, 2 - C + D, C, D).$$

Since $h - k = -f + 2g$, the unique representation in the pair h, k is $y = h - k$. Its data are $-1 + 2 = 1$ and $-1 + 4 = 3$.

Question 4

The smooth functions $u = t^2$ and $v = t^3$ are proposed as a fundamental set for an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients on $\mathbb{R}$.

Tasks

1. Prove that u, v are linearly independent as functions on $\mathbb{R}$.
2. Prove that no such regular normalized equation on $\mathbb{R}$ can have both as solutions. Explain why independence alone is insufficient.
3. Find the normalized equation having both as solutions on $(0, \infty)$ and verify that they form a fundamental set there.
4. On $(0, \infty)$, solve $y(1) = 0$, $y'(1) = 1$ using the pair. Explain why its zero value and slope at zero do not contradict your conclusion about regular equations.

Question 4 – Solution

Strategy. Check both requirements: independence and membership in the solution space of one regular equation on the stated interval.

Step 1: Prove independence. If $At^2 + Bt^3 = 0$ for every real t , then $A + Bt = 0$ for every nonzero t . Evaluating at two distinct nonzero points forces $A = B = 0$.

Step 2: Reject the proposed regular equation. The nonzero function $u = t^2$ has $u(0) = u'(0) = 0$. Uniqueness for continuous normalized coefficients would force it to equal the zero solution on $\mathbb{R}$. Equivalently, substitution at zero would require $u''(0) = 2 = 0$. Thus these independent functions cannot be a fundamental set for any equation of the stated kind.

Step 3: Work on a valid interval. Substituting the two functions for $t > 0$ gives

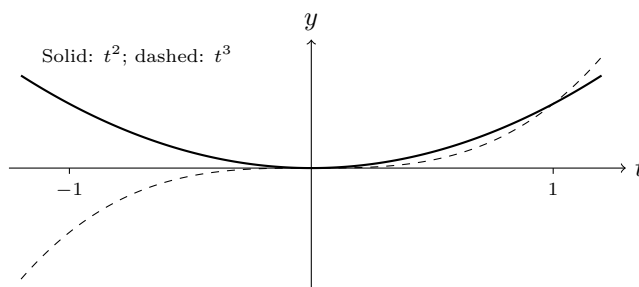
$$2 + 2tp + t^2q = 0, \quad 6t + 3t^2p + t^3q = 0.$$

Subtracting t times the first equation from the second yields $p = -4/t$, then $q = 6/t^2$. Hence

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0.$$

Both residuals vanish, the coefficients are continuous on $(0, \infty)$, and $uv' - u'v = t^4 > 0$ there, so the pair is fundamental.

Step 4: Fit and interpret. The data give $A + B = 0$, $2A + 3B = 1$, so $y = t^3 - t^2$. Its polynomial extension has zero value and slope at zero, but the normalized equation is undefined there. A theorem requiring continuous coefficients on an interval containing zero cannot be applied to that extension.



Question 5

The pair $\cos t, \sin t$ is fundamental for $y'' + y = 0$ on $\mathbb{R}$. Fix $L > 0$ and prescribe two position measurements

$$y(0) = a, \quad y(L) = b, \quad a, b \in \mathbb{R}.$$

Tasks

1. Find the coefficient equations and determine when the measurements select a unique solution.
2. For every exceptional length, give necessary and sufficient conditions for existence and describe all solutions when they exist.
3. At $L = \pi$, describe all solutions for $(a, b) = (1, -1)$ and decide whether any solution exists for $(a, b) = (1, 1)$.
4. Explain why the exceptional lengths do not mean that the fundamental pair has become dependent. Identify which additional measurement at zero would uniquely determine a solution.

Question 5 – Solution

Strategy. Distinguish a fundamental set of functions from the invertibility of a particular measurement map.

Step 1: Solve at ordinary lengths. Every solution is $A \cos t + B \sin t$. The position measurements give

$$A = a, \quad a \cos L + B \sin L = b.$$

If $\sin L \neq 0$, the unique solution is

$$y = a \cos t + \frac{b - a \cos L}{\sin L} \sin t.$$

Thus uniqueness holds exactly at lengths that are not positive integer multiples of π .

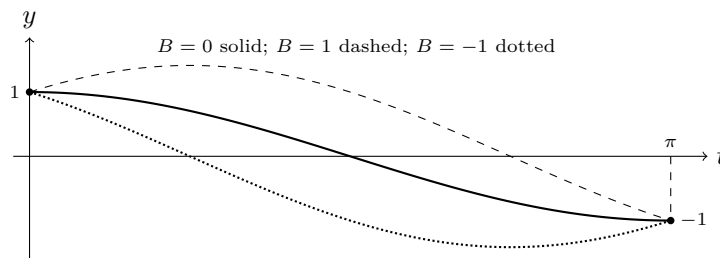
Step 2: Classify exceptional data. If $L = n\pi$, $n = 1, 2, \dots$, the second equation reduces to $b = (-1)^n a$. If this fails, there is no solution. If it holds, B is arbitrary and all solutions are $a \cos t + B \sin t$; there are infinitely many.

Step 3: Apply the classification. At $L = \pi$, the data $(1, -1)$ give

$$y = \cos t + B \sin t, \quad B \in \mathbb{R}.$$

The data $(1, 1)$ fail the necessary condition $b = -a$ and give no solution. The figure shows three members of the compatible family.

Step 4: Explain what failed. The data vectors of $\cos t, \sin t$ using value and slope at zero are $(1, 0)$ and $(0, 1)$, so the pair remains fundamental. At $L = n\pi$, both endpoint positions are insensitive to the coefficient B . Measuring $y'(0) = c$ fixes $B = c$ uniquely; the endpoint data must still be compatible with that solution.



Question 6

Let $D = \mathbb{R} \setminus \{0\}$. On this disconnected domain, consider $y'' = 0$ and the two functions

$$u(t) = 1, \quad v(t) = \begin{cases} -1, & t < 0, \\ 1, & t > 0. \end{cases}$$

A solution means a C^2 function on D ; no matching condition at zero is imposed.

Tasks

1. Verify that both functions solve the equation and are linearly independent on D .
2. Do their constant linear combinations represent every solution on D ? Give a decisive counterexample and compare their independence on each connected component.
3. Find the full solution family on D and an explicit independent spanning set. Prove why at least four functions are needed.
4. Describe every solution satisfying $y(1) = 0$, $y'(1) = 1$. Explain why those data do not determine a unique solution on all of D .

Question 6 – Solution

Strategy. Each connected component has its own integration constants because zero is excluded and no matching rule is imposed.

Step 1: Verify and test independence. Both functions are constant on each component, so both have zero second derivative on D . If $Au + Bv = 0$ on D , the positive side gives $A + B = 0$ and the negative side gives $A - B = 0$. Thus $A = B = 0$.

Step 2: Test spanning. Every combination $Au + Bv$ is constant on each side, so it cannot represent the solution $y = t$. The pair is independent on D but does not span its solution space. On either individual component, $v = u$ or $v = -u$, so the restricted pair is dependent.

Step 3: Exhibit all four freedoms. Integrating separately gives

$$y(t) = \begin{cases} A_- + B_-t, & t < 0, \\ A_+ + B_+t, & t > 0. \end{cases}$$

Let χ_+ equal 1 for $t > 0$ and 0 for $t < 0$, and let $\chi_- = 1 - \chi_+$. An independent spanning set is

$$\{\chi_-, t\chi_-, \chi_+, t\chi_+\}.$$

Restriction to either side proves independence by the independence of $1, t$. Each of the four displayed constants can be chosen freely, so the space has dimension four and no set with fewer than four functions spans it. All four functions are smooth on D .

Step 4: Impose data on one component. On the positive side, $B_+ = 1$ and $A_+ = -1$. Thus $y = t - 1$ for $t > 0$, while A_-, B_- remain arbitrary for $t < 0$. Initial-value uniqueness controls the connected interval containing the initial point; it supplies no information across the excluded point.

Question 7

Consider $y'' + 4y = 0$ on $\mathbb{R}$. Allow complex-valued solutions initially and define

$$z_+ = e^{2it}, \quad z_- = e^{-2it}, \quad i^2 = -1.$$

Then restrict attention to real-valued solutions.

Tasks

1. Verify the two complex solutions and show that they form a fundamental set over the complex numbers.
2. Find the necessary and sufficient relation between complex constants A, B for $Az_+ + Bz_-$ to be real for every real t .
3. Derive the correspondence with real coefficients in $a \cos 2t + b \sin 2t$. Explain why z_+, z_- themselves are not a fundamental set of the real-valued solution space.
4. Solve $y(0) = 1, y'(0) = -2$ in both representations and verify the coefficients.

Question 7 – Solution

Strategy. Specify the scalar field and impose real-valuedness on the entire function, not just its value at one point.

Step 1: Verify complex completeness. Each candidate has second derivative -4 times itself. Their data columns at zero are $(1, 2i)$ and $(1, -2i)$, with determinant $-4i \neq 0$. The initial-data criterion applies over $\mathbb{C}$ as well: real and imaginary parts each satisfy the real equation and its uniqueness theorem.

Step 2: Impose reality. Since $\overline{z_+} = z_-$, the conjugate of $Az_+ + Bz_-$ is $\overline{B}z_+ + \overline{A}z_-$. Independence implies equality to its conjugate exactly when

$$B = \overline{A}.$$

This relation is sufficient too, since the two terms then are conjugates.

Step 3: Convert coordinates. Euler's formula gives

$$Az_+ + Bz_- = (A + B) \cos 2t + i(A - B) \sin 2t.$$

For real a, b , the inverse relations are

$$A = (a - ib)/2, \quad B = (a + ib)/2.$$

They automatically satisfy the reality condition. The exponentials are not themselves real-valued functions on $\mathbb{R}$, so they are not elements of the real-valued solution space. A real fundamental set is $\cos 2t, \sin 2t$.

Step 4: Match the data. The real representation has $a = 1$ and $2b = -2$, hence

$$y = \cos 2t - \sin 2t = \frac{1}{2}(1 + i)e^{2it} + \frac{1}{2}(1 - i)e^{-2it}.$$

Here $A + B = 1$ and $2i(A - B) = -2$, verifying the value and slope. The conjugate coefficients ensure that the resulting solution is real everywhere.

Question 8

Two measured solution profiles are $u(t) = t$ and $v(t) = e^{-t}$. Seek a normalized equation

$$y'' + p(t)y' + q(t)y = 0$$

with both as solutions on a connected open interval containing $t = 0$.

Tasks

1. Use the two profiles to determine p, q wherever the coefficient equations are solvable.
2. Find the largest interval containing zero on which the reconstructed coefficients are continuous, and prove that the profiles form a fundamental set there.
3. Prove that no choice of continuous normalized coefficients on an interval containing -1 can have both profiles as solutions. Does their smoothness resolve this obstruction?
4. Use the fundamental pair to solve $y(0) = 2$, $y'(0) = -1$. Verify the data and explain the distinction between extending this formula and extending the normalized equation.

Question 8 – Solution

Strategy. Treat the two known solutions as simultaneous equations for the unknown coefficients, then check the interval where the result is regular.

Step 1: Reconstruct the equation. Substituting $u = t$ and $v = e^{-t}$ gives

$$p + tq = 0, \quad 1 - p + q = 0.$$

For $t \neq -1$, these have the unique solution

$$p = \frac{t}{t+1}, \quad q = -\frac{1}{t+1}.$$

Thus both residuals vanish in $y'' + [t/(t+1)]y' - y/(t+1) = 0$ wherever these coefficients are defined.

Step 2: Find a fundamental interval. The largest connected open interval containing zero and avoiding the pole is $(-1, \infty)$. The initial-data determinant, or its value at any point of this interval, is

$$uv' - u'v = -(t+1)e^{-t} \neq 0.$$

In particular it equals -1 at zero, so the two solutions form a fundamental set throughout the interval.

Step 3: Locate the obstruction. At $t = -1$, the coefficient equations would require $p - q = 0$ and $1 - p + q = 0$, an immediate contradiction. Even finite pointwise coefficient values cannot satisfy both conditions there. Smoothness of the two functions does not guarantee a common regular normalized equation across that point.

Step 4: Solve and distinguish extensions. In $y = At + Be^{-t}$, the data give $B = 2$ and $A - B = -1$, hence

$$y = t + 2e^{-t} \quad (-1 < t < \infty).$$

The value and slope at zero are $2, -1$, and linearity verifies its equation. The formula is smooth for all real t , but the reconstructed normalized equation is undefined at -1 . Extending a function does not remove that coefficient singularity. Multiplying by $t+1$ instead gives a different, undivided formulation with a vanishing leading coefficient there.

Question 9

For $y'' + y = 0$ on $\mathbb{R}$, let the state at a time s mean the ordered pair $(y(s), y'(s))$. Fix a real reference time s .

Tasks

1. Construct a fundamental pair ϕ_s, ψ_s with state data $(1, 0)$ and $(0, 1)$ at time s . Express any solution in that pair.
2. Derive explicit formulas for the state at $s + h$ in terms of the state (a, b) at s .
3. Show that moving the reference time by h and then k gives the same result as moving it by $h + k$, and that the move by $-h$ reverses the move by h . Prove that $y^2 + (y')^2$ is preserved.
4. For initial state $(3, 4)$ at zero, find the first positive time when the slope is zero and the value then. Describe the direction of travel in the state plane.

Question 9 – Solution

Strategy. Changing the reference time changes the coordinates in a normalized fundamental pair, while the represented solution remains the same.

Step 1: Normalize at the reference time. Set

$$\phi_s(t) = \cos(t - s), \quad \psi_s(t) = \sin(t - s).$$

Both solve the equation and have the required data. Their data determinant is 1, so any solution is $y(t) = a \cos(t - s) + b \sin(t - s)$, where $a = y(s)$ and $b = y'(s)$.

Step 2: Move the state. Evaluation and differentiation give

$$y(s + h) = a \cos h + b \sin h, \quad y'(s + h) = -a \sin h + b \cos h.$$

Step 3: Compose and reverse. Applying the same formulas with increment k yields first component

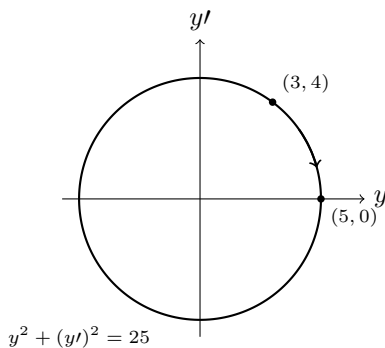
$$a(\cos h \cos k - \sin h \sin k) + b(\sin h \cos k + \cos h \sin k).$$

The addition formulas make this $a \cos(h + k) + b \sin(h + k)$; the second component is $-a \sin(h + k) + b \cos(h + k)$. Taking $k = -h$ gives (a, b) . Squaring the two components and adding cancels the cross terms and gives $a^2 + b^2$.

Step 4: Locate the first turning point. Here $y = 3 \cos t + 4 \sin t$ and $y' = -3 \sin t + 4 \cos t$. The first positive zero of y' is

$$t_* = \arctan(4/3), \quad y(t_*) = 5.$$

Indeed $\cos t_* = 3/5$, $\sin t_* = 4/5$, and the slope is positive before this time. In coordinates (y, y') , the state travels clockwise on the circle of radius 5: at $(5, 0)$ its derivative is $(0, -5)$.



Question 10

For $y'' - y = 0$ on $\mathbb{R}$, fix $L > 0$. Seek solutions ℓ, r with endpoint values

$$\ell(0) = 1, \quad \ell(L) = 0, \quad r(0) = 0, \quad r(L) = 1.$$

Use $\sinh x = (e^x - e^{-x})/2$ and $\cosh x = (e^x + e^{-x})/2$.

Tasks

1. Construct ℓ, r from the exponential fundamental pair and prove that they also form a fundamental set.
2. Use this pair to solve $y(0) = a, y(L) = b$ for arbitrary real a, b . Prove uniqueness.
3. For $a, b \geq 0$ and not both zero, prove positivity on $0 < t < L$ and $y(t) \leq \max(a, b)$ on $[0, L]$. Determine whether $\ell(t) + r(t) = 1$ in the interior.
4. For $a = b = 1$, locate the unique minimum on $[0, L]$ and give its value. Explain why positive interpolation weights do not imply straight-line interpolation.

Question 10 – Solution

Strategy. Choose basis functions adapted to the two measurements, then inspect their positive weights.

Step 1: Build an endpoint basis. Write $r = Ae^t + Be^{-t}$. The data give $A + B = 0$ and $A(e^L - e^{-L}) = 1$, so $A = 1/(2 \sinh L)$ and $B = -A$. Reflecting r about the midpoint gives $\ell(t) = r(L - t)$, hence

$$\ell(t) = \frac{\sinh(L - t)}{\sinh L}, \quad r(t) = \frac{\sinh t}{\sinh L}.$$

Both solve $y'' - y = 0$ and have the stated endpoint values. Their determinant at zero is $\ell(0)r'(0) - \ell'(0)r(0) = 1/\sinh L \neq 0$, so they form a fundamental set on $\mathbb{R}$.

Step 2: Fit the measurements. In this pair the endpoint coefficients are exactly the prescribed values:

$$y = a\ell + br.$$

Every solution has a unique representation, and evaluating at $0, L$ forces its coefficients to be a, b . Thus the endpoint problem has exactly one solution.

Step 3: Bound the positive weights. On $0 < t < L$, both weights are positive. The hyperbolic addition identity gives

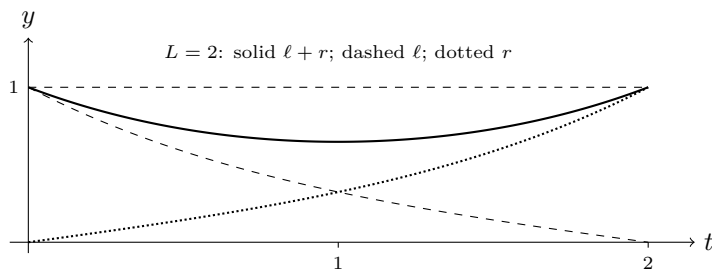
$$\ell(t) + r(t) = \frac{\cosh(t - L/2)}{\cosh(L/2)} \leq 1 \quad (0 \leq t \leq L),$$

with strict inequality in the interior, since $|t - L/2| < L/2$. If $M = \max(a, b)$, then $0 < y$ in the interior and $y \leq M(\ell + r) \leq M$ on the full closed interval. At an endpoint the solution can equal zero if the corresponding datum is zero.

Step 4: Examine equal endpoint data. For $a = b = 1$,

$$y = \frac{\cosh(t - L/2)}{\cosh(L/2)}, \quad y_{\min} = \frac{1}{\cosh(L/2)} \text{ at } t = L/2.$$

The minimum is unique because $\cosh x$ has its unique minimum at zero. The weights are positive but sum to less than one inside; this is not the straight-line interpolation of equal endpoint values.



Question 1

For two real solutions u, v of $y'' + p(t)y' + q(t)y = 0$ on a connected open interval I , define the ordered Wronskian

$$W[u, v](t) = u(t)v'(t) - u'(t)v(t).$$

Assume p, q are continuous and $t_0 \in I$.

Tasks

1. Derive a first-order equation for W and solve it in terms of $W(t_0)$. Your derivation must include the case $W(t_0) = 0$.
2. Prove that W either vanishes everywhere on I or never vanishes there. Explain the consequences for independence of the two solutions.
3. For $(1 + t^2)y'' + 2ty' + (1 + t^2)y = 0$ on $\mathbb{R}$, find $W(t)$ if $W(0) = -3$, without solving the second-order equation.
4. Find the sign and limiting values of this Wronskian. Does its approach to zero at infinity imply that the solutions become dependent at a finite time?

Question 1 – Solution

Strategy. Differentiate the determinant and substitute the common equation before integrating.

Step 1: Derive Abel's identity. Differentiation cancels the mixed terms:

$$W' = uv'' - u''v = u(-pv' - qv) - (-pu' - qu)v = -pW.$$

Multiplication by $\exp(\int_{t_0}^t p(s) ds)$ gives a function with zero derivative. Thus, without dividing by W ,

$$W(t) = W(t_0) \exp\left[-\int_{t_0}^t p(s) ds\right].$$

This includes $W \equiv 0$ when $W(t_0) = 0$.

Step 2: Use the nonzero exponential. The exponential factor is positive and finite at every point of I . A nonzero initial Wronskian stays nonzero with the same sign. If it is zero, the initial-data columns are dependent; a nontrivial constant combination has zero value and slope at t_0 . Regular uniqueness forces that combination to vanish on I . Therefore, for solutions of this common regular equation, $W \equiv 0$ is equivalent to dependence.

Step 3: Normalize the example. Divide by $1 + t^2$, giving $p = 2t/(1 + t^2)$. Integration gives

$$\int_0^t \frac{2s}{1+s^2} ds = \ln(1+t^2), \quad W(t) = -\frac{3}{1+t^2}.$$

Thus the Wronskian is determined without finding either solution. The undivided coefficient $2t$ is not the coefficient to use in Abel's identity.

Step 4: Interpret the limits. We have $W < 0$ everywhere, with $W \rightarrow 0$ from below as $t \rightarrow \pm\infty$. The pair is independent on all of $\mathbb{R}$, and its data columns are independent at every finite time. A limiting zero outside the interval is not a zero at a point where the initial-data criterion applies.

Question 2

Let u, v solve $y'' + 2y' + q(t)y = 0$ on $\mathbb{R}$, where q is continuous, with data

$$(u(0), u'(0)) = (1, 0), \quad (v(0), v'(0)) = (0, 1).$$

Define $f = 2u - v$ and $g = 3u + 4v$. At each time, regard (f, f') and (g, g') as vectors in the value-slope plane.

Tasks

1. For arbitrary constants a, b, c, d , derive the formula relating $W[au + bv, cu + dv]$ to $W[u, v]$.
2. Find $W[f, g](t)$ and decide whether f, g form a fundamental set. Explain the effect of reversing their order.
3. Interpret the Wronskian as oriented parallelogram area. Find the ordinary area at time zero and the first nonnegative time at which it is half that size.
4. Can the two state vectors become parallel at a finite time? Explain what their area tending to zero does and does not imply.

Question 2 – Solution

Strategy. Separate the constant determinant of a basis change from the time evolution supplied by Abel's identity.

Step 1: Expand the determinant. Terms containing uu' or vv' cancel, leaving

$$W[au + bv, cu + dv] = (ad - bc)W[u, v].$$

The constants may be factored through differentiation; this step would require extra terms for time-dependent coefficients.

Step 2: Apply the two determinants. Here $W[u, v](0) = 1$ and $p = 2$, so $W[u, v] = e^{-2t}$. The coefficient determinant is $2 \cdot 4 - (-1) \cdot 3 = 11$, hence

$$W[f, g] = 11e^{-2t} > 0.$$

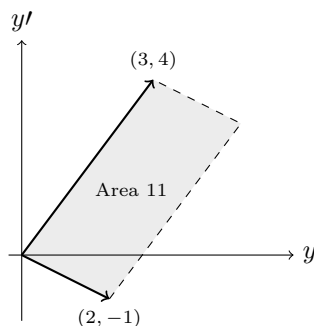
Both combinations solve the equation and are independent, so they are fundamental. Reversing their order negates the Wronskian but leaves independence unchanged.

Step 3: Compute the area. The columns (f, f') , (g, g') have signed area $W[f, g]$ and ordinary area $|W[f, g]|$. At zero they are $(2, -1)$ and $(3, 4)$, with area 11. The half-area equation gives

$$11e^{-2t} = 11/2, \quad t = \frac{1}{2} \ln 2.$$

The area is strictly decreasing for $t \geq 0$, so this is the first such time.

Step 4: Distinguish collapse from a limit. The determinant never vanishes at finite t , so the two state vectors never become parallel there. The area tends to zero as $t \rightarrow \infty$; this does not destroy finite-time independence. Area alone also does not specify the lengths of the two vectors or their angle separately.



Question 3

Consider the functions $u(t) = t^3$ and $v(t) = |t|^3$ on $\mathbb{R}$. A student claims that a Wronskian which is identically zero always proves linear dependence of two C^2 functions.

Tasks

1. Prove that both functions are C^2 on $\mathbb{R}$, explicitly checking the derivatives of v at zero.
2. Compute $W[u, v]$ everywhere and determine whether the two functions are linearly independent on $\mathbb{R}$.
3. Explain precisely why the common regular-equation theorem does not contradict your result. Could both functions solve a normalized equation with continuous coefficients on $\mathbb{R}$?
4. Verify that both solve the undivided equation $t^2y'' - 2ty' = 0$ on $\mathbb{R}$. Compare the ratio v/u on the two sides of zero.

Question 3 – Solution

Strategy. Check the counterexample directly, then identify the hypothesis missing from the student's assertion.

Step 1: Check regularity. For $t \neq 0$, $v' = 3t|t|$ and $v'' = 6|t|$. At zero,

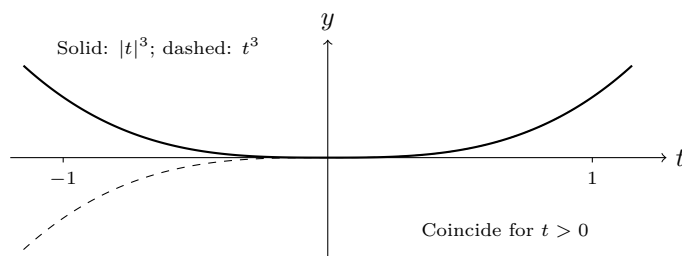
$$v'(0) = \lim_{h \rightarrow 0} |h|^3/h = 0, \quad v''(0) = \lim_{h \rightarrow 0} 3h|h|/h = 0.$$

These agree continuously with the formulas away from zero, so v is C^2 ; u is a polynomial and is smooth.

Step 2: Compare the Wronskian and independence. On $t > 0$, $v = u$; on $t < 0$, $v = -u$. Thus $W = 0$ on each side, and at zero both data columns are zero, so $\boxed{W \equiv 0}$. Yet $Au + Bv = 0$ on all of $\mathbb{R}$ forces $A + B = 0$ on the positive side and $A - B = 0$ on the negative side. Hence $A = B = 0$: the functions are independent on $\mathbb{R}$.

Step 3: Restore the missing hypothesis. The zero-Wronskian implication requires solutions of one common regular homogeneous linear equation on a connected interval, not merely two C^2 functions. The nonzero solution u would have zero value and slope at zero, contradicting regular uniqueness. Thus no normalized equation with continuous coefficients on $\mathbb{R}$ can have both as solutions.

Step 4: Verify a singular equation. For u , the residual is $t^2(6t) - 2t(3t^2) = 0$. On each side $v = \pm u$, so its residual also vanishes; at zero both terms vanish. The leading coefficient is zero there. The ratio v/u is 1 on $t > 0$ and -1 on $t < 0$, and is undefined at the shared zero. Local constant ratios need not join into one global constant.



Question 4

A regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ has a known solution $u = e^t$. A desired companion v must satisfy

$$W[u, v](t) = 2e^{-t^2}, \quad v(0) = 0.$$

Assume p, q are continuous; an unevaluated definite integral is an acceptable exact answer.

Tasks

1. Determine the only possible coefficients p, q .
2. Construct v using the prescribed Wronskian and initial value, and find $v'(0)$.
3. Verify both the Wronskian and the differential equation without evaluating the integral.
4. If the known solution u were not supplied, would the prescribed Wronskian alone determine q ? Justify your answer using initial-data normalization for arbitrary continuous q .

Question 4 – Solution

Strategy. Abel's identity determines the first-derivative coefficient; the known nonzero solution then determines the remaining coefficient.

Step 1: Recover the equation. Since W never vanishes,

$$p = -W'/W = 2t.$$

Substituting $u = e^t$ gives $1 + p + q = 0$, hence

$$y'' + 2ty' - (1 + 2t)y = 0.$$

Step 2: Recover the companion. The identity $W = e^t(v' - v)$ gives $v' - v = 2e^{-t^2-t}$. Multiplying by e^{-t} and integrating from zero yields

$$v(t) = e^t \int_0^t 2e^{-s^2-2s} ds.$$

The integral is smooth on all of $\mathbb{R}$. It gives $v(0) = 0$ and $v'(0) = 2$.

Step 3: Verify by differentiation. Write J for the integral, so $J' = 2e^{-t^2-2t}$. Then $v' = v + 2e^{-t^2-t}$ and $e^t(v' - v) = 2e^{-t^2}$, as prescribed. If $L[y] = y'' + py' + qy$, direct expansion gives

$$uL[v] - vL[u] = W' + pW.$$

Here $L[u] = 0$ and $W' + 2tW = 0$. Since $u = e^t \neq 0$, this proves $L[v] = 0$ everywhere.

Step 4: Identify what the Wronskian omits. With $p = 2t$ and any continuous choice of q , regular existence supplies solutions with data $(1, 0)$ and $(0, 2)$ at zero. Their initial Wronskian is 2, so Abel's identity gives the same $2e^{-t^2}$ for every such q . Thus the Wronskian alone fixes p but not q ; the supplied nonzero seed was essential for the latter.

Question 5

Two independent solutions of a regular equation $y'' + p(t)y' + q(t)y = 0$ on $\mathbb{R}$ have measured ordered Wronskians

$$W(0) = 6, \quad W(2) = 3, \quad W(5) = 12.$$

Assume p, q are continuous and the measurements are exact.

Tasks

1. Find $\int_0^2 p(t) dt$ and $\int_2^5 p(t) dt$, and the average value of p on each interval.
2. Prove that p is positive somewhere in $(0, 2)$, negative somewhere in $(2, 5)$, and zero somewhere in $(0, 5)$.
3. Under the additional assumption that $p(t) = a + bt$, determine a, b and the full Wronskian function. Check all three measurements.
4. For this affine model, find the unique minimum of W on $[0, 5]$. Explain why three Wronskian samples alone do not prove that the true coefficient is affine.

Question 5 – Solution

Strategy. Ratios of Wronskian measurements give integrals of the coefficient, rather than its pointwise values.

Step 1: Extract the averages. Abel's identity gives

$$\int_0^2 p = \ln 2, \quad \int_2^5 p = -\ln 4.$$

The average values are $\boxed{(\ln 2)/2}$ and $\boxed{-(\ln 4)/3}$ respectively.

Step 2: Force a sign change. A continuous function with positive integral must be positive at an interior point; likewise the negative integral requires an interior negative value. Choosing one point from each interval, the intermediate value theorem supplies a zero between them. This conclusion does not specify the value of p at time 2.

Step 3: Fit the affine model. The coefficient equations are $2a + 2b = \ln 2$ and $3a + (21/2)b = -2\ln 2$. Solving gives

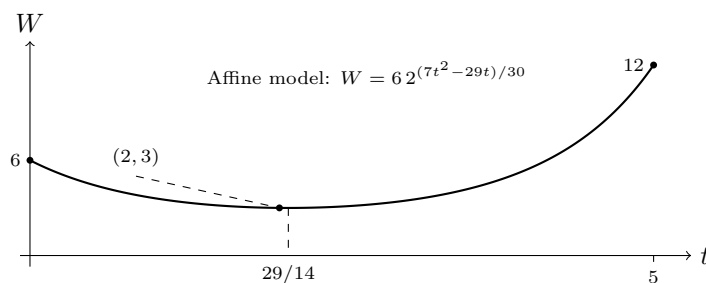
$$p(t) = \frac{29 - 14t}{30} \ln 2, \quad \boxed{W(t) = 6 \cdot 2^{(7t^2 - 29t)/30}}.$$

At $t = 0, 2, 5$, the exponents are 0, -1 , 1, giving 6, 3, 12 as required.

Step 4: Locate and qualify the minimum. Since $W' = -pW$ and $W > 0$, the Wronskian decreases until $t = 29/14$ and increases afterward. Its unique minimum on $[0, 5]$ is

$$\boxed{W(29/14) = 6 \cdot 2^{-841/840}}.$$

Affineness is an extra modeling assumption. For example, adding any multiple of $\cos(2\pi t)$ to this p leaves both measured integrals unchanged, but generally gives a nonaffine coefficient and a different Wronskian between the measurement times.



Question 6

Let u, v be independent real solutions of $y'' + p(t)y' + q(t)y = 0$ on an open interval I , with continuous coefficients. Suppose $a < b$ lie in I and are consecutive zeros of u : $u(a) = u(b) = 0$ and $u(t) \neq 0$ for $a < t < b$.

Tasks

1. Prove that the zeros at a, b are simple and that $v(a), v(b)$ are nonzero.
2. Use the sign of the Wronskian at the endpoints to prove that $v(a)$ and $v(b)$ have opposite signs.
3. Differentiate v/u on (a, b) and use it to prove that v has exactly one zero there.
4. Apply the result to $u = \sin t$, $v = \cos t + 2 \sin t$ between 0 and π , verifying independence and finding the unique zero explicitly.

Question 6 – Solution

Strategy. Use the constant sign of the Wronskian first at the endpoints and then in the derivative of a ratio.

Step 1: Exclude repeated and shared zeros. If u and u' both vanished at an endpoint, regular uniqueness would make u identically zero, contrary to independence. Thus both zeros are simple. Since W never vanishes, $W(a) = -u'(a)v(a)$ and $W(b) = -u'(b)v(b)$ force $v(a), v(b) \neq 0$.

Step 2: Compare endpoint signs. Multiply u by -1 if necessary so that $u > 0$ on (a, b) . Simplicity then implies $u'(a) > 0$ and $u'(b) < 0$. Abel's identity says $W(a), W(b)$ have the same sign. The two endpoint formulas therefore force $v(a), v(b)$ to have opposite signs. Continuity gives at least one zero of v inside.

Step 3: Prove uniqueness of that zero. On the zero-free interval for u ,

$$(v/u)' = \frac{uv' - u'v}{u^2} = \frac{W}{u^2}.$$

The derivative has a fixed nonzero sign, so the ratio is strictly monotone. Its zeros are precisely the zeros of v there. It can vanish at most once; combined with the preceding existence argument, this proves exactly one zero.

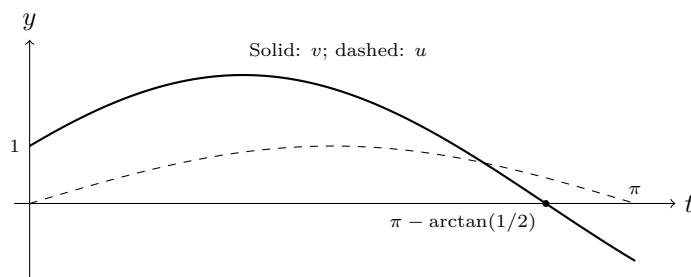
Step 4: Compute the example. Both candidates solve $y'' + y = 0$, and

$$W[\sin t, \cos t + 2 \sin t] = -1.$$

The endpoint values of v are $1, -1$. In $(0, \pi)$, the equation $\cos t + 2 \sin t = 0$ has its root in the second quadrant, with

$$t = \pi - \arctan(1/2).$$

The separation argument proves there are no other roots between these consecutive zeros of $\sin t$.



Question 7

Let p be C^1 and q continuous on an open interval I containing zero, and set $P(t) = \int_0^t p(s) ds$. Consider the substitution $y = e^{-P/2}z$ in $y'' + py' + qy = 0$.

Tasks

1. Derive the differential equation satisfied by z and show that its first-derivative term vanishes.
2. For two transformed solutions, derive the relation between their Wronskian and the original pair's Wronskian. Explain why the transformed Wronskian is constant.
3. Apply the method to $y'' + 2ty' + (t^2 + 3)y = 0$ on $\mathbb{R}$. Construct a fundamental pair normalized to data $(1, 0)$ and $(0, 1)$ at zero and compute its Wronskian.
4. Solve $y(0) = 1$, $y'(0) = 2$ and explain why both members of your fundamental pair tending to zero as $t \rightarrow \infty$ does not contradict independence.

Question 7 – Solution

Strategy. A common nonzero multiplier changes the Wronskian by its square and can remove the first-derivative coefficient.

Step 1: Differentiate the substitution. Put $h = e^{-P/2}$, so $h'/h = -p/2$ and $h''/h = -p'/2 + p^2/4$. Substitution of $y = hz$ gives

$$z'' + (2h'/h + p)z' + (h''/h + ph'/h + q)z = 0,$$

and therefore

$$z'' + [q - p'/2 - p^2/4]z = 0.$$

Step 2: Transform the determinant. Expanding derivatives cancels the terms containing hh' :

$$W[hz_1, hz_2] = h^2 W[z_1, z_2] = e^{-P} W[z_1, z_2].$$

The transformed equation has no first-derivative term, so its Wronskian has derivative zero. The nonzero multiplier preserves independence.

Step 3: Solve the transformed example. Here $P = t^2$ and $q - p'/2 - p^2/4 = t^2 + 3 - 1 - t^2 = 2$. Thus $z'' + 2z = 0$, and the normalized pair is

$$\phi = e^{-t^2/2} \cos(\sqrt{2}t), \quad \psi = e^{-t^2/2} \frac{\sin(\sqrt{2}t)}{\sqrt{2}}.$$

At zero, $h = 1$ and $h' = 0$, so the data remain $(1, 0)$ and $(0, 1)$. The transformed Wronskian is 1, giving $W[\phi, \psi] = e^{-t^2} > 0$ everywhere.

Step 4: Match and interpret. The normalized data give

$$y = \phi + 2\psi = e^{-t^2/2} [\cos(\sqrt{2}t) + \sqrt{2} \sin(\sqrt{2}t)].$$

Its value and slope at zero are 1, 2. Both basis functions decay to zero because their trigonometric factors are bounded and the Gaussian factor decays. Dependence would require an exact constant relation on the interval; the strictly positive finite-time Wronskian rules that out.

Question 8

Suppose p, q are real continuous functions on $\mathbb{R}$, each periodic with period $T > 0$. Consider $y'' + p(t)y' + q(t)y = 0$. A T -periodic solution satisfies $y(t + T) = y(t)$ for every real t .

Tasks

1. Prove that the existence of two independent T -periodic solutions requires $\int_0^T p(t) dt = 0$.
2. If $p \geq 0$ and p is not identically zero, decide whether two such periodic solutions can exist.
3. Is the zero-integral condition sufficient? Analyze $y'' - y = 0$ as a counterexample and justify the absence of nonzero periodic solutions.
4. Does the existence of just one nonzero T -periodic solution force the same integral condition? Analyze $y'' + y' = 0$ and identify a nonperiodic companion.

Question 8 – Solution

Strategy. Periodicity forces the Wronskian to return to its starting value, while Abel's identity specifies its multiplicative change.

Step 1: Compare one full period. Differentiating $y(t + T) = y(t)$ shows that the slope is periodic too. Thus two periodic solutions have $W(T) = W(0)$. Independence gives $W(0) \neq 0$, so Abel's identity implies

$$\exp\left[-\int_0^T p(t) dt\right] = 1.$$

The integral is real; consequently

$$\int_0^T p(t) dt = 0.$$

Step 2: Exclude nontrivial nonnegative damping. By continuity and periodicity, if $p \geq 0$ and is not identically zero, it is positive on some interval within a period. Hence its integral over a period is strictly positive. The necessary condition fails, so two independent T -periodic solutions cannot exist.

Step 3: Disprove sufficiency. For $y'' - y = 0$, $p = 0$ satisfies the integral condition for every T . Every solution is $Ae^t + Be^{-t}$. A continuous periodic function is bounded on $\mathbb{R}$; boundedness as $t \rightarrow \infty$ forces $A = 0$, and boundedness as $t \rightarrow -\infty$ forces $B = 0$. Thus there are no nonzero periodic solutions, much less two independent ones.

Step 4: Test a single periodic solution. In $y'' + y' = 0$, the constant solution $u = 1$ is nonzero and periodic for every $T > 0$, while $p = 1$ has integral $T \neq 0$. A companion is $v = e^{-t}$, with $W[u, v] = -e^{-t} \neq 0$; it is not periodic. One periodic solution does not make a nonzero pair Wronskian periodic and imposes no such zero-integral requirement.

Question 9

On $\mathbb{R}$, consider the original undivided equation

$$t^2 y'' - 2ty' + 2y = 0$$

and the smooth candidates $u = t, v = t^2$. A classical global solution here means a C^2 function on $\mathbb{R}$ satisfying this undivided equation at every point.

Tasks

1. Verify both candidates and compute their Wronskian. Determine their independence on $\mathbb{R}$ and identify every zero of the Wronskian.
2. Apply Abel's identity separately on the two largest regular intervals. Explain why its zero-or-never-zero conclusion cannot be applied across zero.
3. Find every classical global solution by matching the general solutions from the negative and positive sides through zero.
4. Find all global solutions with $y(0) = 0, y'(0) = 1$. Explain both the failure of uniqueness and why no regular normalized equation through zero can have this candidate pair as solutions.

Question 9 – Solution

Strategy. The singular leading coefficient changes which interval hypotheses are available; explicitly match derivatives to determine global solutions.

Step 1: Verify and compute. The residuals are $-2t + 2t = 0$ and $2t^2 - 4t^2 + 2t^2 = 0$. Also

$$W[t, t^2] = t(2t) - 1(t^2) = t^2.$$

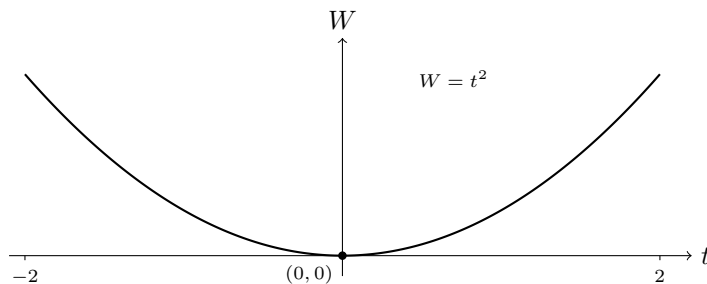
It is zero only at $t = 0$. The functions are independent on $\mathbb{R}$, since $At + Bt^2 \equiv 0$ forces $A = B = 0$.

Step 2: Respect the regular intervals. Division by t^2 gives $p = -2/t$, $q = 2/t^2$, continuous on $(-\infty, 0)$ and $(0, \infty)$. On either interval Abel's identity gives $W = Ct^2$, with the constant fixed within that interval. For this pair it is $C = 1$ on both sides. There is no regular interval containing zero. Abel's conclusion applies separately on the two intervals and makes no assertion across zero.

Step 3: Match all global solutions. The pair is fundamental on either side, so write $A_-t + B_-t^2$ for $t < 0$ and $A_+t + B_+t^2$ for $t > 0$. Continuity forces $y(0) = 0$. Matching first derivatives forces $A_- = A_+$; matching second derivatives forces $B_- = B_+$. Conversely every resulting polynomial solves the equation globally. Thus

$$y = At + Bt^2 \quad \text{on } \mathbb{R}.$$

Step 4: Inspect the singular data. The data give $A = 1$ but leave B arbitrary, so all solutions are $y = t + Bt^2$. Uniqueness fails despite this two-dimensional global family. No regular normalized equation through zero could include $v = t^2$: at zero its value and slope vanish while its second derivative is 2, making the normalized equation impossible there.



Question 10

Let u, v be C^2 real functions on an interval and let θ be a real C^2 function. Define time-dependent combinations

$$f = u \cos \theta + v \sin \theta, \quad g = -u \sin \theta + v \cos \theta.$$

At every time the two-by-two coefficient matrix has determinant 1.

Tasks

1. Derive the actual expression for $W[f, g]$ in terms of u, v, θ and their derivatives. Explain why the constant change-of-basis formula cannot simply be used.
2. Take $u = \cos t$, $v = \sin t$, $\theta = t$ on $\mathbb{R}$. Compute f, g and their Wronskian. Does the pointwise invertibility preserve a fundamental set for $y'' + y = 0$?
3. Keep the same u, v but take $\theta = t/2$. Show that a nonzero resulting Wronskian still does not make the new pair solutions of the original equation.
4. For these trigonometric u, v , classify every real C^2 function θ on $\mathbb{R}$ for which both f, g solve $y'' + y = 0$.

Question 10 – Solution

Strategy. Differentiate the moving coefficients too; pointwise invertibility is not a constant linear transformation of solution functions.

Step 1: Include the extra derivative terms. In the derivatives of f, g , the terms involving u', v' contribute $W[u, v]$, as in a constant rotation. The remaining terms are $\theta'g$ in f' and $-\theta'f$ in g' . Hence

$$W[f, g] = W[u, v] - \theta'(f^2 + g^2) = W[u, v] - \theta'(u^2 + v^2).$$

The additional term is exactly what the constant-coefficient formula omits.

Step 2: Observe a complete collapse. For $u = \cos t, v = \sin t$, the addition formulas give $f = \cos(t - \theta)$, $g = \sin(t - \theta)$. At $\theta = t$, $f = 1$ and $g = 0$, so $W = 0$. They are dependent, and $f'' + f = 1 \neq 0$. Pointwise invertibility of the moving coefficient matrix does not preserve the solution space.

Step 3: Test a nonzero determinant. At $\theta = t/2$, the functions are $\cos(t/2), \sin(t/2)$ and $W = 1/2$. Each has $y'' + y = (3/4)y$, so neither solves the original equation identically. They instead form a fundamental set for $y'' + y/4 = 0$.

Step 4: Classify all valid rotations. Put $\varphi = t - \theta$. The residuals for $f = \cos \varphi, g = \sin \varphi$ are

$$(1 - \varphi'^2) \cos \varphi - \varphi'' \sin \varphi, \quad (1 - \varphi'^2) \sin \varphi + \varphi'' \cos \varphi.$$

Both vanish exactly when $\varphi'^2 = 1$ and $\varphi'' = 0$, since the displayed rotation is invertible. On the connected real line this gives $\varphi = \pm t + C$, hence

$$\theta(t) = C \quad \text{or} \quad \theta(t) = 2t + C, \quad C \in \mathbb{R}.$$

In the second case the final functions are still constant combinations of $\cos t, \sin t$, despite their time-dependent defining coefficients.

Question 1

Let $L[y] = y'' + p(t)y' + q(t)y$ on a connected open interval I , where p, q, g are continuous. Suppose y_p is one solution of $L[y] = g$, and u, v form a fundamental set for $L[y] = 0$.

Tasks

1. Prove that every solution of $L[y] = g$ has the form $y_p + Au + Bv$, with unique constants A, B . Prove both directions.
2. If initial data $y(t_0) = a, y'(t_0) = b$ are prescribed, state the initial data that the homogeneous correction must satisfy and explain uniqueness.
3. For $y'' - 3y' + 2y = 2t^2 - 6t + 2$ on $\mathbb{R}$, verify the supplied particular solution $y_p = t^2$ and find the general solution.
4. Solve the example with $y(0) = 1, y'(0) = 0$, verifying the equation and initial data.

Question 1 – Solution

Strategy. Subtract one known forced solution so that the remaining unknown satisfies a homogeneous equation.

Step 1: Prove the representation. If $L[y] = L[y_p] = g$, linearity gives $L[y - y_p] = 0$. Completeness of u, v then gives $y - y_p = Au + Bv$. Conversely, $L[y_p + Au + Bv] = g + A(0) + B(0) = g$. Independence makes the two constants unique for each solution. Thus

$$y = y_p + Au + Bv.$$

Step 2: Transfer the initial data. The correction $h = y - y_p$ must satisfy

$$h(t_0) = a - y_p(t_0), \quad h'(t_0) = b - y_p'(t_0).$$

Regular homogeneous existence and uniqueness determine h on I . Therefore the forced initial-value problem is unique too. The coefficients in the correction must fit these adjusted data, not automatically the original data.

Step 3: Verify and complete the example. For $y_p = t^2$, the left-hand side is $2 - 6t + 2t^2$, as required. The characteristic polynomial of the associated homogeneous equation is $(r - 1)(r - 2)$, so

$$y = t^2 + Ae^t + Be^{2t}.$$

The exponentials are independent homogeneous solutions, and the representation theorem proves that no further solution freedom is missing.

Step 4: Fit and check. At zero, the equations are $A + B = 1$ and $A + 2B = 0$. They give $A = 2$, $B = -1$, hence

$$y = t^2 + 2e^t - e^{2t}.$$

Its initial value is $2 - 1 = 1$ and its slope is $0 + 2 - 2 = 0$. The polynomial contributes the entire forcing, while each exponential has zero residual, verifying the differential equation everywhere.

Question 2

For $y'' + y = 1$ on $\mathbb{R}$, define

$$y_1 = 1, \quad y_2 = 1 + \cos t, \quad y_3 = 1 + \sin t.$$

Consider constant combinations $y = ay_1 + by_2 + cy_3$, where $a, b, c \in \mathbb{R}$.

Tasks

1. Verify all three solutions and find the necessary and sufficient condition on a, b, c for their combination to solve the same forced equation.
2. Prove that every solution of $y'' + y = 1$ has exactly one representation satisfying that condition.
3. Explain whether the solution set is closed under sums, differences and midpoints. Identify the equation satisfied by the difference of two forced solutions.
4. Find the unique coefficients a, b, c producing $y(0) = 0$, $y'(0) = 2$, and verify the resulting solution.

Question 2 – Solution

Strategy. A fixed nonzero forcing is preserved by affine combinations, whose coefficients sum to one.

Step 1: Track the forcing. Each listed function has $y_j'' + y_j = 1$. Therefore

$$L[ay_1 + by_2 + cy_3] = a + b + c.$$

The combination solves the same equation exactly when $a + b + c = 1$.

Step 2: Describe the full set. Subtracting the particular solution 1 leaves $h'' + h = 0$, so all solutions are $1 + B \cos t + C \sin t$. The proposed combination expands to

$$(a + b + c) + b \cos t + c \sin t.$$

It represents the specified solution exactly with $b = B$, $c = C$, $a = 1 - B - C$. Uniqueness follows from the independence of $1, \cos t, \sin t$: if their combination vanishes, adding its second derivative to itself first forces the constant coefficient to zero, then the two harmonic coefficients vanish.

Step 3: Compare operations. The sum of two solutions has forcing 2, and their difference has forcing 0, so neither operation stays in the set with forcing 1. Their midpoint has forcing $(1 + 1)/2 = 1$ and does stay in the set. The zero function is absent. Thus this is an affine translate of a homogeneous solution space, rather than a vector space itself.

Step 4: Fit the data. In $1 + B \cos t + C \sin t$, the data give $B = -1$, $C = 2$. Hence

$$(a, b, c) = (0, -1, 2), \quad y = 1 - \cos t + 2 \sin t.$$

The coefficients sum to one. Directly, $y'' = \cos t - 2 \sin t$, so $y'' + y = 1$; the initial value and slope are 0, 2.

Question 3

For $y'' + 2y' + y = e^{-t}$ on $\mathbb{R}$, two proposed particular solutions are

$$p(t) = \frac{1}{2}t^2e^{-t}, \quad \tilde{p}(t) = e^{-t}(\frac{1}{2}t^2 + 3 - 2t).$$

Two solvers write $y = p + (A + Bt)e^{-t}$ and $y = \tilde{p} + (\alpha + \beta t)e^{-t}$.

Tasks

1. Verify both particular solutions and explain why their difference must be homogeneous.
2. Derive the exact relation between the two pairs of homogeneous constants. Do the two formulas describe different solution sets?
3. Solve $y(0) = 2$, $y'(0) = -1$ in both representations and verify the data.
4. For the selected solution on $t \geq 0$, determine positivity, monotonicity and its limit. Explain why the particular and homogeneous pieces are not uniquely determined by the trajectory.

Question 3 – Solution

Strategy. Changing the particular solution transfers a homogeneous term between the two pieces without changing the full solution.

Step 1: Verify efficiently. For any twice differentiable F , differentiation gives

$$(Fe^{-t})'' + 2(Fe^{-t})' + Fe^{-t} = e^{-t}F''.$$

Both supplied polynomials have second derivative 1, so both are particular solutions. Their difference $(3 - 2t)e^{-t}$ has zero residual, either by this identity or by subtracting the two forced equations.

Step 2: Match coefficients. Equality of the two representations is equivalent to

$$A = \alpha + 3, \quad B = \beta - 2.$$

This is an invertible reassignment of constants, so the two formulas describe the same complete solution family.

Step 3: Impose the data. In the first representation, $y(0) = A$ and $y'(0) = B - A$, because $p(0) = p'(0) = 0$. Thus $A = 2$, $B = 1$, while the second representation has $\alpha = -1$, $\beta = 3$. Both give

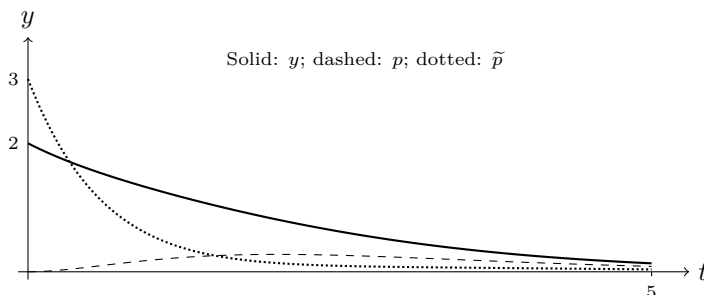
$$y = e^{-t}(2 + t + \tfrac{1}{2}t^2).$$

Its initial value is 2. Differentiation gives $y' = -e^{-t}(1 + \frac{1}{2}t^2)$, whose value at zero is -1 , as required.

Step 4: Interpret the complete trajectory. The derivative follows from subtracting the full polynomial when differentiating its exponential factor:

$$y' = e^{-t}[(1 + t) - (2 + t + \tfrac{1}{2}t^2)] = -e^{-t}(1 + \tfrac{1}{2}t^2).$$

For $t \geq 0$, $y > 0$ and $y' < 0$; the polynomial times e^{-t} tends to zero. The same trajectory admits both decompositions above, so neither piece is unique without a chosen particular solution.



Question 4

For the operator $L[y] = y'' + y$, consider the supplied responses

$$u(t) = \frac{\cos t - \cos 2t}{3}, \quad v(t) = 2(t - \sin t)$$

on $\mathbb{R}$.

Tasks

1. Verify that $L[u] = \cos 2t$, $L[v] = 2t$, and that both responses have zero value and slope at zero.
2. Construct the solution of $y'' + y = \cos 2t + 2t$ with $y(0) = 1$, $y'(0) = -2$. Verify the data.
3. State and prove how both forcing and initial data combine when a constant linear combination of two responses is formed.
4. Let $r = u + \cos t$ and $s = v + \cos t$. Each now has data $(1, 0)$. Correct the naive sum $r + s$ so that it has the combined forcing but only the single prescribed data pair $(1, 0)$.

Question 4 – Solution

Strategy. Superposition combines the initial data as well as the forcing; use a homogeneous correction to achieve the desired data.

Step 1: Verify the supplied responses. We have $u'' = (-\cos t + 4 \cos 2t)/3$, so $u'' + u = \cos 2t$. Also $v' = 2(1 - \cos t)$ and $v'' = 2 \sin t$, so $v'' + v = 2t$. At zero both values and both slopes vanish.

Step 2: Add the forcing and fit the state. The sum $u + v$ handles the combined forcing with zero initial data. Adding the homogeneous solution $\cos t - 2 \sin t$ gives

$$y = u + v + \cos t - 2 \sin t = \frac{4}{3} \cos t - \frac{1}{3} \cos 2t + 2t - 4 \sin t.$$

At zero the value is $4/3 - 1/3 = 1$ and the slope is $2 - 4 = -2$. Linearity verifies the combined forcing, and regular uniqueness proves this is the initial-value solution.

Step 3: Track the full linear map. If $L[y_j] = g_j$ and the initial data are (a_j, b_j) , then $Ay_1 + By_2$ has forcing $Ag_1 + Bg_2$ and data

$$(Aa_1 + Ba_2, Ab_1 + Bb_2).$$

This follows directly from linearity of L , evaluation and differentiation. Ignoring the data part can produce the right equation with the wrong initial state.

Step 4: Remove duplicated initial data. The naive sum $r + s$ has data $(2, 0)$. Subtract one copy of the homogeneous solution $\cos t$:

$$r + s - \cos t = u + v + \cos t.$$

The forcing is unchanged, while the data become $(1, 0)$. This correction differs from Step 2 because the required initial slope here is zero, not -2 .

Question 5

For $y'' - y = 2 \cos t$ on $\mathbb{R}$, a proposed particular solution is $p = -\cos t$. Prescribe real initial data $y(0) = a$, $y'(0) = b$.

Tasks

1. Verify the particular solution and find the full initial-value solution in terms of a, b .
2. Find a necessary and sufficient relation between a, b for boundedness on $[0, \infty)$. For those data, determine the limiting difference $y(t) - p(t)$.
3. Find the unique solution bounded on all of $\mathbb{R}$. Does the selected bounded solution necessarily have a limit as $t \rightarrow \infty$?
4. Compare the trajectories from $(a, b) = (0, -1)$ and $(0, 0)$. Explain why bounded forcing alone does not make every solution bounded.

Question 5 – Solution

Strategy. The difference from a particular solution follows the homogeneous equation, whose growing mode determines boundedness.

Step 1: Verify and fit. Since $p'' = \cos t$, $p'' - p = 2 \cos t$. The general solution is $-\cos t + Ae^t + Be^{-t}$. The data give $A + B = a + 1$, $A - B = b$, hence

$$y = -\cos t + \frac{a+1+b}{2}e^t + \frac{a+1-b}{2}e^{-t}.$$

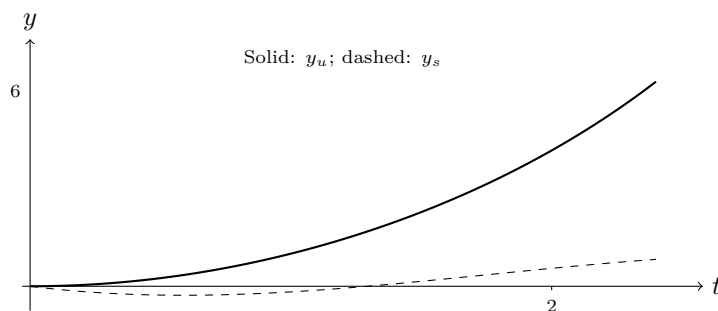
Step 2: Eliminate the growing mode. The bounded oscillatory and decaying terms cannot cancel a nonzero Ae^t as $t \rightarrow \infty$. Thus forward boundedness holds exactly when

$$b = -a - 1.$$

In that case $y = -\cos t + (a+1)e^{-t}$, so $y - p \rightarrow 0$. This convergence is to a time-varying particular solution, not necessarily to a constant.

Step 3: Require both time directions. Boundedness as $t \rightarrow -\infty$ also forces $B = 0$. Thus the unique globally bounded solution is $y = -\cos t$, with data $(-1, 0)$. It has no limit at positive infinity: its values at $2\pi n$ and $(2n+1)\pi$ are -1 and 1 . The forward-bounded family also has these two limiting subsequences because its decaying correction tends to zero.

Step 4: Compare the selected data. Data $(0, -1)$ give $y_s = -\cos t + e^{-t}$, which is bounded forward. Data $(0, 0)$ give $y_u = -\cos t + \cosh t$, which is unbounded forward. Both have the same bounded forcing; the second retains a growing homogeneous component. The finite plot illustrates the distinction; the mode argument proves the global claim.



Question 6

For $y'' + y = \cos t$ on $\mathbb{R}$, a proposed particular solution is $p(t) = \frac{1}{2}t \sin t$. The forcing is bounded in absolute value by one.

Tasks

1. Verify the particular solution and find the full solution family.
2. Solve the initial-value problem with arbitrary data $y(0) = a$, $y'(0) = b$. Identify the zero-data response.
3. Prove that every solution is unbounded above and below on $[0, \infty)$, using two explicit sequences of times. Can a homogeneous correction remove this behavior?
4. For the zero-data response, give exact upper and lower envelope bounds on $t \geq 0$ and the times when they are attained. Explain why envelope contacts need not be stationary points.

Question 6 – Solution

Strategy. Verify the supplied growing response and use exact phase samples to rule out cancellation by bounded homogeneous terms.

Step 1: Verify and complete. Differentiation gives

$$p' = \frac{1}{2} \sin t + \frac{1}{2} t \cos t, \quad p'' = \cos t - \frac{1}{2} t \sin t.$$

Thus $p'' + p = \cos t$ and the general solution is

$$y = A \cos t + B \sin t + \frac{1}{2} t \sin t.$$

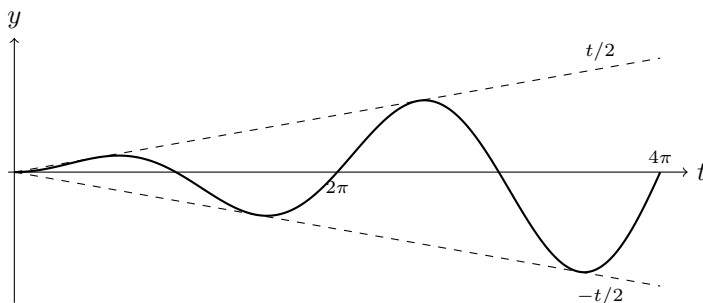
Step 2: Fit the data. Since $p(0) = p'(0) = 0$, the data give $A = a$, $B = b$. In particular the zero-data response is exactly $y_0 = \frac{1}{2} t \sin t$.

Step 3: Prove both unbounded excursions. At $t_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, we have $y(t_n) = b + t_n/2 \rightarrow \infty$. At $s_n = 3\pi/2 + 2\pi n$, we have $y(s_n) = -b - s_n/2 \rightarrow -\infty$. These values do not depend on a . Thus no constant homogeneous correction removes the unbounded behavior, even though the forcing is bounded.

Step 4: Distinguish bounds from extrema. On $t \geq 0$,

$$-t/2 \leq y_0(t) \leq t/2.$$

For $t > 0$, the upper envelope is attained exactly at t_n and the lower at s_n ; both bounds also coincide with y_0 at zero. At a positive upper contact $y'_0 = 1/2$, and at a lower contact $y'_0 = -1/2$. These slopes match those of the respective envelopes and are not zero, so the contacts are not stationary points of y_0 .



Question 7

On $t > 0$, consider

$$y'' = 1/t, \quad y(1) = 0, \quad y'(1) = 0,$$

with supplied particular solution $p(t) = t \ln t - t$. The associated homogeneous equation has the globally smooth solutions $1, t$.

Tasks

1. Verify the particular solution and find every solution on $(0, \infty)$.
2. Solve the initial-value problem and determine its monotonicity, convexity and minimum.
3. Find the right-hand limit at zero and decide whether the selected solution admits continuous, C^1 , or C^2 extension through zero.
4. Can any homogeneous correction to p produce a C^1 extension through zero? Explain why smooth homogeneous solutions do not guarantee continuation of the forced problem.

Question 7 – Solution

Strategy. The forcing can impose endpoint behavior that no homogeneous correction can cancel.

Step 1: Verify and complete. Since $p' = \ln t$ and $p'' = 1/t$, it is a particular solution. Adding all affine homogeneous functions gives

$$y = t \ln t - t + At + B \quad (t > 0).$$

Step 2: Fit and analyze. The slope at 1 gives $A = 0$, and the value then gives $B = 1$. Thus

$$y = t \ln t - t + 1, \quad y' = \ln t, \quad y'' = 1/t > 0.$$

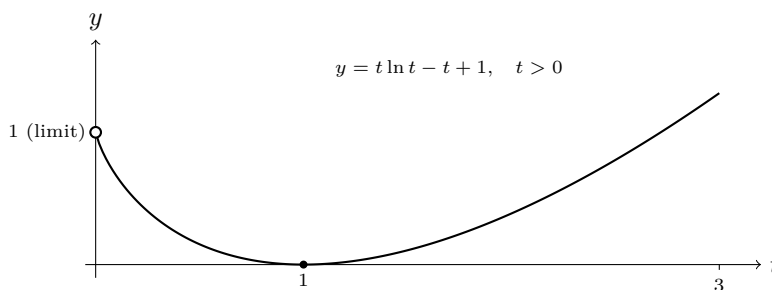
The function decreases on $(0, 1)$, increases on $(1, \infty)$, and is strictly convex. Its unique global minimum is $y(1) = 0$.

Step 3: Test extension of the selected function. Since $t \ln t \rightarrow 0$ as $t \downarrow 0$, we have $y(t) \rightarrow 1$. Setting $y(0) = 1$ and, for example, $y(t) = 1$ for $t < 0$ gives a continuous extension. A finite derivative at zero is impossible because

$$\frac{y(t) - 1}{t} = \ln t - 1 \rightarrow -\infty.$$

Hence there is no C^1 extension and therefore no C^2 extension. The limiting point in the graph is excluded from the original domain.

Step 4: Test every correction. For general A, B , continuity would require $y(0) = B$, but the difference quotient is $\ln t - 1 + A$, again tending to $-\infty$. No finite homogeneous constants cure this. Although $1, t$ extend smoothly, the forcing $1/t$ is undefined at zero. The maximal regular interval containing the initial point is $(0, \infty)$, not all of $\mathbb{R}$.



Question 8

Let u, v satisfy the same forced equation $y'' + p(t)y' + q(t)y = g(t)$ on a regular interval, and write $W = uv' - u'v$. For a concrete example on $\mathbb{R}$, take

$$u = 1 - \cos t, \quad v = 1 - \cos t + \sin t.$$

Tasks

1. Derive the differential identity for W when the common forcing is nonzero.
2. Verify that the example functions solve $y'' + y = 1$. Compute their Wronskian and explain why its zero at zero does not force it to vanish everywhere.
3. Prove that the two example functions are linearly independent as functions, and identify the homogeneous solution obtained from their difference.
4. Classify the constant affine combinations of u, v that solve the same forced equation. Which initial values and slopes at zero can they attain, and do they give the entire forced solution set?

Question 8 – Solution

Strategy. Recompute the Wronskian identity with the forcing included, then distinguish linear independence from a homogeneous fundamental set.

Step 1: Keep the forcing terms. Differentiating and substituting gives

$$W' = uv'' - u''v = u(g - pv' - qv) - v(g - pu' - qu),$$

so

$$W' = -pW + g(u - v).$$

The extra term is absent from Abel's homogeneous identity and generally does not vanish.

Step 2: Verify the example. We have $u'' + u = \cos t + 1 - \cos t = 1$, and adding $\sin t$ preserves this forcing. Direct computation gives

$$W = (1 - \cos t) \cos t - \sin^2 t = \cos t - 1.$$

It is zero at 0 but equals -2 at π . Its derivative is $-\sin t$, matching $g(u - v) = -\sin t$. The homogeneous zero-or-never-zero conclusion is inapplicable to this forced pair.

Step 3: Test independence. If $Au + Bv = 0$, evaluation at π gives $2A + 2B = 0$ and at $\pi/2$ gives $A + 2B = 0$. Hence $A = B = 0$. The difference $v - u = \sin t$ solves the associated homogeneous equation. Independence of the forced functions does not make them a fundamental set of homogeneous solutions.

Step 4: Describe their affine family. The forcing of $Au + Bv$ is $A + B$, so retaining forcing 1 requires $A + B = 1$. Such combinations are

$$u + B \sin t = 1 - \cos t + B \sin t.$$

They have initial value zero and arbitrary slope B . They give exactly the forced solutions with initial value zero, but miss every solution with a different initial value. The full family is $1 + C \cos t + D \sin t$, with two independent initial-data freedoms.

Question 9

Let y solve $y'' + y = 1$ with $y(0) = y'(0) = 0$. On $0 \leq t \leq 1$, consider the approximation $Y(t) = t^2/2$ and the error $E = Y - y$. You may use $0 \leq \sin r \leq r$ for $0 \leq r \leq 1$.

Tasks

1. Compute the residual $R = Y'' + Y - 1$ and derive the differential equation and initial data satisfied by E .
2. Verify by differentiating twice that $\int_0^t \sin(t-s)s^2/2 \, ds$ solves this error initial-value problem. Justify that it equals E .
3. Prove the rigorous error bound $0 \leq E(t) \leq t^4/24$ on $[0, 1]$, including the direction of the approximation error.
4. Find the exact solution y by verification, compute the actual error at 1 to six decimal places, and compare it with the bound. Why does a residual vanishing only at zero not prove exactness?

Question 9 – Solution

Strategy. Treat the approximation residual as a forcing for the error, then bound the resulting integral with a positive kernel.

Step 1: Form the error equation. Since $Y'' = 1$, the residual is $R = t^2/2$. Subtraction gives

$$E'' + E = t^2/2, \quad E(0) = E'(0) = 0.$$

Step 2: Verify the integral. Set $F(t) = \int_0^t \sin(t-s)s^2/2 \, ds$. Leibniz differentiation gives

$$F' = \int_0^t \cos(t-s)s^2/2 \, ds, \quad F'' = t^2/2 - \int_0^t \sin(t-s)s^2/2 \, ds.$$

Thus $F'' + F = t^2/2$ and $F(0) = F'(0) = 0$. The continuous coefficients and forcing give uniqueness, so $F = E$.

Step 3: Bound the error. For $0 \leq s \leq t \leq 1$, the supplied sine inequality gives

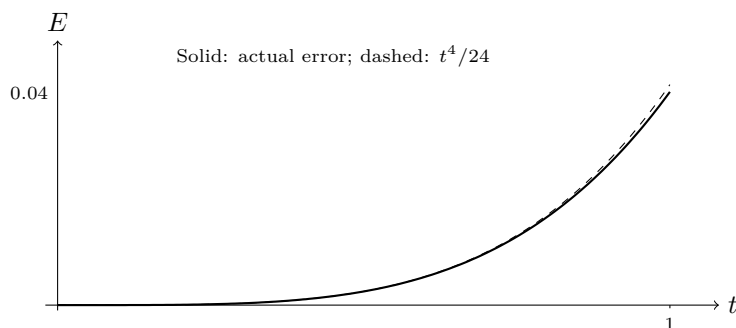
$$0 \leq E(t) \leq \frac{1}{2} \int_0^t (t-s)s^2 \, ds = \frac{1}{2}(t^4/3 - t^4/4) = \boxed{t^4/24}.$$

Hence Y overestimates the solution. The bound vanishes at zero and remains valid across the entire stated interval.

Step 4: Check against the exact response. The function $y = 1 - \cos t$ has the required equation and zero data. Therefore

$$E(1) = \cos 1 - \frac{1}{2} \approx 0.040302 < 1/24 \approx 0.041667.$$

The residual is zero at the initial point but is positive elsewhere on $(0, 1]$. A solution must satisfy its equation at every point, so agreement at one point and in the initial data does not make Y exact.



Question 10

On $0 \leq t \leq 1$, design a forcing g for $y'' + 2y' + y = g(t)$ so that the trajectory has

$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = 1, \quad y'(1) = 0.$$

Restrict the desired trajectory to polynomials of degree at most three.

Tasks

1. Find the unique polynomial meeting the four endpoint requirements.
2. Derive the forcing g that makes this polynomial an exact solution and verify all four requirements.
3. Prove that this forcing changes sign exactly once on $[0, 1]$, locating the zero between $3/4$ and $4/5$. Can a nonnegative forcing realize this same cubic trajectory?
4. Keep the designed forcing but perturb the initial data to $y(0) = \varepsilon$, $y'(0) = 0$. Find the exact position and slope errors at $t = 1$, and determine whether the target state is still reached when $\varepsilon \neq 0$.

Question 10 – Solution

Strategy. Prescribe a trajectory and compute its forcing directly; then use a homogeneous difference equation to analyze initial-data error.

Step 1: Fit the cubic trajectory. Write $y = c_0 + c_1t + c_2t^2 + c_3t^3$. The initial conditions give $c_0 = c_1 = 0$. The final conditions give $c_2 + c_3 = 1$ and $2c_2 + 3c_3 = 0$, so

$$y_*(t) = 3t^2 - 2t^3.$$

The coefficient equations have a unique solution, including among all polynomials of lower degree.

Step 2: Compute the forcing. The derivatives are $y'_* = 6t - 6t^2$ and $y''_* = 6 - 12t$. Thus

$$g = y''_* + 2y'_* + y_* = 6 - 9t^2 - 2t^3.$$

The derivative formulas give zero initial value and slope, and final value $3 - 2 = 1$ with slope $6 - 6 = 0$. The computed forcing verifies the equation identically.

Step 3: Locate the sign change. We have $g(0) = 6$, $g(1) = -5$, and $g' = -18t - 6t^2 < 0$ for $t > 0$. Therefore there is exactly one zero. More precisely,

$$g(3/4) = 3/32 > 0, \quad g(4/5) = -98/125 < 0,$$

so the zero lies strictly between $3/4$ and $4/5$. This same cubic fixes g pointwise, so a nonnegative forcing cannot realize it on the whole interval. This conclusion concerns the specified cubic, not every possible trajectory.

Step 4: Propagate the initial error. The difference $h = y - y_*$ satisfies $h'' + 2h' + h = 0$, with $h(0) = \varepsilon$, $h'(0) = 0$. Hence $h = \varepsilon(1 + t)e^{-t}$ and $h' = -\varepsilon te^{-t}$. At the endpoint,

$$y(1) - 1 = 2\varepsilon/e, \quad y'(1) = -\varepsilon/e.$$

For nonzero ε the target state is missed. The forcing is unchanged; the error evolves entirely through the homogeneous correction.

Question 1

Consider the initial-value problem on $\mathbb{R}$

$$y'' - 3y' + 2y = t^2, \quad y(0) = 0, \quad y'(0) = 0.$$

A student proposes the particular trial $y_p = At^2$ because the forcing contains only a quadratic term.

Tasks

1. Explain why the proposed trial is insufficient. Choose the smallest complete polynomial trial and justify its degree.
2. Determine all coefficients of that trial and verify its residual.
3. Find the full initial-value solution, checking both data.
4. Prove that the polynomial particular solution is unique among polynomials, even though particular solutions in general are not unique.

Question 1 – Solution

Strategy. Differentiation creates lower polynomial powers, so include them before matching every coefficient.

Step 1: Complete the trial. For At^2 , the residual is $2At^2 - 6At + 2A$, which cannot equal t^2 . Use $y_p = at^2 + bt + c$. The nonzero coefficient of y preserves the highest polynomial degree, so degree two is necessary and sufficient here.

Step 2: Match coefficients. Substitution yields

$$2at^2 + (2b - 6a)t + (2a - 3b + 2c) = t^2.$$

Thus $a = 1/2$, $b = 3/2$, $c = 7/4$, giving

$$y_p = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4}.$$

The displayed coefficient equations verify the residual identically.

Step 3: Fit the homogeneous correction. The characteristic roots are 1, 2, so $y = y_p + Ce^t + De^{2t}$. At zero,

$$C + D = -7/4, \quad C + 2D = -3/2.$$

Hence $C = -2$, $D = 1/4$, and

$$y = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4} - 2e^t + \frac{1}{4}e^{2t}.$$

Its value is $7/4 - 2 + 1/4 = 0$ and its slope is $3/2 - 2 + 1/2 = 0$. Adding homogeneous modes does not change the verified forcing.

Step 4: Prove polynomial uniqueness. The difference of two polynomial particular solutions would be a polynomial h with $h'' - 3h' + 2h = 0$. If h were nonzero with degree m and leading coefficient a_m , the leading term of this residual would be $2a_mt^m$, which cannot vanish. Thus $h = 0$. Nonpolynomial homogeneous additions still give infinitely many other particular solutions.

Question 2

On $\mathbb{R}$, solve

$$y'' - 4y' + 4y = e^{2t}(6t + 2), \quad y(0) = 1, \quad y'(0) = 0.$$

Three candidate trial forms are $e^{2t}(At + B)$, $te^{2t}(At + B)$ and $t^2e^{2t}(At + B)$.

Tasks

1. Find the homogeneous roots and explain the resonance multiplicity. Derive the identity obtained by writing $y = e^{2t}F(t)$.
2. Test the first two trial forms and explain exactly why neither can produce the forcing.
3. Use the third trial to find a particular solution and then solve the initial-value problem.
4. Explain why the trial omits the constant and linear powers of F , and why their omission loses no solution freedom. Verify the final equation and data.

Question 2 – Solution

Strategy. Shift the exponential factor out of the operator; the repeated characteristic root reveals the necessary power of t .

Step 1: Identify the repeated root. The characteristic polynomial is $(r - 2)^2$. Product differentiation gives

$$(e^{2t}F)'' - 4(e^{2t}F)' + 4e^{2t}F = e^{2t}F''.$$

Thus the transformed equation is $F'' = 6t + 2$.

Step 2: Reject incomplete shifts. The first trial has $F = At + B$, so $F'' = 0$. The second has $F = At^2 + Bt$, so $F'' = 2A$, which cannot supply the linear term $6t$. Multiplying the usual degree-one trial by only one power of t is still insufficient at a double root.

Step 3: Use the correct trial. With $F = At^3 + Bt^2$, coefficient matching gives $6At + 2B = 6t + 2$, hence $A = B = 1$. The complete solution is

$$y = e^{2t}(C + Dt + t^2 + t^3).$$

The initial value sets $C = 1$, and the slope at zero is $2C + D$, so $D = -2$. Therefore

$$y = e^{2t}(1 - 2t + t^2 + t^3).$$

Step 4: Account for every constant. Constant and linear terms in F have zero second derivative and correspond exactly to e^{2t} and te^{2t} , the homogeneous modes. Leaving them out of the particular trial avoids redundant coefficients; the constants C, D restore them in the general solution. For the final polynomial, $F'' = 2 + 6t$, so the equation holds. Also $F(0) = 1$ and $2F(0) + F'(0) = 2 - 2 = 0$, verifying both data.

Question 3

Consider $y'' + y = t \cos 2t$ on $\mathbb{R}$, with $y(0) = y'(0) = 0$. A proposed trial includes only the cosine term $(at + b) \cos 2t$.

Tasks

1. Explain why a polynomial sine component must also be included, and write the smallest complete trigonometric-polynomial trial.
2. Determine its coefficients by matching both sine and cosine terms.
3. Solve the zero-data initial-value problem and verify the data.
4. Explain why nonresonance with frequency 2 does not guarantee a bounded response to this forcing. Prove that the selected response is unbounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Polynomial differentiation couples the sine and cosine components even when only one appears in the forcing.

Step 1: Complete the trial space. Use

$$y_p = (at + b) \cos 2t + (ct + d) \sin 2t.$$

Frequency 2 differs from the homogeneous frequency 1, so no extra power of t is needed for resonance. Nevertheless, differentiating the proposed cosine-only term twice produces $-4a \sin 2t$, which cannot be ignored.

Step 2: Match all terms. Applying $D^2 + 1$ to the complete trial gives

$$(-3at - 3b + 4c) \cos 2t + (-3ct - 3d - 4a) \sin 2t.$$

Thus $a = -1/3$, $c = 0$, $b = 0$, $d = 4/9$, and

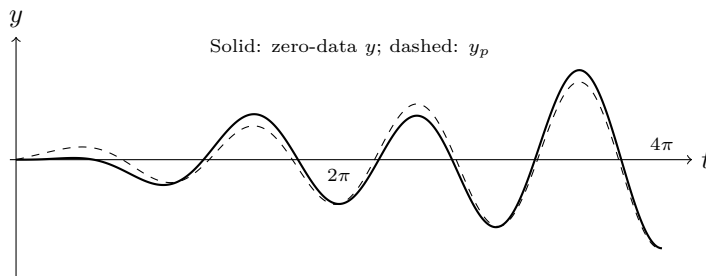
$$y_p = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t.$$

Step 3: Fit the initial state. The particular data are $y_p(0) = 0$ and $y_p'(0) = -1/3 + 8/9 = 5/9$. The homogeneous correction is therefore $-(5/9) \sin t$, giving

$$y = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t - \frac{5}{9} \sin t.$$

Its value and slope at zero vanish, and the correction leaves the verified forcing unchanged.

Step 4: Separate resonance from forcing growth. The forcing itself has a factor t and is unbounded. At $t_n = 2\pi n$, the solution is $y(t_n) = -t_n/3 \rightarrow -\infty$. At $s_n = \pi/2 + 2\pi n$, it is $y(s_n) = s_n/3 - 5/9 \rightarrow \infty$. Thus it is unbounded in both directions despite the absence of frequency resonance. Nonresonance alone is not a boundedness guarantee when the forcing amplitude grows.



Question 4

On $\mathbb{R}$, solve

$$y'' - y = (t + 1)e^t + 2 \cosh t, \quad y(0) = y'(0) = 0,$$

where $\cosh t = (e^t + e^{-t})/2$.

Tasks

1. Rewrite the forcing into distinct exponential families and identify every resonant component.
2. Choose one combined trial with no duplicate unknown coefficients. Explain why matching separate overlapping trials would introduce redundancy.
3. Determine a particular solution by coefficient matching.
4. Add the homogeneous correction, solve the initial-value problem, and verify both the equation and initial data.

Question 4 – Solution

Strategy. Combine overlapping forcing families before choosing the trial, then apply the resonance correction separately to each distinct exponential.

Step 1: Simplify the forcing. The right side becomes

$$(t+2)e^t + e^{-t}.$$

The characteristic roots of $y'' - y = 0$ are $1, -1$, both simple. Therefore each exponential family is resonant once.

Step 2: Choose a nonredundant trial. Use

$$y_p = e^t(at^2 + bt) + cte^{-t}.$$

A separate trial for the e^t inside $2 \cosh t$ would repeat the te^t term already in the first family's trial. Combining the forcing first avoids two coefficients multiplying the same function.

Step 3: Match coefficients. Product differentiation gives

$$(D^2 - 1)(e^t F) = e^t(F'' + 2F'), \quad (D^2 - 1)(e^{-t} G) = e^{-t}(G'' - 2G').$$

Thus the trial produces $e^t(4at + 2a + 2b) - 2ce^{-t}$. Matching gives $a = 1/4$, $b = 3/4$, $c = -1/2$, so

$$y_p = e^t(t^2/4 + 3t/4) - \frac{1}{2}te^{-t}.$$

Step 4: Fit and verify. The particular data are $(0, 1/4)$. With $y = y_p + Ce^t + De^{-t}$, the zero data require $C + D = 0$, $C - D = -1/4$, hence $C = -1/8$, $D = 1/8$. Therefore

$$y = e^t(t^2/4 + 3t/4 - 1/8) + e^{-t}(1/8 - t/2).$$

Its value is $-1/8 + 1/8 = 0$ and its slope is $1/4 - 1/8 - 1/8 = 0$. The two added exponentials are homogeneous, so the coefficient-matched forcing is preserved.

Question 5

For $t \in \mathbb{R}$, consider

$$y'' + 2y' + 5y = e^{-t} \cos 2t, \quad y(0) = y'(0) = 0.$$

Tasks

1. Find the homogeneous roots and choose a correct resonant trial. Derive a useful operator identity with $y = e^{-t}F$.
2. Determine the particular solution and the complete zero-data response.
3. Despite the resonance, prove that every solution tends to zero as $t \rightarrow \infty$.
4. Find a simple envelope for the zero-data response and prove the bound $|y(t)| \leq 1/(4e)$ for $t \geq 0$. Is equality actually attained by the response?

Question 5 – Solution

Strategy. Resonance adds a polynomial factor, but the sign of the exponential rate still controls long-time decay.

Step 1: Identify the resonance. The roots are $-1 \pm 2i$. Thus the exponential-trigonometric forcing matches a simple complex root pair, and a suitable trial is $te^{-t}(A \cos 2t + B \sin 2t)$. Differentiation gives

$$(e^{-t}F)'' + 2(e^{-t}F)' + 5e^{-t}F = e^{-t}(F'' + 4F).$$

Step 2: Match and fit. For $F = t(A \cos 2t + B \sin 2t)$, the transformed residual is $-4A \sin 2t + 4B \cos 2t$. Hence $A = 0$, $B = 1/4$. The particular solution and its first derivative vanish at zero, so

$$y_0 = \frac{1}{4}te^{-t} \sin 2t.$$

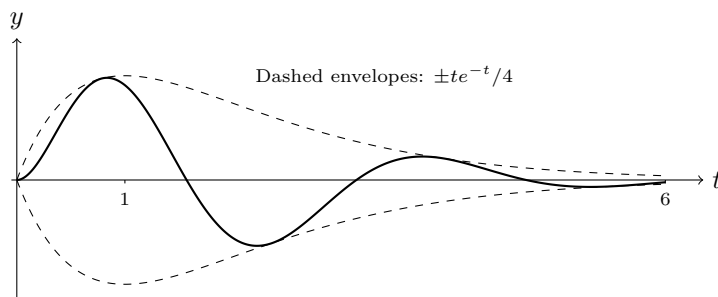
The full family is $e^{-t}[C \cos 2t + D \sin 2t + (t/4) \sin 2t]$. Zero initial data force $C = D = 0$.

Step 3: Prove decay for every solution. For fixed C, D and $t \geq 0$, its absolute value is at most $e^{-t}(|C| + |D| + t/4)$, which tends to zero. The resonant factor t does not overcome the negative exponential rate.

Step 4: Distinguish an envelope bound from an attained maximum. We have $|y_0(t)| \leq te^{-t}/4$. The envelope attains its unique maximum at $t = 1$, since its derivative is $e^{-t}(1 - t)/4$. Thus

$$|y_0(t)| \leq 1/(4e).$$

Equality would require both $t = 1$ and $|\sin 2t| = 1$. But $|\sin 2| < 1$, so the response never attains this bound. It is a valid envelope estimate, not the exact largest displacement.



Question 6

Let $F > 0$ and $\omega \geq 0$. Consider $y'' + 2y' + 5y = F \cos(\omega t)$ on $\mathbb{R}$. For $\omega > 0$, seek a particular response $y_p = A \cos \omega t + B \sin \omega t$ and define its amplitude $R = \sqrt{A^2 + B^2}$. At $\omega = 0$, define R as the absolute value of the constant particular response.

Tasks

1. Derive A, B for $\omega > 0$ and find the constant response at zero frequency.
2. Find the frequency maximizing R on $\omega \geq 0$ and the maximum amplitude.
3. At $\omega = \sqrt{5}$, describe the phase of the response relative to the forcing and compare its amplitude with the maximum.
4. For $\omega > 0$, prove that this is the unique $2\pi/\omega$ -periodic solution and that every other solution approaches it as $t \rightarrow \infty$.

Question 6 – Solution

Strategy. Solve the two coefficient equations, then minimize the amplitude denominator instead of differentiating a complicated square root.

Step 1: Match the harmonic components. The equations are

$$(5 - \omega^2)A + 2\omega B = F, \quad -2\omega A + (5 - \omega^2)B = 0.$$

Writing $\Delta = (5 - \omega^2)^2 + 4\omega^2 > 0$, we obtain

$$A = F(5 - \omega^2)/\Delta, \quad B = 2F\omega/\Delta.$$

At $\omega = 0$, use the constant particular solution $F/5$.

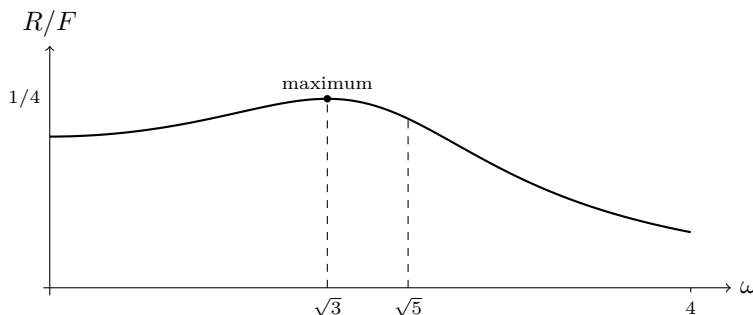
Step 2: Maximize the amplitude. For all $\omega \geq 0$, including this constant-response convention,

$$R = \frac{F}{\sqrt{\omega^4 - 6\omega^2 + 25}} = \frac{F}{\sqrt{(\omega^2 - 3)^2 + 16}}.$$

The denominator is minimized exactly at $\omega = \sqrt{3}$, giving $R_{\max} = F/4$.

Step 3: Locate the quarter-cycle phase lag. At $\omega = \sqrt{5}$, $A = 0$ and $B = F/(2\sqrt{5})$, so $y_p = [F/(2\sqrt{5})] \sin(\sqrt{5}t)$. This is a cosine delayed by phase $\pi/2$. Its amplitude $F/(2\sqrt{5})$ is smaller than $F/4$; the quarter-cycle phase frequency is not the amplitude-maximizing frequency.

Step 4: Prove the steady response is unique. Every difference from y_p is $e^{-t}(C \cos 2t + D \sin 2t)$ and tends to zero forward. If a second solution had the same period $T = 2\pi/\omega$, this difference would be periodic too. For any fixed t , its value equals its value at $t + nT$, whose limit is zero. Hence the difference is identically zero, proving uniqueness of the periodic response.



Question 7

For $\omega > 0$, let y_ω solve

$$y'' + y = \cos(\omega t), \quad y(0) = y'(0) = 0.$$

Tasks

1. Use undetermined coefficients to find y_ω for $\omega \neq 1$, including the homogeneous correction.
2. At $\omega = 1$, choose a resonant trial and solve the same initial-value problem directly.
3. For every fixed t , compute $\lim_{\omega \rightarrow 1} y_\omega(t)$. Explain why the divergent particular coefficient for $\omega \neq 1$ does not imply a divergent fixed-time limit of the full response.
4. Compare boundedness on $[0, \infty)$ in the resonant and nonresonant cases. Can the convergence as $\omega \rightarrow 1$ be uniform on that entire half-line?

Question 7 – Solution

Strategy. Fit the initial data before taking the frequency limit; the homogeneous correction cancels the divergent constant part.

Step 1: Solve off resonance. The trial $A \cos \omega t$ gives $A(1 - \omega^2) = 1$. Fitting zero data adds $-\cos t/(1 - \omega^2)$, so

$$y_\omega(t) = \frac{\cos(\omega t) - \cos t}{1 - \omega^2}, \quad \omega \neq 1.$$

Both the value and slope at zero vanish.

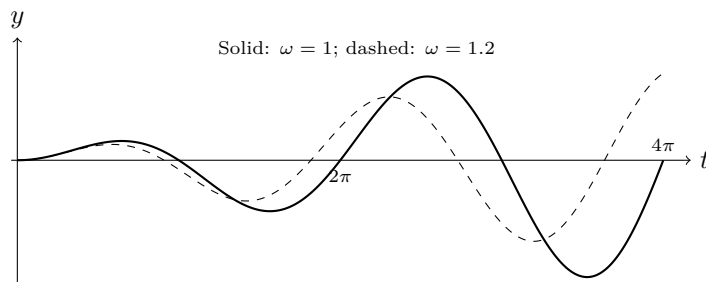
Step 2: Solve at resonance. Use $t(A \cos t + B \sin t)$. Applying $D^2 + 1$ gives $-2A \sin t + 2B \cos t$, hence $A = 0$, $B = 1/2$. Its data are already zero, so $y_1 = (t/2) \sin t$.

Step 3: Take the fixed-time limit. Differentiating numerator and denominator with respect to ω gives

$$\lim_{\omega \rightarrow 1} y_\omega(t) = \frac{-t \sin t}{-2} = \frac{1}{2} t \sin t.$$

This includes $t = 0$. The particular coefficient and the cosine correction separately diverge, but their full numerator vanishes at the same rate as the denominator. They must be combined before taking the limit.

Step 4: Compare long-time behavior. For fixed $\omega \neq 1$, $|y_\omega| \leq 2/|1 - \omega^2|$ on the whole half-line. The resonant response is unbounded, for example at $t = \pi/2 + 2\pi n$. Consequently, for each fixed nonresonant ω , the difference $y_\omega - y_1$ is unbounded on that half-line. Uniform convergence there is impossible, although the fixed-time limit exists. A finite-window plot cannot replace this distinction.



Question 8

Consider the standard finite polynomial–exponential–trigonometric version of undetermined coefficients for constant-coefficient equations. Then examine $y'' + y = 1/(1+t)$ on $I = (-1, \infty)$.

Tasks

1. Which forcings fit the standard finite trial families: $e^{-t}(t^2 + 1)$, $t \sin 3t$, $\cos^2 t$, e^{t^2} , and $1/(1+t)$? Explain any algebraic rewriting needed.
2. Test the rational trial $y_p = A/(1+t)$ for the stated equation and prove that no constant A works.
3. Prove that no finite sum $\sum_{k=1}^N a_k(1+t)^{-k}$ with real coefficients can be a particular solution.
4. Does the failure of these trials imply that an initial-value solution does not exist? State the existence and uniqueness conclusion for any initial point in I .

Question 8 – Solution

Strategy. The standard method depends on closure in a finite differentiation family; a plausible-looking trial outside that family need not close.

Step 1: Classify the forcings. The first two belong to the standard families. The third does too after $\cos^2 t = (1 + \cos 2t)/2$. The functions e^{t^2} and $1/(1+t)$ do not: successive derivatives introduce polynomial factors of ever-increasing degree in the first case and poles of ever-increasing order in the second. The usual finite constant-coefficient matching prescription does not apply to them.

Step 2: Test one inverse power. Put $x = 1 + t > 0$. For $y_p = A/x$,

$$y_p'' + y_p = A/x + 2A/x^3.$$

Equality to $1/x$ would require $A(x^2 + 2) = x^2$ for all $x > 0$. Matching x^2 forces $A = 1$, while matching the constant term forces $2A = 0$, a contradiction.

Step 3: Exclude any finite inverse-power sum. If such a sum were nonzero, choose its largest index N with $a_N \neq 0$. Its second derivative contains

$$a_N N(N+1)x^{-N-2}.$$

No other differentiated term, undifferentiated term or forcing term has that most singular power. Equivalently, multiply the proposed equation by x^{N+2} and let $x \downarrow 0$: the left side tends to $a_N N(N+1) \neq 0$, while the right side tends to zero. The zero sum also fails to give the nonzero forcing.

Step 4: Separate method failure from nonexistence. On I , the normalized coefficients and the forcing $1/(1+t)$ are continuous. For every $t_0 \in I$ and every finite value and slope at t_0 , the regular linear theorem gives exactly one solution on I . A failed finite ansatz is a limitation of the trial class, not a proof that the differential equation lacks solutions.

Question 9

An unknown forcing in $y'' - 3y' + 2y = Ae^{3t} + Bt$ has two real constants A, B . An exact response on $\mathbb{R}$ has measured data

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y^{(3)}(0) = 0.$$

Tasks

1. Determine A, B using the equation and its derivative at zero before solving for y .
2. Choose a complete nonresonant trial and compute a particular solution for the identified forcing.
3. Fit the homogeneous correction and verify all four measured quantities.
4. Explain the distinct roles of the four measurements. Once A, B are fixed, are four freely prescribed initial quantities available for this second-order equation?

Question 9 – Solution

Strategy. Use acceleration and its derivative to identify the forcing, then use value and slope to select the trajectory.

Step 1: Identify the forcing parameters. At zero, the equation gives $1 = A$. Differentiating once gives $y^{(3)} - 3y'' + 2y' = 3Ae^{3t} + B$, so at zero $-3 = 3A + B$. Therefore

$$A = 1, \quad B = -6.$$

Step 2: Match a particular solution. Neither 3 nor 0 is a root of $(r-1)(r-2)$, so use $y_p = ce^{3t} + dt + k$. The residual is $2ce^{3t} + 2dt + (2k - 3d)$. Matching $e^{3t} - 6t$ gives

$$y_p = \frac{1}{2}e^{3t} - 3t - \frac{9}{2}.$$

Step 3: Fit and verify. Write $y = y_p + Ce^t + De^{2t}$. Since $y_p(0) = -4$ and $y'_p(0) = -3/2$, the zero data give $C + D = 4$, $C + 2D = 3/2$. Hence

$$y = \frac{1}{2}e^{3t} + \frac{13}{2}e^t - \frac{5}{2}e^{2t} - 3t - \frac{9}{2}.$$

At zero its value and slope are $1/2 + 13/2 - 5/2 - 9/2 = 0$ and $3/2 + 13/2 - 5 - 3 = 0$. Its second and third derivatives there are $9/2 + 13/2 - 10 = 1$ and $27/2 + 13/2 - 20 = 0$. The coefficient matching already verifies the forcing everywhere.

Step 4: Interpret the information count. Before calibration there are two unknown forcing parameters and two trajectory constants. The four measurements determine those four quantities here. Once the forcing is fixed, only the value and slope are freely assignable: the equation and its derivative determine the higher derivatives. Arbitrary additional acceleration measurements would be compatibility tests, not extra solution freedoms.

Question 10

For a real parameter c , consider on $\mathbb{R}$

$$y'' - y = e^{-t} + ce^{-2t}, \quad y(0) = y'(0) = 0.$$

Both forcing components tend to zero as $t \rightarrow \infty$.

Tasks

1. Choose resonant and nonresonant trials as appropriate and find the full zero-data solution in terms of c .
2. Find the unique value of c making this solution bounded on $[0, \infty)$.
3. For that tuned value, prove the sign of the solution for every $t > 0$ and find its limiting value. You may use the strict inequality $e^{-t} > 1 - t$ for $t > 0$.
4. If the tuned parameter is replaced by $c + \delta$ with $\delta \neq 0$, determine the leading long-time term and its sign. Explain why decaying forcing alone does not guarantee a decaying response.

Question 10 – Solution

Strategy. Determine which forcing combination cancels the growing homogeneous mode after the initial data are fitted.

Step 1: Match and fit. The root -1 is simple, while -2 is not a root. Use $y_p = ate^{-t} + be^{-2t}$. Since $(D^2 - 1)(te^{-t}) = -2e^{-t}$ and $(D^2 - 1)e^{-2t} = 3e^{-2t}$, we obtain $a = -1/2$, $b = c/3$. Fitting the zero data gives

$$y = \left(\frac{1}{4} + \frac{c}{6}\right)e^t + \left(-\frac{1}{4} - \frac{c}{2}\right)e^{-t} - \frac{1}{2}te^{-t} + \frac{c}{3}e^{-2t}.$$

The value is zero because the constant coefficients sum to zero; the slope is zero because the derivative contributions sum to zero.

Step 2: Cancel the unstable mode. Every term except the e^t term tends to zero. Thus boundedness holds exactly when $1/4 + c/6 = 0$, giving $c = -3/2$. The tuned response is

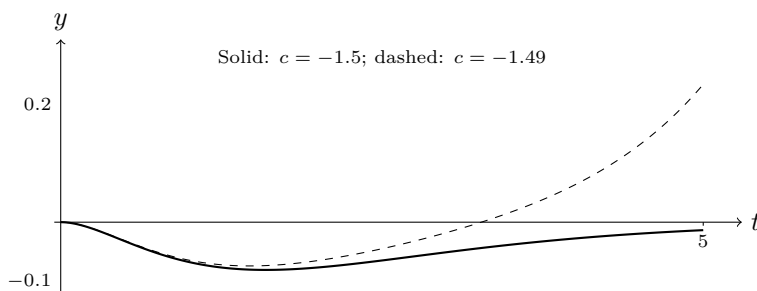
$$y_* = \frac{1}{2}e^{-t}(1 - t - e^{-t}).$$

Step 3: Prove sign and decay. For $t > 0$, the supplied strict exponential inequality gives $1 - t - e^{-t} < 0$, so $y_* < 0$. All its terms decay, hence $y_* \rightarrow 0$ from below. It has zero initial value and slope despite remaining strictly negative afterward.

Step 4: Examine imperfect cancellation. Replacing the tuned value by $-3/2 + \delta$ makes the growing-mode coefficient $\delta/6$. More explicitly, the change in solution is

$$\delta[e^t/6 - e^{-t}/2 + e^{-2t}/3].$$

Consequently $y(t)/e^t \rightarrow \delta/6$: the response tends to positive infinity for $\delta > 0$ and negative infinity for $\delta < 0$. The forcing still decays; the initial-data adjustment excites a growing homogeneous component unless the cancellation is exact.



Question 1

On $t > 0$, consider

$$t^2 y'' - 2ty' + 2y = t^3, \quad y(1) = y'(1) = 0.$$

Two homogeneous solutions are $y_1 = t$ and $y_2 = t^2$.

Tasks

1. Normalize the equation and compute the Wronskian. Derive the two equations for the parameter derivatives using the auxiliary condition $u_1' y_1 + u_2' y_2 = 0$.
2. Use variation of parameters to find a particular solution and then fit the initial data.
3. Check the final solution in the original, unnormalized equation and at the initial point.
4. A student inserts t^3 , rather than the normalized forcing, into the parameter formulas. Find the resulting particular expression and its actual forcing in the original equation. Explain the error.

Question 1 – Solution

Strategy. Divide by the leading coefficient before solving for the parameter derivatives; check the answer in the original equation.

Step 1: Normalize and derive. The normalized forcing is $g(t) = t$, and $W = t(2t) - t^2 = t^2 \neq 0$. With $y = u_1t + u_2t^2$, impose $tu'_1 + t^2u'_2 = 0$. Differentiating after this cancellation gives

$$tu'_1 + t^2u'_2 = 0, \quad u'_1 + 2tu'_2 = t.$$

Thus $u'_1 = -t$ and $u'_2 = 1$.

Step 2: Integrate and fit. Choose $u_1 = -t^2/2$, $u_2 = t$; constants of integration contribute only homogeneous terms. Hence $y_p = t^3/2$ and

$$y = Ct + Dt^2 + t^3/2.$$

The data give $C + D = -1/2$ and $C + 2D = -3/2$, so $C = 1/2$, $D = -1$. Therefore

$$y = \frac{1}{2}t(t-1)^2.$$

Step 3: Verify the original equation. For the operator $L[y] = t^2y'' - 2ty' + 2y$, one has $L[t] = L[t^2] = 0$ and $L[t^3] = 2t^3$. Thus $L[y] = t^3$. Also $y(1) = 0$ and $y' = (3t^2 - 4t + 1)/2$, so $y'(1) = 0$. The normalized coefficients are continuous on $(0, \infty)$, giving uniqueness there.

Step 4: Diagnose the missing division. Using $g = t^3$ instead gives $u'_1 = -t^3$, $u'_2 = t^2$ and

$$\tilde{y}_p = -t^5/4 + t^5/3 = t^5/12.$$

Since $L[t^5] = (20 - 10 + 2)t^5 = 12t^5$, this expression satisfies $L[\tilde{y}_p] = t^5$, not t^3 . Adding homogeneous terms cannot repair its forcing. The parameter formulas apply to the normalized equation; omitting the division multiplies the intended original forcing by t^2 .

Question 2

Solve on $I = (-\pi/2, \pi/2)$:

$$y'' + y = \sec t, \quad y(0) = y'(0) = 0.$$

Use the homogeneous pair $\cos t, \sin t$.

Tasks

1. Derive the parameter derivatives and evaluate the integrals with lower endpoint zero.
2. Simplify the initial-value solution and verify its equation and both data.
3. Determine the limits of y and y' as $t \rightarrow (\pi/2)^-$. Can the solution extend as a C^1 function through that endpoint?
4. Explain why a finite limit of y does not enlarge the interval on which this initial-value problem has a classical solution.

Question 2 – Solution

Strategy. The forcing is singular at the interval endpoints. Keep its domain visible even when the solution itself has a finite limit.

Step 1: Integrate the parameter system. Here $W = 1$. The equations $u'_1 \cos t + u'_2 \sin t = 0$ and $-u'_1 \sin t + u'_2 \cos t = \sec t$ give

$$u'_1 = -\tan t, \quad u'_2 = 1.$$

Since $\cos t > 0$ on I , integration from zero yields $u_1 = \ln(\cos t)$, $u_2 = t$.

Step 2: Verify the response. The zero lower endpoints fit both data, and

$$y = \cos t \ln(\cos t) + t \sin t.$$

Differentiation simplifies to

$$y' = -\sin t \ln(\cos t) + t \cos t, \quad y'' = -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t \sin t.$$

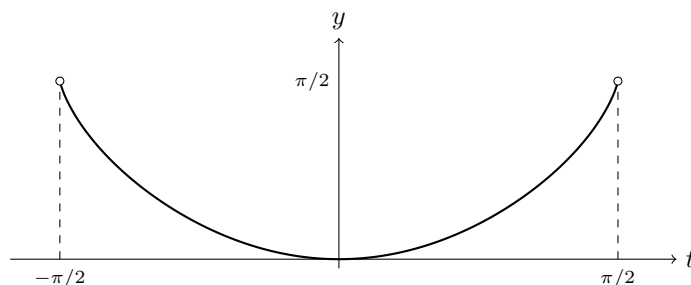
Therefore $y'' + y = (\sin^2 t + \cos^2 t)/\cos t = \sec t$, and both data vanish.

Step 3: Examine the endpoint. Using $x \ln x \rightarrow 0$ as $x \rightarrow 0^+$, we obtain

$$\lim_{t \rightarrow (\pi/2)^-} y(t) = \pi/2, \quad \lim_{t \rightarrow (\pi/2)^-} y'(t) = +\infty.$$

Thus even a C^1 extension is impossible. The graph approaches a finite height with an unbounded positive slope; the endpoint is excluded.

Step 4: Keep the equation's domain. The right side $\sec t$ is undefined at $\pi/2$ and $-\pi/2$. A continuous assignment to y at an endpoint would not define the differential equation there. The maximal open interval containing zero on which this IVP is classical is exactly I .



Question 3

On $\mathbb{R}$, consider

$$y'' - 2y' + y = \frac{e^t}{1+t^2}, \quad y(0) = y'(0) = 0.$$

Use $y_1 = e^t$ and $y_2 = te^t$.

Tasks

1. Compute the Wronskian and use variation of parameters to solve the initial-value problem.
2. Independently verify the result by writing $y = e^t F(t)$ and simplifying the differential equation for F .
3. Prove that the selected response is strictly positive for every $t > 0$.
4. Find $\lim_{t \rightarrow \infty} y(t)/(te^t)$ and interpret the effect of the factor $1/(1+t^2)$ in the forcing.

Question 3 – Solution

Strategy. The repeated homogeneous root is handled by the Wronskian; an exponential substitution provides an independent residual check.

Step 1: Compute and integrate. The Wronskian is $e^t(1+t)e^t - te^{2t} = e^{2t}$. Hence

$$u_1' = -\frac{t}{1+t^2}, \quad u_2' = \frac{1}{1+t^2}.$$

Integrating from zero gives $u_1 = -\frac{1}{2}\ln(1+t^2)$ and $u_2 = \arctan t$. Therefore

$$y = e^t[t \arctan t - \frac{1}{2}\ln(1+t^2)].$$

Zero integration endpoints ensure $y(0) = y'(0) = 0$.

Step 2: Check by a different calculation. Writing $y = e^t F$ gives $y'' - 2y' + y = e^t F''$. For the displayed bracket,

$$F' = \arctan t, \quad F'' = \frac{1}{1+t^2}, \quad F(0) = F'(0) = 0.$$

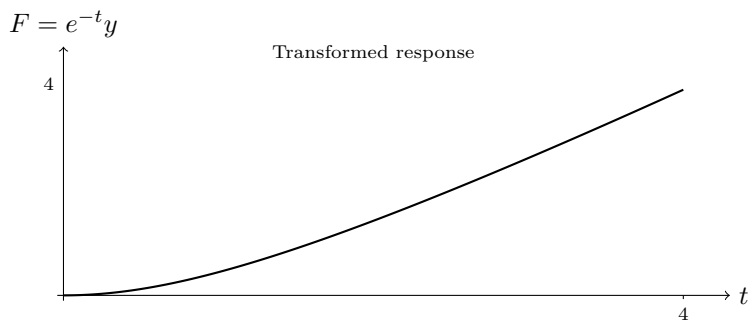
These identities verify the equation and both data independently. All coefficients and the forcing are continuous on $\mathbb{R}$, so this is the unique global IVP solution.

Step 3: Prove the sign. For $t > 0$, $F(t) = \int_0^t \arctan s \, ds > 0$. Equivalently, $F(t) = \int_0^t (t-s)/(1+s^2) \, ds$ has a positive integrand in the interior. Since $e^t > 0$, the response is strictly positive.

Step 4: Compare the growth rate. For $t > 0$,

$$\frac{y}{te^t} = \arctan t - \frac{\ln(1+t^2)}{2t} \rightarrow \boxed{\pi/2}.$$

Although the forcing is smaller than e^t by a factor tending to zero, the zero-data response is asymptotic to $(\pi/2)te^t$. The graph shows the transformed response $F = e^{-t}y$, whose slope tends to $\pi/2$; it is not a plot of y itself.



Question 4

Let g be continuous on $\mathbb{R}$. For $y'' + y = g(t)$, compare the homogeneous bases

$$(y_1, y_2) = (\cos t, \sin t), \quad (z_1, z_2) = (\cos t + \sin t, \cos t - \sin t).$$

In both parameter constructions, set the two parameter values to zero at $t = 0$.

Tasks

1. Derive the parameter derivatives for both bases, including the sign of each Wronskian.
2. Prove directly that both constructions give the same integral formula for the zero-data response.
3. If the lower endpoint is changed from zero to a , express the difference between the two particular solutions as a homogeneous combination.
4. Apply the formula to $g(t) = e^{t^2}$ with $y(0) = 2$, $y'(0) = -1$. Verify the solution without assuming that its integrals have elementary antiderivatives.

Question 4 – Solution

Strategy. A basis changes the parameter functions, but the integral response with fixed initial data must remain the same.

Step 1: Compute both parameter systems. For the first basis $W = 1$, so $u'_1 = -g \sin t$, $u'_2 = g \cos t$. The constant change-of-basis determinant is -2 , hence $W_z = -2$. Thus

$$v'_1 = \frac{1}{2}g(\cos t - \sin t), \quad v'_2 = -\frac{1}{2}g(\cos t + \sin t).$$

The negative Wronskian is essential to these signs.

Step 2: Compare the reconstructed functions. For the first basis, the integrand multiplying $g(s)$ is $\sin t \cos s - \cos t \sin s = \sin(t - s)$. For the second, it is

$$\frac{1}{2}[(\cos t + \sin t)(\cos s - \sin s) - (\cos t - \sin t)(\cos s + \sin s)],$$

which expands to the same expression. Therefore both give

$$Y_0(t) = \int_0^t \sin(t-s)g(s) ds.$$

This also holds for negative t using oriented integrals.

Step 3: Change the lower endpoint. Set $Y_a(t) = \int_a^t \sin(t-s)g(s) ds$. Then

$$Y_a - Y_0 = \cos t \int_0^a \sin s g(s) ds - \sin t \int_0^a \cos s g(s) ds.$$

Both coefficients are constants in t . Changing the lower endpoint changes a particular solution only by a homogeneous term; the data must be fitted again.

Step 4: Use a nonelementary forcing. The requested solution is

$$y = 2 \cos t - \sin t + \int_0^t \sin(t-s)e^{s^2} ds.$$

Leibniz differentiation gives $Y'_0 = \int_0^t \cos(t-s)e^{s^2} ds$ and $Y''_0 = e^{t^2} - Y_0$. Also $Y_0(0) = Y'_0(0) = 0$. Thus the equation and data hold. A definite-integral formula is a complete exact solution even without an elementary antiderivative.

Question 5

On $t > 0$, solve

$$ty'' - (t+1)y' + y = t^2, \quad y(1) = y'(1) = 0.$$

The functions $t+1$ and e^t solve the homogeneous equation. Also consider the original undivided equation at $t=0$.

Tasks

1. Normalize the equation, compute the Wronskian, and obtain a particular solution by variation of parameters.
2. Fit the initial data and check the result directly in the undivided equation.
3. Does this selected function extend smoothly through zero and still satisfy the undivided equation? Find its value and slope there.
4. Prove that these same value and slope at zero do not select a unique smooth solution of the undivided equation near zero.

Question 5 – Solution

Strategy. Use variation of parameters where the leading coefficient is nonzero, then examine the original equation separately at its singular point.

Step 1: Normalize and integrate. The normalized forcing is $g = t$, and $W = (t+1)e^t - e^t = te^t$. Hence

$$u_1' = -1, \quad u_2' = (t+1)e^{-t}.$$

Choose $u_1 = -t$, $u_2 = -(t+2)e^{-t}$, giving

$$y_p = -t(t+1) - (t+2) = -t^2 - 2t - 2.$$

Thus $y = C(t+1) + De^t - t^2 - 2t - 2$ on $(0, \infty)$.

Step 2: Fit and verify. The conditions are $2C + De = 5$ and $C + De = 4$, so $C = 1$, $D = 3/e$ and

$$y = 3e^{t-1} - t^2 - t - 1.$$

Its derivatives are $y' = 3e^{t-1} - 2t - 1$, $y'' = 3e^{t-1} - 2$. Substitution cancels the exponential terms; the polynomial terms yield

$$-2t + (t+1)(2t+1) - t^2 - t - 1 = t^2.$$

Both data at one vanish. The normalized problem is regular on $(0, \infty)$, where its solution is unique.

Step 3: Continue the selected function. The formula is analytic on $\mathbb{R}$, and the same residual identity holds there. At zero,

$$y(0) = y'(0) = 3/e - 1.$$

The undivided equation reduces to $-y'(0) + y(0) = 0$, which these values satisfy. This extension does not require dividing by t at zero.

Step 4: Demonstrate nonuniqueness at zero. The nonzero homogeneous function $h = e^t - (t+1)$ is analytic, with $h(0) = h'(0) = 0$. For every real k , $y + kh$ solves the undivided equation and has the same two data at zero. These are distinct functions since $h''(0) = 1$. The Wronskian vanishes at zero precisely where the leading coefficient vanishes; the regular normalized uniqueness theorem does not apply there.

Question 6

Let $T > 0$ and let g be continuous on $[0, T]$. Consider

$$y'' - y = g(t), \quad y(0) = y'(0) = 0.$$

Use the homogeneous basis e^t, e^{-t} .

Tasks

1. Derive a single definite-integral formula for the response by variation of parameters, and verify it by differentiation.
2. If $|g(t)| \leq M$ with $M > 0$, find a sharp bound for $|y(t)|$ on $[0, T]$ and exhibit forcing that attains it.
3. If $g \geq 0$, prove $y \geq 0$ and give a precise condition guaranteeing $y(t) > 0$ at a specified time $t > 0$.
4. For $g(t) = (1 + \cos 2t)/2$, find an elementary response and compare it with the bound for $M = 1$.

Question 6 – Solution

Strategy. The sign of the integral kernel gives comparison and error bounds without evaluating each forcing integral.

Step 1: Derive the response kernel. Here $W = -2$, so $u'_1 = e^{-t}g/2$ and $u'_2 = -e^tg/2$. Integrating from zero and combining gives

$$y(t) = \int_0^t \sinh(t-s)g(s) ds.$$

Differentiation gives $y' = \int_0^t \cosh(t-s)g(s) ds$ and $y'' = g(t) + y(t)$. Both initial data vanish.

Step 2: Prove sharpness. For $0 \leq s \leq t$, $\sinh(t-s) \geq 0$. Consequently

$$|y(t)| \leq M \int_0^t \sinh(t-s) ds = M(\cosh t - 1).$$

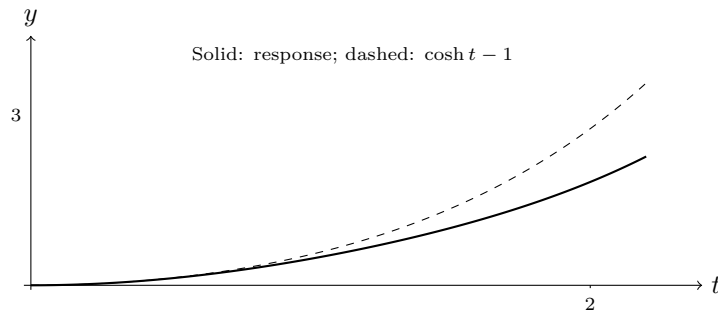
Constant forcing $g = M$ gives equality for every t , and $g = -M$ gives equality in magnitude. The bound is therefore sharp for this forcing class; it also bounds response error when g denotes a forcing error.

Step 3: Prove positivity carefully. If $g \geq 0$, the integral is nonnegative. At fixed $t > 0$, it is strictly positive exactly when g is not identically zero on $[0, t]$. Indeed, continuity then gives a positive subinterval lying before t , where the kernel is strictly positive. Forcing that is positive only at later times cannot make the response positive at this time.

Step 4: Evaluate a comparison example. For $g = (1 + \cos 2t)/2$, an elementary particular solution is $-1/2 - \cos(2t)/10$. Fitting zero data gives

$$y = \frac{3}{5} \cosh t - \frac{1}{2} - \frac{1}{10} \cos 2t.$$

Direct differentiation yields $y'' - y = (1 + \cos 2t)/2$ and zero data. Since $0 \leq g \leq 1$, the kernel proves $0 < y(t) < \cosh t - 1$ for $t > 0$: both g and $1 - g$ are nonzero on every such interval.



Question 7

On $t \geq 0$, consider the family

$$y'' - y = \frac{1}{1+t^2}, \quad y(0) = 0, \quad y'(0) = v.$$

Write $f(t) = 1/(1+t^2)$ and $I = \int_0^\infty e^{-s} f(s) ds$. An answer containing a convergent definite integral is exact.

Tasks

1. Use variation of parameters to express y_v as an integral. Prove that I converges.
2. Find the only possible value of v for which y_v is bounded on $[0, \infty)$.
3. For that value, rewrite the response using the positive kernel $K(t, s) = \sinh(\min\{t, s\})e^{-\max\{t, s\}}$. Prove boundedness and determine its sign for $t > 0$.
4. Prove that the bounded response tends to zero, and determine the long-time behavior when the initial slope differs from the selected value.

Question 7 – Solution

Strategy. Cancel the growing mode using a convergent integral, then prove the resulting tail expression is bounded.

Step 1: Construct the family. Variation of parameters with e^t, e^{-t} gives

$$y_v = v \sinh t + \int_0^t \sinh(t-s)f(s) ds.$$

Since $0 < f \leq 1$, $0 < I \leq \int_0^\infty e^{-s} ds = 1$, with strict inequality $I < 1$.

Step 2: Identify the necessary slope. Expanding the hyperbolic sine yields

$$e^{-t}y_v = \frac{v}{2}(1 - e^{-2t}) + \frac{1}{2} \int_0^t e^{-s} f(s) ds - \frac{1}{2} e^{-2t} \int_0^t e^s f(s) ds.$$

The last term has magnitude at most $(e^{-t} - e^{-2t})/2$. Thus $e^{-t}y_v \rightarrow (v + I)/2$. Boundedness requires $v_* = -I$.

Step 3: Prove sufficiency and sign. Splitting the integral at $s = t$ and substituting $v = -I$ gives

$$y_*(t) = - \int_0^\infty K(t,s)f(s) ds, \quad K(t,s) = \begin{cases} e^{-t} \sinh s, & s \leq t, \\ e^{-s} \sinh t, & s \geq t. \end{cases}$$

This identity follows by grouping the e^t and e^{-t} terms above. Moreover,

$$\int_0^\infty K(t,s) ds = e^{-t}(\cosh t - 1) + e^{-t} \sinh t = 1 - e^{-t}.$$

Therefore $|y_*| \leq 1 - e^{-t} < 1$. For $t > 0$, the kernel and forcing are positive for $s > 0$, so $y_*(t) < 0$.

Step 4: Prove decay and sensitivity. Given $\varepsilon > 0$, choose A with $f(s) \leq \varepsilon$ for $s \geq A$. For $t > A$, the integral over $[0, A]$ has magnitude at most $e^{-t} \int_0^A \sinh s ds \rightarrow 0$; the remaining magnitude is at most $\varepsilon \int_A^\infty K \leq \varepsilon$. Hence $y_* \rightarrow 0$. If $v = v_* + \delta$, then $y_v = y_* + \delta \sinh t$, so $y_v/e^t \rightarrow \delta/2$. It tends to $+\infty$ for $\delta > 0$ and $-\infty$ for $\delta < 0$.

Question 8

For $t \geq 0$, solve $y'' + y = g(t)$ with zero initial value and slope. First use the continuous finite-duration forcing

$$g_2(t) = \begin{cases} \sin 2t, & 0 \leq t \leq \pi, \\ 0, & t > \pi. \end{cases}$$

Later compare it with g_3 , defined in the same way using $\sin 3t$.

Tasks

1. Derive the integral response and solve explicitly for g_2 on $[0, \pi]$.
2. Find the response after π , checking the matching value and slope. Does turning off the forcing return the solution to rest?
3. For any continuous forcing supported in $[0, \pi]$, derive the two integral conditions equivalent to $y(t) = 0$ for every $t \geq \pi$.
4. Test g_3 against these conditions and find its response. Explain how a nonzero forcing can leave no later motion.

Question 8 – Solution

Strategy. After the forcing ends, its two weighted integrals become the coefficients of the remaining homogeneous oscillation.

Step 1: Solve while the first forcing acts. Variation of parameters gives $y(t) = \int_0^t \sin(t-s)g(s) ds$. For g_2 on $[0, \pi]$, evaluation gives

$$y_2(t) = \frac{2 \sin t - \sin 2t}{3}.$$

Its value and slope at zero vanish, and $y_2'' + y_2 = \sin 2t$, independently verifying the integral evaluation.

Step 2: Match at the shutoff time. At π , $y_2 = 0$ and $y_2' = -4/3$. The subsequent homogeneous response is

$$y_2(t) = -\frac{4}{3} \sin(t - \pi) = \frac{4}{3} \sin t, \quad t \geq \pi.$$

It matches both data. Since $g_2(\pi) = 0$, $y'' = g_2 - y$ also matches. The resulting function is a classical C^2 solution, but it does not return to rest.

Step 3: Derive the cancellation conditions. For any such forcing and $t \geq \pi$,

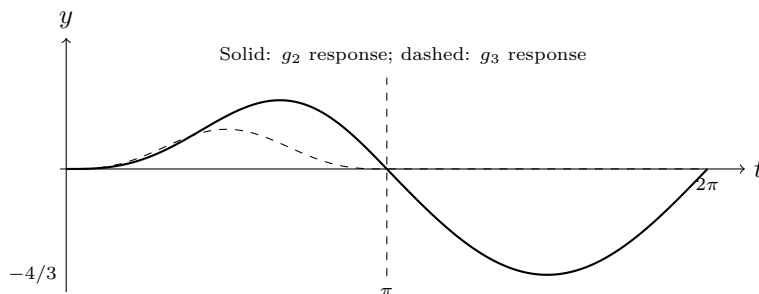
$$y(t) = \sin t \int_0^\pi \cos s g(s) ds - \cos t \int_0^\pi \sin s g(s) ds.$$

Independence of sine and cosine shows that the response vanishes identically after π exactly when both displayed integrals are zero. Cancellation of just one moment is insufficient.

Step 4: Construct a return to rest. For g_3 , product-to-sum identities give $\int_0^\pi \cos s \sin 3s ds = 0$ and $\int_0^\pi \sin s \sin 3s ds = 0$. Explicitly,

$$y_3(t) = \begin{cases} (3 \sin t - \sin 3t)/8, & 0 \leq t \leq \pi, \\ 0, & t \geq \pi. \end{cases}$$

On the first interval it has residual $\sin 3t$ and zero initial data; at π its value and slope are zero. The forcing's signed contributions cancel both final-state components, despite a nonzero response during the forcing interval.



Question 9

Let f be continuous on $[0, \pi]$. Seek C^2 solutions of

$$y'' + y = f(t), \quad y(0) = y(\pi) = 0.$$

Use variation of parameters; no general boundary-value theory is required.

Tasks

1. Write every solution satisfying the first endpoint condition, using its free initial slope A .
2. Derive a necessary and sufficient condition on f for the second endpoint condition. Explain uniqueness or nonuniqueness when the condition holds.
3. Apply the criterion to $f = 1$ and $f = \cos t$, and find all solutions in the latter case.
4. For $f = \cos t$, impose the additional condition $\int_0^\pi y(t) dt = 0$. Determine whether it selects a unique solution and find it.

Question 9 – Solution

Strategy. Variation of parameters supplies all initial-value responses; evaluating the integral at the second endpoint reveals the obstruction.

Step 1: Keep the free slope. The homogeneous pair is $\cos t, \sin t$, with $W = 1$. Integrating the parameter derivatives from zero and imposing $y(0) = 0$ gives every candidate:

$$y(t) = A \sin t + \int_0^t \sin(t-s)f(s) ds, \quad y'(0) = A.$$

Differentiating this formula verifies the equation and shows no candidate has been lost.

Step 2: Evaluate the second endpoint. Since $\sin \pi = 0$ and $\sin(\pi-s) = \sin s$, the remaining condition is

$$\int_0^\pi \sin s f(s) ds = 0.$$

If it fails, no choice of A helps. If it holds, every real A works: there are infinitely many solutions, differing by $A \sin t$. Thus this endpoint pair never selects exactly one solution for a compatible forcing.

Step 3: Compare the two forcings. For $f = 1$, the integral is 2, so there is no solution. For $f = \cos t$, it is $\int_0^\pi \sin s \cos s ds = 0$. Evaluation of the response integral, or a direct residual check, gives

$$y(t) = (A + t/2) \sin t, \quad A \in \mathbb{R}.$$

Indeed, $(t \sin t/2)'' + t \sin t/2 = \cos t$, and both endpoints vanish.

Step 4: Apply the extra measurement. The needed integrals are $\int_0^\pi \sin t dt = 2$ and, by integration by parts, $\int_0^\pi t \sin t dt = \pi$. Thus

$$0 = 2A + \pi/2, \quad A = -\pi/4, \quad y(t) = (t/2 - \pi/4) \sin t.$$

The coefficient of A in this measurement is nonzero, so it selects exactly one member of the compatible family. This does not contradict the earlier nonuniqueness: the added integral condition supplies new information.

Question 10

For $t \geq 0$, solve

$$y'' + y = e^{-t^2}, \quad y(0) = y'(0) = 0.$$

Define the convergent integrals $C = \int_0^\infty e^{-s^2} \cos s \, ds$ and $S = \int_0^\infty e^{-s^2} \sin s \, ds$. No special-function evaluations of C or S are required.

Tasks

1. Use variation of parameters to express the exact solution through two finite definite integrals, and verify its initial data and residual.
2. Show that the solution approaches the periodic function $P(t) = C \sin t - S \cos t$ in the sense that $y(t) - P(t) \rightarrow 0$.
3. Prove the quantitative estimate $|y(t) - P(t)| \leq e^{-t^2}/(2t)$ for every $t > 0$.
4. Prove $S > 0$ by pairing successive positive and negative half-waves of sine. Use this to show that y does not tend to zero, even though the forcing does.

Question 10 – Solution

Strategy. Keep the variation-of-parameters integrals exact, then bound their tails to identify the surviving oscillation.

Step 1: Write and check the exact response. With $W = 1$, the parameter derivatives are $u_1' = -e^{-t^2} \sin t$ and $u_2' = e^{-t^2} \cos t$. Hence

$$y(t) = \sin t \int_0^t e^{-s^2} \cos s \, ds - \cos t \int_0^t e^{-s^2} \sin s \, ds.$$

Equivalently, $y = \int_0^t \sin(t-s)e^{-s^2} \, ds$. Differentiating gives $y' = \int_0^t \cos(t-s)e^{-s^2} \, ds$ and $y'' = e^{-t^2} - y$; both initial data vanish.

Step 2: Isolate the tails. Absolute convergence follows from the integrability of e^{-s^2} (for $s \geq 1$, it is at most e^{-s}). Subtracting the full integrals gives

$$y(t) - P(t) = - \int_t^\infty \sin(t-s)e^{-s^2} \, ds.$$

Thus $|y - P| \leq \int_t^\infty e^{-s^2} \, ds \rightarrow 0$.

Step 3: Obtain an explicit error bound. For $s \geq t > 0$, $1 \leq s/t$. Consequently

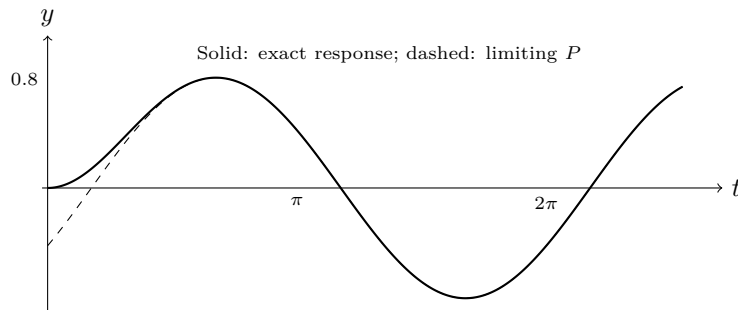
$$|y - P| \leq \frac{1}{t} \int_t^\infty s e^{-s^2} \, ds = \boxed{\frac{e^{-t^2}}{2t}}.$$

Combining the tails before estimating avoids an unnecessary sum of two bounds.

Step 4: Prove a nonzero surviving oscillation. Absolute convergence permits pairing intervals. Their k th pair contributes

$$\int_0^\pi \sin u [e^{-(2k\pi+u)^2} - e^{-((2k+1)\pi+u)^2}] \, du > 0, \quad k \geq 0.$$

Therefore $S > 0$. Along $t = 2n\pi$, $y(t) \rightarrow -S$; along $t = (2n+1)\pi$, $y(t) \rightarrow S$. The solution has no limit and does not decay to zero. A decaying forcing can leave a persistent homogeneous oscillation.



Question 1

A 2 kg mass stretches an ideal vertical spring by 0.196 m at static equilibrium. Use $g = 9.8 \text{ m/s}^2$ and neglect damping. Let x be downward displacement from equilibrium. At $t = 0$, the mass is 0.05 m below equilibrium and moving upward at 0.20 m/s.

Tasks

1. Determine the spring constant and derive the equation for x from Newton's law, starting with extension measured from the spring's natural length.
2. Solve the initial-value problem and find its oscillation amplitude and angular frequency.
3. Find the first positive time the mass crosses equilibrium and its signed velocity then.
4. Use mechanical energy to verify the crossing speed. Find the maximum spring extension from natural length and explain why gravity is absent from the final equation for x .

Question 1 – Solution

Strategy. Distinguish extension from natural length from displacement about equilibrium, and keep the upward initial velocity negative.

Step 1: Establish the coordinate and model. Write $z = \ell + x$ for downward extension, where $\ell = 0.196$ m. The static balance is $k\ell = mg$, so $k = 19.6/0.196 = 100$ N/m. Newton's law gives

$$2x'' = mg - k(\ell + x) = -100x, \quad x(0) = 0.05, \quad x'(0) = -0.20.$$

Thus $\omega_0 = \sqrt{k/m} = 5\sqrt{2}\text{ s}^{-1}$.

Step 2: Solve and find the amplitude. Fitting both data gives

$$x(t) = \frac{1}{20} \cos(5\sqrt{2}t) - \frac{1}{25\sqrt{2}} \sin(5\sqrt{2}t).$$

The amplitude satisfies $R^2 = x(0)^2 + [x'(0)/\omega_0]^2 = 1/400 + 1/1250 = 33/10000$. Hence $R = \sqrt{33}/100$ m. Differentiation verifies $x'' + 50x = 0$ and the two data.

Step 3: Locate the first crossing. The first zero lies in $0 < \omega_0 t < \pi/2$, where the motion is upward. Solving $x = 0$ gives

$$t_* = \frac{1}{5\sqrt{2}} \arctan \frac{5\sqrt{2}}{4}.$$

At that crossing, $x' = -\omega_0 R = -\sqrt{66}/20$ m/s. The negative sign identifies upward travel; using an arbitrary branch of arctangent could miss the first crossing.

Step 4: Check energy and physical extension. Relative to equilibrium, the conserved energy is $E = mx'^2/2 + kx^2/2$. Initially,

$$E = \frac{1}{2}(2)(0.20)^2 + \frac{1}{2}(100)(0.05)^2 = 0.165 \text{ J}.$$

At $x = 0$, $E = x'^2$, giving $|x'| = \sqrt{0.165} = \sqrt{66}/20$ m/s. The maximum natural-length extension is $\ell + R = 0.196 + \sqrt{33}/100$ m, about 0.25345 m. Gravity has not been neglected: its constant force cancels the spring's equilibrium force after shifting the coordinate.

Question 2

A mass of 1.5 kg vibrates freely about equilibrium with viscous damping:

$$mx'' + cx' + kx = 0, \quad c > 0, \quad k > 0.$$

It is released from rest at $x(0) = 0.08$ m. Measured successive positive displacement maxima are separated by π s, and each is $e^{-\pi/2}$ times the preceding one. Treat these measurements as exact.

Tasks

1. Infer the decay rate and damped angular frequency, explaining why same-sign peaks must be used. Determine c and k with units.
2. Find the response and verify that its positive maxima really have the measured spacing and ratio.
3. Determine the ratio of mechanical energies at successive positive maxima. Why is it not the displacement ratio?
4. If the mass had not been supplied, identify exactly which combinations of m, c, k these measurements determine, and exhibit the remaining ambiguity.

Question 2 – Solution

Strategy. Peak spacing determines the oscillation frequency; peak ratios determine exponential decay. A known mass converts these rates into physical coefficients.

Step 1: Infer the coefficients. An underdamped response has form $e^{-\alpha t}(A \cos \omega_d t + B \sin \omega_d t)$, with $\alpha = c/(2m)$ and $\omega_d^2 = k/m - \alpha^2$. Same-sign peaks are one damped period apart, so

$$\omega_d = \frac{2\pi}{\pi} = 2 \text{ s}^{-1}, \quad e^{-\alpha\pi} = e^{-\pi/2} \implies \alpha = \frac{1}{2} \text{ s}^{-1}.$$

Thus $\boxed{c = 3/2 \text{ N s/m}}$ and $\boxed{k = 51/8 \text{ N/m}}$. Adjacent positive and negative extrema would be only half a period apart.

Step 2: Reconstruct and verify the peaks. The initial value gives $A = 0.08$, and $x'(0) = 0$ gives $B = 0.02$. Hence

$$\boxed{x = 0.08e^{-t/2}[\cos 2t + \frac{1}{4} \sin 2t]}, \quad x' = -0.17e^{-t/2} \sin 2t.$$

The characteristic roots are $-1/2 \pm 2i$, verifying the equation. Extrema occur at $t = n\pi/2$; positive maxima occur at $t = n\pi$, where $x = 0.08e^{-n\pi/2}$. The derivative changes from positive to negative at each interior positive maximum, and the release at zero starts a decrease.

Step 3: Compare energies. At an extremum, velocity is zero, so $E = kx^2/2$. Successive positive-peak energies therefore have ratio

$$\boxed{e^{-\pi}},$$

not $e^{-\pi/2}$. Energy depends quadratically on displacement at a turning point.

Step 4: State the identification limit. Without a known mass, only $c/m = 2\alpha = 1 \text{ s}^{-1}$ and $k/m = \alpha^2 + \omega_d^2 = 17/4 \text{ s}^{-2}$ are determined. Multiplying m, c, k by the same positive constant leaves the normalized equation, the complete free response, and all peak measurements unchanged. The three coefficients cannot then be identified separately.

Question 3

A mass-spring-damper system has $m = 1$ kg, $c = 2$ Ns/m and $k = 5$ N/m. It starts at equilibrium with velocity 1 m/s in the positive direction, with no external forcing. Define

$$E(t) = \frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2.$$

Tasks

1. Find the displacement and velocity, including the damping regime.
2. Derive the energy balance directly from the equation. Does decreasing energy imply that $|x|$ must decrease from the start?
3. Find the first positive turning time, its displacement, and the energy dissipated up to that time.
4. Explain why $E' = 0$ at a turning point does not mean damping has ceased permanently. Find the total energy dissipated as $t \rightarrow \infty$.

Question 3 – Solution

Strategy. Track energy in both position and velocity. Displacement can grow while total mechanical energy falls.

Step 1: Solve the free motion. The equation is $x'' + 2x' + 5x = 0$, with roots $-1 \pm 2i$, so the motion is underdamped. The data give

$$x = \frac{1}{2}e^{-t} \sin 2t, \quad x' = e^{-t}[\cos 2t - \frac{1}{2} \sin 2t].$$

These expressions have $x(0) = 0$, $x'(0) = 1$ and residual zero.

Step 2: Derive the dissipation law. Multiplying the equation by x' gives

$$E' = x'(mx'' + kx) = -cx'^2 = \boxed{-2x'^2 \leq 0}.$$

Nevertheless, $x'(0) = 1$, so x and $|x|$ initially increase from zero. Kinetic energy is partly converted into spring energy while the damper removes energy. Monotonic total energy is not monotonic displacement.

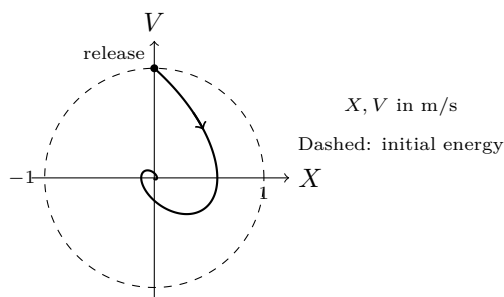
Step 3: Find the first turning point. The first root of $x' = 0$ satisfies $\tan 2t = 2$, in $0 < 2t < \pi/2$. Set $a = \frac{1}{2} \arctan 2$. Then

$$\boxed{t_* = a, \quad x(t_*) = e^{-a}/\sqrt{5} \text{ m.}}$$

The velocity changes from positive to negative, so this is a positive maximum. Initially $E(0) = 1/2 \text{ J}$. At the turning point,

$$E(a) = \frac{5}{2}x(a)^2 = \frac{1}{2}e^{-2a} \text{ J}, \quad \boxed{E(0) - E(a) = \frac{1}{2}(1 - e^{-2a}) \text{ J.}}$$

Step 4: Interpret zero instantaneous loss. At a turning point, $x' = 0$, so damper power is instantaneously zero. But $x \neq 0$ there and $x'' = -5x \neq 0$, so motion resumes immediately. The displayed response and velocity both tend to zero; hence $E \rightarrow 0$ and total dissipated energy is $1/2 \text{ J}$. In the figure, $X = \sqrt{k/m}x$ and $V = x'$ both have velocity units, and $E = m(X^2 + V^2)/2$.



Question 4

A critically damped system has $m = 1$ kg, $c = 6$ N s/m and $k = 9$ N/m. Its initial data are $x(0) = a > 0$ and $x'(0) = v$, with a in meters and v in meters per second. There is no forcing.

Tasks

1. Derive and verify the response for arbitrary a, v .
2. Classify exactly which initial velocities cause a finite equilibrium crossing. Find the crossing time when one exists.
3. Classify exactly which initial velocities make the displacement stay positive and decrease monotonically toward zero.
4. For $a = 0.10$ m and $v = -0.60$ m/s, find the crossing and the subsequent minimum. Use the result to assess the claim that critical damping always prevents overshoot.

Question 4 – Solution

Strategy. Critical damping fixes the repeated root, but the sign of the linear factor still depends on the initial velocity.

Step 1: Fit the repeated-root solution. The characteristic polynomial is $(r + 3)^2$. Thus

$$x(t) = e^{-3t}[a + (v + 3a)t], \quad x'(t) = e^{-3t}[v - 3(v + 3a)t].$$

Here the numerical rate 3 has units s^{-1} . The two data follow immediately, and substitution gives $x'' + 6x' + 9x = 0$.

Step 2: Classify crossings. Since $e^{-3t} > 0$, a positive-time zero occurs exactly when $v + 3a < 0$. In that case it is unique and

$$t_c = \frac{a}{-v - 3a}.$$

The linear factor changes sign there, so this is a crossing, not a touch. If $v \geq -3a$, the factor remains positive for all finite $t \geq 0$, including the equality case $x = ae^{-3t}$.

Step 3: Classify monotone positive return. For $v \geq -3a$, the displacement is positive. Its derivative is nonpositive for all $t \geq 0$ exactly when $v \leq 0$, because $v - 3(v + 3a)t$ then begins nonpositive and decreases. Thus the required range is

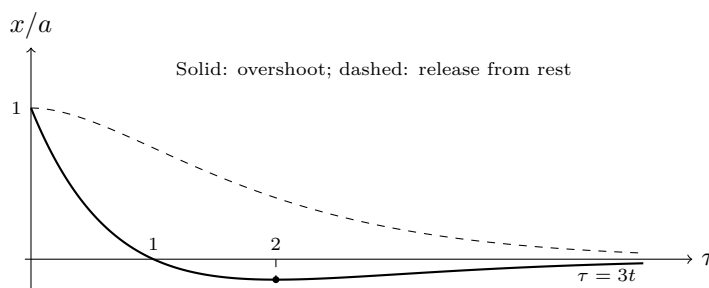
$$-3a \leq v \leq 0.$$

For $v > 0$, the mass first moves farther from equilibrium. For $v < -3a$, it crosses equilibrium and cannot be a positive monotone return.

Step 4: Exhibit overshoot. For the stated data, $x = 0.10e^{-3t}(1 - 3t)$ and $x' = e^{-3t}(-0.60 + 0.90t)$. Therefore

$$t_c = 1/3 \text{ s}, \quad t_{\min} = 2/3 \text{ s}, \quad x_{\min} = -0.10e^{-2} \text{ m}.$$

The derivative changes from negative to positive at the minimum. Critical damping eliminates sustained oscillations, but sufficiently negative initial velocity still causes one overshoot. The comparison uses $\tau = 3t$ and normalized displacement x/a .



Question 5

A 1 kg mass on a spring with $k = 4 \text{ N/m}$ is released from $x(0) = a = 0.01 \text{ m}$, with $x'(0) = 0$. Compare viscous damping coefficients $c = 4$ and $c = 5 \text{ Ns/m}$; write their responses as x_4, x_5 .

Tasks

1. Determine the damping regimes and find both exact responses.
2. Prove that each response stays positive and strictly decreases for $t > 0$.
3. Prove $x_5(t) > x_4(t)$ for every $t > 0$. You may form the equation for $w = x_5 - x_4$ and derive its zero-data integral response.
4. Explain the implication for the first time the displacement reaches any fixed level ε with $0 < \varepsilon < a$. For general $c > 4$, examine the slow characteristic root as $c \rightarrow \infty$.

Question 5 – Solution

Strategy. Compare complete release responses, rather than assuming a larger damping force always gives a faster return.

Step 1: Solve both release problems. For $c = 4$, the repeated root is -2 , giving critical damping. For $c = 5$, roots $-1, -4$ give overdamping. Fitting the data yields

$$x_4 = a(1 + 2t)e^{-2t}, \quad x_5 = \frac{a}{3}(4e^{-t} - e^{-4t}).$$

Their characteristic forms verify the equations; both have value a and slope zero at the release.

Step 2: Check positivity and decrease. The first expression is positive, and the second is positive because $4e^{-t} > e^{-4t}$. Their derivatives are

$$x_4' = -4ate^{-2t} < 0, \quad x_5' = \frac{4a}{3}(e^{-4t} - e^{-t}) < 0 \quad (t > 0).$$

Both tend to zero.

Step 3: Prove the comparison for every time. Applying $D^2 + 5D + 4$ to $w = x_5 - x_4$ gives

$$w'' + 5w' + 4w = -x_4', \quad w(0) = w'(0) = 0.$$

Variation of parameters with e^{-t}, e^{-4t} yields

$$w(t) = \int_0^t \frac{e^{-(t-s)} - e^{-4(t-s)}}{3} [-x_4'(s)] ds.$$

For $0 < s < t$, both factors are strictly positive. Hence $x_5(t) > x_4(t)$ for every $t > 0$. This integral proof avoids relying on a finite plot or only on large-time rates.

Step 4: Interpret settling and heavy damping. Each response reaches any $\varepsilon \in (0, a)$ exactly once, by continuity and strict decrease. At the time $x_4 = \varepsilon$, one still has $x_5 > \varepsilon$, so the more heavily damped response reaches the level later. For general $c > 4$ in this SI model,

$$r_{\text{slow}} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} = \frac{-2k}{c + \sqrt{c^2 - 4mk}} \rightarrow 0^-.$$

The release excites this mode with coefficient $-ar_{\text{fast}}/(r_{\text{slow}} - r_{\text{fast}}) > 0$. Excessive damping therefore introduces a slow return, with $r_{\text{slow}} \sim -k/c$; damping strength alone is not a measure of settling speed.

Question 6

An undamped 1 kg mass on a 25 N/m spring is initially at rest at equilibrium. For $t \geq 0$, apply the force $2 \cos 4t$ N.

Tasks

1. Derive and solve the forced initial-value problem.
2. Express the response as a product that displays beats. Identify the carrier angular frequency and the period of the nonnegative amplitude envelope.
3. At $t = 2\pi$ s, find both displacement and velocity. Does this mean the mass stays at rest afterward? Justify your answer using the equation.
4. Replace the forcing by $2 \cos 5t$ N and find the new zero-data response. Explain the difference between beats and exact resonance.

Question 6 – Solution

Strategy. The homogeneous correction needed to fit rest data creates the nearby-frequency interference; exact resonance requires a different particular form.

Step 1: Solve the nonresonant problem. Newton's law gives $x'' + 25x = 2 \cos 4t$, $x(0) = x'(0) = 0$. Since $25 - 16 = 9$, a particular response is $(2/9) \cos 4t$. Fitting zero data gives

$$x = \frac{2}{9}(\cos 4t - \cos 5t).$$

Its second derivative plus $25x$ equals $2 \cos 4t$, and both initial data vanish.

Step 2: Separate the two time scales. The cosine-difference identity gives

$$x = \frac{4}{9} \sin(9t/2) \sin(t/2).$$

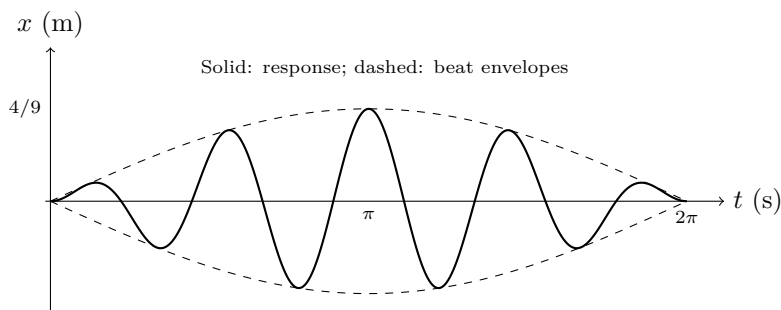
The carrier angular frequency is $9/2 \text{ s}^{-1}$. The nonnegative envelope is $A(t) = (4/9)|\sin(t/2)| \text{ m}$, with period $2\pi \text{ s}$. Its zeros occur at $t = 2n\pi$, and its largest value is $4/9 \text{ m}$, attained by x at $t = \pi$. The envelope period uses the absolute value; the signed slow sine has period 4π .

Step 3: Distinguish a rest state from remaining at rest. Here $x' = \frac{2}{9}(-4 \sin 4t + 5 \sin 5t)$, so $x(2\pi) = x'(2\pi) = 0$. Yet the applied force then equals 2 N ; the equation gives $x''(2\pi) = 2 \text{ m/s}^2$. The mass immediately accelerates again. Reaching the equilibrium rest state while a nonzero force persists does not keep it there.

Step 4: Solve at exact resonance. For $x'' + 25x = 2 \cos 5t$, use $x_p = At \sin 5t$. Its residual is $10A \cos 5t$, so $A = 1/5$ and the data are already zero:

$$x_{\text{res}} = \frac{1}{5}t \sin 5t.$$

At $t_n = (\pi/2 + 2n\pi)/5$, it equals $t_n/5 \rightarrow \infty$, whereas the beating response is bounded by $4/9$. Beats arise from interference of distinct frequencies; resonance produces a growing factor when the forcing matches the natural frequency.



Question 7

A mass-spring-damper system has $m = 2$ kg, $c = 4$ N s/m and $k = 20$ N/m. Forcing is $F_0 \cos \omega t$, where $F_0 > 0$ is fixed and $\omega > 0$. Consider its steady periodic response. Let X be displacement amplitude and V be velocity amplitude.

Tasks

1. Derive the cosine and sine coefficients of the steady response and express X and V in terms of ω .
2. Find the frequency that maximizes displacement amplitude and its maximum value.
3. Independently find the frequency that maximizes velocity amplitude and its maximum value.
4. At the velocity-maximizing frequency, determine the displacement phase relative to the applied force. Explain why a request for the 'resonant frequency' needs a specified response quantity.

Question 7 – Solution

Strategy. Different measured quantities introduce different frequency factors; optimize their amplitudes separately.

Step 1: Match harmonic coefficients. Set $x = A \cos \omega t + B \sin \omega t$. The equation $2x'' + 4x' + 20x = F_0 \cos \omega t$ gives

$$(20 - 2\omega^2)A + 4\omega B = F_0, \quad -4\omega A + (20 - 2\omega^2)B = 0.$$

Writing $\Delta = (20 - 2\omega^2)^2 + 16\omega^2 > 0$, we obtain

$$A = \frac{F_0(20 - 2\omega^2)}{\Delta}, \quad B = \frac{4F_0\omega}{\Delta}, \quad \boxed{X = \frac{F_0}{\sqrt{\Delta}}, \quad V = \omega X.}$$

The homogeneous roots are $-1 \pm 3i$, so all initial-data transients decay and this periodic response is the long-time motion.

Step 2: Maximize displacement. Complete the square:

$$\Delta = 4[(\omega^2 - 8)^2 + 36].$$

It has its unique minimum for $\omega > 0$ at $\omega_X = 2\sqrt{2}\text{s}^{-1}$. Therefore $\boxed{X_{\max} = F_0/12}$ in meters when F_0 is expressed in newtons.

Step 3: Maximize velocity. Divide the amplitude denominator by ω :

$$V = \frac{F_0}{\sqrt{(20/\omega - 2\omega)^2 + 16}}.$$

This is largest exactly when $20/\omega - 2\omega = 0$, giving

$$\boxed{\omega_V = \sqrt{10}\text{s}^{-1}, \quad V_{\max} = F_0/4}$$

in meters per second when F_0 is in newtons. The spring and inertial contributions to this denominator cancel there.

Step 4: Interpret the phase and the two maxima. At $\omega = \sqrt{10}$, $A = 0$ and $B = F_0/(4\sqrt{10}) > 0$. Thus $x = [F_0/(4\sqrt{10})] \sin(\sqrt{10}t)$, a displacement lag of $\pi/2$ behind the cosine force. Its displacement amplitude is smaller than $F_0/12$, even though its velocity amplitude is largest. The displacement peak, velocity peak and undamped natural frequency need not all coincide; the response quantity must be named.

Question 8

A 1 kg mass is connected to a moving support by a spring with $k = 9 \text{ N/m}$ and a parallel viscous damper with $c = 2 \text{ N s/m}$. Let $b(t) = B \cos \omega t$ be support displacement and $x(t)$ the mass's absolute displacement, measured in the same positive direction about equilibrium. Here $B > 0$ and $\omega > 0$. Spring and damping forces depend on relative displacement and relative velocity.

Tasks

1. Derive the equations for absolute displacement x and relative displacement $z = x - b$.
2. Find the steady amplitude ratios X/B and Z/B , where X and Z are the amplitudes of x and z .
3. Determine exactly when the absolute amplitude is smaller than the support amplitude, including the equality frequency.
4. Find the low- and high-frequency limits of both ratios. Explain why good isolation of the mass does not imply small relative spring deformation.

Question 8 – Solution

Strategy. The damper responds to velocity relative to the support. Distinguish motion of the mass from deformation of the suspension.

Step 1: Derive the two models. Newton's law gives $x'' = -9(x - b) - 2(x' - b')$. Therefore

$$x'' + 2x' + 9x = 2b' + 9b, \quad \boxed{z'' + 2z' + 9z = -b'' = B\omega^2 \cos \omega t.}$$

The inertial term in the relative equation comes from $x'' = z'' + b''$; it is not an extra spring force.

Step 2: Compute both amplitude ratios. Let $\Delta = (9 - \omega^2)^2 + 4\omega^2$. Matching sine and cosine in the relative equation gives

$$z = A \cos \omega t + D \sin \omega t, \quad A = \frac{B\omega^2(9 - \omega^2)}{\Delta}, \quad D = \frac{2B\omega^3}{\Delta}.$$

Thus $Z/B = \omega^2/\sqrt{\Delta}$. Since $x = z + B \cos \omega t$, its cosine coefficient is $B + A$, not its amplitude $B + Z$. Simplifying $(B + A)^2 + D^2$ gives

$$\boxed{\frac{X}{B} = \frac{\sqrt{81 + 4\omega^2}}{\sqrt{\Delta}}, \quad \frac{Z}{B} = \frac{\omega^2}{\sqrt{\Delta}}.}$$

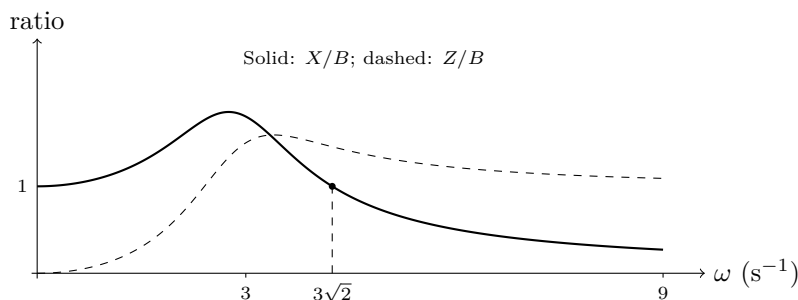
The homogeneous roots $-1 \pm 2\sqrt{2}i$ ensure the initial-data transients decay.

Step 3: Find the isolation threshold. Since $\Delta > 0$, $X/B < 1$ exactly when

$$\Delta - (81 + 4\omega^2) = \omega^2(\omega^2 - 18) > 0.$$

For $\omega > 0$, this means $\boxed{\omega > 3\sqrt{2} \text{ s}^{-1}}$. Equality holds at $3\sqrt{2}$; below that positive frequency the absolute motion is amplified.

Step 4: Interpret the limiting motions. As $\omega \rightarrow 0^+$, $X/B \rightarrow 1$ and $Z/B \rightarrow 0$: the mass follows the slowly moving support. As $\omega \rightarrow \infty$, $X/B \rightarrow 0$ but $Z/B \rightarrow 1$. In fact $A/B \rightarrow -1$ and $D/B \rightarrow 0$, so $z \approx -b$ while $x \approx 0$. An almost stationary mass can still have relative deformation comparable to the base motion.



Question 9

A 1 kg mass with $c = 2 \text{ N s/m}$ and $k = 5 \text{ N/m}$ obeys $x'' + 2x' + 5x = F(t)$ in SI units. An actuator must move it from $x(0) = 0.10 \text{ m}$, $x'(0) = 0$ to $x(1) = x'(1) = 0$, then turn off for $t > 1$. Before $t = 0$, it is held motionless at $x = 0.10 \text{ m}$ by a constant force.

Tasks

1. Find the unique cubic trajectory satisfying the four prescribed endpoint data on $[0, 1]$.
2. Compute its required force. Examine the force and acceleration when it joins the held state at zero and the rest state at one.
3. Instead require zero acceleration at both endpoints as well. Find the unique polynomial trajectory of degree at most five satisfying all six conditions.
4. Compute the new force and verify that it joins the pre-motion holding force and the zero post-motion force continuously. Show that the mass remains at rest after one second.

Question 9 – Solution

Strategy. Position and velocity matching ensure a continuous state, but continuous actuator force also requires compatible acceleration.

Step 1: Fit the cubic. Write $x = a_0 + a_1t + a_2t^2 + a_3t^3$. The initial conditions give $a_0 = 0.10$, $a_1 = 0$; the final conditions give $a_2 + a_3 = -0.10$, $2a_2 + 3a_3 = 0$. Hence

$$x_c = 0.10(1 - 3t^2 + 2t^3).$$

The nonsingular two-equation system makes this cubic unique.

Step 2: Check the actuator joins. Differentiating gives $x'_c = 0.60(t^2 - t)$ and $x''_c = 1.20t - 0.60$. Thus

$$F_c = x''_c + 2x'_c + 5x_c = t^3 - 0.30t^2 - 0.10 \quad \text{N.}$$

The holding force before zero is $5(0.10) = 0.50$ N. Yet $F_c(0) = -0.10$ N and $F_c(1) = 0.60$ N, so both joins require force jumps. Acceleration jumps from zero to -0.60 initially and from 0.60 to zero finally. The joined trajectory is C^1 and piecewise C^2 , not globally C^2 .

Step 3: Add acceleration matching. For a polynomial of degree at most five, the initial conditions fix $a_0 = 0.10$, $a_1 = a_2 = 0$. The endpoint conditions become

$$a_3 + a_4 + a_5 = -0.10, \quad 3a_3 + 4a_4 + 5a_5 = 0, \quad 6a_3 + 12a_4 + 20a_5 = 0.$$

Solving gives $a_3 = -1$, $a_4 = 1.5$, $a_5 = -0.6$, uniquely. Therefore

$$x_q = 0.10(1 - 10t^3 + 15t^4 - 6t^5).$$

Step 4: Verify force continuity and rest. Substitution into the model yields

$$F_q = 0.50 - 6t + 12t^2 - 5t^3 + 1.5t^4 - 3t^5 \quad \text{N.}$$

It has $F_q(0) = 0.50$ and $F_q(1) = 0$, matching the constant forces outside the motion interval. All six trajectory conditions hold, so the joined motion is C^2 . For $t > 1$, zero forcing and zero value and slope at one have the unique solution $x = 0$. The improvement is continuous force and acceleration; no claim of a force-minimizing trajectory is needed.

Question 10

A 1 kg mass with damping $c = 2 \text{ N s/m}$ and spring constant $k = 9 \text{ N/m}$ is driven by $F(t) = 6 \cos \omega t \text{ N}$, with $\omega > 0$. Consider only the steady periodic response. For a periodic quantity h , define its cycle average by

$$\langle h \rangle = \frac{1}{T} \int_0^T h(t) dt, \quad T = 2\pi/\omega.$$

Tasks

1. Derive the instantaneous energy balance and the relation between average input power and average damper dissipation.
2. Find the steady response coefficients and derive an explicit formula for average input power as a function of ω .
3. Find the unique positive frequency maximizing average input power and the maximum power.
4. At that frequency, find the instantaneous response, stored mechanical energy, and work supplied in one period. Explain how positive work per cycle can coexist with constant stored energy.

Question 10 – Solution

Strategy. The work done by the actuator splits into stored energy and damping loss. Periodicity removes the net storage change over a cycle.

Step 1: Derive the power balance. For $E = x'^2/2 + 9x^2/2$, multiplying $x'' + 2x' + 9x = F$ by x' gives

$$E' = Fx' - 2x'^2.$$

The steady energy is periodic, so integrating over one period gives $\langle Fx' \rangle = 2\langle x'^2 \rangle$. This is an average statement for general ω .

Step 2: Compute the mean input. Write $x = A \cos \omega t + B \sin \omega t$. Coefficient matching yields

$$\Delta = (9 - \omega^2)^2 + 4\omega^2, \quad A = \frac{6(9 - \omega^2)}{\Delta}, \quad B = \frac{12\omega}{\Delta}.$$

Since $x' = -\omega A \sin \omega t + \omega B \cos \omega t$, averaging the products gives

$$\bar{P} = \langle 6 \cos \omega t x' \rangle = 3\omega B = \frac{36\omega^2}{\Delta} \quad \text{W.}$$

Also $2\langle x'^2 \rangle = \omega^2(A^2 + B^2) = 36\omega^2/\Delta$, independently checking the energy balance.

Step 3: Maximize average power. For $\omega > 0$,

$$\bar{P} = \frac{36}{(9/\omega - \omega)^2 + 4} \leq 9 \text{ W.}$$

Equality holds uniquely when $\omega^2 = 9$. Thus $\omega = 3 \text{ s}^{-1}$, $\bar{P}_{\max} = 9 \text{ W}$.

Step 4: Account for the work at the maximum. At this frequency $A = 0$, $B = 1$, so $x = \sin 3t \text{ m}$ and $x' = 3 \cos 3t \text{ m/s}$. Therefore

$$E = \frac{9}{2}(\cos^2 3t + \sin^2 3t) = \boxed{4.5 \text{ J}}.$$

The instantaneous input and damping powers are both $18 \cos^2 3t \text{ W}$: here $E' = 0$ at every instant, not just on average. Over $T = 2\pi/3 \text{ s}$, the supplied work is $9T = \boxed{6\pi \text{ J}}$, exactly the energy dissipated. Constant stored energy does not mean zero energy throughput.

